Journal of Algebra 481 (2017) 91–119

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Defect of an extension, key polynomials and local uniformization

Jean-Christophe San Saturnino

Université Toulouse III Paul Sabatier, Institut de Mathématiques de Toulouse, 118 route de Narbonne, 31062 Toulouse cedex 9, France

A R T I C L E   I N F O

Article history:
Received 24 December 2014
Available online 15 March 2017
Communicated by Steven Dale Cutkosky

MSC:
13A18
12J10
12J20
14B05
14E15

Keywords:
Defect
Valued fields
Local uniformization
Key polynomials

A B S T R A C T

In this article, we prove that the defect of all simple extension of valued field is the product of the effective degrees of the complete set of key polynomials associated with. As a consequence, we obtain a local uniformization theorem for valuations of rank 1 centered on an equicharacteristic quasi-excellent local domain satisfying some inductive assumptions of lack of defect.

0. Introduction

In order to obtain a local uniformization theorem, F.-V. Kuhlmann systematized the study of the defect of an extension of valued fields. We know that this defect is a power of p, the characteristic of the residual field of the valuation. Another approach of the

E-mail address: san@math.ups-tlse.fr.
URL: http://www.math.univ-toulouse.fr/~san/.

http://dx.doi.org/10.1016/j.jalgebra.2017.02.023

local uniformization problem for a valuation of rank 1 consists in studying the set of key polynomials associated with the valuation. The key polynomials have been developed in [10] and [3], a link between the two approaches is given in [7]. When the set of key polynomials does not have limit key polynomials, in [9] it is proved that we can obtain a local uniformization theorem for valuations of rank 1. In particular, it is also proved that, if the residual field of the valuation is of characteristic zero, we have local uniformization of the valuation, result known since Zariski.

In [11], a link has been found between the defect and the key polynomials, namely the jump of the valuation. In the context of [3], this notion coincides with the degree of a limit key polynomial over the degree of a previous key polynomial whose degree is the same as those of an infinite family of key polynomials.

In this paper, we aim at studying more precisely the link between the defect and the key polynomials. We will express the defect as a product of effective degrees and will generalize the result of [11]. The effective degree is closely related to the Newton polygon associated with a key polynomial. We will use this result in order to prove a local uniformization theorem for valuations of rank 1 centered on an equicharacteristic quasi-excellent local domain satisfying some inductive assumptions of defectless of the quotient field.

In the first section, we recall some definitions about the center of a valuation and the graded algebras associated with a valuation.

In the second section we define the defect of a valuation for a valuation ν of a field K having a unique extension μ of a field L. Roughly speaking, the defect is the number obtained when equality holds in the inequality:

$$[L : K] \leq [\Gamma_\mu : \Gamma_\nu][k_\mu : k_\nu],$$

where Γ_ν (resp. Γ_μ) is the value group of ν (resp. μ) and k_ν (resp. k_μ) is the residual field of ν (resp. μ). We know that it is a power of p, the characteristic of k_ν.

In the third section we first recall the definitions of key polynomials given in [3] or [9]. From these definitions, we give some characterizations of a complete set of key polynomials, which is very useful to determine whether a set of polynomials is a complete or a 1-complete set of key polynomials. For a key polynomial Q, we also define the effective degree of a polynomial P corresponding to this key polynomial. When we write the standard expansion of P in terms of Q, the largest power of Q for which the valuation of the monomials in Q is maximal is the effective degree of P. Next, we prove that some graded algebras associated with a valuation are euclidean for the effective degree. We end this section by recalling that the sequence of effective degrees is decreasing and so, that it admits a constant value. If this value is 0 or 1, the set of key polynomial considered is 1-complete.

In the fourth section, we first prove that the defect of an extension of a valuation is the product of the effective degrees of the limit key polynomials provided by the valuation. The proof uses the notion of key polynomials of W. Mahboub (see [8]), so it is true for valuation of any rank. The main idea lies in the fact that the Newton polygon has only one face if the field is henselian. We compute some defects with four examples of M. Vaquié, W. Mahboub and S.D. Cutkosky.

In the next section we prove that there are no limit key polynomials for valuation of rank 1 over a defectless field.

In the final section we generalize the results of [9] and prove a local uniformization theorem for a valuation of rank 1, with some inductive assumptions of local uniformization in lower dimension and no defect for a well-defined extension. From there we can prove a second time the local uniformization for a valuation of rank 1 in characteristic zero.

The author thanks M. Vaquié who suggested to study the precise link between the defect and the limit key polynomial, as well as B. Teissier for his advice on clarification and structuration of the notion of key polynomials. The author also thanks W. Mahboub for his ability to provide examples of key polynomials verifying or not some properties. I gratefully thank M. Spivakovsky for our discussions and careful readings.

Notation. Let ν be a valuation of a field K. We denote $R_\nu = \{f \in K \mid \nu(f) \geqslant 0\}$. It is a local ring whose maximal ideal is given by $m_\nu = \{f \in K \mid \nu(f) > 0\}$. We then denote by $k_\nu = R_\nu/m_\nu$ the residue field of R_ν and $\Gamma_\nu = \nu(K^*)$.

For a field K, we will denote by $\overline{K}$ an algebraic closure of K, by K^{sep} a separable closure of K and by $Aut(\overline{K}|K)$ the group of automorphisms of $\overline{K}|K$.

If R is a ring and I an ideal of R, we will denote by $\widehat{R}^I$ the *I-adic completion* of R. When $(R, \mathfrak{m})$ is a local ring, we will say the *completion* of R instead of the $\mathfrak{m}$-adic completion of R and will denote it by $\widehat{R}$.

For all $P \in Spec(R)$, we denote by $\kappa(P) = R_P/PR_P$ the residue field of R_P.

For $\alpha \in \mathbb{Z}^n$ and $u = (u_1, ..., u_n)$ a n-uplet of elements of R, we write:

$$u^\alpha = u_1^{\alpha_1} ... u_n^{\alpha_n}.$$

For $P, Q \in R[X]$ with $P = \sum_{i=0}^{n} a_i Q^i$ and $a_i \in R[X]$ such that the degree of a_i is strictly less than Q, we write:

$$d_Q^\circ(P) = n.$$

If $Q = X$, we will note simply $d^\circ(P)$ instead of $d_X^\circ(P)$.

Finally, if R is a domain, we denote by $Frac(R)$ its quotient field.

1. Center of a valuation, graded algebra associated and saturation

In this section, Γ denotes a totally ordered commutative group.

Definition 1.1. Let R be a ring and P a prime ideal. A valuation $\nu : R \to \Gamma \cup \{\infty\}$ **centered on P** is given by a minimal prime ideal P_∞ of R included in P and a valuation of the quotient field of R/P_∞ centered on P/P_∞. The ideal P_∞ is the support of the valuation, namely $P_\infty = \nu^{-1}(\infty)$.

If R is a local ring with maximal ideal $\mathfrak{m}$, we will say that ν is **centered on *R*** when ν is centered on $\mathfrak{m}$.

Definition 1.2. Let R be a ring and $\nu : R \to \Gamma \cup \{\infty\}$ a valuation centered on a prime ideal of R. For all $\alpha \in \nu(R \setminus \{0\})$, we define the ideals:

$$\mathcal{P}_\alpha = \{f \in R \mid \nu(f) \geqslant \alpha\};$$
$$\mathcal{P}_{\alpha,+} = \{f \in R \mid \nu(f) > \alpha\}.$$

We define **the graded algebra of *R* associated with *ν*** by:

$$gr_\nu(R) = \bigoplus_{\alpha \in \nu(R\setminus\{0\})} P_\alpha / P_{\alpha,+}.$$

The algebra $gr_\nu(R)$ is a domain. For $f \in R \setminus \{0\}$, we denote by $in_\nu(f)$ its image in $gr_\nu(R)$.

When R is a local domain, we give another definition of graded algebra.

Definition 1.3. Let R be a local domain, $K = Frac(R)$ and $\nu : K^* \twoheadrightarrow \Gamma \cup \{\infty\}$ a valuation of K centered on R. For all $\alpha \in \Gamma$, we define the R_ν-submodules of K as follows:

$$P_\alpha = \{f \in K \mid \nu(f) \geqslant \alpha\} \cup \{0\};$$
$$P_{\alpha,+} = \{f \in K \mid \nu(f) > \alpha\} \cup \{0\}.$$

We define **the graded algebra associated at *ν*** by:

$$G_\nu = \bigoplus_{\alpha \in \Gamma} P_\alpha / P_{\alpha,+}.$$

For $f \in K^*$, we write $in_\nu(f)$ its image in G_ν.

Remark 1.4. We have the natural embedding:

$$gr_\nu(R) \hookrightarrow G_\nu.$$

Definition 1.5. Let G be a graded algebra without zero divisors. We call **saturation of *G*** the graded algebra G^* defined by:

$$G^* = \left\{ \frac{f}{g} \;\middle|\; f, g \in G,\ g \text{ homogeneous},\ g \neq 0 \right\}.$$

We say that G is **saturated** when $G = G^*$.

Remark 1.6. For all graded algebra G, we have:

$$G^* = (G^*)^* .$$

In particular, G^* is always saturated.

Example 1.7. Let ν be a valuation centered on a local ring R. Then:

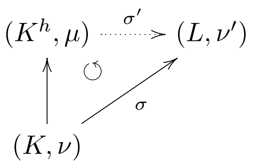
$$G_\nu = (gr_\nu(R))^* .$$

In particular, G_ν is saturated.

