Chapter 1

The complex exponential function

This is a *very important* function !

1.1 The series

For any $z \in \mathbf{C}$, we define:

$$\exp(z) := \sum_{n\geq 0} \frac{1}{n!} z^n = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots$$

On the closed disk

$$\overline{\mathrm{D}}(0,R) := \{z \in \mathbf{C} \mid |z| \leq R\},$$

one has $\left|\frac{1}{n!} z^n\right| \leq \frac{1}{n!} R^n$ and we know that the series $\sum_{n\geq 0} \frac{1}{n!} R^n$ converges for any $R > 0$. Therefore, $\exp(z)$ is a normally convergent series of continuous functions, and $z \mapsto \exp(z)$ is *a continuous function from* $\mathbf{C}$ *to* $\mathbf{C}$.

Theorem 1.1.1 *For any $a, b \in \mathbf{C}$, one has $\exp(a+b) = \exp(a)\exp(b)$.*

Proof. - We just show the calculation, but this should be justified by arguments from real analysis (absolute convergence implies commutative convergence):

$$\begin{aligned}
\exp(a+b) &= \sum_{n\geq 0} \frac{1}{n!} (a+b)^n \\
&= \sum_{n\geq 0} \frac{1}{n!} \sum_{k+l=n} \frac{(k+l)!}{k!l!} a^k b^l \\
&= \sum_{\substack{n\geq 0 \\ k+l=n}} \frac{1}{n!} \frac{(k+l)!}{k!l!} a^k b^l \\
&= \sum_{k,l\geq 0} \frac{1}{k!l!} a^k b^l \\
&= \exp(a)\exp(b).
\end{aligned}$$

□

We now give a list of basic, easily proved properties. First, the effect of complex conjugation:

$$\forall z \in \mathbf{C}\,,\ \overline{\exp(z)} = \exp(\overline{z}).$$

Since obviously $\exp(0) = 1$, on draws from the theorem:

$$\forall z \in \mathbf{C}\,,\ \exp(z) \in \mathbf{C}^* \text{ and} \exp(-z) = \frac{1}{\exp(z)}\cdot$$

Also, $z \in \mathbf{R} \Rightarrow \exp(z) \in \mathbf{R}^*$ and then, writing $\exp(z) = \big(\exp(z/2)\big)^2$, one sees that $\exp(z) \in \mathbf{R}_+^*$.

Last, if $z \in \mathrm{i}\mathbf{R}$ (pure imaginary), then $\overline{z} = -z$, so putting $w := \exp(z)$, one has $\overline{w} = w^{-1}$ so that $|w| = 1$. In other words, exp sends $\mathrm{i}\mathbf{R}$ to the unit circle $\mathbf{U} := \{z \in \mathbf{C} \mid |z| = 1\}$.

Summarizing, if $x := \mathrm{Re}(z)$ and $y := \mathrm{Im}(z)$, then exp sends z to $\exp(z) = \exp(x)\exp(\mathrm{i}y)$, where $\exp(x) \in \mathbf{R}_+^*$ and $\exp(\mathrm{i}y) \in \mathbf{U}$.

Exercice 1.1.2 For $z \in \mathbf{C}$, define $\cos(z) := \dfrac{\exp(z) + \exp(-z)}{2}$ and $\sin(z) := \dfrac{\exp(z) - \exp(-z)}{2\mathrm{i}}$, so that cos is an even function, sin is an odd function and $\exp(z) = \cos(z) + \mathrm{i}\sin(z)$. Translate the property of theorem 1.1.1 into properties of cos and sin.

