VANISHING CYCLES

AND CARTAN EIGENVECTORS

Laura Brillon, Vadim Schechtman and Alexander Varchenko

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Introduction

0.1. Let $A(R)$ be the Cartan matrix of a finite root system R. The coordinates of its eigenvectors have an important meaning in the physics of integrables systems: namely, these numbers appear as the masses of particles (or, dually, as the energy of solitons) in affine Toda field theories, cf. [F], [D].

Historically, the first example of the system of type E_8 appeared in the pioneering papers [Z] on the 2D critical Ising model in a magnetic field.

The aim of this note is a study of these numbers, and their q-deformations, using the motivation coming from the singularity theory.

Let us suppose that R is simply laced, i.e. of type A, D, or E. These root systems are in one-to-one correspondence with (classes of) simple singularities $f : \mathbb{C}^N \longrightarrow \mathbb{C}$, cf. [AGV]. Under this correspondence, the root lattice $Q(R)$ is identified with the lattice of vanishing cycles, and the Cartan matrix $A(R)$ is the intersection matrix with respect to a *distinguished base*. The action of the Weyl group on $Q(R)$ is realized by Gauss - Manin monodromies - this is the Picard - Lefschetz theory (for some details see §1 below).

Remarkably, this geometric picture provides a finer structure: namely, the symmetric matrix $A = A(R)$ comes equiped with a decomposition

$$A = L + L^t \tag{0.1}$$

where L is a nondegenerate triangular "Seifert form", or "variation matrix". The matrix

$$C = -L^{-1}L^t \tag{0.2}$$

represents a Coxeter element of R; geometrically it is the operator of "classical monodromy".

We call the relation (0.1) - (0.2) between the Cartan matrix and the Coxeter element the *Cartan/Coxeter correspondence.*

Incidentally, in a particular case (corresponding to a bipartition of the Dynkin graph) this relation is equivalent to an observation by R.Steinberg, cf. [Stein], cf. 2.4 below. It enables one to relate the eigenvectors of A and C, cf. Theorem 2.5.

A decomposition (0.1) will be called *a polarization* of the Cartan matrix A. In 2.3 below we introduce an operation of *Sebastiani - Thom*, or *joint* product $A * B$ of Cartan matrices (or of polarized lattices) A and B. With respect to this operation the Coxeter eigenvectors factorize very simply.

In this note we will mainly concentrate on the example of E_8; this lattice decomposes into three "quarks":

$$E_8 = A_4 * A_2 * A_1 \tag{0.3}$$

This decomposition is the main message from the singularity theory, and we discuss it in detail in this note.

We use (0.3) and the Cartan/Coxeter correspondence to obtain some expressions for all Cartan eigenvectors of E_8; this is the first main result of this note, see 3.9 below.

(An elegant expression for all the Cartan eigenvectors of all finite root systems was obtained by P.Dorey, cf. [D] (a), Table 2 on p. 659.)

0.2. In the paper [Giv] A.Givental has proposed a q-twisted version of the Picard - Lefschetz theory, which gave rise to a q-deformation of A,

$$A(q) = L + qL^t.$$

In the last section, §4, we calculate the eigenvalues and eigenvectors of $A(q)$ in terms of the eigenvalues and eigenvectors of A. This is the second main result of this note.

It turns out that if λ is an egenvalue of A then

$$\lambda(q) = 1 + (\lambda - 2)\sqrt{q} + q \tag{0.4}$$

will be an eigenvalue of $A(q)$. The coordinates of the corresponding eigenvector $v(q)$ are obtained from the coordinates of $v = v(1)$ by multiplication by appropriate powers of q; this is related to the fact that the Dynkin graph of A is a tree, cf. 4.9.

For an example of E_8, see (4.5.1).

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§1. Recollections from singularity theory

Here we recall some classical constructions and statements, cf. [AGV].

This section plays a motivational role: formally the results of the following sections are purely algebraic, and do not depend on it, however, one could not arrive at them without this geometric motivation.

1.1. Let $f: (\mathbb{C}^N, 0) \longrightarrow (\mathbb{C}, 0)$ be the germ of a holomorphic function with an isolated critical point at 0, with $f(0) = 0$. We will be interested only in polynomial functions (from the list below, cf. 1.4), so $f \in \mathbb{C}[x_1, \ldots, x_N]$. The *Milnor ring* of f is defined by

$$\mathrm{Miln}(f, 0) = \mathbb{C}[[x_1, \ldots, x_N]]/(\partial_1 f, \ldots, \partial_N f)$$

where $\partial_i := \partial/\partial x_i$; it is a finite-dimensional commutative $\mathbb{C}$-algebra. (In fact, it is a Frobenius, or, equivalently, a Gorenstein algebra.) The number

$$\mu := \dim_{\mathbb{C}} \mathrm{Miln}(f, 0)$$

is called the multiplicity or Milnor number of $(f, 0)$.

A *Milnor fiber* is

$$V_z = f^{-1}(z) \cap \bar{B}_\rho$$

where

$$\bar{B}_\rho = \{(x_1, \ldots, x_N)| \ \sum |x_i|^2 \leq \rho\}$$

for $1 \gg \rho \gg |z| > 0$.

For z belonging to a small disc $D_\epsilon = \{z \in \mathbb{C}| \ |z| < \epsilon\}$, the space V_z is a complex manifold with boundary, homotopically equivalent to a bouquet $\vee S^{N-1}$ of μ spheres, [M].

The family of free abelian groups

$$Q(f; z) := \tilde{H}_{N-1}(V_z; \mathbb{Z}) \overset{\sim}{=} \mathbb{Z}^\mu, \ z \in \overset{\bullet}{D}_\epsilon := D_\epsilon \setminus \{0\}, \tag{1.1.1}$$

($\tilde{H}$ means that we take the reduced homology for $N = 1$), carries a flat Gauss - Manin conection.

Take $t \in \mathbb{R}_{>0} \cap \overset{\bullet}{D}_\epsilon$; the lattice $Q(f; t)$ does not depend, up to a canonical isomorphism, on the choice of t. Let us call this lattice $Q(f)$. The linear operator

$$T(f) : Q(f) \overset{\sim}{\longrightarrow} Q(f) \tag{1.1.2}$$

induced by the path $p(\theta) = e^{i\theta} t, \ 0 \leq \theta \leq 2\pi$, is called the classical monodromy of the germ $(f, 0)$.

In all the examples below $T(f)$ has finite order h. The eigenvalues of $T(f)$ have the form $e^{2\pi ik/h}$, $k \in \mathbb{Z}$. The set of suitable chosen k's for each eigenvalue are called the *spectrum* of our singularity.