2. Defectless fields

Definition 2.1. ([11], Définition 1.2) A valued field (K, ν) is said to be **henselian** if for all algebraic extension $L|K$, there exists a unique valuation μ of L which extends ν.

We call **henselization of (K, ν)** all extension (K^h, μ) of (K, ν) such that (K^h, μ) is henselian and for all henselian valued field (L, ν') and all immersion $\sigma : (K, \nu) \hookrightarrow (L, \nu')$, there exists a unique immersion $\sigma' : (K^h, \mu) \hookrightarrow (L, \nu')$ which extends σ.



Remark 2.2. All the henselizations of a given valued field are isomorphic, it is its smaller henselian extensions. Moreover, the henselization (K^h, μ) of (K, ν) is an immediate extension, that is $\Gamma_\nu = \Gamma_\mu$ and $k_\nu = k_\mu$ (for a proof see Theorem 7.42 of [6]).

The henselization of a given valued field (K, ν) can be constructed explicitly. We give here the construction proposed by Kuhlmann in [6], Chapter 7. Consider μ to be an extension of ν on $\overline{K}$. Write:

$$G^d(\overline{K}|K, \mu) = \{\sigma \in Aut(\overline{K}|K) \mid \forall\, \alpha \in \overline{K},\ \mu(\sigma(\alpha)) = \mu(\alpha)\},$$
$$K^{h(\mu)} = \{\alpha \in K^{sep} \mid \forall\, \sigma \in G^d(\overline{K}|K, \mu), \sigma(\alpha) = \alpha\}.$$

$(K^{h(\mu)}, \mu)$ is an henselization of (K, ν), it is an algebraic separable immediate extension of (K, ν).

Definition 2.3. Let K be a field with a valuation ν. Consider L a finite extension of K and write $\mu_1, ..., \mu_g$ the valuations of L which extend ν. For $1 \leqslant i \leqslant g$, choose a valuation $\overline{\mu}_i$ of $\overline{K}$ whose restriction on L is μ_i. We denote by $K^{h(\overline{\mu}_i)}$ the field constructed previously.

It is an henselization of (K, ν). Moreover, it is a subfield of $L^{h(\overline{\mu}_i)} := L.K^{h(\overline{\mu}_i)}$ which is itself an henselization of (L, μ_i). We define the **defect of the extension $L|K$ in μ_i** by:

$$d_{L|K}(\mu_i, \nu) = \frac{\left[L^{h(\overline{\mu}_i)} : K^{h(\overline{\mu}_i)}\right]}{e_i f_i},$$

where $e_i = [\Gamma_{\mu_i} : \Gamma_\nu]$ and $f_i = [k_{\mu_i} : k_\nu]$.

Remark 2.4. By a theorem of Ostrowski (Lemma 11.17 of [6]), we can show that $d_{L|K}(\mu_i, \nu) = p^{a_i}$, where $a_i \in \mathbb{N}$ and $p = char(k_\nu)$.

Proposition 2.5. *([6], Lemma 7.46) Let K be a field with a valuation ν. Consider L a finite extension of K and denote by $\mu_1, ..., \mu_g$ the valuations of L which extend ν. Then:*

$$[L : K] = \sum_{i=1}^{g} d_i e_i f_i,$$

where $d_i = d_{L|K}(\mu_i, \nu)$, $e_i = [\Gamma_{\mu_i} : \Gamma_\nu]$ and $f_i = [k_{\mu_i} : k_\nu]$.

Definition 2.6. Let K be a field with a valuation ν. Consider L a finite extension of K and denote by $\mu_1, ..., \mu_g$ the valuations of L which extend ν. We call **global defect of the extension $L|K$ on ν** the quotient:

$$d_{L|K}(\nu) = \frac{[L : K]}{\sum\limits_{i=1}^{g} e_i f_i},$$

where $e_i = [\Gamma_{\mu_i} : \Gamma_\nu]$ and $f_i = [k_{\mu_i} : k_\nu]$.

We say that L is a **defectless extension** of K when $d_{L|K}(\nu) = 1$.

Remark 2.7. If the extension $L|K$ is normal, then the global defect is a power of p, where $p = char(k_\nu)$, because $d_{L|K}(\nu) = d_{L|K}(\mu_i, \nu)$, for all $i \in \{1, ..., g\}$ (see [6], Lemma 11.3); otherwise it is a rational number.

Remark 2.8. $d_{L|K}(\nu) = 1$ if and only if for all $i \in \{1, ..., g\}, d_{L|K}(\mu_i, \nu) = 1$.

Definition 2.9. A field K is said to be **defectless** if all finite extension of K is defectless.

Proposition 2.10. *([6], Theorem 11.23) All field with a valuation ν such that $char(k_\nu) = 0$ is a defectless field.*

3. Key polynomials and effective degree

Let $K \hookrightarrow K(x)$ be a simple transcendental extension of fields. Let μ' be a valuation of $K(x)$ and denote $\mu := \mu'_{|\,K}$. Denote by G the group of values of μ' and G_1 the smallest non-zero isolated subgroup of G. Assume that the rank of μ is 1 and $\mu'(x) > 0$. Finally, for $\beta \in G$, recall that:

$$P'_\beta = \{f \in K(x) \mid \mu'(f) \geqslant \beta\} \cup \{0\},$$

$$P'_{\beta,+} = \{f \in K(x) \mid \mu'(f) > \beta\} \cup \{0\},$$

$$G_{\mu'} = \bigoplus_{\beta \in G} P'_\beta / P'_{\beta,+},$$

and $in_{\mu'}(f)$ is the image of $f \in K(x)$ in $G_{\mu'}$.

Definition 3.1. A **complete set of key polynomials** for μ' is a well ordered collection:

$$\mathbf{Q} = \{Q_i\}_{i \in \Lambda} \subset K[x]$$

such that, for all $\beta \in G$, the additive group $P'_\beta \cap K[x]$ is generated by the products under the form $a \prod_{j=1}^{s} Q_{i_j}^{\gamma_j}$, where $a \in K$ and such that $\sum_{j=1}^{s} \gamma_j \mu'\left(Q_{i_j}\right) + \mu(a) \geqslant \beta$.

The set is said to be **1-complete** if the previous condition occurs for all $\beta \in G_1$ and if for all $i \in \Lambda$, we have $\mu'(Q_i) \in G_1$.

Theorem 3.2. *([3], Theorem 7.11) There exists a collection* $\mathbf{Q} = \{Q_i\}_{i \in \Lambda}$ *which is a* 1*-complete set of key polynomials.*

By Theorem 3.2, we know that there exists a 1-complete set of key polynomials $\mathbf{Q} = \{Q_i\}_{i \in \Lambda}$ and that the order type of Λ is at most $\omega \times \omega$. If K is defectless, we will see that the order type of Λ is at most ω and there is no limit key polynomial. In particular, this is the case when $char(k_\mu) = 0$. For all $i \in \Lambda$, denote $\beta_i = \mu'(Q_i)$.

Let $l \in \Lambda$, write:

$$\alpha_i = d^{\circ}_{Q_{i-1}}(Q_i),\ \forall\, i \leqslant l;$$

$$\boldsymbol{\alpha}_{\boldsymbol{l+1}} = \{\alpha_i\}_{i \leqslant\, l};$$

$$\mathbf{Q}_{l+1} = \{Q_i\}_{i \leqslant\, l}.$$

We also use the notation $\overline{\gamma}_{l+1} = \{\gamma_i\}_{i \leqslant\, l}$, where the γ_i are all zero, except for a finite number and $\mathbf{Q}_{l+1}^{\overline{\gamma}_{l+1}} = \prod_{i \leqslant\, l} Q_i^{\gamma_i}$.

Definition 3.3. A multiindex $\overline{\gamma}_{l+1}$ is said to be **standard with respect to $\boldsymbol{\alpha}_{\boldsymbol{l+1}}$** if $0 \leqslant \gamma_i < \alpha_{i+1}$, for $i \leqslant l$.

An **l-standard monomial in $\boldsymbol{Q_{l+1}}$** is a product of the form $c_{\overline{\gamma}_{l+1}}\mathbf{Q}_{l+1}^{\overline{\gamma}_{l+1}}$, where $c_{\overline{\gamma}_{l+1}} \in K$ and $\overline{\gamma}_{l+1}$ is standard with respect to $\boldsymbol{\alpha_{l+1}}$.

An **l-standard expansion not involving $\boldsymbol{Q_l}$** is a finite sum $\sum\limits_{\beta} S_\beta$ of l-standard monomials not involving Q_l, where β ranges over a some finite subset of G_+ and $S_\beta = \sum\limits_j d_{\beta,j}$ is a sum of standard monomials of value β such that $\sum\limits_j in_{\mu'}(d_{\beta,j}) \neq 0$.

Definition 3.4. Let $f \in K[x]$ and $i \leqslant l$. An **i-standard expansion of f** is an expression of the form:

$$f = \sum_{j=0}^{s_i} c_{j,i} Q_i^j,$$

where $c_{j,i}$ is an i-standard expansion not involving Q_i.

Remark 3.5. Such expansion exists by Euclidean division and is unique, in the sense of that the $c_{j,i} \in K[x]$ are unique. More precisely, if $i \in \mathbb{N}$, we can prove by induction that the i-standard expansion is unique.