1.2 The function exp is C-derivable

Lemma 1.2.1 *If $|z| \leq R$, then $|\exp(z) - 1 - z| \leq \dfrac{e^R}{2}|z|^2$.*

Proof. - $|\exp(z) - 1 - z| = \dfrac{z^2}{2}\left(1 + \dfrac{z}{6} + \dfrac{z^2}{24} + \cdots\right)$ and $\left|1 + \dfrac{z}{6} + \dfrac{z^2}{24} + \cdots\right| \leq 1 + \dfrac{R}{6} + \dfrac{R^2}{24} + \cdots \leq e^R$.
□

Theorem 1.2.2 *For any fixed $z_0 \in \mathbf{C}$:*

$$\lim_{h \to 0} = \exp(z_0).$$

Proof. - $\dfrac{\exp(z_0 + h) - \exp(z_0)}{h} = \exp(z_0)\dfrac{\exp(h) - 1}{h}$ and, after the lemma, $\dfrac{\exp(h) - 1}{h} \to 1$ when $h \to 0$. □

Therefore, exp is derivable with respect to the complex variable: we say that it is $\mathbf{C}$-derivable (we shall change terminology later) and that its $\mathbf{C}$-derivative is itself, which we write $\dfrac{d\exp(z)}{dz} = \exp(z)$ or $\exp' = \exp$.

Corollary 1.2.3 *On* $\mathbf{R}$, exp *restricts to the usual real exponential function; that is, for $x \in \mathbf{R}$, $\exp(x) = e^x$.*

Proof. - The restricted function $\exp : \mathbf{R} \to \mathbf{R}$ sends 0 to 1 and it is its own derivative, so it is the usual real exponential function. □

For this reason, for now on, we shall put $e^z := \exp(z)$ when z is an arbitraty complex number.

Corollary 1.2.4 *For* $y \in \mathbf{R}$, *one has* $\exp(iy) = \cos(y) + i\sin(y)$.

Proof. - Put $f(y) := \exp(\mathrm{i}y)$ and $g(y) := \cos(y) + \mathrm{i}\sin(y)$. These functions satisfy $f(0) = g(0) = 1$ and $f' = \mathrm{i}f$, $g' = \mathrm{i}g$. Therefore the function $h := f/g$ which is well defined from $\mathbf{R}$ to C satisfies $h(0) = 1$ and $h' = 0$, so that it is constant equal to 1. □

Note that this implies the famous formula of Euler $e^{\mathrm{i}\pi} = -1$.

Corollary 1.2.5 *For* $x, y \in \mathbf{R}$, *one has* $e^{x+iy} = e^x(\cos y + i\sin y)$.

Corollary 1.2.6 *The exponential map* $\exp : \mathbf{C} \to \mathbf{C}^*$ *is surjective.*

Proof. - Any $w \in \mathbf{C}^*$ can be written $w = r(\cos\theta + \mathrm{i}\sin\theta)$, so $w = \exp(\ln(r) + \mathrm{i}\theta)$. □

One can find a proof which does not use trigonometry in the preliminary chapter of Rudin.

Exercice 1.2.7 Let $a, b > 0$ and $U := \{z \in \mathbf{C} \mid -a < \mathrm{Re}(z) < a \text{and} -b < \mathrm{Im}(z) < b\}$ (thus, an open rectangle under the identification of $\mathbf{C}$ with $\mathbf{R}$). Assuming $b < \pi$, describe the image $V := \exp(U) \subset \mathbf{C}^*$ and define an inverse map $V \to U$.

The exponential viewed as a map $\mathbf{R}^2 \to \mathbf{R}^2$. It will be useful to consider functions $f : \mathbf{C} \to \mathbf{C}$ as functions $\mathbf{R}^2 \to \mathbf{R}^2$, under the usual identification of $\mathbf{C}$ with $\mathbf{R}^2$: $x + \mathrm{i}y \leftrightarrow (x, y)$. In this way, f is described by $(x,y) \mapsto F(x,y) := \big(A(x,y), B(x,y)\big)$, where $\begin{cases} A(x,y) := \mathrm{Re}\big(f(x+\mathrm{i}y)\big), \\ B(x,y) := \mathrm{Im}\big(f(x+\mathrm{i}y)\big). \end{cases}$

In the case where f is the exponential function exp, we compute easily:

$$\begin{cases} A(x,y) = e^x\cos(y), \\ B(x,y) = e^x\sin(y), \end{cases} \implies F(x,y) = \big(e^x\cos(y), e^x\sin(y)\big).$$