1.2. Morse deformations. The $\mathbb{C}$-vector space $\mathrm{Miln}(f, 0)$ may be identified with the tangent space to the base B of the miniversal defomation of f. For

$$\lambda \in B^0 = B \setminus \Delta$$

where $\Delta \subset B$ is an analytic subset of codimension 1, the corresponding function $f_\lambda : \mathbb{C}^N \longrightarrow \mathbb{C}$ has μ nondegenerate Morse critical points with distinct critical values, and the algebra $\mathrm{Miln}(f_\lambda)$ is semisimple, isomorphic to $\mathbb{C}^\mu$.

Let $0 \in B$ denote the point corresponding to f itself, so that $f = f_0$, and pick $t \in \mathbb{R}_{>0} \cap \overset{\bullet}{D}_\epsilon$ as in 1.1.

Afterwards pick $\lambda \in B^0$ close to 0 in such a way that the critical values $z_1, \ldots z_\mu$ of f_λ have absolute values $<< t$.

As in 1.1, for each

$$z \in \tilde{D}_\epsilon := D_\epsilon \setminus \{z_1, \ldots z_\mu\}$$

the Milnor fiber V_z has the homotopy type of a bouquet $\vee S^{N-1}$ of μ spheres, and we will be interested in the middle homology

$$Q(f_\lambda; z) = \tilde{H}_{N-1}(V_z; \mathbb{Z}) \overset{\sim}{=} \mathbb{Z}^\mu$$

($\tilde{H}$ means that we take the reduced homology for $N = 1$).

The lattices $Q(f_\lambda; z)$ carry a natural bilinear product induced by the cup product in the homology which is symmetric (resp. skew-symmetric) when N is odd (resp. even).

The collection of these lattices, when $z \in \tilde{D}_\epsilon$ varies, carries a flat Gauss - Manin connection.

Consider an "octopus"

$$Oct(t) \subset \mathbb{C}$$

with the head at t: a collection of non-intersecting paths p_i ("tentacles") connecting t with z_i and not meeting the critical values z_j otherwise. It gives rise to a base

$$\{b_1, \ldots, b_\mu\} \subset Q(f_\lambda) := Q(f_\lambda; t)$$

(called "distinguished") where b_i is the cycle vanishing when being transferred from t to z_i along the tentacle p_i, cf. [Gab], [AGV].

The Picard - Lefschetz formula describe the action of the fundamental group $\pi_1(\tilde{D}_\epsilon; t)$ on $Q(f_\lambda)$ with respect to this basis. Namely, consider a loop γ_i which turns around z_i along the tentacle p_i, then the corresponding transformation of

$Q(f_\lambda)$ is the reflection (or transvection) $s_i := s_{b_i}$, cf. [Lef], Théorème fondamental, Ch. II, p. 23.

The loops γ_i generate the fundamental group $\pi_1(\tilde{D}_\epsilon)$. Let

$$\rho : \ \pi_1(\tilde{D}_\epsilon; t) \longrightarrow GL(Q(f_\lambda))$$

denote the monodromy representation. The image of ρ, denoted by $G(f_\lambda)$ and called the *monodromy group of* f_λ, lies inside the subgroup
$O(Q(f_\lambda)) \subset GL(Q(f_\lambda))$ of linear transformations respecting the above mentioned bilinear form on $Q(f_\lambda)$.

The subgroup $G(f_\lambda)$ is generated by $s_i, 1 \le i \le \mu$.

As in 1.1, we have the monodromy operator

$$T(f_\lambda) \in G(f_\lambda),$$

the image by ρ of the path $p \subset \tilde{D}_\epsilon$ starting at t and going around all points $z_1, \ldots, z_\mu$.

This operator $T(f_\lambda)$ is now a product of μ simple reflections

$$T(f_\lambda) = s_1 s_2 \ldots s_\mu,$$

- this is because the only critical value 0 of f became μ critical values $z_1, \ldots, z_\mu$ of f_λ.

One can identify the relative (reduced) homology $\tilde{H}_{N-1}(V_t, \partial V_t; \mathbb{Z})$ with the dual group $\tilde{H}_{N-1}(V_t; \mathbb{Z})^*$, and one defines a map

$$\mathrm{var} : \tilde{H}_{N-1}(V_t, \partial V_t; \mathbb{Z}) \longrightarrow \tilde{H}_{N-1}(V_t; \mathbb{Z}),$$

called a *variation operator*, which translates to a map

$$L : Q(f_\lambda)^* \xrightarrow{\sim} Q(f_\lambda)$$

("Seifert form") such that the matrix $A(f_\lambda)$ of the bilinear form in the distinguished basis is

$$A(f_\lambda) = L + (-1)^{N-1} L^t,$$

and

$$T(f_\lambda) = (-1)^{N-1} L L^{-t}.$$

SIGNS !!!

A choice of a path q in B connecting 0 with λ, enables one to identify $Q(f)$ with $Q(f_\lambda)$, and $T(f)$ will be identified with $T(f_\lambda)$.

The image $G(f)$ of the monodromy group $G(f_\lambda)$ in $GL(Q(f)) \cong GL(Q(f_\lambda))$ is called the monodromy group of f; it does not depend on a choice of a path q.

1.3. Sebastiani - Thom factorization. If $g \in \mathbb{C}[y_1, \ldots, y_M]$ is another function, the sum, or **join** of two singularities $f \oplus g: \mathbb{C}^{N+M} \longrightarrow \mathbb{C}$ is defined by

$$(f \oplus g)(x, y) = f(x) + g(y)$$

Obviously we can identify

$$\mathrm{Miln}(f \oplus g) \stackrel{\sim}{=} \mathrm{Miln}(f) \otimes \mathrm{Miln}(g)$$

Note that the function $g(y) = y^2$ is a unit for this operation.

It follows that the singularities $f(x_1, \ldots, x_N)$ and

$$f(x_1, \ldots, x_N) + x_{M+1}^2 + \ldots + x_{N+M}^2$$

are "almost the same". In order to have good signs (and for other purposes) it is convenient to add some squares to a given f to get $N \equiv 3 \mod (4)$.

