LOCALIZATION OF DELIGNE GROUPOIDS

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1. Deligne functor

1.1. We fix a field k of characteristic 0. Let L be a dg Lie algebra over k, A — an Artin local k-algebra over k with the residue field k. Following Deligne, define a groupoid $\mathcal{G}(L; A)$ as follows, cf. [GM1], Sec. 2.

Let $\mathbf{m}$ be the maximal ideal of A. The Lie algebra $L_{\mathbf{m}} := L \otimes \mathbf{m}$ is nilpotent; hence so is $L^0_{\mathbf{m}}$. Let $G^0_{\mathbf{m}}$ be the coresponding Lie group. This group acts on $L^1_{\mathbf{m}}$ by the following rule. Given $g = \exp(\lambda) \in G^0_{\mathbf{m}}$ where $\lambda \in L^0_{\mathbf{m}}$, and $\alpha \in L^1_{\mathbf{m}}$ we set

$$g \circ \alpha = \exp(\mathrm{ad}(\lambda))(\alpha) + \frac{\mathrm{Id} - \exp(\mathrm{ad}(\lambda))}{\mathrm{ad}(\lambda)}(d\lambda) \tag{1}$$

Consider the map

$$Q_A : L^1_{\mathbf{m}} \longrightarrow L^2_{\mathbf{m}}$$

defined as $Q_A(\alpha) = d\alpha + \frac{1}{2}[\alpha, \alpha]$. One shows that the action (1) respects the subspace $\ker(Q_A) \subset L^1_{\mathbf{m}}$.

By definition, $\mathcal{O}b(\mathcal{G}(L; A)) = \ker(Q_A)$, and

$$\mathrm{Hom}_{\mathcal{G}(L;A)}(\alpha, \beta) = \{g \in G^0_{\mathbf{m}} | g \circ \alpha = \beta\}.$$

This way we get a (2?)-functor

$$\mathcal{G}(L) : \mathcal{A}rtin_k \longrightarrow \mathcal{G}roupoids,\ A \mapsto \mathcal{G}(L; A)$$

from the category of Artin local k-algebras with the residue field k to the (2?)-category of groupoids.

1.2. Consider the composition of $\mathcal{G}(L)$ with the functor of the set of connected components

$$\pi_0 : \mathcal{G}roupoids \longrightarrow \mathcal{S}ets.$$

We get

$$\pi_0(\mathcal{G}(L)) : \mathcal{A}rtin_k \longrightarrow \mathcal{S}ets$$

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1.2.1. **Conjecture.** *The functor $\pi_0(\mathcal{G}(L))$ is prorepresentable by a complete local k-algebra*

$$R_L = H_0^{Lie}(L)$$

Recall that by definition

$$H_i^{Lie}(L) = H^{-i}(C(L))$$

where

$$C : \mathcal{D}glie_k \longrightarrow \mathcal{C}ocomcoalg_k$$

is the Quillen chain functor.

If $H^0(L) = 0$ this seems to be proved in [GM2], Sect. 1 and 3.11.

1.2.2. **Question.** Is the functor $\pi_1(\mathcal{G}(L))$ prorepresentable? Maybe by $H_1^{Lie}(L)$?

1.3. One should certainly understand better the nature of the functor $\mathcal{G}$.

It is quite possible that one can express $\mathcal{G}(L; A)$ in terms of the cocommutative coalgebra $C(L_A)$.

If this is so, one could immediately generalize the definition of $\mathcal{G}(L)$ to *S-(homotopy) Lie algebras* — this is important in the sequel.

1.4. In fact, one could figure out this generalization directly. Let L be an S-Lie algebra. The analogue of the mapping Q_A is the mapping

$$Q_A(\alpha) = d\alpha + \frac{1}{2!}[\alpha, \alpha] + \frac{1}{3!}[\alpha, \alpha, \alpha] + \dots$$

where we use "higher brackets"

$$[\cdot, \dots, \cdot] : (L^1)^{\otimes i} \longrightarrow L^2.$$

Now suppose that L^0 is an honest Lie algebra (which is true in interesting cases). It is quite probable that the action of $G^0_{\mathbf{m}}$ respects $\ker(Q_A)$.

Then one proceeds exactly as in the original definition.

1.5. Actually, maybe the above definition of $\mathcal{G}(L)$ is right only in the assumtion $H^0(L) = 0$ (we allways suppose that $H^i(L) = 0$ for $i < 0$).

Let us consider $\pi_0(\mathcal{G}(L; A))$. It looks like a "cohomology group": first one takes $\ker(Q_A)$, and then factorizes by the action of $G^0_{\mathbf{m}}$. (This operation also resembles "a symplectic reduction").

Note that groupoids $\mathcal{G}(L; A)$ depend, so to say, only on $H^1(L)$ and $H^2(L)$, see [GM1], 2.4.