We are going to compare the differential of the map F with the $\mathbf{C}$-derivative of the exponential map. On the one hand, the differential $dF(x,y)$ is the linear map defined by the relation:

$$F(x+u, y+v) = F(x,y) + dF(x,y)(u,v) + o(u,v),$$

where $o(u,v)$ is small compared to the norm of (u,v) when $(u,v) \to (0,0)$. Actually, $dF(x,y)$ can be expressed using partial derivatives:

$$dF(x,y)(u,v) = \Big(\frac{\partial A(x,y)}{\partial x}u + \frac{\partial A(x,y)}{\partial y}v, \frac{\partial B(x,y)}{\partial x}u + \frac{\partial B(x,y)}{\partial y}v\Big).$$

Therefore, it is described by the Jacobian matrix:

$$JF(x,y) = \begin{pmatrix} \frac{\partial A(x,y)}{\partial x} & \frac{\partial A(x,y)}{\partial y} \\ \frac{\partial B(x,y)}{\partial x} & \frac{\partial B(x,y)}{\partial y} \end{pmatrix}$$

On the side of the complex function $f := \exp$, putting $z := x + \mathrm{i}y$ and $h := u + \mathrm{i}v$, we write:

$$f(z+h) = f(z) + hf'(z) + o(h), \text{ that is } \exp(z+h) = \exp(z) + h\exp(z) + o(h)$$

Here, the linear part is $f'(z)h = \exp(z)h$, so we draw the conclusion that (under our correspondance of $\mathbf{C}$ with $\mathbf{R}^2$):

$$hf'(z) \longleftrightarrow dF(x,y)(u,v),$$

that is, comparing real and imaginary parts:

$$\begin{cases} \dfrac{\partial A(x,y)}{\partial x}u + \dfrac{\partial A(x,y)}{\partial y}v = \mathrm{Re}(f'(z))u - \mathrm{Im}(f'(z))v, \\ \dfrac{\partial B(x,y)}{\partial x}u + \dfrac{\partial B(x,y)}{\partial y}v = \mathrm{Im}(f'(z))u + \mathrm{Re}(f'(z))v. \end{cases}$$

Since it must be true for all u, v, we conclude that:

$$JF(x,y) = \begin{pmatrix} \dfrac{\partial A(x,y)}{\partial x} & \dfrac{\partial A(x,y)}{\partial y} \\ \dfrac{\partial B(x,y)}{\partial x} & \dfrac{\partial B(x,y)}{\partial y} \end{pmatrix} = \begin{pmatrix} \mathrm{Re}(f'(z)) & -\mathrm{Im}(f'(z)) \\ \mathrm{Im}(f'(z)) & \mathrm{Re}(f'(z)) \end{pmatrix}$$

As a consequence, the Jacobian determinant $\det JF(x,y)$ is equal to $|f'(z)|^2$ and thus vanishes if and only if $f'(z) = 0$: in the case of the exponential function, it vanishes nowhere.

Exercice 1.2.8 Verify these formulas when $A(x,y) = e^x \cos(y)$, $B(x,y) = e^x \sin(y)$ and $f'(z) = \exp(x + \mathrm{i}y)$.

1.3 The exponential function as a covering map

From equation $e^{x+\mathrm{i}y} = e^x(\cos y + \mathrm{i}\sin y)$, one sees that $e^z = 1 \Leftrightarrow z \in 2\mathrm{i}\pi\mathbf{Z}$, *i.e.* $\exists k \in \mathbf{Z} \,:\, z = 2\mathrm{i}\pi k$. It follows that $e^{z_1} = e^{z_2} \Leftrightarrow z_2 - z_1 \in 2\mathrm{i}\pi\mathbf{Z}$, *i.e.* $\exists k \in \mathbf{Z} \,:\, z_2 = z_1 + 2\mathrm{i}\pi k$. We shall write this relation: $z_2 \equiv z_1 \pmod{2\mathrm{i}\pi\mathbf{Z}}$ or more shortly $z_2 \equiv z_1 \pmod{2\mathrm{i}\pi}$.