Definition 3.6. Let $f \in K[x]$, $i \leqslant l$ and $f = \sum\limits_{j=0}^{s_i} c_{j,i} Q_i^j$ be an i-standard expansion of f. We define the **i-truncation of $\boldsymbol{\mu'}$**, denoted by μ'_i, as being the following pseudo-valuation:

$$\mu'_i(f) = \min_{0 \leqslant j \leqslant s_i} \{ j\mu'(Q_i) + \mu'(c_{j,i}) \}.$$

Remark 3.7. We can prove that μ'_i is a valuation. Moreover, we have:

$$\forall\, f \in K[x],\ i \in \Lambda,\ \mu'_i(f) \leqslant \mu'(f).$$

Proposition 3.8. *Let $\{Q_i\}_{i\in\Lambda} \subset K[x]$. Denote by $H = \{f \in K[x] \mid \mu'(f) \notin G_1\}$, and for a graded algebra G, denote by G^* its saturation (see Definition 1.5). Consider the following assertions:*

1. *The set $\{Q_i\}_{i\in\Lambda}$ is 1-complete.*
2. *For all $f \in K[x] \setminus H$, there exists $i \in \Lambda$ such that, $\mu'_i(f) = \mu'(f)$.*
3. *For all $i \in \Lambda$, $Q_i \notin H$ and there exists $h \in H$ monic or zero such that the set $\{Q_i\}_{i\in\Lambda} \cup \{h\}$ is complete.*
4. *There exists $h \in H$ monic or zero such that $G_{\mu'} = (G_\mu [\{in_{\mu'}(Q_i)\}_{i\in\Lambda}, in_{\mu'}(h)])^*$.*

Then 1. $\Leftrightarrow$ 2. $\Leftrightarrow$ 3. *and* 1. $\Rightarrow$ 4.

Proof. Let us show that $\{Q_i\}_{i\in\Lambda}$ is 1-completed if and only if for all $f \in K[x] \setminus H$ there exists $i \in \Lambda$ such that $\mu'_i(f) = \mu'(f)$.

Assume $\{Q_i\}_{i\in\Lambda}$ is 1-completed, and consider $f \in K[x]$ such that $\mu'(f) \in G_1$. Denote $\beta = \mu'(f)$. Then $P'_\beta \cap K[x]$ is generated by the products of the form $a\prod_{j=1}^{s} Q_{i_j}^{\gamma_j}$, $a \in K$, with $\sum_{j=1}^{s} \gamma_j \mu'\left(Q_{i_j}\right) + \mu(a) \geqslant \beta = \mu'(f)$. Thus $\mu'_{i_s}(f) \geqslant \mu'(f)$. Moreover, by definition, for all $i \in \Lambda$, we have $\mu'_i(f) \leqslant \mu'(f)$. We conclude that $\mu'_{i_s}(f) = \mu'(f)$. Conversely, if all $f \in K[x]$ such that $\mu'(f) \in G_1$ satisfies $\mu'_i(f) = \mu'(f)$, for some $i \in \Lambda$, then, f writes as a sum of monomials in $\boldsymbol{Q}_{i+1}$ of value at least $\mu'(f) \geqslant \beta$. So f lies in the ideal generated by all there monomials. We can show in the same way that $\{Q_i\}_{i\in\Lambda}$ is completed if and only if for all $f \in K[x]$ and for all $i \in \Lambda$, we have $\mu'_i(f) = \mu'(f)$.

Let us show that $\{Q_i\}_{i\in\Lambda}$ is 1-completed if and only if for all $i \in \Lambda$, $Q_i \notin H$ and there exists $h \in H$ monic or zero such that the set $\{Q_i\}_{i\in\Lambda} \cup \{h\}$ is completed.

If $\{Q_i\}_{i\in\Lambda}$ is completed then, by definition, for all $i \in \Lambda$ we have $\mu'(Q_i) \in G_1$ and take $h = 0$. If $\{Q_i\}_{i\in\Lambda}$ is not completed, there exists an element $h \in K[x]$ such that $\mu'(h) \notin G_1$ and $\mu'_i(h) < \mu'(h)$, for all $i \in \Lambda$. Take h of minimal degree and monic satisfying the previous inequality. For $f \in K[x]$, consider the standard expansion in terms of h: $f = \sum_{j=0}^{s} c_j h^j$. Denote $m = \min\{j \in \{1,...,s\} \mid c_j \neq 0\}$. Denote by μ'_h the truncated valuation in terms of h (because $\mu'(h) \notin G_1$). Then:

$$\mu'_h(f) = \mu'_h(c_m h^m) = \mu'(c_m h^m) = \mu'(f).$$

The converse is obvious.

Finally, if $\{Q_i\}_{i\in\Lambda}$ is 1-completed, then there exists $h \in H$ monic or zero such that the set $\{Q_i\}_{i\in\Lambda} \cup \{h\}$ is completed, that is $P'_\beta \cap K[x]$ is generated by elements of the form $a\prod_{j=1}^{s} g_j^{\gamma_j}$ where $g_j \in \{Q_i\}_{i\in\Lambda} \cup \{h\}$. So we deduce that $G_{\mu'} = (G_\mu\left[\{in_{\mu'}(Q_i)\}_{i\in\Lambda}, in_{\mu'}(h)\right])^*$. □

Remark 3.9. Note that 4. $\not\Rightarrow$ 1. Indeed, consider Example 2.2 of [5], if k is a field and $\iota : k(u,v) \hookrightarrow k\left[[t]\right]$ a monomorphism such that $\iota(v) = t$ and $\iota(u) = \sum_{i\geqslant 1} c_i t^i$ with $c_i \in k^*$. Assume that $Q_\infty = u - \sum_{i\geqslant 1} c_i v^i$ is transcendental over $k(u,v)$. Denote by μ' the valuation of $k(u,v)$ induced by the t-adic valuation of $k\left[[t]\right]$ through ι, and by μ the restriction of μ' on $K = k(v)$. The set of key polynomials associated with the extension $K \hookrightarrow K(u)$ is $\{Q_i\}_{i\geqslant 1}$ with $Q_1 = u$ and $Q_i = u - \sum_{j=1}^{i-1} c_i v^i$ and $G_{\mu'} = k_\mu[in_{\mu'}(u)]$.

In the definition below, we use the terminology of [12] with the notation of [3].

Definition 3.10. Let $h \in K[x]$. Consider its i-standard expansion $h = \sum_{j=0}^{s_i} c_{j,i} Q_i^j$. We call the ***i*-th effective degree of h** the natural number:

$$\delta_i(h) = \max\{j \in \{0, ..., s_i\} \mid j\beta_i + \mu'(c_{j,i}) = \mu'_i(h)\}.$$

By convention, $\delta_i(0) = -\infty$.

Remark 3.11. If we denote:

$$in_i(h) = \sum_{j \in S_i(h,\beta_i)} in_{\mu'}(c_{j,i})X^j \in G_\mu[in_{\mu'}(\mathbf{Q}_i), X],$$

where $S_i(h, \beta_i) = \{j \in \{0, ..., s_i\} \mid j\beta_i + \mu'(c_{j,i}) = \mu'_i(h)\}$, then $\delta_i(h) = d^\circ_X(in_i(h))$.

Moreover, if $in_{\mu'_i}(h) = in_{\mu'_i}(g)$ then $\delta_i(h) = \delta_i(g)$. The definition of the effective degree extends naturally to $gr_{\mu'_i}(K[x])$.

Remark 3.12. Remind that, by Proposition 5.2 of [3], for all ordinal number $l \in \Lambda$, the sequence $(\delta_{l+i}(h))_{i\in\mathbb{N}^*}$ is non-increasing. Thus there exists $i_0 \in \mathbb{N}^*$ such that $\delta_{l+i_0}(h) = \delta_{l+i_0+i}(h)$, for all $i \geqslant 1$ and we denote this common value by $\delta_{l+\omega}(h)$ or $\delta_{l+\omega}$ if no confusion is possible. Note that $\delta_{l+\omega}$ could be equal to 0.

The next three lemmas generalize Lemmas 2.12, 2.13 and 2.14 of [2]. The proofs are almost the same.

Lemma 3.13. *Let $l \in \Lambda$. Then for all $f, g \in K[x]$ we have:*

$$\delta_l(fg) = \delta_l(f) + \delta_l(g).$$

Proof. It is a direct consequence of Remark 3.11. □

Lemma 3.14. *For all $h \in K[x]$, $\delta_i(h) = 0$ if and only if $in_{\mu'_i}(h)$ is a unit of $gr_{\mu'_i}(K[x])$.*

Proof. If $\delta_i(h) = 0$ then $\mu'_i(h) = \mu'(h)$ and $in_{\mu'_i}(h) = in_{\mu'_i}(c_{0,i})$ in $gr_{\mu'_i}(K[x])$. Since the polynomial Q_i is irreducible in $K[x]$ and $d^\circ(c_{0,i}) < d^\circ(Q_i)$, the polynomials Q_i and $c_{0,i}$ are coprime in $K[x]$. Then, there exists $U, V \in K[x]$ such that $UQ_i + Vc_{0,i} = 1$. We deduce that $\mu'_i(Vc_{0,i}) = \mu'_i(1) < \mu'_i(UQ_i)$ and that h is a unit of $gr_{\mu'_i}(K[x])$.

