PROBS AND SHEAVES

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Contents

§1. Some abstract nonsense: braided categories and braided bialgebras

PROP: products and permutations (Adams/MacLane, 1965)

PROB: products and braids

1.1. Braided categories. We will work with **k**-linear categories where **k** is a field.

A symmetric monoidal category $\mathcal{C}$ is equipped with a tensor product

$$(A, B) \mapsto A \otimes B,$$

an object $\mathbf{1}$, and natural isomorphisms:

unit

$$\mathbf{1} \otimes A \xrightarrow{\sim} A$$

associativity

$$a = a(A, B, C) : \; A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \otimes C$$

and commutativity (R-matrix)

$$c = c(A, B) = R(A, B) = R_{AB} : \; A \otimes B \xrightarrow{\sim} B \otimes A$$

satisfying some axioms.

In particular:

a satisfies the Stasheff pentagon aciom;

c satisfies a hexagon property (or Yang - Baxter identity)

$$R_{BC} R_{AC} R_{AB} = R_{AB} R_{AC} R_{BC} : \; A \otimes B \otimes C \xrightarrow{\sim} C \otimes B \otimes A \qquad (YB)$$

and a reflection property

$$R_{AB} R_{BA} = \mathrm{Id}_{A \otimes B} \qquad (R)$$

It follows that on any $A^{\otimes n}$ the symmetric group S_n acts.

In a braided category we forget about the axiom (R). This implies that on each $A^{\otimes n}$ only the braid group Br_n acts. See [JS].

Tensor functors $F : \mathcal{C} \longrightarrow \mathcal{C}'$

1.2. Braided bialgebras. Inside a braided category we have a notion of a bialgebra A:

it has a unit $\eta : \mathbf{1} \longrightarrow A$, counit $\epsilon : A \longrightarrow \mathbf{1}$, multiplication $\mu : A^{\otimes 2} \longrightarrow A$ and comultiplication $\Delta : A \longrightarrow A^{\otimes 2}$

(some people call it a Hopf algebra; other people require an antipode for a Hopf algebra; we will not need it).

Main axiom: compatibility of μ with Δ:

μ should be a morphism of coalgebras, or, equivalently, Δ should be a morphism of algebras.

To make sense of it one should define a structure of an algebra on $A^{\otimes 2}$: this is done using the R-matrix (sic!).

1.3. Mother of all braided bialgebras. A universal braided category $BBialg$ with the universal B bialgebra inside it, cf. [H], 6.1.

Universal property:

a braided bialgebra A in a braided category C is the same as a tensor functor

$$f_A : \ BBialg \longrightarrow C, \ f_A(B) = A.$$

It is generated as a tensor category by one object B, so it has objects $B^{\otimes n}, \ n \geq 0, \ B^0 = \mathbf{1}$.

Morphisms are generated by:

$$\mu : B^{\otimes 2} \longrightarrow B, \ \Delta : \ B \longrightarrow B^{\otimes 2},$$

$$\eta : \ \mathbf{1} \longrightarrow B, \ \epsilon : B \longrightarrow \mathbf{1},$$
$$c^{\pm} = R^{\pm} : \ B^{\otimes 2} \xrightarrow{\sim} B^{\otimes 2}$$

subject to the relations which make of B a bialgebra.

1.4. $\mathbb{N}$-graded bialgebras. An $\mathbb{N}$-graded bialgebra in a braided category $\mathcal{C}$ is a collection of objects $A_\bullet = \{A_n\}, n \geq 0, A_0 = \mathbf{1}$ together with morphisms

$$\mu : A_n \otimes A_m \longrightarrow A_{n+m},\ \Delta :\ A_{n+m} \longrightarrow A_n \otimes A_m$$

satisfying the usual axioms.