Theorem 1.3.1 *The map* $\exp : \mathbf{C} \to \mathbf{C}^*$ *is a* covering map*, that is: for any* $w \in \mathbf{C}^*$*, there is a neighborhood* $V \subset \mathbf{C}^*$ *of* w *such that* $\exp^{-1}(V) = \bigsqcup U_k$ *(disjoint union), where each* $U_k \subset \mathbf{C}$ *is an open set and* $\exp : U_k \to V$ *is an homeomorphism (a bicontinuous bijection).*

Proof. - Choose a particular $z_0 \in \mathbf{C}$ such that $\exp(z_0) = w_0$. Choose an open neighborhood U_0 of z_0 such that, for any $z', z'' \in U_0$, one has $|z'' - z'| < 2\pi$. Then exp maps bijectively U_0 to $V := \exp(U_0)$. Moreover, one has $\exp^{-1}(V) = \bigsqcup U_k$ where k runs in $\mathbf{Z}$ and the $U_k = U_0 + 2\mathrm{i}\pi k$ are open sets. It remains to show that V is an open set. The most generalizable way is to use the local inversion theorem, since the Jacobian determinant vanishes nowhere. Another way is to choose an open set as in exercice 1.2.7. □

The fact that exp is a covering map is a very important topological property and it has many consequences.

Corollary 1.3.2 (Path lifting property) *Let $a < b$ in $\mathbf{R}$ and let $\gamma : [a,b] \to \mathbf{C}^*$ be a continuous path with origin $\gamma(a) = w_0 \in \mathbf{C}^*$. Let $z_0 \in \mathbf{C}$ be such that $\exp(z_0) = w_0$. Then, there exists a unique* lifting, *a continuous path $\overline{\gamma} : [a,b] \to \mathbf{C}^*$ such that $\forall t \in [a,b]$, $\exp\overline{\gamma}(t) = \gamma(t)$ and subject to the initial condition $\overline{\gamma}(a) = z_0$.*

Exercice 1.3.3 If one chooses another $z_0' \in \mathbf{C}$ such that $\exp(z_0') = w_0$, one gets another lifting $\overline{\gamma}' : [a,b] \to \mathbf{C}^*$ such that $\forall t \in [a,b]$, $\exp\overline{\gamma}'(t) = \gamma(t)$ and subject to the initial condition $\overline{\gamma}'(a) = z_0'$. Show that there is some *constant* $k \in \mathbf{Z}$ such that $\forall t \in [a,b]$, $\overline{\gamma}'(t) = \overline{\gamma}(t) + 2i\pi k$.

Corollary 1.3.4 (Index of a loop with respect to a point) *Let $\gamma : [a,b] \to \mathbf{C}^*$ be a continuous* loop, *that is $\gamma(a) = \gamma(b) = w_0 \in \mathbf{C}^*$. Then, for any lifting $\overline{\gamma}$ of γ, one has $\overline{\gamma}(b) - \overline{\gamma}(a) = 2i\pi n$ for some $n \in \mathbf{Z}$. The number n is the same for all the liftings, it depends only on the loop γ: it is the* index of γ around 0, *written $I(0,\gamma)$.*

Actually, another property of covering maps (the "homotopy lifting property) allows one to conclude that $I(0,\gamma)$ does not change if γ is continuously deformed within $\mathbf{C}^*$: it only depends on the "homotopy class" of γ (see the topology course).

Example 1.3.5 If $\gamma(t) = e^{nit}$ on $[0,2\pi]$, then all liftings of γ have the form $\overline{\gamma}(t) = nit + 2i\pi k$ for some $k \in \mathbf{Z}$ and one finds $I(0,\gamma) = n$.

