CARTAN EIGENVECTORS, TODA MASSES, AND THEIR q-DEFORMATIONS

Vadim Schechtman

Talk at

Kavli Institute for the Physics and Mathematics of the Universe,

Kashiwa-no-ha, October 2015

This is a report on a joint work with Laura Brillon and Alexander Varchenko, cf. [BS], [BSV].

Plan

1. Cartan eigenvectors and Toda masses.
2. Vanishing cycles, Sebastiani - Thom product, and E_8.
3. Givental's q-deformations.

§1. Cartan eigenvectors and Toda masses

Let $\mathfrak{g}$ be a simple finite dimensional complex Lie algebra ; $(\,,\,)$ will denote the Killing form on $\mathfrak{g}$. We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$; let $R \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, $\{\alpha_1, \ldots, \alpha_r\} \subset R$ a base of simple roots,

$$\mathfrak{g} = (\oplus_{\alpha<0}\mathfrak{g}_\alpha) \oplus \mathfrak{h} \oplus (\oplus_{\alpha>0}\mathfrak{g}_\alpha)$$

the root decomposition. Let

$$\theta = \sum_{i=1}^{r} n_i\alpha_i$$

be the longest root ; we set

$$\alpha_0 := -\theta, n_0 := 1.$$

The number

$$h = \sum_{i=0}^{r} n_i$$

is the Coxeter number of $\mathfrak{g}$; set $\zeta = \exp(2\pi i/h)$.

For each $\alpha \in R$ choose a base vector $E_\alpha \in \mathfrak{g}_\alpha$.

Let $A = (\langle \alpha_i, \alpha_j^\vee \rangle)_{i,j=1}^r$ be the Cartan matrix of R.

The eigenvalues of A are

$$\lambda_i = 2(1 - \cos(2k_i\pi/h)),\ 1 \leq i \leq r.$$

where

$$1 = k_1 < k_2 < \ldots < k_r = h - 1$$

are the exponents of R.

The coordinates of the eigenvectors of A have an important meaning in the physics of integrable systems : namely, these numbers appear as the masses of particles (or, dually, as the energy of solitons) in affine Toda field theories, cf. [F], [D].

Historically, the first example of the system of type E_8 appeared in the pioneering papers [Z] on the 2D critical Ising model in a magnetic field.

The aim of this talk is a study of these numbers, and of their q-deformations.

Principal element and principal gradation. Let $\rho^\vee \in \mathfrak{h}$ be defined by

$$\langle \alpha_i, \rho^\vee \rangle = 1, \ i = 1, \dots, r.$$

Let G denote the adjoint group of $\mathfrak{g}$, and

$$\exp : \ \mathfrak{g} \longrightarrow G$$

the exponential map.

We set

$$P := \exp(2\pi i \rho^\vee / h) \in G.$$

Thus, Ad P defines a $\mathbb{Z}/h\mathbb{Z}$-grading on $\mathfrak{g}$,

$$\mathfrak{g} = \oplus_{k=0}^{h-1} \mathfrak{g}_k, \ \mathfrak{g}_k = \{x \in \mathfrak{g} | \ \mathrm{Ad}_P(x) = \zeta^k x\}.$$

We have $\mathfrak{g}_0 = \mathfrak{h}$.

Fix complex numbers $m_i \neq 0$, $i = 0, \ldots, r$, $m_0 = 1$ and define an element

$$E = \sum_{i=0}^{r} m_i E_{\alpha_i},$$

We have $E \in \mathfrak{g}_1$; Kostant calls E a *cyclic element*.

We define, with Konstant, [K], the subspace

$$\mathfrak{h}' := Z(E) \subset \mathfrak{g}$$

It is proven in [K], Thm. 6.7, that $\mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{g}$, called *the Cartan subalgebra in apposition to* $\mathfrak{h}$ *with respect to the principal element* P.

The subspace $\mathfrak{h}' \cap \mathfrak{g}_i$ is nonzero iff $i \in \{k_1, k_2, \ldots, k_r\}$ where $1 = k_1 < k_2 < \ldots < k_r = h - 1$ are the *exponents* of ˇ. We have $k_i + k_{r+1-i} = h$.