1.5. Mother of all $\mathbb{N}$-graded bialgebras is a category $NBBialg$ together with a $\mathbb{N}$-graded bialgebra (NBB) $B_\bullet$ inside it such that for a braided category $\mathcal{C}$ the category $NBB(C)$ is equivalent to the category of tensor functors

$$NBBialg \longrightarrow \mathcal{C}$$

It is generated as a braided category by a collection of objects

$$B_0 = \mathbf{1}, B_1, B_2, \ldots$$

So, its objects are symbols

$$B_{\mathbf{n}} = B_{n_1} \otimes \ldots \otimes B_{n_k}$$

defined for all sequences

$$\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k,\ k \in \mathbb{N}.$$

Define the weight

$$m(\mathbf{n}) := \sum h_i = m.$$

Maps in $NBBialg$ are compositions of various multiplications and comultiplications

$$\mu :\ B_{\mathbf{n}} \longrightarrow B_{\mathbf{n}'},\ \Delta :\ B_{\mathbf{n}'} \longrightarrow B_{\mathbf{n}}$$

where $m(\mathbf{n}) = m(\mathbf{n}')$ (there are no nontrivial morphisms between objects of different weights).

For any $m \in \mathbb{N}$ we will denote by $NBBialg_m \subset NBBialg$ the full subcategory with objects $B_{\mathbf{n}}$ of weight m.

See a simplest nontrivial example in 2.1 below.

§2. Relation to perverse sheaves

on symmetric powers of the complex affine line

2.1. Example. Perverse sheaves on a disc and $NBBialg_2$.

2.1.1. Recall $\mathcal{M}(\mathbb{C};0) = Perv(\mathbb{C},0) \subset \mathcal{D}(\mathbb{C},0)$.

Let $Constr(\mathbb{C},0)$ denote the category of constructible sheaves on $A^1 = \mathbb{C}$ smooth over $\mathbb{C}^*$. So an object $\mathcal{F}$ of it is a couple of finite dimensional vector spaces

$$\mathcal{F}_0 = \Gamma(\mathcal{F}; D(0;2)) = i_0^*\mathcal{F},\ \mathcal{F}_1 = \Gamma(\mathcal{F}; D(1;1/2)) = i_1^*\mathcal{F},$$

and morphisms

$$T:\ \mathcal{F}_1 \xrightarrow{\sim} \mathcal{F}_1$$

(monodromy around 0), and

$$\gamma:\ \mathcal{F}_0 \longrightarrow \mathcal{F}_1^T \subset \mathcal{F}_1$$

("generalisation").

Let $\mathcal{D}(A^1,0)$ be its derived category (all sheaves will be with values in a $Vect^f(\mathbf{k})$).

Functors Ψ, Φ.

(a) Usual fibers

For $\mathcal{F} \in \mathcal{D}(A^1,0)$ consider complexes

$$i_1^*\mathcal{F} = R\Gamma(D(1;1/2);\mathcal{F})$$

and

$$i_0^*\mathcal{F} = R\Gamma(\mathbb{C};\mathcal{F})$$

The complex $i_1^*\mathcal{F}$ is equpped with an automorphism T - the monodromy around 0.

We have the obvious restriction ("generalization") map

$$g : i_0^*\mathcal{F} \longrightarrow i_1^*\mathcal{F}$$

which lands in fact in the invariants of T.

(b) Hyperbolic fibers

We define

$$\Psi(\mathcal{F}) := i_1^*\mathcal{F}[1]$$

("nearby cycles"), and

$$\Phi(\mathcal{F}) := Cone(g)[1]$$

("vanishing cycles").

So we have a canonical map

$$u : \ \Psi(\mathcal{F}) \longrightarrow \Phi(\mathcal{F})$$

From the data above one can extract a map

$$v : \ \Phi(\mathcal{F}) \longrightarrow \Psi(\mathcal{F}),$$

"variation", such that

$$vu = 1 - T$$

(exercise).

Hyperbolic, or "perverse" , or Lefschetz, sheaves

We define

$$\mathcal{M}(\mathbb{C}, 0) \subset \mathcal{D}(\mathbb{C}, 0)$$

as the full sucategory of complexes $\mathcal{F}$ such that

$$\Psi(\mathcal{F}), \ \Phi(\mathcal{F}) \in \text{Vect}(\mathbf{k}) \subset \mathcal{D}(\mathbb{C}, 0).$$

The category $\mathcal{M}(\mathbb{C}, 0)$ is equivalent to a category of quadruples

$$(\Phi, \Psi, v : \Phi \longrightarrow \Psi, \ u : \Psi \longrightarrow \Phi), \Phi, \Psi \in Vect^f(\mathbf{k})$$

such that

$$T_\Psi := 1 - vu$$

is invertible.