Conversely, it is sufficient to apply the Lemma 3.13 and the Remark 3.11. □

Lemma 3.15. *For all $i \in \Lambda$, the ring $gr_{\mu'_i}(K[x])$ is euclidean for δ_i. That is to say:*

$$\forall g, h \in K[x],\ h \neq 0,\ \exists\, Q, R \in K[x],\ g = hQ + R \text{ in } gr_{\mu'_i}(K[x]) \text{ and } 0 \leqslant \delta_i(R) < \delta_i(h).$$

Proof. Denote $h = \sum_{j=0}^{s_i} c_{j,i}Q_i^j$, we can assume without loss of generality that $c_{j,i} = 0$ for $j > \delta_i(h)$, because $\mu'(c_{j,i}Q_i^j) > \mu'_i(c_{j,i}Q_i^j)$ and so $in_{\mu'_i}\left(\sum_{j=\delta_i(h)+1}^{s_i} c_{j,i}Q_i^j\right) = 0$. We can also assume that $c_{\delta_i(h),i} = 1$ by Lemma 3.14. Note that $d^\circ(h) = \delta_i(h).d^\circ(Q_i)$.

By euclidean division in $K[x]$, there exist $P, Q \in K[x]$ such that $g = hQ + P$ with $0 \leqslant d\,^\circ(P) < d\,^\circ(h)$. Let $P = \sum_{j=0}^{t_i} p_{j,i} Q_i^j$ be the i-standard expansion of P. If we write $R = \sum_{j=0}^{\delta_i(P)} p_{j,i} Q_i^j$, we obtain that $h = gQ + R$ in $gr_{\mu'_i}(K[x])$ and:

$$\delta_i(P).d\,^\circ(Q_i) \leqslant d\,^\circ(p_{\delta_i(P),i}) + \delta_i(P).d\,^\circ(Q_i) = d\,^\circ(R) \leqslant d\,^\circ(P) < d\,^\circ(h) = \delta_i(h).d\,^\circ(Q_i).$$

Thus, $\delta_i(R) = \delta_i(P) < \delta_i(h)$. □

The construction of the key polynomials is recursive (see [7,3] and [4]). For $l \in \mathbb{N}^*$, we construct a set of key polynomials $\mathbf{Q}_{l+1} = \{Q_i\}_{1 \leqslant i \leqslant l}$; the following alternative holds:

(1) $\exists\, l_0 \in \mathbb{N},\ \beta_{l_0} \notin G_1$;
(2) $\forall\, l \in \mathbb{N},\ \beta_l \in G_1$.

In case (1), we stop the construction. The set $\mathbf{Q}_{l_0} = \{Q_i\}_{1 \leqslant i \leqslant l_0 - 1}$ is by definition a 1-complete set of key polynomials and $\Lambda = \{1, ..., l_0 - 1\}$. Note that the set $\mathbf{Q}_{l_0+1}$ is a complete set of key polynomials.

In case (2), if K is defectless, we will prove in Proposition 5.1 that the set $\mathbf{Q}_\omega = \{Q_i\}_{i \geqslant 1}$ is infinite and that $\Lambda = \mathbb{N}^*$. The next propositions will guarantee that the set of key polynomials so obtained is also 1-complete.

The next lemma, very useful, allows us to note that there does not exist any increasing bounded sequence with valuations of rank 1.

Lemma 3.16. *Let ν be a valuation of rank 1 centered on a local noetherian ring R. Denote by P_∞ the support of ν. Then, $\nu(R \setminus P_\infty)$ does not contain any increasing bounded infinite sequence.*

Remark 3.17. From Definition 1.1, the data of a valuation ν centered on a local ring $(R, \mathfrak{m})$ is the data of a minimal prime ideal P_∞ of R (the support of the valuation) and of a valuation ν' of the quotient field of R/P_∞ such that $R/P_\infty \subset R_{\nu'}$ and $\mathfrak{m}/P_\infty = R/P_\infty \cap m_{\nu'}$.

Proof. Let $(\beta_i)_{i \geqslant 1}$ be an increasing infinite sequence of $\nu(R \setminus P_\infty)$ bounded by β. This sequence corresponds to a decreasing infinite sequence of ideals of R/P_β. It is sufficient to prove that R/P_β is of finite length. Denote by $\mathfrak{m}$ the maximal ideal of R, $\nu(\mathfrak{m}) = \min\{\nu(R \setminus P_\infty) \setminus \{0\}\}$ and let Γ be the value group of ν. Note that the group $\nu(R \setminus P_\infty)$ is archimedean. Indeed, by contradiction, if $\nu(R \setminus P_\infty)$ is not archimedean, there exist $\alpha,\ \beta \in \nu(R \setminus P_\infty)$, $\beta \neq 0$ such that, for all $n \geqslant 1$, $n\beta \leqslant \alpha$. In particular, the set:

$$\{\gamma \in \Gamma \mid \exists\, n \in \mathbb{N} \setminus \{0\},\ -n\beta < \gamma < n\beta\}$$

is a non-trivial isolated subgroup of Γ.

We deduce that there exists $n \in \mathbb{N}$ such that:

$$\beta \leqslant n\nu(\mathfrak{m}).$$

Thus, $\mathfrak{m}^n \subset P_\beta$ and therefore, there exists a surjective map:

$$R/\mathfrak{m}^n \twoheadrightarrow R/P_\beta. \qquad \square$$

Proposition 3.18. *([3], Proposition 3.30) Assume that we have constructed an infinite set of key polynomials $\mathbf{Q}_\omega = \{Q_i\}_{i\geqslant 1}$ such that, for all $i \in \mathbb{N}^*$, $\beta_i \in G_1$. Assume further that the sequence $\{\beta_i\}_{i\geqslant 1}$ is not bounded in G_1. Then, the set of key polynomials $\mathbf{Q}_\omega$ is* 1-*complete.*

Proof. It is sufficient to prove that, for all $\beta \in G_1$ and for all $h \in K[x]$ such that $\mu'(h) = \beta$, h lies in the R_μ-submodule of $K[x]$ generated by all the monomials of the form $a \prod\limits_{j=1}^{s} Q_{i_j}^{\gamma_j}$, $a \in K$, such that $\mu'\left(a \prod\limits_{j=1}^{s} Q_{i_j}^{\gamma_j}\right) \geqslant \beta$.

Therefore consider $h \in K[x]$ such that $\mu'(h) \in G_1$. Write $h = \sum\limits_{j=0}^{d} h_j x^j$. We can assume, without loss of generality, that:

$$\forall\, j \in \{0, ..., d\},\ \mu(h_j) \geqslant 0.$$

Otherwise it is sufficient to multiply h by an element of K appropriately chosen.

Since the sequence $\{\beta_i\}_{i\geqslant 1}$ is not bounded in G_1, there exists $i_0 \in \mathbb{N}^*$ such that:

$$\mu'(h) < \beta_{i_0}.$$

Denote by $h = \sum\limits_{j=0}^{s_{i_0}} c_{j,i_0} Q_{i_0}^j$ the i_0-standard expansion of h. This expansion is obtained by euclidean division, in view of the choice made on the coefficients of h and, since the sequence $\left\{\frac{\beta_i}{d^\circ(Q_i)}\right\}_{i\geqslant 1}$ is increasing (it is sufficient to take the $(i-1)$-standard expansion of Q_i). We prove easily that:

$$\forall\, j \in \{0, ..., s_{i_0}\},\ \mu\left(c_{j,i_0}\right) \geqslant 0.$$

Recall that, by construction of the key polynomials, for $j \in \{0, ..., s_{i_0}\}$, we have $\mu'_{i_0}\left(c_{j,i_0}\right) = \mu'\left(c_{j,i_0}\right)$. We then deduce that:

$$\forall\, j \in \{1, ..., s_{i_0}\},\ \mu'\left(c_{j,i_0} Q_{i_0}^j\right) = \mu'_{i_0}\left(c_{j,i_0} Q_{i_0}^j\right) > \mu'(h).$$

Thus, $\mu'(h) = \mu'\left(c_{0,i_0}\right)$ and so, h is a sum of monomials in $\mathbf{Q}_{i_0+1}$ of valuation at least $\mu'(h)$ (in particular, $\mu'_{i_0}(h) = \mu'(h)$). $\square$

We now consider two cases:

(1) $\sharp\{i \geqslant 1 \mid \alpha_i > 1\} = +\infty$;
(2) $\sharp\{i \geqslant 1 \mid \alpha_i > 1\} < +\infty$.

In case (1), from Proposition 3.19, we can prove that the infinite set of key polynomials is always 1-complete, independently of the characteristic of k_μ. In case (2), if the effective degree is always 1, and if the set of key polynomials $\mathbf{Q}_\omega = \{Q_i\}_{i\geqslant 1}$ is not complete, we will prove in Proposition 3.20 that the sequence $\{\beta_i\}_{i\geqslant 1}$ is never bounded. In that case, Proposition 3.18, we deduce that the set of key polynomials $\mathbf{Q}_\omega = \{Q_i\}_{i\geqslant 1}$ is also 1-complete.