Set

$$\mathfrak{h}'^{(i)} := \mathfrak{h}' \cap \mathfrak{g}_{k_i}, \ 1 \leq i \leq r;$$

these are the subspaces of dimension 1.

Pick a nonzero vector $e^{(i)} \in \mathfrak{h}'^{(i)}$ for all $1 \leq i \leq r$, for example $e^{(1)} = E$.

The operators $\mathrm{ad}_{e^{(i)}} \mathrm{ad}_{e^{(h-i)}}$ preserve $\mathfrak{h}$; let

$$\tilde{M}^{(i)} := \mathrm{ad}_{e^{(i)}} \mathrm{ad}_{e^{(h-i)}} |_{\mathfrak{h}} : \ \mathfrak{h} \longrightarrow \mathfrak{h}$$

denote its restriction to $\mathfrak{h}$.

Theorem. *For each* $1 \leq i \leq r$ *there exists a unique operator* $M^{(i)} \in \mathfrak{gl}(\mathfrak{h})$ *whose square is equal to* $\tilde{M}^{(i)}$ *such that the column vector of its eigenvalues in the approriate numbering*

$$\mu^{(i)} := (\mu_1^{(i)}, \dots, \mu_r^{(i)})^t$$

is an eigenvector of the Cartan matrix A *with eigenvalue*

$$\lambda_i := 2(1 - \cos(2k_i\pi/h)).$$

The operators $M^{(1)}, \dots, M^{(r)}$ *commute with each other.*

The proof is based on a relation between the Cartan matrix A and the Coxeter element of our root system which will be discussed later.

Relation to affine Toda field theories

Consider a classical field theory whose fields are smooth functions $\phi : X \longrightarrow \mathfrak{h}$ where $X = \mathbb{R}^2$ ("space - time"), with coordinates x_1, x_2.

The Lagrangian density $\mathcal{L}_e(\phi)$ of the theory depends on an element $e \in \mathfrak{h}'$ where $\mathfrak{h}'$ is a Cartan algebra in apposition to $\mathfrak{h}$.

We fix a $\mathbb{C}$-antilinear Cartan involution $* : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$ such that $\mathfrak{h}'^{(i)*} = \mathfrak{h}'^{(r-i)}$.

We set

$$\mathcal{L}_e(\phi) = \frac{1}{2}\sum_{a=1}^{2}(\partial_a\phi, \partial_a\phi) - m^2(\mathrm{Ad}_{\exp(\phi)}(e), e^*).$$

Here $\partial_a := \partial/\partial x_a$, $(,)$ denotes the Killing form on $\mathfrak{g}$.

The Euler - Lagrange equations of motion are

$$\mathcal{D}_e(\phi) := \Delta\phi + m^2[\mathrm{Ad}_{\exp(\phi)}(e), e^*] = 0, \quad (EL)$$

where $\Delta\phi = \sum_{a=1}^{2} \partial_a^2 \phi$. It is a system of r nonlinear differential equations of the second order.

The linear approximation to the nonlinear equation (EL) is a Klein - Gordon equation

$$\Delta_e \phi := \Delta\phi + m^2 \, \mathrm{ad}_e \, \mathrm{ad}_{e^*}(\phi) = 0 \qquad (ELL)$$

It admits r "normal mode" solutions

$$\phi_j(x_1, x_2) = e^{i(k_j x_1 + \omega_j x_2)} y_j, \;\; k_j^2 + \omega_j^2 = m^2 \mu_j^2,$$

$1 \leq j \leq r$, where μ_j^2 are the eigenvalues of the square mass operator

$$M_e^2 := \mathrm{ad}_e \, \mathrm{ad}_{e^*} \; : \;\; \mathfrak{h} \longrightarrow \mathfrak{h}$$

and y_j are the corresponding eigenvectors.

In other words, (ELL) decouples into r equations describing scalar particles of masses μ_j, which explains the name "masses" for them.

Due to commutativity of $\mathfrak{h}'$, for all $e, e' \in \mathfrak{h}'$,

$$[\Delta_e, \Delta_{e'}] = 0.$$

§2. Vanishing cycles, Sebastiani - Thom product, and E_8.