The fundamental Sebastiani - Thom theorem, [ST], says that there exists a natural isomorphism of lattices

$$Q(f \oplus g) \stackrel{\sim}{=} Q(f) \otimes_{\mathbb{Z}} Q(g),$$

and under this identification the full monodromy decomposes as

$$T_{f \oplus g} = T_f \otimes T_g$$

Thus, if

$$\mathrm{Spec}(T_f) = \{e^{\mu_p \cdot 2\pi i/h_1}\},\ \mathrm{Spec}(T_f) = \{e^{\nu_q \cdot 2\pi i/h_2}\}$$

then

$$\mathrm{Spec}(T_{f \oplus g}) = \{e^{(\mu_p h_2 + \nu_q h_1) \cdot 2\pi i/h_1 h_2}\}$$

1.4. Simple singularities. Cf. [AGV] (a), 15.1. They are:

$$x^{n+1},\ n \geq 1, \tag{A_n}$$

$$x^2 y + y^{n-1},\ n \geq 4 \tag{D_n}$$

$$x^4 + y^3 \tag{E_6}$$

$$xy^3 + x^3 \tag{E_7}$$

$$x^5 + y^3 \tag{E_8}$$

Their names come from the following facts:

— their lattices of vanishing cycles may be identified with the corresponding root lattices;

— the monodromy group is identified with the corresponding Weyl group;

— the classical monodromy T_f is a Coxeter element, therefore its order h is equal to the Coxeter number, and

$$\mathrm{Spec}(T_f) = \{e^{2\pi i k_1/h}, \ldots, e^{2\pi i k_r/h}\}$$

where the integers

$$1 = k_1 < k_2 < \ldots < k_r = h - 1,$$

are the exponents of our root system.

We will discuss the case of E_8 in some details below.

§2. Cartan - Coxeter correspondence and join product

2.1. Lattices, polarization, Coxeter elements.

Let us call *a lattice* a pair (Q, A) where Q is a free abelian group, and

$$A : Q \times Q \longrightarrow \mathbb{Z}$$

a symmetric bilinear map ("Cartan matrix"). We shall identify A with a map

$$A : Q \longrightarrow Q^\vee := Hom(Q, \mathbb{Z}).$$

A polarized lattice is a triple (Q, A, L) where (Q, A) is a lattice, and

$$L : \ Q \overset{\sim}{\longrightarrow} Q^\vee$$

("variation", or "Seifert matrix") is an isomorphism such that

$$A = A(L) := L + L^\vee \qquad (2.1.1)$$

where

$$L^\vee : Q = Q^{\vee\vee} \overset{\sim}{\longrightarrow} Q^\vee$$

is the conjugate to L.

The *Coxeter automorphism* of a polarized lattice is defined by

$$C = C(L) = -L^{-1}L^\vee \in GL(Q). \qquad (2.1.2)$$

We shall say that the operators A and C are in a *Cartan - Coxeter correspondence.*

2.1.1. Example. Let (Q, A) be a lattice, and $\{e_1, \ldots, e_n\}$ an ordered $\mathbb{Z}$-base of Q. With respect to this base A is expressed as a symmetric matrix $A = (a_{ij}) = A(e_i, e_j) \in \mathfrak{gl}_n(\mathbb{Z})$. Let us suppose that all a_{ii} are even. We define the matrix of L to be the unique upper triangular matrix (ℓ_{ij}) such that $A = L + L^t$ (in patricular $\ell_{ii} = a_{ii}/2$; in our examples we will have $a_{ii} = 2$.) We will call L the *standard polarization* associated to an ordered base. $\square$

Polarized lattices form a groupoid:

an isomorphosm of polarized lattices $f : \ (Q_1, A_1, L_1) \overset{\sim}{\longrightarrow} (Q_2, A_2, L_2)$ is by definition an isomorphism of abelian groups $f : Q_1 \overset{\sim}{\longrightarrow} Q_2$ such that

$$L_1(x, y) = L_2(f(x), f(y))$$

(and whence $A_1(x, y) = A_2(f(x), f(y))$).

2.2. Lemma. (i) (orthogonality)

$$A(x,y)=A(Cx,Cy).$$

(ii) (gauge transformations) *For any* $P\in GL(Q)$

$$A(P^\vee LP)=P^\vee A(L)P,\ C(P^\vee LP)=P^{-1}C(L)P.$$

$\square$

2.3. Join product. Suppose we are given two polarized lattices (Q_i,A_i,L_i), $i=1,2$.

Set $Q=Q_1\otimes Q_2$, whence

$$L:=L_1\otimes L_2:Q\overset{\sim}{\longrightarrow}Q^\vee,$$

and define

$$A:=A_1*A_2:=L+L^\vee:Q\overset{\sim}{\longrightarrow}Q^\vee$$

The triple (Q,A,L) will be called the **join**, or **Sebastiani - Thom**, product of the polarized lattices Q_1 and Q_2, and denoted by Q_1*Q_2.

Obviously

$$C(L)=-C(L_1)\otimes C(L_2)\in GL(Q_1\otimes Q_2).$$

It follows that if if $\mathrm{Spec}(C(L_i))=\{e^{2\pi ik_i/h_i},\ k_i\in K_i\}$ then

$$\mathrm{Spec}(C(L))=\{-e^{2\pi i(k_1/h_1+k_2/h_2)},\ (k_1,k_2)\in K_1\times K_2\} \tag{2.3.1}$$

2.4. Black/white decomposition and a Steinberg's theorem. Cf. [Stein], [C], [BS]. Let $\alpha_1,\dots,\alpha_r$ be a base of simple roots of a finite reduced irreducible root system R. Suppose that R is simply laced.

Let

$$A=(a_{ij})=(\langle\alpha_i,\alpha_j^\vee\rangle)$$

be the Cartan matrix; suppose that R is simply laced, so that A is symmetric.

Choose a black/white coloring of the set of vertices of the corresponding Dynkin graph in such a way that any two neighbouring vertices have different colours.

Let us choose an ordering of simple roots in such a way that the first r roots are black, and the last $r-p$ roots are white. In this base A has a block form

$$A=\begin{pmatrix}2I_p & X\\ X^t & 2I_{r-p}\end{pmatrix}$$

Consider a Coxeter element

$$C=s_1s_2\dots s_r=C_BC_W, \tag{2.4.1}$$

where

$$C_B = \prod_{i=1}^{p} s_i,\ C_W = \prod_{i=p+1}^{r} s_i.$$

Here s_i denotes the simple reflection corresponding to the root α_i.

The matrices of C_B, C_W with respect to the base $\{\alpha_i\}$ are

$$C_B = \begin{pmatrix} -I & -X \\ 0 & I \end{pmatrix}, C_W = \begin{pmatrix} I & 0 \\ -X^t & -I \end{pmatrix},$$

so that

$$C_B + C_W = 2I - A. \tag{2.4.2}$$

This is an observation due to R.Steinberg, cf. [Stein], p. 591.

2.4.1. Remark: a matrix quadratic polynomial. We can express the basic relations (2.4.1), (2.4.2) by saying that C_B, C_W are the "roots" of a matrix quadratic polynomial

$$t^2 + (A-2)t + C = (t - C_B)(t - C_W),$$

$\square$.

We can also rewrite this as follows. Set

$$L = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}.$$

Then $A = L + L^t$, and one checks easily that

$$C = -L^{-1}L^t, \tag{2.4.3}$$

so we are in the situation 2.1. This explains the name "Cartan - Coxeter coresspondence".