1.4 The exponential of a matrix

For a complex vector $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbf{C}^n$, we define $||X||_\infty := \max\limits_{1 \leq i \leq n} (|x_i|)$. Then, fot a complex square matrix $A = (a_{i,j})_{1 \leq i,j \leq n} \in \mathrm{Mat}_n(\mathbf{C})$, define the subordinated norm:

$$|||A|||_\infty := \sup_{\substack{X \in \mathbf{C}^n \\ X \neq 0}} \frac{||AX||_\infty}{||X||_\infty} = \max_{1 \leq i \leq n} \sum_{j=1}^{n} \left| a_{i,j} \right|.$$

Then, for the identity matrix, $|||I_n|||_\infty = 1$; and, for a product, $|||AB|||_\infty \leq |||A|||_\infty \, |||B|||_\infty$. It follows easily that $\left|\left|\left|\frac{1}{k!}A^k\right|\right|\right|_\infty \leq \frac{1}{k!} |||A|||_\infty^k$ for all $k \in \mathbf{N}$, so that the series $\sum\limits_{k \geq 0} \frac{1}{k!} A^k$ converges absolutely for any $A \in \mathrm{Mat}_n(\mathbf{C})$. It actually converges normally on all compacts and therefore define a continuous map $\exp : \mathrm{Mat}_n(\mathbf{C}) \to \mathrm{Mat}_n(\mathbf{C})$, $A \mapsto \sum\limits_{k \geq 0} \frac{1}{k!} A^k$. We shall also write for short $e^A := \exp(A)$. In the case $n = 1$, the notation is consistent.

Examples 1.4.1 (i) For a diagonal matrix $A := \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$, one has $\frac{1}{k!} A^k = \mathrm{Diag}(\lambda_1^k/k!, \ldots, \lambda_n^k/k!)$, so that $\exp(A) = \mathrm{Diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})$.
(ii) If A is an upper triangular matrix with diagonal $D := \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$, then $\frac{1}{k!} A^k$ is an upper triangular matrix with diagonal $\frac{1}{k!} D^k$, so that $\exp(A)$ is an upper triangular matrix with diagonal $\exp(D) = \mathrm{Diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})$. Similar relations hold for lower triangular matrices.

(iii) Take $A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $A^2 = I_2$, so that $\exp(A) = aI_2 + bA = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, where $a = \sum_{k \geq 0} \frac{1}{(2k)!}$ and $b = \sum_{k \geq 0} \frac{1}{(2k+1)!}$.

The same kind of calculations as for the exponential map gives the rules $\exp(0_n) = I_n$; $\exp(\overline{A}) = \overline{\exp(A)}$; and:

$$AB = BA \Longrightarrow \exp(A+B) = \exp(A)\exp(B) = \exp(B)\exp(A).$$

Remark 1.4.2 The condition $AB = BA$ is required to use the Newton binomial formula. If we take for instance $A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $A := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, then $AB \neq BA$. We have $A^2 = B^2 = 0$, so that $\exp(A) = I_2 + A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\exp(B) = I_2 + B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, thus $\exp(A)\exp(B) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. On the other hand, $A + B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the previous example gave the value of $\exp(A+B)$, which was different.

It follows from the previous rules that $\exp(-A) = \big(\exp(A)\big)^{-1}$ so that exp actually sends $\mathrm{Mat}_n(\mathbf{C})$ to $\mathrm{GL}_n(\mathbf{C})$. Now there are rules more specific to matrices. For the transpose, using the fact that ${}^t(A^k) = ({}^tA)^k$, and also the continuity of $A \mapsto {}^tA$ (this is required to go to the limit in the infinite sum), we see that $\exp({}^tA) = {}^t(\exp(A))$. Last, if $P \in \mathrm{GL}_n(\mathbf{C})$, from the relation $(PAP^{-1})^n = PA^nP^{-1}$ (and also from the continuity of $A \mapsto PA^nP^{-1}$), we deduce the very useful equality:

$$P\exp(A)P^{-1} = \exp(PAP^{-1}).$$

Now any complex matrix A is conhugate to an upper triangular matrix T having the eigenvalues of A on the diagonal; using the exaamples above, one concludes that if A has eigenvalues $\lambda_1, \ldots, \lambda_n$, then $\exp(A)$ has eigenvalues $e^{\lambda_1}, \ldots, e^{\lambda_n}$:

$$\mathrm{Sp}(e^A) = e^{\mathrm{Sp}(A)}.$$

Note that this implies $\mathrm{Tr}(e^A) = e^{\det A}$.