2.1. Here we recall some classical constructions from the singularity theory.

Let $f: (\mathbb{C}^N, 0) \longrightarrow (\mathbb{C}, 0)$ be the germ of a holomorphic function with an isolated critical point at 0, with $f(0) = 0$.

A *Milnor fiber* is

$$V_z = f^{-1}(z) \cap \bar{B}_\rho$$

where

$$\bar{B}_\rho = \{(x_1, \ldots, x_N) |\ \sum |x_i|^2 \leq \rho\}$$

for $1 \gg \rho \gg |z| > 0$.

For z belonging to a small disc $D_\epsilon = \{z \in \mathbb{C} |\ |z| < \epsilon\}$, the space V_z is a complex manifold with boundary, homotopically equivalent to a bouquet $\vee S^{N-1}$ of μ spheres, where

$$\mu = \dim_{\mathbb{C}} \operatorname{Miln}(f, 0),$$

$$\mathrm{Miln}(f,0) = \mathbb{C}[[x_1,\dots,x_N]]/(\partial_1 f,\dots,\partial_N f).$$

The family of free abelian groups

$$Q(f;z) := \tilde{H}_{N-1}(V_z;\mathbb{Z}) \overset{\sim}{=} \mathbb{Z}^\mu,\ z \in \overset{\bullet}{D}_\epsilon := D_\epsilon \setminus \{0\},$$

($\tilde{H}$ means that we take the reduced homology for $N = 1$), carries a flat Gauss - Manin conection.

Take $t \in \mathbb{R}_{>0} \cap \overset{\bullet}{D}_\epsilon$; the lattice $Q(f;t)$ does not depend, up to a canonical isomorphism, on the choice of t. Let us call this lattice $Q(f)$. The linear operator

$$T(f) : Q(f) \overset{\sim}{\longrightarrow} Q(f)$$

induced by the path $p(\theta) = e^{i\theta}t,\ 0 \le \theta \le 2\pi$, is called the classical monodromy of the germ $(f,0)$.

2.2. Morse deformations. The $\mathbb{C}$-vector space $\mathrm{Miln}(f,0)$ may be identified with the tangent space to the base B of the miniversal defomation of f. For

$$\lambda \in B^0 = B \setminus \Delta$$

where $\Delta \subset B$ is an analytic subset of codimension 1, the corresponding function $f_\lambda : \mathbb{C}^N \longrightarrow \mathbb{C}$ has μ nondegenerate Morse critical points with distinct critical values, and the algebra $\mathrm{Miln}(f_\lambda)$ is semisimple, isomorphic to $\mathbb{C}^\mu$.

Let $0 \in B$ denote the point corresponding to f itself, so that $f = f_0$, and pick $t \in \mathbb{R}_{>0} \cap \dot{D}_\epsilon$ as in 1.1.

Afterwards pick $\lambda \in B^0$ close to 0 in such a way that the critical values $z_1, \ldots z_\mu$ of f_λ have absolute values $<< t$.

As in 2.1, for each

$$z \in \tilde{D}_\epsilon := D_\epsilon \setminus \{z_1, \ldots z_\mu\}$$

the Milnor fiber V_z has the homotopy type of a bouquet $\vee S^{N-1}$ of μ spheres, and we will be interested in the middle homology

$$Q(f_\lambda; z) = \tilde{H}_{N-1}(V_z; \mathbb{Z}) \overset{\sim}{=} \mathbb{Z}^\mu$$

The lattices $Q(f_\lambda; z)$ carry a natural bilinear product induced by the cup product in the homology which is symmetric (resp. skew-symmetric) when N is odd (resp. even).

The collection of these lattices, when $z \in \tilde{D}_\epsilon$ varies, carries a flat Gauss - Manin connection.

Consider an "octopus"

$$Oct(t) \subset \mathbb{C}$$

with the head at t : a collection of non-intersecting paths p_i ("tentacles") connecting t with z_i and not meeting the critical values z_j otherwise. It gives rise to a base

$$\{b_1, \ldots, b_\mu\} \subset Q(f_\lambda) := Q(f_\lambda; t)$$

(called "distinguished") where b_i is the cycle vanishing when being transferred from t to z_i along the tentacle p_i, cf. [Gab], [AGV].