2.5. Eigenvectors' correspondence. The following corollary was obtained by the physicists, cf. [F], or [BS], Corollary 3.7.

Theorem. *In the notations of* 2.3, *a vector*

$$x = \sum x_i\alpha_i$$

is an eigenvector of A with the eigenvalue $2(1-\cos\theta)$ *iff the vector*

$$x_c := \sum e^{\pm i\theta/2}x_i$$

where the sign in $e^{\pm i\theta/2}$ is plus if i is a white vertex, and minus otherwise, is an eigenvector of c with eigenvalue $e^{2i\theta}$.

$\square$

§3. Sebastiani - Thom factorization of E_8

3.1. Root systems A_n. We consider the Dynkin graph of A_n with the obvious numbering of the vertices.

The Coxeter number $h = n + 1$, the set of exponents:

$$\mathrm{Exp}(A_n) = \{1, 2, \ldots, n\}$$

The eigenvalues of any Coxeter element are $e^{i\theta_k}$, and the eigenvalues of the Cartan matrix $A(A_n)$ are $2 - 2\cos\theta_k$, $\theta_k = 2\pi k/h$, $k \in \mathrm{Exp}(A_n)$.

An eigenvector of $A(A_n)$ with the eigenvalue $2 - 2\cos\theta$ has the form

$$x(\theta) = (\sum_{k=0}^{n-1} e^{i(n-1-2k)\theta}, \sum_{k=0}^{n-2} e^{i(n-2-2k)\theta}, \ldots, 1) \tag{3.1.1}$$

Denote by $C(A_n)$ the Coxeter element

$$C(A_n) = s_1 s_2 \ldots s_n$$

Its eigenvector with the eigenvalue $e^{2i\theta}$ is:

$$X_{C(A_n)} = (\sum_{k=0}^{n-j} e^{2ik\theta})_{1 \le j \le n}$$

For example, for $n = 4$:

$$C_{A_4} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

and

$$X_{C(A_4)} = \begin{pmatrix} 1 + e^{2i\theta} + e^{4i\theta} + e^{6i\theta} \\ 1 + e^{2i\theta} + e^{4i\theta} \\ 1 + e^{2i\theta} \\ 1 \end{pmatrix}$$

is an eigenvector with eigenvalue $e^{2i\theta}$.

Similarly, for $n = 2$:

$$C_{A_2} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \; X_{C(A_2)} = \begin{pmatrix} 1 + e^{2i\gamma} \\ 1 \end{pmatrix}$$

3.2. E_8 **versus** $A_4 * A_2 * A_1$**: elementary analysis.**

The ranks:

$$r(E_8) = 8 = r(A_4)r(A_2)r(A_1);$$

the Coxeter numbers:

$$h(E_8) = h(A_4)h(A_2)h(A_1) = 5 \cdot 3 \cdot 2 = 30.$$

It follows that

$$|R(E_8)| = 240 = |R(A_4)||R(A_2)||R(A_1)|.$$

The exponents of E_8 are:

$$1, 7, 13, 19, 11, 17, 23, 29.$$

All these numbers, except 1, are primes, and these are all primes ≤ 30, not dividing 30.

They may be determined from the formula

$$\frac{i}{5} + \frac{j}{3} + \frac{1}{2} = \frac{30 + k(i,j)}{30},\ 1 \leq i \leq 4,\ 1 \leq j \leq 2,$$

so

$$k(i,1) = 1 + 6(i-1) = 1, 7, 13, 19;$$
$$k(i,2) = 1 + 10 + 6(i-1) = 11, 17, 23, 29.$$

This shows that the exponents of E_8 are the same as the exponents of $A_4 * A_2 * A_1$.

The following theorem is more delicate.

3.3. Theorem (Gabrielov). *There exists a polarization of the root lattice* $Q(E_8)$ *and an isomorphism of polarized lattices*

$$\Gamma : Q(A_4) * Q(A_2) * Q(A_1) \xrightarrow{\sim} Q(E_8). \qquad (3.3.1)$$

In the left hand side $Q(A_n)$ means the root lattice of A_n with the standard Cartan matrix and the standard polarization

$$A(A_n) = L(A_n) + L(A_n)^t$$

where the Seifert matrix $L(A_n)$ is upper triangular.

In the process of the proof, given in 3.4 - 3.6 below, the isomorphism Γ will be written down explicitly.

3.4. Beginning of the proof.

For $n = 4, 2, 1$, we consider the bases of simple roots $e_1, \ldots, e_n$ in $Q(A_n)$, with scalar products given by the Cartan matrices $A(A_n)$.

The tensor product of three lattices

$$Q_* = Q(A_4) \otimes Q(A_2) \otimes Q(A_1)$$

will be equipped it with the "factorizable" basis in the lexicographic order:

$$(f_1, \ldots, f_8) := (e_1 \otimes e_1 \otimes e_1, e_1 \otimes e_2 \otimes e_1, e_2 \otimes e_1 \otimes e_1, e_2 \otimes e_2 \otimes e_1,$$
$$e_3 \otimes e_1 \otimes e_1, e_3 \otimes e_2 \otimes e_1, e_4 \otimes e_1 \otimes e_1, e_4 \otimes e_2 \otimes e_1).$$

Introduce a scalar product (x, y) on Q_* given, in the basis $\{f_i\}$, by the matrix

$$A_* = A_4 * A_2 * A_1.$$

3.5. Gabrielov - Picard - Lefschetz transformations α_m, β_m.

Let $(Q, (,))$ be a lattice of rank r. We introduce the following two sets of transformations $\{\alpha_m\}, \{\beta_m\}$ on the set $Bases - cycl(Q)$ of cyclically ordered bases of Q.

If $x = (x_i)_{i \in \mathbb{Z}/r\mathbb{Z}}$ is a base, and $m \in \mathbb{Z}/r\mathbb{Z}$, we set

$$(\alpha_m(x))_i = \begin{cases} x_{m+1} + (x_{m+1}, x_m)x_m & \text{if } i = m \\ x_m & \text{if } i = m+1 \\ x_i & \text{otherwise} \end{cases}$$

and

$$(\beta_m(x))_i = \begin{cases} x_m & \text{if } i = m-1 \\ x_{m-1} + (x_{m-1}, x_m)x_m & \text{if } i = m \\ x_i & \text{otherwise} \end{cases}$$

We define also a transformation γ_m by

$$(\gamma_m(x))_i = \begin{cases} -x_m & \text{if } i = m \\ x_i & \text{otherwise} \end{cases}$$

3.6. Passage from $A_4 * A_2 * A_1$ to E_8. Consider the base $f = \{f_1, \ldots f_8\}$ of the lattice $Q_* := Q(A_4) \otimes Q(A_2) \otimes Q(A_1)$ described in 3.4, and apply to it the following transformation

$$G' = \gamma_2\gamma_1\beta_4\beta_3\alpha_3\alpha_4\beta_4\alpha_5\alpha_6\alpha_7\alpha_1\alpha_2\alpha_3\alpha_4\beta_6\beta_3\alpha_1, \tag{3.6.1}$$

cf. [Gab], Example 3.