Example 1.4.3 Let $A := \begin{pmatrix} 0 & -\pi \\ \pi & 0 \end{pmatrix}$. Then A is diagonalisable with spectrum $\mathrm{Sp}(A) = \{\mathrm{i}\pi, -\mathrm{i}\pi\}$. Thus, $\exp(A)$ is diagonalisable with spectrum $\{-1, -1\}$. Therefore, $\exp(A) = -I_2$.

Exercice 1.4.4 Compute $\exp\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$ in two ways: by diagonalisation as in the example above; by direct calculation as in a previous example. Deduce from this the value of $\exp\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

1.5 Application to differential equations

Let $A \in \mathrm{Mat}_n(\mathbf{C})$ be fixed. Then, $z \mapsto e^{zA}$ is a $\mathbf{C}$-derivable function from $\mathbf{C}$ to the complex linear space $\mathrm{Mat}_n(\mathbf{C})$; this simply means that each coefficient is a $\mathbf{C}$-derivable function from $\mathbf{C}$ to itself.

Derivating our matrix-valued function coefficientwise, we find:

$$\frac{d}{dz}e^{zA} = Ae^{zA} = e^{zA}A.$$

Indeed, $\dfrac{e^{(z+h)A} - e^{zA}}{h} = e^{zA}\dfrac{e^{hA} - I_n}{h} = \dfrac{e^{hA} - I_n}{h}e^{zA}$ and $\dfrac{e^{hA} - I_n}{h} = A + \dfrac{h}{2}A^2 + \cdots$

Now consider the vectorial differential equation:

$$\frac{d}{dz}X(z) = AX(z),$$

where $X : \mathbf{C} \to \mathbf{C}^n$ is searched as a $\mathbf{C}$-derivable vector-valued function, and again derivation is performed coefficientwise. We solve this by changing of unknown function: $X(z) = e^{zA}Y(z)$. Then, applying Leibniz rule for derivation: $(fg)' = f'g + fg'$ (it works the same for $\mathbf{C}$-derivation), we find:

$$X' = AX \Longrightarrow e^{zA}Y' + Ae^{zA}Y = Ae^{zA}Y \Longrightarrow e^{zA}Y' = 0 \Longrightarrow Y' = 0.$$

Therefore, $Y(z)$ is a constant function. If now we fix $z_0 \in \mathbf{C}$, $X_0 \in \mathbf{C}^n$ and we adress the Cauchy problem:

$$\begin{cases} \dfrac{d}{dz}X(z) = AX(z), \\ X(z_0) = X_0, \end{cases}$$

we see that the unique solution is $X(z) := e^{(z-z_0)A}X_0$.

An important theoretical consequence is the following. Call $Sol(A)$ the set of solutions of $\dfrac{d}{dz}X(z) = AX(z)$. This is obviously a complex linear space. What we proved is that the map $X \mapsto X(z_0)$ from $Sol(A)$ to $\mathbf{C}^n$, which is obviously linear, is also bijective. Therefore, it is an isomorphism of $Sol(A)$ with $\mathbf{C}^n$. (This is a very particular case of the Cauchy theorem for *complex* differential equations.)

Example 1.5.1 To solve the *second order scalar equation* (with constant coefficients) $f'' + pf' + qf = 0$ $(p, q \in \mathbf{C})$, we introduce the vector valued function $X(z) := \begin{pmatrix} f(z) \\ f'(z) \end{pmatrix}$ and find that our scalar equation is actually equivalent to the vector equation:

$$X' = AX, \text{ where } A := \begin{pmatrix} 0 & 1 \\ -q & p \end{pmatrix}.$$

Therefore, the solution will be searched in the form $X(z) := e^{(z-z_0)A}X_0$, where z_0 may be chosen at will or else imposed by initial conditions.

Exercice 1.5.2 Compute $e^{(z-z_0)A}$ and solve the problem with initial conditions $f(0) = a$, $f'(0) = b$. There will be a discussion according to whether $p^2 - 4q = 0$ or $\neq 0$.