The Picard - Lefschetz formula describe the action of the fundamental group $\pi_1(\tilde{D}_\epsilon; t)$ on $Q(f_\lambda)$ with respect to this basis. Namely, consider a loop γ_i which turns around z_i along the tentacle p_i, then the corresponding transformation of $Q(f_\lambda)$ is the reflection (or transvection) $s_i := s_{b_i}$, cf. [Lef], Théorème fondamental, Ch. II, p. 23.

The loops γ_i generate the fundamental group $\pi_1(\tilde{D}_\epsilon)$. Let

$$\rho : \ \pi_1(\tilde{D}_\epsilon; t) \longrightarrow GL(Q(f_\lambda))$$

denote the monodromy representation. The image of ρ, denoted by $G(f_\lambda)$, is called the *monodromy group of* f_λ.

The subgroup $G(f_\lambda)$ is generated by $s_i, 1 \leq i \leq \mu$.

As in 2.1, we have the monodromy operator

$$T(f_\lambda) \in G(f_\lambda),$$

the image by ρ of the path $p \subset \tilde{D}_\epsilon$ starting at t and going around all points $z_1, \ldots, z_\mu$.

This operator $T(f_\lambda)$ is now a product of μ simple reflections

$$T(f_\lambda) = s_1 s_2 \ldots s_\mu.$$

One can identify the relative (reduced) homology $\tilde{H}_{N-1}(V_t, \partial V_t; \mathbb{Z})$ with the dual group $\tilde{H}_{N-1}(V_t; \mathbb{Z})^*$, and one defines a map

$$\text{var} : \tilde{H}_{N-1}(V_t, \partial V_t; \mathbb{Z}) \longrightarrow \tilde{H}_{N-1}(V_t; \mathbb{Z}),$$

called a *variation operator*, which translates to a map

$$L : Q(f_\lambda)^* \xrightarrow{\sim} Q(f_\lambda)$$

("Seifert form") such that the matrix $A(f_\lambda)$ of the bilinear form in the distinguished basis is

$$A(f_\lambda) = L + (-1)^{N-1} L^t,$$

and

$$T(f_\lambda) = (-1)^{N-1} L L^{-t}.$$

A choice of a path q in B connecting 0 with λ, enables one to identify $Q(f)$ with $Q(f_\lambda)$, and $T(f)$ will be identified with $T(f_\lambda)$.

2.3. Sebastiani - Thom factorization. If $g \in \mathbb{C}[y_1, \ldots, y_M]$ is another function, the sum, or **join** of two singularities $f \oplus g : \mathbb{C}^{N+M} \longrightarrow \mathbb{C}$ is defined by

$$(f \oplus g)(x, y) = f(x) + g(y)$$

The fundamental Sebastiani - Thom theorem, [ST], says that there exists a natural isomorphism of lattices

$$Q(f \oplus g) \stackrel{\sim}{=} Q(f) \otimes_{\mathbb{Z}} Q(g),$$

and under this identification the full monodromy decomposes as

$$T_{f \oplus g} = T_f \otimes T_g$$

2.4. Examples : simple singularities.

$$x^{n+1}, \; n \geq 1, \qquad (A_n)$$

$$x^5 + y^3 + z^2 \qquad (E_8)$$

Their names come from the following facts :

— their lattices of vanishing cycles may be identified with the corresponding root lattices ;

— the monodromy group is identified with the corresponding Weyl group ;

— the classical monodromy T_f is a Coxeter element, therefore its order h is equal to the Coxeter number, and

$$\mathrm{Spec}(T_f) = \{e^{2\pi i k_1/h}, \ldots, e^{2\pi i k_r/h}\}$$

where the integers

$$1 = k_1 < k_2 < \ldots < k_r = h - 1,$$

are the exponents of our root system.

We will discuss the case of E_8 in some details below.

