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Dear Barry,

the text below contains a couple of "conjectures" for characteristic 0 case, together with a sketch of possible proof. I am quite optimistic here.

It is possible that one can modify the statement so that it would work in char. p case. However, one should think more about that.

Yours,

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1.

1.1. Let us fix a ground field k of characteristic 0. Let G be a group, V a finite dimensional k-vector space,

$$\rho : G \longrightarrow \mathrm{GL}(V)$$

a representation.

Let $\mathcal{A}rtin_k$ denote a category of artinian local k-algebras with the residue field k. Let

$$\mathcal{D}ef_\rho : \mathcal{A}rtin_k \longrightarrow \mathcal{G}roupoids$$

denote a functor which assigns to an algebra A a groupoid $\mathcal{D}ef_\rho(A)$ whose objects are deformations of ρ to a representation

$$\rho_A : G \longrightarrow \mathrm{GL}(V_A)$$

where $V_A = V \otimes_k A$, and morphisms — isomorphisms inducing identity on ρ.

We denote by

$$\mathrm{Def}_\rho : \mathcal{A}rtin_k \longrightarrow \mathcal{S}ets$$

the composition of $\mathcal{D}ef_\rho$ with the functor π_0 of the set of connected components.

Let

$$\mathrm{Ad}_\rho : G \longrightarrow \mathrm{GL}(\mathfrak{gl}(\mathfrak{V}))$$

denote the adjoint representation $\mathrm{Ad}_\rho(g)(f)(x) = gf(g^{-1}x)$. So $\mathfrak{gl}(\mathfrak{V})$ is a G-module.

Let us suppose that $H^0(G; \mathfrak{gl}(\mathfrak{V})) = 0$. (It is probable also that one should impose some finiteness conditions on $H^\bullet(G, \mathfrak{gl}(\mathfrak{V}))$ in the sequel.)

Then the functor Def_ρ is prorepresentable by a complete local k-algebra R. We want to describe R.

1.2. Note that $\mathfrak{gl}(\mathfrak{V})$ is a Lie algebra in the category of G-modules, i.e. the bracket

$$[\ ,\ ] : \mathfrak{gl}(\mathfrak{V}) \times \mathfrak{gl}(\mathfrak{V}) \longrightarrow \mathfrak{gl}(\mathfrak{V})$$

is G-equivariant.

Let $C^\bullet(G;\mathfrak{gl}(\mathfrak{V}))$ denote a standard cochain complex of G with coefficients in $\mathfrak{gl}(\mathfrak{V})$. One can construct a skew symmetric "bracket"

$$b : \Lambda^2\, C^\bullet(G;\mathfrak{gl}(\mathfrak{V})) \longrightarrow \mathfrak{C}^\bullet(\mathfrak{G};\mathfrak{gl}(\mathfrak{V})) \tag{1}$$

which would satisfy the Jacobi identity up to a homotopy. It will induce a structure of a graded Lie algebra on the cohomology $H^\bullet(G;\mathfrak{gl}(\mathfrak{V}))$. Namely, given $\phi^n \in C^n(G;\mathfrak{gl}(\mathfrak{V}),\ \phi^{\mathfrak{m}} \in \mathfrak{C}^{\mathfrak{m}}(\mathfrak{G};\mathfrak{gl}(\mathfrak{V}))$, we define $b(\phi^n \wedge \phi^m) \in C^{n+m}(G;\mathfrak{gl}(\mathfrak{V}))$ by the skew symmetrization of the Alexander-Witney multiplication:

$$b(\phi^n \wedge \phi^m)(g_1,\dots,g_{n+m}) = \frac{1}{2}([\phi^n(g_1,\dots,g_n),\ {}^{g_1\cdots g_n}\phi^m(g_{n+1},\dots,g_{n+m})] + (-1)^{m+n+1}[\phi^m(g_1,\dots,g_m),\ {}^{g_1\cdots g_m}\phi^n(g_{m+1},\dots,g_{n+m})])$$