Proposition 3.19. *([3], Corollary 5.8) Assume that we have constructed an infinite set of key polynomials $\mathbf{Q}_\omega = \{Q_i\}_{i\geqslant 1}$ such that, for all $i \in \mathbb{N}^*$, $\beta_i \in G_1$. Furthermore, assume that the set $\{i \geqslant 1 \mid \alpha_i > 1\}$ is infinite. Then, $\mathbf{Q}_\omega$ is a 1-complete set of key polynomials.*

Proof. Let $h \in K[x]$. Like in the proof of Proposition 3.18, it is sufficient to show that $\mu'_i(h) = \mu'(h)$ for a $i \geqslant 1$. However, if we write:

$$\delta_i(h) = \max S_i(h, \beta_i),$$

where:

$$S_i(h,\beta_i) = \{j \in \{0,...,s_i\} \mid j\beta_i + \mu'(c_{j,i}) = \mu'_i(h)\},$$

$$h = \sum_{j=0}^{s_i} c_{j,i} Q_i^j,$$

by (1) of Proposition 5.2 of [3], we have:

$$\alpha_{i+1}\delta_{i+1}(h) \leqslant \delta_i(h),\ \forall\, i \geqslant 1.$$

We deduce that if $\delta_i(h) > 0$ and $\alpha_{i+1} > 1$, then:

$$\delta_{i+1}(h) < \delta_i(h), \forall\, i \geqslant 1.$$

The set $\{i \geqslant 1 \mid \alpha_i > 1\}$ is infinite and the previous inequality does not occur infinitely, we conclude that there exists $i_0 \geqslant 1$ such that $\delta_{i_0}(h) = 0$ and so $\mu'_{i_0}(h) = \mu'(h)$. $\square$

From now, we assume that we have constructed an infinite set of key polynomials $\mathbf{Q}_\omega = \{Q_i\}_{i\geqslant 1}$ such that $\alpha_i = 1$, for some i sufficiently large. Thus for this i, we have:

$$Q_{i+1} = Q_i + z_i,$$

where z_i is an i-standard homogeneous expansion, of value β_i, which does not contain Q_i. More we assume that $\beta_i \in G_1$ and that there exists $h \in K[x]$ such that, for all $i \geqslant 1$:

$$\mu'_i(h) < \mu'(h).$$

Denote by Q_ω the monic polynomial of smaller degree possible satisfying the previous inequality. Like in the proof of the Proposition 3.19, we denote:

$$\delta_i(Q_\omega) = \max S_i(Q_\omega, \beta_i),$$

where:

$$S_i(Q_\omega, \beta_i) = \{j \in \{0, ..., s_i\} \mid j\beta_i + \mu'(c_{j,i}) = \mu'_i(Q_\omega)\},$$
$$Q_\omega = \sum_{j=0}^{s_i} c_{j,i} Q_i^j.$$

By (1) of Proposition 5.2 of [3], we have the following inequality:

$$\alpha_{i+1}\delta_{i+1}(Q_\omega) \leqslant \delta_i(Q_\omega), \; \forall \, i \geqslant 1.$$

Since $\alpha_i = 1$ for i sufficiently large, there exists $\delta_\omega \in \mathbb{N}^*$ such that $\delta_\omega = \delta_i(Q_\omega)$, for i sufficiently large.

Proposition 3.20. *([3], Proposition 6.8) Under the previous assumptions, if $\delta_\omega = 1$ then the sequence $\{\beta_i\}_{i\geqslant 1}$ is not bounded in G_1.*

4. Defect and effective degree

In this section, we do not assume any assumption about the rank of the valuation. We use the construction of key polynomials of W. Mahboub (see [8]). Recall that the order type of the set of key polynomials is at most $\omega \times \omega$.

Lemma 4.1. *Let $(K, \mu) \hookrightarrow (L, \mu')$ be a finite and simple valued field. Let $\{Q_i\}_{i\in\Lambda}$ be the well ordered set of key polynomials associated with this extension. Then there exists $n_0 \in \mathbb{N}^*$ such that $\Lambda \leqslant \omega n_0$, i.e. $\{Q_i\}_{i\in\Lambda}$ has a finite number of limit key polynomials.*

Proof. Let $P \in K[x]$ monic and irreducible such that $L = K[x]/(P)$. Denote also by μ' the pseudo-valuation of $K[x]$ for which the set of key polynomials is associated with, so that for all $Q \in (P)$, $\mu'(Q) = \infty$. Assume that the set $\{Q_{\omega i}\}_{i\geqslant 1}$ is infinite. Since $\alpha_{\omega i} > 1$, for all $i \geqslant 1$, the sequence $\{d\,^\circ(Q_{\omega i})\}_{i\geqslant 1}$ is increasing. Then there exists $n_0 \in \mathbb{N}^*$ such that $d\,^\circ(P) \leqslant d\,^\circ(Q_{\omega i})$, for all $i \geqslant n_0$.

Let $i \geqslant n_0$. By euclidean division there exist $S_i, R_i \in K[x]$ such that $Q_{\omega i} = PS_i + R_i$ with $R_i = 0$ or $0 \leqslant d\,^\circ(R_i) < d\,^\circ(P) \leqslant d\,^\circ(Q_{\omega i})$. If $R_i \neq 0$ then $\mu'(Q_{\omega i}) = \mu'(R_i)$, which belies the minimality of the degree of a limit key polynomial. If $R_i = 0$, then $Q_{\omega i} \in (P)$ and the $Q_{\omega i}$ are not key polynomials for $i > n_0$. $\square$

Lemma 4.2. *Let (K, μ) be an henselian field and L be a finite and simple extension of K. By definition, μ extends uniquely to L and this extension corresponds to a (pseudo-)valuation in $K[x]$ denoted by μ'. Consider $\{Q_i\}_{i\in\Lambda}$ the well ordered set of key polynomials associated with μ' and $n_0 \in \mathbb{N}^*$ the smallest integer such that $\Lambda \leqslant \omega n_0$. Then there exists an index $i_0 \in \Lambda$ having a predecessor, such that:*

$$[L : K] = d^{\circ}(Q_{\omega(n_0-1)+i_0})d_{\omega n_0},$$

and where:

$$d_{\omega n_0} = \begin{cases} \delta_{\omega n_0} & \text{if} \quad \Lambda = \omega n_0 \ \text{and} \ \sharp\{i \geqslant 1 \mid \alpha_{\omega(n_0-1)+i} = 1\} = +\infty \\ 1 & \text{if} \quad \Lambda < \omega n_0 \ \text{or} \ \Lambda = \omega n_0 \ \text{and} \ \sharp\{i \geqslant 1 \mid \alpha_{\omega(n_0-1)+i} = 1\} < +\infty. \end{cases}$$

Proof. By assumption, $L = K[x]/(P(x))$ with $P \in K[x]$ irreducible and monic.

Assume that Λ is an ordinal having an immediate predecessor. Let us denote it by $\omega(n_0 - 1) + n$, $n \in \mathbb{N}^*$. By definition, since $\Lambda < \omega n_0$, $d_{\omega n_0} = 1$. By construction of key polynomials, $P = Q_{\omega(n_0-1)+n}$. Thus, $i_0 = n$ and:

$$[L : K] = d^{\circ}(P) = d^{\circ}(Q_{\omega(n_0-1)+i_0})d_{\omega n_0}.$$

Assume now that Λ is a limit ordinal, denote it by ωn_0.

If $\sharp\{i \geqslant 1|\alpha_{\omega(n_0-1)+i} = 1\} < +\infty$, then $d_{\omega n_0} = 1$. Like in the proof of Proposition 3.19, there exists $i_0 \in \mathbb{N}^*$ minimal such that $\mu'_{\omega(n_0-1)+i_0}(P) = \mu'(P)$. Thus, if we write the $\omega(n_0 - 1) + i_0$-standard expansion of P, we note that $Q_{\omega(n_0-1)+i_0}$ belongs to the center of the valuation μ', which is the ideal (P). Note that, by definition and by construction of the key polynomials and by choice of i_0, we have $d^{\circ}(P) \leqslant d^{\circ}(Q_{\omega(n_0-1)+i_0})$. Since $Q_{\omega(n_0-1)+i_0} \in (P)$, we conclude that $Q_{\omega(n_0-1)+i_0} = cP$, $c \in K^*$ and so:

$$[L : K] = d^{\circ}(P) = d^{\circ}(Q_{\omega(n_0-1)+i_0})d_{\omega n_0}.$$

If $\sharp\{i \geqslant 1 \mid \alpha_{\omega(n_0-1)+i} = 1\} = +\infty$, then $d_{\omega n_0} = \delta_{\omega n_0}$. In that case, $P = Q_{\omega n_0}$. Take $i_0 \in \mathbb{N}^*$ sufficiently large such that $\alpha_{\omega(n_0-1)+i} = 1$ and $\delta_{\omega(n_0-1)+i}(P) = \delta_{\omega(n_0-1)+i+1}(P)$ for all $i \geqslant i_0$. Recall that this common value is denoted by $\delta_{\omega n_0}$. By Proposition 2.12 of [12], since K is henselian, the Newton polygon $\Delta_{\omega(n_0-1)+i}(P)$ have a unique side of slope $\beta_{\omega(n_0-1)+i}$. This is equivalent to:

$$\begin{aligned}\mu'(c_{0,\omega(n_0-1)+i}) &= \mu'(c_{s_{\omega(n_0-1)+i},\omega(n_0-1)+i}) + s_{\omega(n_0-1)+i}\beta_{\omega(n_0-1)+i} \\ &\leqslant \mu'(c_{j,\omega(n_0-1)+i}) + j\beta_{\omega(n_0-1)+i},\end{aligned}$$

with $P = \sum_{j=0}^{s_{\omega(n_0-1)+i}} c_{j,\omega(n_0-1)+i} Q^j_{\omega(n_0-1)+i}$ and $0 \leqslant j \leqslant s_{\omega(n_0-1)+i}$. By Corollary 3.23 of [3], $\beta_{\omega(n_0-1)+i}$ determine always a side of $\Delta_{\omega(n_0-1)+i}(P)$ which only has one, so:

$$\mu'_{\omega(n_0-1)+i}(P) = \mu'(c_{0,\omega(n_0-1)+i}) = \mu'(c_{s_{\omega(n_0-1)+i},\omega(n_0-1)+i}) + s_{\omega(n_0-1)+i}\beta_{\omega(n_0-1)+i}.$$