Exercise. Prove that the last condition is equivalent to:

$$T_\phi = 1 - uv$$

is invertible.

2.1.2. Let $A_\bullet$ be an $\mathbb{N}$-graded bialgebra. It follows from the axioms of a bialgebra that for any $z \in A_1^{\otimes 2}$

$$\Delta\mu(z) = z + R(z).$$

It follows that $NBBialg_2$ has two objects, B_2 and $B_1^{\otimes 2}$, and the set of maps admits two generators

$$\mu : B_1^{\otimes 2} \longrightarrow B_2,\ \Delta : B_2 \longrightarrow B_1^{\otimes 2},$$

with

$$1 - \Delta\mu = -R_{B_1^{\otimes 2}}.$$

We see that

$$\mathcal{M}(\mathbb{C}, 0) \xrightarrow{\sim} Funct(NBBialg_2, Vect^f(\mathbf{k})).$$

Note that

$$\mathcal{M}(\mathbb{C}, 0) \stackrel{\sim}{=} Perv(Sym^2 A^1, \Delta)$$

Our main theorem is a generalisation of the above.

2.1.3. Factorizable sheaves.

Let us call a factorizable perverse sheaf over $(\mathbb{C}, 0)$ a collection

$A_1, A_2 \in Vect^f(\mathbf{k})$,

$u : A_1^{\otimes 2} \longrightarrow A_2,\ v :\ A_2 \longrightarrow A_1^{\otimes 2}$

such that $vu - \mathrm{Id}$ is invertible. They form a category $\mathcal{M}^{fact}(\mathbb{C}, 0)$ equipped with an obvious functor

$$\mathcal{M}^{fact}(\mathbb{C}, 0) \longrightarrow \mathcal{M}(\mathbb{C}, 0).$$

This category is equivalent to the category of *tensor* functors

$$Funct^{tens}(NBBialg_2, Vect^f(\mathbf{k})).$$

2.2. Hochschild - Tate complex for the disc. We have a commutative square

$$\begin{array}{ccc} A_1 \otimes A_1 & \stackrel{(1,R)}{\longrightarrow} & (A_1 \otimes A_1)^2 \\ \mu \downarrow & & \downarrow + \\ A_2 & \stackrel{\Delta}{\longrightarrow} & A_1 \otimes A_1 \end{array}$$

whence a complex concentrated in degrees $[-1, 1]$

$$0 \longrightarrow A_1 \otimes A_1 \longrightarrow A_2 \oplus (A_1 \otimes A_1)^2 \longrightarrow A_1 \otimes A_1 \longrightarrow 0 \qquad (HT)$$

quasiisomorphic to A_2. This is nothing else but Beilinson's monad, cf [B], no. 3, p. 46.

2.3. Theorem. *For every $m \geq 0$ we have an equivalence of categories*

$$Funct(NBBialg_m, Vect^f(\mathbf{k})) \stackrel{\sim}{\longrightarrow} Perv(Sym^m A^1, \Delta)$$

where to the right we have the category of perverse sheaves smooth along the diagonal stratification.

For an arbitrary m the correspondind Hochschild - Tate complex (HT) is concentrated in degrees $[-m, m]$, and is quasiisomorphic to A_m.

§3. Contingency tables and Yanus sheaves

Contingency: eventuality, chance; la contingence

They appeared in the work of Karl Pearson of 1904, [Pe]. See a good review in [DG].

Karl Pearson was one of three great statisticians; two other ones are Student (William Gosset, a Head Brewer of Guinness), and Sir Ronald Fisher.

3.1. Contingency matrices, or tables.

3.1.1. Definition. A contingency matrix is a rectangular matrix $A = (a_{ij})$ with $a_{ij} \in \mathbb{N} = \mathbb{Z}_{\geq 0}$. We suppose that there is no row or column containing only zeros.