Let $\check{C}^\bullet(G;\mathfrak{gl}(\mathfrak{V}))$ denote a cosimplicial space whose associated complex is $C^\bullet(G;\mathfrak{gl}(\mathfrak{V}))$. Thus,

$$\check{C}^n(G;\mathfrak{gl}(\mathfrak{V})) = \mathrm{Hom}_{\mathcal{S}\mathfrak{ets}}(\mathfrak{G}^n, \mathfrak{gl}(\mathfrak{V})),$$

cofaces and codegeneracies are defined in a usual way. $\check{C}^\bullet(G;\mathfrak{gl}(\mathfrak{V}))$ is a cosimplicial Lie algebra.

In [HS1], 5.2.12, a certain functor

$$\Omega : \Delta\mathcal{L}ie \longrightarrow \mathcal{D}glie$$

from the category of cosimplicial Lie algebras to the category of dg Lie algebras is defined. (It is certain "de Rham" complex of compatible polynomial differential forms à la Sullivan.)

Let us introduce the notation

$$R\Gamma^{Lie}(G;\mathfrak{gl}(\mathfrak{V})) = \Omega(\check{\mathfrak{C}}^\bullet(\mathfrak{G};\mathfrak{gl}(\mathfrak{V})))$$

We have a natural map *of complexes*

$$I : R\Gamma^{Lie}(G;\mathfrak{gl}(\mathfrak{V})) \longrightarrow \mathfrak{C}^\bullet(\mathfrak{G};\mathfrak{gl}(\mathfrak{V})) \tag{2}$$

which is a quasi-isomorphism and is compatible with brackets up to a homotopy. (I is essentially an integration.)

Given a dg Lie algebra Lie $\mathfrak{g}$, let $C(\mathfrak{g})$ denote the standard Chevalley-Eilenberg-Quillen complex of $\mathfrak{g}$ (see for example *loc.cit.,* 2.2). It is a cocommutative dg coalgebra. We denote

$$H_i^{Lie}(\mathfrak{g}) = H^{-i}(C(\mathfrak{g})).$$

Now we can formulate

1.3. **Conjecture 1.** *We have a natural isomorphism*

$$\kappa : R^* \cong H_0^{Lie}(R\Gamma^{Lie}(G;\mathfrak{gl}(\mathfrak{V}))) \tag{3}$$

Here R^* denotes the space of continuous functionals $R \longrightarrow k$, k being considered in the discrete topology.

Both sides of (3) come up with natural filtrations and κ is compatible with them.

The statement is completely similar to [HS1], 1.3.1. Note however that we do not require the smoothness of R here.

We want to deduce this conjecture from a more general statement.

2. $\mathcal{D}ef_\rho$ and Deligne functor

2.1. Let $\mathfrak{g}$ be a dg Lie algebra over k. We suppose that $\mathfrak{g}$ lives in non-negative degrees and that $H^0(\mathfrak{g}) = 0$.

Deligne (see [GM], [D]) defines a functor

$$\mathcal{G}_\mathfrak{g} : \mathcal{A}rtin_k \longrightarrow \mathcal{G}roupoids$$

as follows.

For $A \in \mathcal{A}rtin_k$ let $\mathfrak{m}$ be its maximal ideal. Consider a dg Lie algebra $\mathfrak{g}_\mathfrak{m} = \mathfrak{g} \otimes \mathfrak{m}$. By definition,

$$\mathrm{Ob}(\mathcal{G}_\mathfrak{g}(A)) = \{\phi \in \mathfrak{g}^1_\mathfrak{m} | d\phi + \frac{1}{2}[\phi, \phi] = 0\}.$$

Let $G(\mathfrak{g}^0_\mathfrak{m})$ be the Lie group associated with a nilpotent Lie algebra $\mathfrak{g}^0_\mathfrak{m}$. It acts by conjugation on $\mathfrak{g}_\mathfrak{m}$.