We deduce, by definition of $\delta_{\omega(n_0-1)+i}(P)$, that:

$$\delta_{\omega(n_0-1)+i}(P) = s_{\omega(n_0-1)+i}.$$

But, for $i \geqslant i_0$, $d^\circ(Q_{\omega(n_0-1)+i}) = d^\circ(Q_{\omega(n_0-1)+i_0})$ and $\delta_{\omega(n_0-1)+i}(P) = \delta_{\omega n_0}$. Thus:

$$\delta_{\omega n_0} = s_{\omega(n_0-1)+i} = \frac{d^\circ(P)}{d^\circ(Q_{\omega(n_0-1)+i_0})}. \qquad \square$$

Corollary 4.3. *With the assumptions of Lemma 4.2, we have:*

$$\prod_{j=1}^{n_0} d_{\omega j} = d_{L|K}(\mu', \mu).$$

Proof. Recall that we denote $\alpha_{\omega j} = d^\circ_{Q_{\omega(j-1)+i_0}}(Q_{\omega j})$ with i_0 a ωj-inessential index and $j \in \mathbb{N}^*$. In [11], this number corresponds to the jump of order j denoted by $s^{(j)}$. By Proposition 3.4.4 of [8], we can always assume that $\delta_{\omega j} = \alpha_{\omega j}$, for all $j < n_0$. By Proposition 2.9 of [11] we have:

$$d^\circ(Q_{\omega(n_0-1)+i_0}) = [\Gamma_{\mu'_{\omega(n_0-1)+i_0}} : \Gamma_\mu][k_{\mu'_{\omega(n_0-1)+i_0}} : k_\mu] \prod_{j=1}^{n_0-1} \alpha_{\omega j}.$$

We verify after that whether $\Lambda = \omega(n_0-1) + i_0$ or if $\Lambda = \omega n_0$, we always have:

$$[\Gamma_{\mu'} : \Gamma_\mu] = [\Gamma_{\mu'_{\omega(n_0-1)+i_0}} : \Gamma_\mu],$$
$$[k_{\mu'} : k_\mu] = [k_{\mu'_{\omega(n_0-1)+i_0}} : k_\mu].$$

Thus, by applying Lemma 4.2, we obtain:

$$[L : K] = \left([\Gamma_{\mu'} : \Gamma_\mu][k_{\mu'} : k_\mu] \prod_{j=1}^{n_0-1} \alpha_{\omega j} \right) d_{\omega n_0}.$$

Since $d_{\omega j} = \delta_{\omega j} = \alpha_{\omega j}$, we concluded:

$$d_{L|K}(\mu', \mu) = \left(\prod_{j=1}^{n_0-1} \alpha_{\omega j} \right) d_{\omega n_0} = \left(\prod_{j=1}^{n_0-1} d_{\omega j} \right) d_{\omega n_0} = \prod_{j=1}^{n_0} d_{\omega j}. \qquad \square$$

Corollary 4.4. *With the assumptions of Lemma 4.2, we have:*

$$\delta_{\omega n_0} = \alpha_{\omega n_0}.$$

Proof. By Corollary 2.10 of [11], the defect is the total jump, applying Corollary 4.3 we get the following equality:

$$\prod_{j=1}^{n_0} \alpha_{\omega j} = d_{L|K}(\mu', \mu) = \prod_{j=1}^{n_0} d_{\omega j}.$$

Such $\delta_{\omega j} = \alpha_{\omega j} \neq 0$, for all $j < n_0$, we deduce that:

$$\delta_{\omega n_0} = \alpha_{\omega n_0}. \qquad \square$$

Corollary 4.5. *Let (K, μ) be a valued field and L be a finite and simple extension of K. Denote $\mu^{(1)}, ..., \mu^{(g)}$ the different extensions of μ on L, they correspond to a (pseudo-)valuation of $K[x]$ denoted in the same fashion. Consider $\{Q_l^{(i)}\}_{l\in\Lambda^{(i)}}$ the set of key polynomials associated with $\mu^{(i)}$ and $n_0^{(i)} \in \mathbb{N}^*$ the smallest integer such that $\Lambda^{(i)} \leqslant \omega n_0^{(i)}$, $1 \leqslant i \leqslant g$. Then we have:*

$$d_{L|K}(\mu^{(i)}, \mu) = \prod_{j=1}^{n_0^{(i)}} d_{\omega j}^{(i)}.$$

We deduce that:

$$[L : K] = \sum_{i=1}^{g} e_i f_i d_{\omega}^{(i)} d_{\omega 2}^{(i)} ... d_{\omega n_0^{(i)}}^{(i)},$$

where $e_i = \left[\Gamma_{\mu^{(i)}} : \Gamma_\mu\right]$ and $f_i = \left[k_{\mu^{(i)}} : k_\mu\right]$.

Proof. By Proposition 2.5, we know that:

$$[L : K] = \sum_{i=1}^{g} e_i f_i d_i,$$

where $d_i = d_{L|K}(\mu^{(i)}, \mu)$, $e_i = \left[\Gamma_{\mu^{(i)}} : \Gamma_\mu\right]$ and $f_i = \left[k_{\mu^{(i)}} : k_\mu\right]$. Applying Corollary 4.3 to the fields $L^{h(\mu^{(i)})}$ and $K^{h(\mu^{(i)})}$ we obtain the announced equalities. $\square$

Corollary 4.6. *Under the assumptions of Corollary 4.5, if we denote $p = car(k_\mu)$, then, for $1 \leqslant i \leqslant g$ and $1 \leqslant j \leqslant n_0^{(i)}$, there exists $e_{\omega j}^{(i)} \in \mathbb{N}$ such that:*

$$\delta_{\omega j}^{(i)} = p^{e_{\omega j}^{(i)}}.$$

Proof. It is a direct consequence of Corollary 4.5 and Remark 2.4. □

Example 4.7. We study the Example 3.2 of [11]. Let k be an algebraic closed field of characteristic 0. Consider $K = k(y)$ endowed with its y-adic valuation denoted by μ. The polynomial:

$$P = x^4 + y^2x^3 + y^3(y^2 - 2)x^2 - y^5x + y^6$$

is irreducible in $K[x]$. The valuation extends only on two valuations $\mu^{(1)}$ and $\mu^{(2)}$ over the field $L = K[x]/(P)$ given by the sequences of key polynomials $\{Q_i^{(1)}\}_{i\in\Lambda^{(1)}}$ and $\{Q_i^{(2)}\}_{i\in\Lambda^{(2)}}$ such that:

$$\begin{aligned}
&Q_1^{(1)} = x && \beta_1^{(1)} = 3/2 \\
&Q_2^{(1)} = x^2 - y^3 && \beta_2^{(1)} = 7/2 \\
&Q_i^{(1)} = Q_2^{(1)} + (y^2 - u_{i-3})x && \beta_i^{(1)} = 3/2 + i \;;\quad i \geqslant 3
\end{aligned}$$

and:

$$\begin{aligned}
&Q_1^{(2)} = x && \beta_1^{(2)} = 3/2 \\
&Q_2^{(2)} = x^2 - y^3 && \beta_2^{(2)} = 9/2 \\
&Q_i^{(2)} = Q_2^{(2)} + u_{i-2}x && \beta_i^{(2)} = 3/2 + (i+1) \;;\quad i \geqslant 3
\end{aligned}$$

where $u_0 = 0$, $u_l = \sum_{j=1}^{l} c_j y^{j+2}$, for all $l \geqslant 1$, $c_1 = c_2 = 1$ and $c_j = 4^{j-2}\dfrac{3...(2j-3)}{6...(2j)}$ for all $j \geqslant 3$. Following the notation of the Corollary 4.5, with $\Lambda^{(1)} = \Lambda^{(2)} = \omega$, and $\alpha_i^{(1)} = \alpha_i^{(2)} = 1$ for all $i \geqslant 2$, we deduce that:

$$d_\omega^{(1)} = d_\omega^{(2)} = 1,$$

because $p = 1$ if $car(k) = 0$. Moreover, since a field of characteristic zero is defectless, we have:

$$d_{L|K}(\mu^{(1)}, \mu) = 1 = d_\omega^{(1)}$$

and:

$$d_{L|K}(\mu^{(2)}, \mu) = 1 = d_\omega^{(2)}.$$

Finally, since $\Gamma_\mu = \mathbb{Z}$, $\Gamma_{\mu^{(1)}} = \Gamma_{\mu^{(2)}} = (1/2)\mathbb{Z}$, and $k_\mu = k_{\mu^{(1)}} = k_{\mu^{(2)}} = k$ we deduce that:

$$e_1 = e_2 = 2 \text{ and } f_1 = f_2 = 1.$$

We obtain again the result of Corollary 4.5, that is to say:

$$4 = [L : K] = e_1 f_1 d_\omega^{(1)} + e_2 f_2 d_\omega^{(2)} = 2 \times 1 \times 1 + 2 \times 1 \times 1.$$