Then the base $G'(f)$ has the intersection matrix given by the Dynkin graph of E_8, with the ordering indicated on Figure 1 below.

$$\begin{array}{ccccccccccccc} 1 & - & 2 & - & 3 & - & 5 & - & 6 & - & 7 & - & 8 \\ & & & & | & & & & & & & & \\ & & & & 4 & & & & & & & & \end{array}$$

Fig. 1. Gabrielov's ordering of E_8.

This concludes the proof of Theorem 3.3. $\square$

3.7. The induced map of root sets. By definition, the isomorphism of lattices Γ, (3.3.1), induces a bijection between the bases

$$g: \ \{f_1, \ldots, f_8\} \xrightarrow{\sim} \{\alpha_1, \ldots, \alpha_8\} \subset R(E_8).$$

where in the right hand side we have the base of simple roots, and a map

$$G: \ R(A_4) \times R(A_2) \times R(A_1) \longrightarrow R(E_8), \ G(x, y, z) = \Gamma(x \otimes y \otimes z)$$

of sets of the same cardinality 240 which is not a bijection however: its image consists of 60 elements.

Note that the set of vectors $\alpha \in Q(E_8)$ with $(\alpha, \alpha) = 2$ coincides with the root system $R(E_8)$, cf. [Serre], Première Partie, Ch. 5, 1.4.3.

3.8. Passage to Bourbaki ordering.

The isomorphism G' (3.6.1) is given by a matrix $G' \in GL_8(\mathbb{Z})$ such that

$$A_G(E_8) = G'^t A_* G'$$

where we denoted

$$A_* = A(A_4) * A(A_2) * A(A_1),$$

the factorized Cartan matrix, and A_G denotes the Cartan matrix of E_8 with respect to the numbering of roots indicated on Fig. 1.

Now let us pass to the numbering of vertices of the Dynkin graph of type E_8 indicated in [B] (the difference with Gabrielov's numeration is in three vertices $2, 3$, and 4).

$$\begin{array}{ccccccccccccc} 1 & - & 3 & - & 4 & - & 5 & - & 6 & - & 7 & - & 8 \\ & & & & | & & & & & & & & \\ & & & & 2 & & & & & & & & \end{array}$$

Fig. 2. Bourbaki ordering of E_8.

The Gabrielov's Coxeter element (the full monodromy) in the Bourbaki numbering looks as follows:

$$C_G(E_8) = s_1 \circ s_3 \circ s_4 \circ s_2 \circ s_5 \circ s_6 \circ s_7 \circ s_8$$

3.9. Lemma. *Let $A(E_8)$ be the standard Cartan matrix of E_8 from* [B]:

$$A(E_8) = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Then

$$A(E_8) = G^t A_* G$$

and

$$C_G(E_8) = G^{-1} C_* G$$

where

$$C_* = C(Q(A_4) * Q(A_2) * Q(A_1)) = C(A_4) \otimes C(A_2) \otimes C(A_1),$$

is the factorized Coxeter element, and

$$G = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (3.8.1)$$

Here

$$G = G'P$$

where P is the permutation matrix of passage from the Gabrielov's ordering on Fig. 1 to the Bourbaki ordering on Fig. 2.

3.9. Cartan eigenvectors of E_8. To obtain the Cartan eigenvectors of E_8, one should pass from $C_G(E_8)$ to a "black/white" coxeter element (as in 2.4)

$$C_{BW}(E_8) = s_1 \circ s_4 \circ s_6 \circ s_8 \circ s_2 \circ s_3 \circ s_5 \circ s_7$$

Any two Coxeter elements are conjugate in the Weyl group $W(E_8)$.

The elements $C_G(E_8)$ and $C_{BW}(E_8)$ are conjugate by the following element of $W(E_8)$:

$$C_G(E_8) = w^{-1} C_{BW}(E_8) w$$

where

$$w = s_7 \circ s_5 \circ s_3 \circ s_2 \circ s_6 \circ s_4 \circ s_5 \circ s_1 \circ s_3 \circ s_2 \circ s_4 \circ s_1 \circ s_3 \circ s_2 \circ s_1 \circ s_2$$

This expression for w can be obtained using an algorithm described in [C].

Thus, if x_* is an eigenvector of $C_*(E_8)$ then

$$x_{BW} = wG^{-1}x_*$$

is an eigenvector of $C_{BW}(E_8)$. But we know the eigenvectors of $C_*(E_8)$, they are all factorizable.

This provides the eigenvectors of $C_{BW}(E_8)$, which in turn have very simple relation to the eigenvectors of $A(E_8)$, due to Theorem 2.5.

Conclusion: an expression for the eigenvectors of $A(E_8)$.

Let $\theta = \frac{a\pi}{5},\ 1 \le a \le 4,\ \gamma = \frac{b\pi}{3},\ 1 \le b \le 2,\ \delta = \frac{\pi}{2}$,

$$\alpha = \theta + \gamma + \delta = 1 + \frac{k\pi}{30},$$

$$k \in \{1, 7, 11, 13, 17, 19, 23, 29\}.$$

The 8 eigenvalues of $A(E_8)$ have the form

$$\lambda(\alpha) = \lambda(\theta, \gamma) = 2 - 2\cos\alpha$$

An eigenvector of $A(E_8)$ with the eigenvalue $\lambda(\theta, \gamma)$ is

$$X_{E_8}(\theta, \gamma) = \begin{pmatrix} \cos(\gamma + \theta - \delta) + \cos(\gamma - 3\theta - \delta) + \cos(\gamma - \theta - \delta) \\ \cos(2\gamma + 2\theta) \\ \cos(2\gamma) + \cos(2\gamma + 2\theta) + \cos(2\gamma - 2\theta) + \cos(4\theta) + \cos(2\theta) \\ \cos(\gamma + 3\theta - \delta) + \cos(\gamma + \theta - \delta) + \cos(-\gamma + 3\theta - \delta) \\ 2\cos(2\gamma) + 2\cos(2\gamma + 2\theta) + \cos(2\gamma - 2\theta) + \cos(2\gamma + 4\theta) + \cos(4\theta) + 2\cos(2\theta) + 1 \\ \cos(\gamma + 3\theta - \delta) + \cos(\gamma + \theta - \delta) \\ \cos(2\gamma) + \cos(2\theta - 2\delta) \\ \cos(\gamma - \theta - \delta) \end{pmatrix}$$