An arrow in $\mathcal{G}_\mathfrak{g}(A)$ is a couple $(\gamma \in G_\mathfrak{g}(A), \phi \in \mathrm{Ob}(\mathcal{G}_\mathfrak{g}(A))$; it acts from ϕ to $\gamma\phi\gamma^{-1} - d\gamma \cdot \gamma^{-1}$.

2.2. **Lemma.** *If $\mathfrak{g} \longrightarrow \mathfrak{g}'$ is a map of graded Lie algebras which is a quasi-isomorphism as a map of complexes then the induced natural transformation $\mathcal{G}_\mathfrak{g} \longrightarrow \mathcal{G}_{\mathfrak{g}'}$ is an equivalence.*

2.3. **Lemma.** *The functor $A \mapsto \pi_0\mathcal{G}_\mathfrak{g}(A)$ is pro-representable by a complete local k-algebra $H^{Lie}_0(\mathfrak{g})^*$.*

(It seems that one should impose here some finiteness conditions on $H^\bullet(\mathfrak{g})$).

2.4. **Conjecture 2.** *There exists a natural equivalence*

$$\mathfrak{K} : \mathcal{D}ef_\rho \xrightarrow{\sim} \mathcal{G}_{R\Gamma^{Lie}(G;\mathfrak{gl}(\mathfrak{V}))} \tag{4}$$

The sketch of a possible proof will be given in the next Section.

Note that 1.3 follows at once from 2.4 and 2.3.

3. Homotopy Lie algebras

3.1. To construct the map $\mathfrak{K}$, we first define another incarnation for $R\Gamma(G; \mathfrak{gl}(\mathfrak{V}))$.

Let us denote for brevity $\mathfrak{g}^\bullet = C^\bullet(G; \mathfrak{gl}(\mathfrak{V}))$. Although it is not an honest dg Lie algebra, it is proven in [HS2] that this complex has a structure of a *Sugawara Lie algebra*[1]. This means the following.

Let us consider the doubly graded space

$$D(\mathfrak{g}^\bullet) = S^\bullet(\mathfrak{g}^\bullet[1])$$

[1]in *loc.cit.* we used the name "homotopy Lie algebra"; here we use the name suggested by Drinfeld

here $S^\bullet$ denotes the symmetric algebra and we forget for a moment the differential in $\mathfrak{g}^\bullet$. The bigrading is defined as follows:

$$D(\mathfrak{g}^\bullet)^{pq} = (S^{-p}(\mathfrak{g}^\bullet[1]))^{p+q} = (\Lambda^{-p}(\mathfrak{g}^\bullet))^q$$

Let $C(\mathfrak{g}^\bullet)$ denote the corresponding simply graded space. Being a symmetric algebra, $C(\mathfrak{g}^\bullet)$ is naturally a graded cocommutative coalgebra.

By definition (due to Drinfeld), a structure of a *Sugawara Lie algebra* on $\mathfrak{g}^\bullet$ is a differential d on $C(\mathfrak{g}^\bullet)$ of degree 1 making the later a dg coalgebra.

We can decompose d into a sum

$$d = \sum_{r\in\mathbb{Z}} d_r,$$

d_r having bidegree $(r, -r+1)$. We require that $d_r = 0$ for $r < 0$.

The equation $d^2 = 0$ means in fact an infinite number of equations:

$$d_0^2 = 0, \tag{5}$$

$$d_0 d_1 + d_1 d_0 = 0, \tag{6}$$

$$d_0 d_2 + d_1 d_1 + d_2 d_0 = 0, \tag{7}$$

etc. Due to the compatibility of d with the comultiplication, d_r is uniquely determined by $d_0, \dots, d_{r-1}$ and by its component

$$[\ ,\ ]_{r+1} : \Lambda^{r+1}(\mathfrak{g}^\bullet) \longrightarrow \mathfrak{g}^\bullet[-r+1]. \tag{8}$$

We require that d_0 should coincide with the differential coming from the differential in $\mathfrak{g}^\bullet$. d_1 can be chosen to coincide with the bracket (1), up to a sign.