Example 4.8. We study the example of W. Mahboub given in [7]. Let k be a field of characteristic $p > 2$. Denote by ν the z-adic valuation of $k(z)$ and by μ the valuation of $K = k(z, y)$ defined by the set of key polynomials $\{Q_{y,i}\}_{i \geqslant 1}$ given in [7] extending ν. For $e \in \mathbb{N}^*$, write:

$$f = x^{p^e} - y^2 - z.$$

We want to find all the valuations of $L = K[x]/(f)$ which extend μ. Consider the valuation μ' of L defined by the set of key polynomials $\{Q_i\}_{i \in \Lambda}$:

$$\begin{aligned} Q_1 &= x & \beta_1 &= 1 - \frac{1}{4p} \\ Q_i &= x^{p^{e-1}} - \sum_{j=0}^{i} h_j & \beta_i &= 1 - \frac{1}{2^{2i} p^i} \quad ; \quad i \geqslant 2 \end{aligned}$$

where $h_0 = 0$ and $h_j = \dfrac{Q_{y,2j+1}^2}{z^{2^{2j} p^j - 1}}$, for all $j \geqslant 1$. We deduced that, for all $i \geqslant 1$:

$$f = (Q_i + \sum_{j=0}^{i} h_j)^p - y^2 - z = Q_i^p + \sum_{j=0}^{i} h_j^p - Q_{y,2} = Q_i^p + \frac{Q_{y,2i}}{z^{2^{2i-2} p^i - p}}.$$

But $\mu(h_i^p) = \mu\left(\dfrac{Q_{y,2i}}{z^{2^{2i-2} p^i - p}}\right)$, so:

$$in_i(f) = X^p - in_\mu(h_i^p) = (X - in_\mu(h_i))^p.$$

Thus, for all $i \geqslant 1$, $\delta_i(f) = p$ and so $\delta_\omega = p$. Moreover we are in the case where $\Lambda = \omega$ and $\alpha_i = 1$, for all $i \geqslant 3$. We deduce by Corollary 4.5, that:

$$[k_{\mu'} : k_\mu] = p^{e-1};$$
$$d_{L|K}(\mu', \mu) = d_\omega = \delta_\omega = p.$$

Since $[L : K] = p^e$, by Corollary 4.5, we deduced that μ' is the unique valuation extending μ on L.

Example 4.9. Let k be a field of characteristic $p > 0$. We denote $K = k\left((y, z)\right)$. Consider the valuation $\nu : K^* \to \mathbb{Z}^2_{lex}$, trivial on k, defined by $\nu(z) = (1, 0)$ and $\nu(y) = (0, 1)$. Consider the polynomial $P \in K[w]$ defined as follows:

$$P(w) = w^p - z^{p^2 - p} w + y^p z^{p^2}.$$

Write $L = K[w]/(P)$. Let us find all the valuations $\nu^{(i)}$ of L extending ν and all the corresponding sets of key polynomial $\{Q_j^{(i)}\}_{j\in\Lambda^{(i)}}$.

Finding a valuation of L extending ν means finding a pseudo-valuation of $K[w]$ of kernel P. Let μ be such a pseudo-valuation. The only possible values of w are:

$$\mu(w) = \begin{cases} (p,p) \\ (p,1) \\ (p,0) \end{cases}.$$

But if $\mu(w) = (p,1)$, then $\mu(w^p) = \mu(y^p z^{p^2}) = (p^2, p) > \mu(z^{p^2-p}w) = (p^2, 1)$. The monomial w^p is not of minimal value in P. The two only possible values of w are:

$$\mu(w) = \begin{cases} (p,p) \\ (p,0) \end{cases}.$$

1) Assume that $\mu(w) = (p,p)$.

We are in the case of Example 5.3.2 of [8]. In that case, we prove that the key polynomials are:

$$\begin{aligned} Q_1^{(1)} &= w, \\ Q_j^{(1)} &= w - z^p u_j, \end{aligned}$$

where $u_j = \sum_{i=1}^{i} y^{p^i}$ and $\mu(Q_j^{(1)}) = (p, p^j)$, for all $j \geqslant 2$. Moreover the polynomial P is a limit key polynomial. Finally, the polynomial $in_j(P)$ is irreducible for all $j \geqslant 1$, the valuation extends uniquely since $\mu(w) = (p,p)$.

Thus, a first way to extend ν over L consists in considering the valuation $\nu^{(1)}$ defined by the key polynomials $Q_j^{(1)}$, $j \geqslant 1$ and $Q_\omega^{(1)} = P$.

Write $u = \sum_{i\geqslant 1} y^{p^i}$ the limit of the sequence $(u_j)_j$ in $k[[y,z]]$. Note that u is solution of the equation $X - X^p = y^p$. The other solutions of the equation are the $u - l$ with $l \in \{1, ..., p-1\}$. By writing $\alpha_l = z^p(u - l)$, for $l \in \{0, ..., p-1\}$, it is easy to prove that:

$$\prod_{l=0}^{p-1} (w - \alpha_l) = w^p - z^{p^2-p} w + y^p z^{p^2}.$$

Thus, the roots of P are $\alpha_0, ..., \alpha_{p-1}$, and if we write $Q_\infty^{(1)} = w - \alpha_0 = w - z^p u$, then:

$$P(w) = Q_\infty^{(1)}(w) \prod_{l=1}^{p-1} (w - \alpha_l).$$

Finally, since $in_j(P) = -z^{p^2-p}(X - y^{p^i} z^p)$, for all $j \geqslant 1$ and $\Lambda^{(1)} = \omega$, we deduce:

$$d_{L|K}(\nu^{(1)}, \nu) = d_\omega^{(1)} = \delta_\omega^{(1)} = 1.$$

2) Assume that $\mu(w) = (p, 0)$.
Denote $Q_1^{(2)} = w$. Then:

$$in_1(P) = X\left(X^{p-1} - z^{p^2-p}\right).$$

Write $Q_2^{(2)} = \left(Q_1^{(2)}\right)^{p-1} - z^{p^2-p}$. Then:

$$P = Q_1^{(2)} Q_2^{(2)} + y^p z^{p^2}.$$

The only choice for the valuation of $Q_2^{(2)}$ is:

$$\mu(Q_2^{(2)}) = (p^2 - p, p) > (p, 0) = \mu(Q_1^{(2)}).$$

In this case, the polynomial $in_2(P)$ is given by:

$$in_2(P) = in_\mu(Q_1^{(2)})\left(X + \frac{in_\mu(y^p z^{p^2})}{in_\mu(Q_1^{(2)})}\right).$$

Note that $\mu(u) = \mu(y^p) = (0, p)$; then $\mu(\alpha_0) = \mu(z^p u) = (p, p)$. By writing $h = \sum_{m=1}^{p-1} \alpha_0^{p-m} w^{m-1}$, we note that $h \in K[w]$ represents $\frac{in_\mu(y^p z^{p^2})}{in_\mu(Q_1^{(2)})}$ because $\mu(h) = \mu(w^{p-2}\alpha_0) = (p^2 - p, p)$. Then denote by:

$$Q_3^{(2)} = Q_2^{(2)} + h.$$

But, $(w - \alpha_0)h = \alpha_0(w^{p-1} - \alpha_0^{p-1})$. So:

$$(w - \alpha_0)Q_3^{(2)} = w^p - z^{p^2-p}w + \alpha_0 z^{p^2-p} - \alpha_0^p.$$

Since, $\alpha_0 z^{p^2-p} - \alpha_0^p = uz^{p^2} - u^p z^{p^2} = (u - u^p)z^{p^2} = y^p z^{p^2}$ we deduce that:

$$Q_\infty^{(1)} Q_3^{(2)} = (w - \alpha_0)Q_3^{(2)} = P.$$

The sequence of key polynomials stops. The second way to extend ν on L consists in considering the valuation $\nu^{(2)}$ defined with the key polynomials $Q_1^{(2)}$, $Q_2^{(2)}$ and $Q_3^{(2)}$.