One can simplify it as follows:

$$X_{E_8}(\theta, \gamma) = \begin{pmatrix} 2\cos(4\theta)\cos(\gamma - \theta - \delta) \\ -\cos(2\gamma + 2\theta) \\ 2\cos^2(\theta) \\ -2\cos(\gamma)\cos(3\theta - \delta) - \cos(\gamma + \theta - \delta) \\ -2\cos(2\gamma + 3\theta)\cos(\theta) + \cos(2\gamma) \\ -2\cos\theta\cos(\gamma + 2\theta - \delta) \\ -2\cos(\gamma + \theta - \delta)\cos(\gamma - \theta + \delta) \\ -\cos(\gamma - \theta - \delta) \end{pmatrix} \tag{3.9.1}$$

3.10. Perron - Frobenius and all that. The Perron - Frobenius eigenvector corresponds to the eigenvalue

$$2 - 2\cos\frac{\pi}{30},$$

and may be chosen as

$$v_{PF} = \begin{pmatrix} 2\cos\frac{\pi}{5}\cos\frac{11\pi}{30} \\ \cos\frac{\pi}{15} \\ 2\cos^2\frac{\pi}{5} \\ 2\cos\frac{2\pi}{30}\cos\frac{\pi}{30} \\ 2\cos\frac{4\pi}{15}\cos\frac{\pi}{5}+\frac{1}{2} \\ 2\cos\frac{\pi}{5}\cos\frac{7\pi}{30} \\ 2\cos\frac{\pi}{30}\cos\frac{11\pi}{30} \\ \cos\frac{11\pi}{30} \end{pmatrix}$$

If we order its coordinates in the increasing order, we get

$$v_{PF<} = \begin{pmatrix} \cos\frac{11\pi}{30} \\ 2\cos\frac{\pi}{5}\cos\frac{11\pi}{30} \\ 2\cos\frac{\pi}{30}\cos\frac{11\pi}{30} \\ \cos\frac{\pi}{15} \\ 2\cos\frac{\pi}{5}\cos\frac{7\pi}{30} \\ 2\cos^2\frac{\pi}{5} \\ 2\cos\frac{4\pi}{15}\cos\frac{\pi}{5}+\frac{1}{2} \\ 2\cos\frac{2\pi}{30}\cos\frac{\pi}{30} \end{pmatrix}$$

Zamolodchikov gives in [Z] the following expression for a PF vector:

$$v_{Zam}(m) = \begin{pmatrix} m \\ 2m\cos\frac{\pi}{5} \\ 2m\cos\frac{\pi}{30} \\ 4m\cos\frac{\pi}{5}\cos\frac{7\pi}{30} \\ 4m\cos\frac{\pi}{5}\cos\frac{2\pi}{15} \\ 4m\cos\frac{\pi}{5}\cos\frac{\pi}{30} \\ 8m\cos^2\frac{\pi}{5}\cos\frac{7\pi}{30} \\ 8m\cos^2\frac{\pi}{5}\cos\frac{2\pi}{15} \end{pmatrix}$$

Setting $m = \cos\frac{11\pi}{30}$, we have indeed :

$$v_{PF<} = v_{Zam}(\cos\frac{11\pi}{30})$$

3.11. Another form of the eigenvectors' matrix. As was noticed in [BS], the coordiantes of all eigenvectors of $A(E_8)$ may be obtained from the coordinates of the PF vector by some permutations and sign changes.

Namely, if $(z_1, \ldots, z_8)$ is a PF vector then the other eigenvectors are the columns of the matrix

$$Z = \begin{pmatrix} z_1 & z_7 & z_4 & z_2 & z_2 & z_4 & z_7 & z_1 \\ z_2 & z_1 & -z_7 & -z_4 & z_4 & z_7 & -z_1 & -z_2 \\ z_3 & z_6 & z_5 & z_8 & -z_8 & -z_5 & -z_6 & -z_3 \\ z_4 & z_2 & -z_1 & -z_7 & -z_7 & -z_1 & z_2 & z_4 \\ z_5 & -z_8 & -z_3 & z_6 & -z_6 & z_3 & z_8 & -z_5 \\ z_6 & -z_5 & -z_8 & z_3 & z_3 & -z_8 & -z_5 & z_6 \\ z_7 & -z_4 & z_2 & -z_1 & z_1 & -z_2 & z_4 & -z_7 \\ z_8 & -z_3 & z_6 & -z_5 & -z_5 & z_6 & -z_3 & z_8 \end{pmatrix}$$

However, these eigenvectors differ from the ones given by the formula (3.9.1): the latter ones are proportional to the former ones.

§4. Givental's q-deformations

4.1. Definition. *Let (Q, A, L) be a polarized lattice. We define a q-deformed Cartan matrix by*

$$A(q) = L + qL^t.$$

This definition is inspired by the q-deformed Picard - Lefschetz theory developed by Givental, [Giv].

4.2. "Black/white" q-deformation. Let

$$A = \begin{pmatrix} 2 & B \\ C & 2 \end{pmatrix}$$

be a block matrix, and $v = \begin{pmatrix} x \\ y \end{pmatrix}$ its eigenvector with the eigenvalue λ. This means that

$$By = (\lambda - 2)x, \ Cx = (\lambda - 2)y. \tag{4.2.1}$$

Consider the matrix

$$A(q) = \begin{pmatrix} 1+q & B \\ qC & 1+q \end{pmatrix},$$

and let us look for its eigenvector in the form $v = \begin{pmatrix} x \\ by \end{pmatrix}$ with eigenvalue μ. This would mean that

$$By = \frac{\mu - q - 1}{b} x, \ Cx = \frac{b(\mu - q - 1)}{q} x. \tag{4.2.2}$$

Comparing (4.2.1) and (4.2.2) we conclude that

$$\frac{\mu'}{b} = \frac{b\mu'}{q}, \ \mu' = \mu - q - 1,$$

whence $b = \sqrt{q}$, and that

$$\lambda - 2 = \frac{\mu'}{\sqrt{q}}.$$

Conclusion: *the vector*

$$v(q) = \begin{pmatrix} x \\ q^{1/2} y \end{pmatrix}$$

is an eigenvector of $A(q)$ with eigenvalue

$$\lambda(q) = 1 + (\lambda - 2) q^{1/2} + q. \tag{4.2.3}$$

For another approach to the same values see 4.10 below.

4.2.1. A generalization. More generally, let

$$M = \begin{pmatrix} 2I & A & 0 \\ C & 2I & B \\ 0 & D & 2I \end{pmatrix}$$

be a block Jacobi matrix which admits an eigenvector $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ for the eigenvalue $\lambda = 2 - 2\cos\theta$.