2.5. Lattices, polarization, Coxeter elements.

Let us call *a lattice* a pair (Q, A) where Q is a free abelian group, and

$$A : Q \times Q \longrightarrow \mathbb{Z}$$

a symmetric bilinear map ("Cartan matrix"). We shall identify A with a map

$$A : Q \longrightarrow Q^{\vee} := Hom(Q, \mathbb{Z}).$$

A polarized lattice is a triple (Q, A, L) where (Q, A) is a lattice, and

$$L : \ Q \xrightarrow{\sim} Q^{\vee}$$

("variation", or "Seifert matrix") is an isomorphism such that

$$A = A(L) := L + L^{\vee}$$

where

$$L^\vee : Q = Q^{\vee\vee} \xrightarrow{\sim} Q^\vee$$

is the conjugate to L.

The *Coxeter automorphism* of a polarized lattice is defined by

$$C = C(L) = -L^{-1}L^\vee \in GL(Q).$$

We shall say that the operators A and C are in a *Cartan - Coxeter correspondence*.

Example. Let (Q, A) be a lattice, and $\{e_1, \ldots, e_n\}$ an ordered $\mathbb{Z}$-base of Q. With respect to this base A is expressed as a symmetric matrix $A = (a_{ij}) = A(e_i, e_j) \in \mathfrak{gl}_n(\mathbb{Z})$. Let us suppose that all a_{ii} are even. We define the matrix of L to be the unique upper triangular matrix (ℓ_{ij}) such that $A = L + L^t$ (in patricular $\ell_{ii} = a_{ii}/2$; in our examples we will have $a_{ii} = 2$.)

2.6. Join product. Suppose we are given two polarized lattices (Q_i, A_i, L_i), $i = 1, 2$.

Set $Q = Q_1 \otimes Q_2$, whence

$$L := L_1 \otimes L_2 : Q \xrightarrow{\sim} Q^\vee,$$

and define

$$A := A_1 * A_2 := L + L^\vee : Q \xrightarrow{\sim} Q^\vee$$

The triple (Q, A, L) will be called the **join**, or **Sebastiani - Thom**, product of the polarized lattices Q_1 and Q_2, and denoted by $Q_1 * Q_2$.

Obviously

$$C(L) = -C(L_1) \otimes C(L_2) \in GL(Q_1 \otimes Q_2).$$

It follows that if if $\mathrm{Spec}(C(L_i)) = \{e^{2\pi i k_i/h_i},\ k_i \in K_i\}$ then

$$\mathrm{Spec}(C(L)) = \{-e^{2\pi i(k_1/h_1 + k_2/h_2)},\ (k_1, k_2) \in K_1 \times K_2\}$$

2.7. Recall that E_8 corresponds to the singularity

$$f(x, y, z) = z^5 + y^3 + x^2$$

E_8 versus $A_4 * A_2 * A_1$: elementary analysis.

The ranks :

$$r(E_8) = 8 = r(A_4)r(A_2)r(A_1);$$

the Coxeter numbers :

$$h(E_8) = h(A_4)h(A_2)h(A_1) = 5 \cdot 3 \cdot 2 = 30.$$

It follows that

$$|R(E_8)| = 240 = |R(A_4)||R(A_2)||R(A_1)|.$$

The exponents of E_8 are :

$$1, 7, 13, 19, 11, 17, 23, 29.$$

All these numbers, except 1, are primes, and these are all primes ≤ 30, not dividing 30.

Occasionally they form a group

$$U(\mathbb{Z}/30\mathbb{Z}).$$

They may be determined from the formula

$$\frac{i}{5} + \frac{j}{3} + \frac{1}{2} = \frac{30 + k(i,j)}{30}, \ 1 \leq i \leq 4, \ 1 \leq j \leq 2,$$

so

$$k(i,1) = 1 + 6(i-1) = 1, 7, 13, 19;$$

$$k(i,2) = 1 + 10 + 6(i-1) = 11, 17, 23, 29.$$

This shows that the exponents of E_8 are the same as the exponents of $A_4 * A_2 * A_1$.

The following theorem is more delicate :

2.8. Theorem (Gabrielov). *There exists a polarization of the root lattice $Q(E_8)$ and an isomorphism of polarized lattices*

$$\Gamma : Q(A_4) * Q(A_2) * Q(A_1) \xrightarrow{\sim} Q(E_8).$$

In fact, this isomorphism is given by an explicit (but complicated) formula.