We may regard A as a bipartite directed graph.

Weight:

$$\Sigma A = \sum_{i,j} a_{ij} \in \mathbb{N}$$

3.1.2. *Vertical and horizontal fusions*

Given $M = A$ as above and $1 \leq j \leq m-1$ we define an $n \times (m-1)$ matrix $\Sigma_j^h(M)$ whose j-th column is the result of "fusing" the j-th and $(j+1)$-th column of M, i.e. the new

$$\Sigma_j^h(M)_{ij} = a_{ij} + a_{i,j+1}$$

Similarly we define vertically fused $(n-1) \times m$ matrices $\Sigma_i^v(M)$, $1 \leq i \leq n-1$.

We write $M \leq_h M'$ (resp $M \leq_v M''$) if M is obtained from M' (resp. from M'') by a composition of horizontal (resp. vertical) fusions.

A fusion is called *anodyne* if the nonzero elements of the fused matrix are the same as for the original one.

We write

$$A \leq_h B,\ A \leq_v C$$

All contingency matrices form a bicategory CM.

3.1.3. Definition. A category QCM as a $\mathbf{k}$-linear category with the same objects as CM, i.e. the contingency matrices, and arrows generated by

$$h_{AB}: \ A \longrightarrow B \text{ for } A \leq_h B$$

and

$$v_{CA}: \ C \longrightarrow A \text{ for } A \leq_v C,$$

subject to relations:

(a) transitivity wrt horizontal and vertical arrows

(b) mixed axiom:

$$h_{AB}v_{CA} = \sum_{D: B\leq_v D, C\leq_h D} v_{DB}h_{CD}$$

3.2. Braided category $\mathfrak{CM}$ and a NBB inside it.

The set of anodyne arrows $An \subset Mor\ QCM$ has the Ore property. We define

$$\mathfrak{CM} = QCM[An^{-1}].$$

For any $n \geq 0$ let $\mathfrak{CM}_n \subset \mathfrak{CM}$ denote the full subcategory of matrices of weight n.

3.2.1. (Partial) braided structure. Introduce an operation

$$\otimes: \ (M, N) \mapsto M \otimes N = (M\ N)$$

by concatenation of matrices; it is defined when M and N have the same number of rows.

Define braidings using the anodyne morphisms:

$$(M\ N) \longrightarrow \begin{pmatrix} 0 & N \\ M & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} N \\ M \end{pmatrix} \longrightarrow \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix} \longrightarrow (N\ M)$$

Inside $\mathfrak{CM}$ we have 1×1 matrices

$$(n) \in \mathfrak{CM}_n, \ n \geq 1.$$

Each contingency table is isomorphic to a tensor product of them.

3.3. Example: weight 2. The category $\mathfrak{CM}_2$ has 5 objects:

$$(2), (1\ 1), \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We have

$$Funct(\mathfrak{CM}_2, Vect^f(\mathbf{k})) \cong \mathcal{M}(\mathbb{C}, 0).$$

(see a picture with 9 contingency matrices)

3.4. Main equivalence and Yanus sheaves.

Theorem. *We have an equivalence of categories*

$$e: \ NBBialg \overset{\sim}{\longrightarrow} \mathfrak{CM}, \ e(B_n) = (n)$$

compatible with tensor products. It induces for each n an equivalence

$$e: NBBialg_n \overset{\sim}{\longrightarrow} \mathfrak{CM}_n.$$

Thus,

$$Perv(Sym^n(\mathbb{C})) \cong Funkt(\mathfrak{CM}_n, Vect^f(\mathbf{k}))$$

A functor

$$\mathfrak{CM}_n \longrightarrow Vect^f(\mathbf{k})$$

is *Yanus* (cf. [P]): it is covariant with respect to horizontal contractions and contravariant with respect to vertical ones.

3.4. Interpretation using double cosets.

$$Ob\ \mathfrak{CM}_n = \coprod_{I,J\subset[n]} S_n\backslash(S_I \times S_J)$$

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