In any case, $\mathcal{G}$ takes quasi-isomorphisms into equivalences, so for every $A \in \mathcal{A}rtin_k$ we get

$$\mathcal{G}(A) : \mathcal{H}olie_k \longrightarrow \mathcal{G}roupoids.$$

1.5.1. Maybe one could define "∞-groupoids" $\mathcal{G}^\infty(L; A)$ which comprise "all" the information about L.

Maybe the corresponding functors

$$\pi_i(\mathcal{G}^\infty(L)) : \mathcal{A}rtin_k \longrightarrow \mathcal{S}ets$$

are prorepresented by $H_i^{Lie}(L)$?

And of course the Deligne functor should be the truncation $\tau_{\leq 1}\mathcal{G}^\infty$.

2. Localization

2.1. Let X be a topological space, $\mathcal{L}$ a sheaf of dg k-Lie algebras over X. Applying the Deligne functor, we get a bi-(2-)functor

$$\mathcal{G}(\mathcal{L}) : \mathcal{S}ite(X) \times \mathcal{A}rtin_k \longrightarrow \mathcal{G}roupoids, \ (U, A) \mapsto \mathcal{G}(\Gamma(U, \mathcal{L}); A)$$

where $\mathcal{S}ite(X)$ denotes the opposite of the category of open subsets of X.

If $A \in \mathcal{A}rtin_k$ is fixed, we get a functor

$$\mathcal{G}(\mathcal{L}; A) : \mathcal{S}ite(X) \longrightarrow \mathcal{G}roupoids.$$

It seems that these functors in general do not form a "champ". Let us pass to associated champs. We get functors:

$$\mathcal{G}(\mathcal{L})^\natural : \mathcal{S}ite(X) \times \mathcal{A}rtin_k \longrightarrow \mathcal{G}roupoids \tag{2}$$

2.2. It seems that "in good cases" there exists an open covering $X = \cup U_i$ such that $\mathcal{G}(\mathcal{L})|_{U_i} = \mathcal{G}(\mathcal{L})^\natural|_{U_i}$.

(Example. X a scheme, $\mathcal{L}$—$\mathcal{O}_X$-quasicoherent, U_i affine.)

Suppose this to be true and fix such a covering $\mathcal{U} = \{U_i\}$. Let us consider the corresponding Čech complex

$$\check{C}(\mathcal{U}, \mathcal{L})$$

— it is a cosimplicial dg Lie algebra; let $\mathcal{L}^\natural(X)$ denote "the" corresponding S-Lie algebra. Actually it is nothing but $R\Gamma^{Lie}(X, \mathcal{L})$.

2.2.1. **Main conjecture.** *We have a natural in A equivalence of groupoids*

$$\mathcal{G}(\mathcal{L})^\natural(X, A) \cong \mathcal{G}(\mathcal{L}^\natural(X), A)$$

3. Application to deformations

3.1. Let X be a smooth scheme over k. We will study smooth deformations of X. Let

$$\mathrm{Def}_X : \mathcal{A}rtin_k \longrightarrow \mathcal{G}roupoids$$

denote the functor of infinitesimal deformations.

3.1.1. **Conjecture.** *Suppose X is affine, $X = \mathrm{Spec}(B)$. There exists a (natural in B) dg Lie algebra L_B and an equivalence of functors*

$$\mathrm{Def}_X \xrightarrow{\sim} \mathcal{G}(L).$$

One has $H^i(L_B) = 0$ for $i \neq 0$ and

$$H^0(L_B) = \mathrm{Der}_k(B).$$

(In the case of non-smooth X one should consider a cotangent complex.)

If this is true, we get a sheaf of dg Lie algebras $\mathcal{L}_X$ over X for affine X, with $\mathcal{H}^0(\mathcal{L}_X) \cong \mathcal{T}_X$.

3.2. Now suppose X be arbirtary. Let

$$\mathrm{Def}_X^{\sim} : \mathcal{S}ite(X) \times \mathcal{A}rtin_k \longrightarrow \mathcal{G}roupoids$$

be the "presheaf" corresponding to Def_X.

3.2.1. **Conjecture.** $\mathrm{Def}_X^{\sim}$ is a champ.

Now from 2.2.1 it follows:

Theorem (conditional). *We have a natural equivalence of functors $\mathcal{A}rtin_k \longrightarrow \mathcal{G}roupoids$*

$$\mathrm{Def}_X \xrightarrow{\sim} \mathcal{G}(R\Gamma^{Lie}(X, \mathcal{T}_X))$$

Hence, by 1.2.1 we get

Corollary (conditional). *Let R_X be a complete local ring of the universal infinitesimal deformation. We have a natural isomorphism*

$$R_X \cong H_0^{Lie}(R\Gamma^{Lie}(X, \mathcal{T}_X))$$

References

[GM1] W. Goldman, J. Millson, The deformation theory of representations of fundamental groups of compact Kähler manifolds, *Publ. IHES*, **67** (1988), 43-96.

[GM2] W. Goldman, J. Millson, The homotopy invariance of the Kuranishi space, *Ill. J. Math.*, **34** (1990), 337-367.