Using a relation between the Cartan/Coxeter correspondence discussed above, one can obtain

2.9. Corollary : an expression for the eigenvectors of $A(E_8)$.

Let $\theta = \frac{a\pi}{5},\ 1 \leq a \leq 4,\ \gamma = \frac{b\pi}{3},\ 1 \leq b \leq 2,\ \delta = \frac{\pi}{2},$

$$\alpha = \theta + \gamma + \delta = 1 + \frac{k\pi}{30},$$

$$k \in \{1, 7, 11, 13, 17, 19, 23, 29\}.$$

The 8 eigenvalues of $A(E_8)$ have the form

$$\lambda(\alpha) = \lambda(\theta, \gamma) = 2 - 2\cos\alpha$$

An eigenvector of $A(E_8)$ with the eigenvalue $\lambda(\theta, \gamma)$ is

$$X_{E_8}(\theta,\gamma) = -\begin{pmatrix} 2\cos(4\theta)\cos(\gamma-\theta-\delta) \\ -\cos(2\gamma+2\theta) \\ 2\cos^2(\theta) \\ -2\cos(\gamma)\cos(3\theta-\delta)-\cos(\gamma+\theta-\delta) \\ -2\cos(2\gamma+3\theta)\cos(\theta)+\cos(2\gamma) \\ -2\cos\theta\cos(\gamma+2\theta-\delta) \\ -2\cos(\gamma+\theta-\delta)\cos(\gamma-\theta+\delta) \\ -\cos(\gamma-\theta-\delta) \end{pmatrix} \tag{2.9.1}$$

The **Perron - Frobenius** eigenvector corresponds to the eigenvalue

$$2 - 2\cos\frac{\pi}{30},$$

and may be chosen as

$$v_{PF} = \begin{pmatrix} 2\cos\frac{\pi}{5}\cos\frac{11\pi}{30} \\ \cos\frac{\pi}{15} \\ 2\cos^2\frac{\pi}{5} \\ 2\cos\frac{2\pi}{30}\cos\frac{\pi}{30} \\ 2\cos\frac{4\pi}{15}\cos\frac{\pi}{5} + \frac{1}{2} \\ 2\cos\frac{\pi}{5}\cos\frac{7\pi}{30} \\ 2\cos\frac{\pi}{30}\cos\frac{11\pi}{30} \\ \cos\frac{11\pi}{30} \end{pmatrix}$$

2.10. Another form of the eigenvectors' matrix. The coordiantes of all eigenvectors of $A(E_8)$ may also be obtained from the coordinates of the PF vector by some permutations and sign changes.

Namely, if $(z_1, \ldots, z_8)$ is a PF vector then the other eigenvectors are the columns of the matrix

$$Z = \begin{pmatrix} z_1 & z_7 & z_4 & z_2 & z_2 & z_4 & z_7 & z_1 \\ z_2 & z_1 & -z_7 & -z_4 & z_4 & z_7 & -z_1 & -z_2 \\ z_3 & z_6 & z_5 & z_8 & -z_8 & -z_5 & -z_6 & -z_3 \\ z_4 & z_2 & -z_1 & -z_7 & -z_7 & -z_1 & z_2 & z_4 \\ z_5 & -z_8 & -z_3 & z_6 & -z_6 & z_3 & z_8 & -z_5 \\ z_6 & -z_5 & -z_8 & z_3 & z_3 & -z_8 & -z_5 & z_6 \\ z_7 & -z_4 & z_2 & -z_1 & z_1 & -z_2 & z_4 & -z_7 \\ z_8 & -z_3 & z_6 & -z_5 & -z_5 & z_6 & -z_3 & z_8 \end{pmatrix}$$

The group of permutations involved is isomorphic to

$$U(\mathbb{Z}/30\mathbb{Z})/\{\pm 1\}.$$

These eigenvectors differ from the ones given by the formula (2.9.1) : the latter ones are proportional to the former ones.