Then its q-deformation

$$M(q) = \begin{pmatrix} (1+q)I & A & 0 \\ qC & (1+q)I & D \\ 0 & qD & (1+q)I \end{pmatrix}$$

admits an eigenvector

$$v(q) = \begin{pmatrix} x \\ q^{\frac{1}{2}} y \\ qz \end{pmatrix}$$

with the eigenvalue

$$\lambda(q) = 1 + q - 2q^{\frac{1}{2}} \cos\theta.$$

This can be generalized to the block Jacobi matrices of any size.

4.3. Remark (M.Finkelberg). The expression (4.2.3) resembles the number of points of an elliptic curve X over a finite field $\mathbb{F}_q$. To better appreciate this resemblance, note that in all our examples λ will have the form

$$\lambda = 2 - 2\cos\theta,$$

so if we set

$$\alpha = \sqrt{q}e^{i\theta}$$

("a Frobenius root") then $|\alpha| = \sqrt{q}$, and

$$\lambda(q) = 1 - \alpha - \bar{\alpha} + q,$$

cf. [IR], Chapter 11, §1, [Kn], Chapter 10, Theorem 10.5.

So, the Coxeter eigenvalues $e^{2i\theta}$ may be seen as analogs of "Frobenius roots of an elliptic curve over $\mathbb{F}_1$".

4.4. Standard deformation for A_n. Let us consider the following q-deformation of $A = A(A_n)$:

$$A(q) = \begin{pmatrix} 1+q & -1 & 0 & \dots & 0 \\ -q & 1+q & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -q & 1+q \end{pmatrix}$$

Then

$$\mathrm{Spec}(A(q)) = \{\lambda(q) := 1 + (\lambda - 2)\sqrt{q} + q |\ \lambda \in \mathrm{Spec}(A(1))\}.$$

If $x = (x_1, \dots, x_n)$ is an eigenvector of $A = A(1)$ with eigenvalue λ then

$$x(q) = (x_1, q^{1/2}x_2, \dots, q^{(n-1)/2}x_n)$$

is an eigenvector of $A(q)$ with eigenvalue $\lambda(q)$.

4.5. Standard deformation for E_8. A q-deformation:

$$A_{E_8}(q) = \begin{pmatrix} 1+q & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+q & 0 & -1 & 0 & 0 & 0 & 0 \\ -q & 0 & 1+q & -1 & 0 & 0 & 0 & 0 \\ 0 & -q & -q & 1+q & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q & 1+q & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q & 1+q & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -q & 1+q & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -q & 1+q \end{pmatrix}$$

Its eigenvalues are

$$\lambda(q) = 1 + q + (\lambda - 2)\sqrt{q} = 1 + q - 2\sqrt{q}\cos\theta$$

where $\lambda = 2 - 2\cos\theta$ is an eigenvalue of $A(E_8)$.

If $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ is an eigenvector of $A(E_8)$ for the eigenvalue λ, then

$$X = (x_1, \sqrt{q}x_2, \sqrt{q}x_3, qx_4, q\sqrt{q}x_5, q^2x_6, q^2\sqrt{q}x_7, q^3x_8) \tag{4.5.1}$$

is an eigenvector of $A_{E_8}(q)$ for the eigenvalue $\lambda(q)$.

We see the patterns here:

(i) The eigenvalues are always $\lambda(q)$, as in the black/white case, so the q-deformed Cartan matrices for different orderings are conjugate.

(ii) The coordinates of the q-deformed eigenvectors are equal to $q^? \times$ the coordinates of the original eigenvector.

For an explanation, see 4.7, 4.8 below.

4.6. Examples of conjugation.

(a) Let

$$A(A_4; q) = \begin{pmatrix} 1+q & -1 & 0 & 0 \\ -q & 1+q & -1 & 0 \\ 0 & -q & 1+q & -1 \\ 0 & 0 & -q & 1+q \end{pmatrix}$$

be the q-deformation of the standard $A(A_4)$, and

$$A_{BW}(A_4; q) = \begin{pmatrix} 1+q & 0 & -1 & 0 \\ 0 & 1+q & -1 & -1 \\ -q & -q & 1+q & 0 \\ 0 & -q & 0 & 1+q \end{pmatrix}$$

that of the "black/white" one. Then

$$A(A_4; q) = P(q) \cdot A_{BW}(A_4; q) \cdot P^{-1}(q),$$

with

$$P(q) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \text{ and } P^{-1}(q) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}$$

The matrix $P(q)$ is a deformation of a permutation matrix.

(b) A similar example for E_8:

$$A(E_8; q) = P(q) \cdot A_{BW}(E_8; q) \cdot P^{-1}(q),$$

with

$$P(q) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & q^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2 \\ 0 & 0 & 0 & q^3 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In fact, always

$$P(q) = D(q)P(1) = P(1)D'(q)$$

where $P(1)$ is a permutation matrix, and $D(q), D'(q)$ are diagonal matrices, with some natural powers q^n on the diagonal.

Below follows an explanation of what is going on.

4.7. Conjugacy of different q-deformations.

Let $A = (a_{ij}) \in \mathfrak{gl}_r(\mathbb{C})$ be a symmetric matrix, and

$$A = L + L^t,$$

the standard polarization, with L upper triangular. Thus, $L = (\ell_{ij})$, with $\ell_{ii} = a_{ii}/2$, and

$$\ell_{ij} = \begin{cases} a_{ij} & \text{if } i < j \\ 0 & \text{if } i > j \end{cases}$$

For a bijection

$$\sigma : \{1, \ldots, r\} \xrightarrow{\sim} \{1, \ldots, r\},$$

define a matrix $L_\sigma = (\ell^\sigma_{ij})$ with $\ell^\sigma_{ii} = a_{ii}/2$, and

$$\ell^\sigma_{ij} = \begin{cases} a_{ij} & \text{if } \sigma(i) < \sigma(j) \\ 0 & \text{if } \sigma(i) > \sigma(j) \end{cases}$$

Obviously

$$A = L_\sigma + L_\sigma^t,$$

i.e. we have got a different, "σ - twisted", polarization of A.

Consider the q-deformations of A corresponding to these two polarizations:

$$A(q) = L + qL^t$$

and

$$A(\sigma; q) = L_\sigma + qL_\sigma^t.$$

Let us ask a

4.7.1. Question. *Find a diagonal matrix of the form*

$$D = D(\sigma; q) = \mathrm{Diag}(q^{n_1}, \ldots, q^{n_r}) \tag{4.7.1}$$

such that

$$A(\sigma; q) = D^{-1}A(q)D. \tag{4.7.2}$$

This equation means that

$$A(\sigma; q)_{ij} = q^{-n_i+n_j}A(q)_{ij}$$

for all i, j. This is an overdertermined system of $r(r-1)/2$ equations on r variables n_i, not solvable in general.