It is usefull to imagine maps (8) as "higher brackets". For example, $[\ ,\ ]_3$ is a homotopy between the Jacobi identity for $[\ ,\ ]_2$ and 0.

So, $C(\mathfrak{g}^\bullet)$ becomes a cocommutative *dg*-coalgebra.

This structure is not unique but it is unique in some "homotopy" sense.

3.2. It is quite probable that one can define Deligne groupoid for Sugawara Lie algebras too. Namely, we set

$$\text{Ob } \mathcal{G}_{\mathfrak{g}^\bullet}(A) = \{\phi \in \mathfrak{g}^1_{\mathfrak{m}} | d\phi + \sum_{i=2}^{\infty} \frac{1}{i!}[\phi, \dots, \phi]_i = 0\}; \tag{9}$$

maps being defined using the adjoint action of g^0_{m} on $\mathfrak{g}^\bullet$. Here we use the fact that g^0_{m} is an honest Lie algebra and it acts honestly (not up to a homotopy) on $g^\bullet_{\mathrm{m}}$.

3.3. Suppose this is true. For $A \in Artin_k$, I would like to define a functor

$$L : \mathcal{D}ef_\rho(A) \longrightarrow \mathcal{G}_{\mathfrak{g}^\bullet}(A)$$

as follows.

Suppose we have $(\rho_A : G \longrightarrow \mathrm{GL}(V_A)) \in \mathrm{Ob}\ \mathcal{D}ef_\rho(A)$. We have an obvious embedding $i : \mathrm{GL}(k) \hookrightarrow \mathrm{GL}(A)$; denote

$$\bar{\rho}_A(g) = \rho_A(g) \cdot \rho(g)^{-1}.$$

These elements lie in the nilpotent group

$$\mathrm{GL}(A; \mathfrak{m}) := \mathrm{Ker}\ (\mathrm{GL}(A) \longrightarrow \mathrm{GL}(k))$$

and satisfy a cocycle condition

$$\bar{\rho}_A(g_1 g_2) = \bar{\rho}_A(g_1) \cdot {}^{g_1}\bar{\rho}(g_2) \tag{10}$$

where we set

$${}^g x = \rho(g) x \rho(g)^{-1}$$

for $g \in G, x \in \mathrm{GL}(A; \mathfrak{m})$.

Now I would like to define L on objects by the formula

$$L(\rho_A) = \log\ \bar{\rho}_A \in \mathfrak{g}^1_{\mathfrak{m}} = \mathrm{Hom}_{Sets}(G, \mathfrak{gl}_{\mathfrak{m}}). \tag{11}$$

The hope is that one can introduce higher brackets in such a way that, when we apply log to (10) and use Campbell-Hausdorf, we get exactly "a higher Maurer-Cartan equation" (9), so that $L(\rho_A)$ in fact belongs to Ob $\mathcal{G}_{\mathfrak{g}^\bullet}(A)$.

For example, the beginning of the Campbell-Hausdorf formula

$$\log(e^X \cdot e^Y) = X + Y + \frac{1}{2}[X, Y] + \dots$$

corresponds to the bracket $[\ ,\ ]_2$ — one can check this easily. This seems to prove "conjectures" for 2-jets.

Or else, given higher brackets, one could try to find a power series for L (not necessarily logarithm).

If this is true, then L could be *an isomorphism* (and not only an equivalence).

3.4. It seems certain that

Claim. *The map (2) may be extended to a quasi-isomorphism*

$$\mathfrak{I} : C(R\Gamma^{\mathcal{L}ie}(G; \mathfrak{gl}(\mathfrak{V})) \longrightarrow \mathfrak{C}(\mathfrak{g}^\bullet) \tag{12}$$

($\mathfrak{I}$ consists of "higher integrations").

Consequently it induces an equivalence of Deligne groupoids. Combining this with the previous argument we get the statement of Conjecture 2.

References

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