§3. Givental's q-deformations

In the paper [Giv] A.Givental studies the vanishing cycles of mutlivalued of multivalued functions of the form

$$f(x_1, \ldots, x_n)^q,$$

and develops a q-analog of the Picard - Lefschetz theory.

Motivated by his theory, we suggest a

3.1. Definition. *Let (Q, A, L) be a polarized lattice. We define a q-deformed Cartan matrix by*

$$A(q) = L + qL^t.$$

Let $A = (a_{ij}) \in \mathfrak{gl}_r(\mathbb{C})$ be a symmetric matrix, and

$$A = L + L^t,$$

the standard polarization, with L upper triangular. Thus, $L = (\ell_{ij})$, with $\ell_{ii} = a_{ii}/2$, and

$$\ell_{ij} = \begin{cases} a_{ij} & \text{if } i < j \\ 0 & \text{if } i > j \end{cases}$$

Let us assign to A its "Dynkin graph" $\Gamma(A)$ having $\{1, \ldots, r\}$ as the set of vertices, vertices i and j being connected by an edge iff $a_{ij} \neq 0$.

3.2. Theorem. *Let us suppose that $\Gamma(A)$ is a tree. Then :*

(i) *The eigenvalues of $A(q)$ have the form*

$$\lambda(q) = 1 + (\lambda - 2)q^{1/2} + q \qquad (3.2.1)$$

where λ is an eigenvalue of A.

(ii) *If*

$$x = (x_1, \ldots, x_r)$$

is an eigenvector of A with the eigenvalue λ then the eigenvector $x(q)$ of $A(q)$ with the eignevalue $\lambda(q)$ has the form

$$x(q) = (q^{n_1}x_1, \ldots, q^{n_r}x_r),$$

with $n_i \in \frac{1}{2}\mathbb{Z}$.

3.3. Remark (M.Finkelberg). The expression (3.2.1) resembles the number of points of an elliptic curve X over a finite field $\mathbb{F}_q$. To appreciate better this resemblance, note that in all our examples λ has the form

$$\lambda = 2 - 2\cos\theta,$$

so if we set

$$\alpha = \sqrt{q}e^{i\theta}$$

("a Frobenius root") then $|\alpha| = \sqrt{q}$, and

$$\lambda(q) = 1 - \alpha - \bar{\alpha} + q \text{ " } = \text{ " } |X(\mathbb{F}_q)|$$

3.4. Example.

$$A_{E_8}(q) = \begin{pmatrix} 1+q & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+q & 0 & -1 & 0 & 0 & 0 & 0 \\ -q & 0 & 1+q & -1 & 0 & 0 & 0 & 0 \\ 0 & -q & -q & 1+q & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q & 1+q & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q & 1+q & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -q & 1+q & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -q & 1+q \end{pmatrix}$$

Its eigenvalues are

$$\lambda(q) = 1 + q + (\lambda - 2)\sqrt{q} = 1 + q - 2\sqrt{q}\cos\theta$$

where $\lambda = 2 - 2\cos\theta$ is an eigenvalue of $A(E_8)$.

If $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ is an eigenvector of $A(E_8)$ for the eigenvalue λ, then

$$X = (x_1, \sqrt{q}x_2, \sqrt{q}x_3, qx_4, q\sqrt{q}x_5, q^2x_6, q^2\sqrt{q}x_7, q^3x_8) \qquad (4.5.1)$$

is an eigenvector of $A_{E_8}(q)$ for the eigenvalue $\lambda(q)$.

References

[BS] L.Brillon, V.Schechtman, Coxeter element and particle masses, http ://www.math.univ-toulouse.fr/ schechtman/masse3.pdf

[BSV] L.Brillon, V.Schechtman, A. Varchenko, Vanishing cycles and Cartan eigenvectors, http ://xxx.lanl.gov/abs/1509.05591.

[Giv] A.Givental, Twisted Picard - Lefschetz formulas, *Funct. Anal. Appl.*, **22** (1988), 12 - 22.

[K] B.Kostant, The principal three-dimensional subgroup and the Betty numbers of a complex simple Lie group, *Amer. J. Math.* **LXXXI** (1959), 973 - 1032.