However, we have something positive to say.

Let $A = (a_{ij})$ be any matrix having the property

$$a_{ij} \neq 0 \text{ implies } a_{ji} \neq 0 \tag{4.7.3}$$

Let us assign to A its "Dynkin graph" $\Gamma(A)$ having $[r] := \{1, \ldots, r\}$ as the set of vertices, vertices i and j being connected iff $a_{ij} \neq 0$.

Now let us return to our symmetric matrix A.

4.7.1. Tree lemma. *If* $\Gamma(A)$ *is a tree then for each* $\sigma \in S_r = \mathrm{Aut}([r])$ *there exists* $D = D(\sigma; q)$ *such that* (4.7.2) *holds true.*

A proof will be given in 4.9 below (the reader may also wish to take it as an excercise).

For the moment we assume this assertion.

4.8. Now let us turn a permutation matrix P on.

Define a permutation matrix $P = P(\sigma) = (p_{ij}) \in GL_r(\mathbb{Z})$, by

$$p_{ij} = \delta_{i,\sigma(j)}.$$

Then its inverse

$$P^{-1} = (p'_{ij}),\ p'_{ij} = \delta_{\sigma(i),j} = \delta_{i,\sigma^{-1}(j)} = P^t \tag{4.8.1}$$

Define

$$A' = P^{-1}AP = (a'_{ij}).$$

Then

$$a'_{ij} = a_{\sigma(i),\sigma(j)}. \tag{4.8.2}$$

It follows that A' is symmetric as well.

Decompose

$$A' = L' + L'^t, \tag{4.8.3}$$

with L' upper triangular.

Define a q-deformation

$$A'(q) = L' + qL'^t,$$

Let us look for a matrix $P(q)$ of the form

$$P(q) = PD(q),$$

$D(q) = D(\sigma; q)$ being as in (4.7.1), such that

$$A'(q) = P(q)^{-1}A(q)P(q). \tag{4.8.4}$$

(Note that

$$D(q)P = PD'(q)$$

where

$$D'(q) = \mathrm{diag}(q^{n'_1}, \dots, q^{n'_r}),\ n'_i = n_{\sigma^{-1}(i)},$$

cf. (4.8.2).)

The matrix A' has two polarizations: the first, the standard one, (4.8.3), and the second, the σ-twisted one:

$$A' = L_\sigma + L_\sigma^t,\ L_\sigma := P^{-1}LP.$$

So, we are in the situation 4.7, and we are looking for a diagonal matrix D.

If we are lucky, and D exists, for example, according to the Tree lemma 4.7.1, if $\Gamma(A)$ is a tree, then the problem (4.8.4) is solved.

This is the case for the finite Cartan matrices of types A, D or E.

4.9. Homology of the Dynkin graph, the diagonal conjugacy, and a proof of Tree Lemma. Under the assumptions of Lemma 4.7.1, pick any orientation on the Dynkin graph $\Gamma = \Gamma(A)$; let $\vec{\Gamma}$ denote the oriented graph thus obtained.

Consider a cochain complex

$$O \longrightarrow C^0(\vec{\Gamma}; \mathbb{C}^*) \xrightarrow{d} C^1(\vec{\Gamma}; \mathbb{C}^*) \longrightarrow 0$$

We have $\mathrm{Coker}(d) = H^1(\vec{\Gamma}; \mathbb{C}^*) = 0$ since Γ is a tree. This means the following:

(i) *For any collection of numbers* $\{b_{ij} \in \mathbb{C}^*, (i,j) \in [r]^2, i \neq j\}$ *such that*

$$b_{ji} = b_{ij}^{-1}, \tag{4.9.1}$$

there exists a collection $\{c_i \in \mathbb{C}^*, i \in [r]\}$ *such that*

$$b_{ij} = c_j / c_i. \tag{4.9.2}$$

Moreover, we can choose the numbers c_i *in such a way that they are products of some* b_{ij}.

To prove the last assertion, let us choose a trunk of our tree Γ, and partially order its vertices by taking the minimal vertex i_0 to be the beginning of the trunk, and then going "upstairs". This defines an orientation on Γ. Now, given a 1-cochain (b_{ij}), we set $c_{i_0} = 1$ and then define the other c_i one by one, by going upstairs, and using as a definition

$$c_j = b_{ij} c_i, \ \ i < j.$$

Obviously, the numbers c_i defined in such a way, are products of b_{ab}.

□

If we don't suppose Γ to be a tree then to get a solution of (4.9.2), in addition to (4.9.1) one should impose on $\{b_{ij}\}$ one more condition:

for any non-contractible loop $i_1 \longrightarrow i_2 \longrightarrow \ldots i_k \longrightarrow i_1$ in Γ,

$$b_{i_1 i_2} b_{i_2 i_3} \ldots b_{i_k i_1} = 1. \tag{4.9.1a}$$

(One could restrict to the loops representing some set of generators of $H_1(\vec{\Gamma}; \mathbb{Z})$.)

Let us return to the conditions of Lemma 4.7.1.

(ii) *Let* $A' = (a'_{ij})$ *be another matrix with* $\Gamma(A') = \Gamma(A)$ *such that* $a'_{ii} = a_{ii}$ *for all* i, *and*

$$a'_{ij}/a_{ij} = a_{ji}/a'_{ji}$$

for all $i \neq j$. *Then there exists a diagonal matrix*

$$D = \mathrm{Diag}(c_1, \ldots, c_r)$$

such that $A' = D^{-1} A D$.

This is a corollary of (i), namely, set $b_{ij} = a'_{ij}/a_{ij}$.

Now we can prove the Tree Lemma. In fact, two matrices $A(q)$ and $A_\sigma(q)$ satisfy the conditions of (ii), with $b_{ij} = 1, q$, or q^{-1}.

We can choose c_i to be the integer powers of q due to the last assertion of (i).

4.10. Another way to compute the eigenvalues and eigenvectors of $A(q)$. Let A be a symmetric generalized Cartan matrix, and $A(q)$ its standard q-deformation.

We can apply 4.9 (ii) to the matrices $A(q)$ and a symmetric

$$A'(q) = \sqrt{q}A + (1-\sqrt{q})^2 I$$

So, there exists a diagonal matrix D as above such that

$$A(q) = D^{-1}A'(q)D.$$

But the eigenvalues of $A'(q)$ are obviously

$$\lambda(q) = \sqrt{q}\lambda + (1-\sqrt{q})^2 = 1 + (\lambda - 2)\sqrt{q} + q.$$

If v is an eigenvector of A for λ then v is an eigenvector of $A'(q)$ for $\lambda(q)$, and Dv will be an eigenvector of $A(q)$ for $\lambda(q)$.

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