Finally, since $\Lambda^{(2)} = 3 < \omega$, we deduce that:

$$d_{L|K}(\nu^{(2)}, \nu) = d_\omega^{(2)} = 1.$$

Then we obtain:

$$p = [L : K] = d^\circ(P) = d^\circ(Q_1^{(1)})d_\omega^{(1)} + d^\circ(Q_3^{(2)})d_\omega^{(2)} = 1 + (p - 1).$$

Example 4.10. We study an example proposed by S.D. Cutkosky in [1]. Let k be a field of characteristic $p > 0$. Denote $K = k(u, v)$ and $L = K[y]/(f)$ where:

$$f = y^{p^2+1} + y^{p^2} - yv^p + u - v^p.$$

We define a valuation ν over K such that $\nu(u) = p$ with:

$$R_\nu = \bigcup_{i \geqslant 0} A_i,$$

where $A_i = k\,[u_i, v_i]_{(u_i,v_i)}$, and:

$$\begin{cases} u_0 = u \\ v_0 = v \end{cases};\ \begin{cases} u_{pi+j} = \dfrac{u_{pi}}{v_{pi+j}^j} \\ v_{pi+j} = v_{pi} \end{cases},1 \leqslant j < p\,;\ \begin{cases} u_{p(i+1)} = v_{pi} \\ v_{p(i+1)} = \dfrac{u_{pi}}{v_{pi}^p} - \gamma_{p(i+1)} \end{cases};\ \gamma_{p(i+1)} \in k^*.$$

Then we have $\nu(v_{pi}) = \dfrac{1}{p^i}$ and $\nu(u_{pi}) = \dfrac{1}{p^{i-1}}$. Note that, for $i \geqslant 0$:

$$v_{p(i-1)} - \gamma_{p(i+1)}v_{pi}^p = v_{pi}^p v_{p(i+1)},$$

where we write $v_{-p} = u$. Then:

$$\nu(v_{p(i-1)} - \gamma_{p(i+1)}v_{pi}^p) = \frac{1}{p^{i-1}} - \frac{1}{p^{i+1}}.$$

We want to extend ν on L. There exists a first valuation denoted by ν^*, given by the sequence of key polynomials $\{Q_l\}_{l\in\Lambda}$:

$$Q_1 = y,\ \beta_1 = \nu^*(Q_1) = \frac{1}{p};$$

$$Q_l = Q_{l-1} - \gamma_{p+2l-3}^{\frac{1}{p}} \prod_{j=0}^{l-2} v_{p(2j+1)},\ \beta_l = \nu^*(Q_l) = \sum_{j=0}^{l-1} \frac{1}{p^{2j+1}}.$$

Note that $\lim\limits_{l\to+\infty} \beta_l = \dfrac{p}{p^2 - 1}$, so there exists a first limit key polynomial which is:

$$Q_\omega = y^p - v,\ \beta_\omega = \nu^*(Q_\omega) = 1.$$

We continue to define the set of key polynomials, for $i \geqslant 1$, by:

$$Q_{\omega+i} = Q_{\omega+i-1} + \gamma_{p+2i-2}^{\frac{1}{p}} \prod_{j=0}^{i-1} v_{p(2j)},\ \beta_{\omega+i} = \nu^*(Q_{\omega+i}) = \sum_{j=0}^{i} \frac{1}{p^{2j}}.$$

Note that $\lim\limits_{i\to+\infty} \beta_{\omega+i} = 1 + \dfrac{1}{p^2-1}$, so there exists a second limit key polynomial which is:

$$Q_{\omega 2} = y^{p^2} + u - v^p = Q_\omega^p + u,\ \beta_{\omega 2} = \nu^*(Q_{\omega 2}) = p + \frac{1}{p}.$$

We end the set of key polynomials with, for $n \geqslant 1$:

$$Q_{\omega 2+n} = Q_{\omega 2+n-1} + (-1)^n u y^n,\ \beta_{\omega 2+n} = \nu^*(Q_{\omega 2+n}) = p + \frac{n+1}{p}.$$

The last and third limit key polynomial is $Q_{\omega 3} = f$ and $\beta_{\omega 3} = +\infty$. Note that for all $n \geqslant 0$, we have:

$$f = (y+1)Q_{\omega 2+n} + (-1)^{n+1} u y^{n+1}.$$

So, in $K[[y]]$, $f = (y+1)Q_\infty$, where $Q_\infty = y^{p^2} + u - v^p - uy \sum\limits_{n\geqslant 1} (-1)^n y^n$. Note that for $l \geqslant 1$, we have:

$$y^p - v = Q_l^p - v_{p(2(l-1))} \prod_{j=0}^{l-2} v_{p(2j+1)}^p,$$

so that:

$$in_l(y^p - v) = X^p - in_\nu \left(v_{p(2(l-1))} \prod_{j=0}^{l-2} v_{p(2j+1)}^p \right) = \left(X - in_\nu \left(\prod_{j=0}^{l-2} v_{p(2j+1)} \right) \right)^p.$$

We deduce that for all $l \geqslant 1$, we have:

$$\alpha_l = 1,$$
$$d_\omega = \delta_\omega = \delta_l(y^p - v) = p = \alpha_\omega.$$

In the same way, for $i \geqslant 1$:

$$y^{p^2} + u - v^p = Q_\omega^p + u = Q_{\omega+i}^p - v_{p(2i-1)} \prod_{j=0}^{i-2} v_{p(2j)}^p,$$

$$in_{\omega+i}(y^{p^2} + u - v^p) = X^p - in_\nu \left(v_{p(2i-1)} \prod_{j=0}^{i-2} v_{p(2j)}^p \right) = \left(X - in_\nu \left(\prod_{j=0}^{i-1} v_{p(2j)} \right) \right)^p.$$

We deduce that for all $i \geqslant 1$:

$$\alpha_{\omega+i} = 1,$$
$$d_{\omega 2} = \delta_{\omega 2} = \delta_{\omega 2+i}(y^{p^2} + u - v^p) = p = \alpha_{\omega 2}.$$

Finally, for all $n \geqslant 0$:

$$in_{\omega 2+n}(f) = X + in_\nu\left((-1)^{n+1} u y^{n+1}\right).$$

We deduce that for all $n \geqslant 1$:

$$\alpha_{\omega 2+n} = 1,$$
$$d_{\omega 3} = \delta_{\omega 3} = \delta_{\omega 3+n}(f) = 1.$$

By Corollary 4.3:

$$d_{L|K}(\nu^*, \nu) = d_\omega d_{\omega 2} d_{\omega 3} = p \times p \times 1 = p^2.$$

Note that $[L : K] = p^2 + 1$, so there exists a second valuation $\nu^{(2)}$ of L who extends ν and which is defectless. By the equality $f = (y+1)Q_\infty$ in $K[[y]]$, the valuation satisfies $\nu^{(2)}(y) = 0$ and so this is the trivial valuation.

5. Key polynomials and defectless fields

Consider $K \hookrightarrow K(x)$ a simple transcendental field extension. Let μ' be a valuation of $K(x)$, write $\mu := \mu'_{|K}$. We denote by G the value group of μ' and by G_1 the smallest isolated non-zero subgroup of G. We assume that μ is of rank 1, $\mu'(x) > 0$.

Proposition 5.1. *If K is defectless, then there exists a 1-complete set of key polynomials $\{Q_i\}_{i \in \Lambda}$ such that Λ is either a finite set or $\mathbb{N}^*$. In particular, there is no limit key polynomial for valuations of rank 1 over defectless fields.*

Proof. Let us apply the process of [3]. If there exists $i_0 \in \mathbb{N}$, such that $\beta_{i_0} \notin G_1$, we write $\Lambda = \{1, ..., i_0 - 1\}$ and, by definition, $\{Q_i\}_{i \in \Lambda}$ is 1-completed. Otherwise, for all $i \in \mathbb{N}$, $\beta_i \in G_1$ and we write $\Lambda = \mathbb{N}^*$. If $\sharp\{i \geqslant 1 \mid \alpha_i > 1\} = +\infty$, by Proposition 3.19, the set $\{Q_i\}_{i \in \Lambda}$ is 1-completed. If $\sharp\{i \geqslant 1 \mid \alpha_i > 1\} < +\infty$, denote by Q_ω the monic polynomial of smallest degree such that, for all $i \geqslant 1$:

$$\mu'_i(Q_\omega) < \mu'(Q_\omega).$$

Since K is defectless, the extension $K \hookrightarrow L = K[x]/(Q_\omega)$ is defectless. Consider μ' as the composition of a valuation $\mu^{(1)}$ of value group G_1 centered on $K[x]/(Q_\omega)$ and a valuation θ of value group G/G_1 centered on $K[x]_{(Q_\omega)}$. The set of key polynomials for

$\mu^{(1)}$ is the same as the set of key polynomials for μ' except that $\mu^{(1)}(Q_\omega) = \infty$. Thus, $d_{L|K}(\mu^{(1)}, \mu) = 1$. By Corollary 4.5, we deduce that:

$$\delta_\omega = d_{L|K}(\mu^{(1)}, \mu) = 1.$$

By Proposition 3.20, we conclude that the sequence $\{\beta_i\}_{i \geqslant 1}$ is unbounded in G_1 and so, by Proposition 3.18, the set $\{Q_i\}_{i \in \Lambda}$ is a 1-completed set of key polynomials. □

Corollary 5.2. *If $car(k_\mu) = 0$, there exists a 1-completed set of key polynomials $\{Q_i\}_{i \in \Lambda}$ such that Λ is finite or equal to $\mathbb{N}^*$. In particular, there is no limit key polynomial for valuations of rank 1 whose residual field is of characteristic zero.*

Proof. Apply Proposition 2.10 and Proposition 5.1. □

6. Local uniformization of quasi-excellent local domain without defect

We extend here the results of the section 8 of [9] for a valuation satisfying some inductive assumption about defect. More precisely, in [9], in order to obtain a theorem of monomialization, the valuation needs to have a complete set of key polynomials without limit key polynomial: This is the case if the valuation is defectless. As a corollary, we find the local uniformization in characteristic zero.

Let $(R, \mathfrak{m}, k)$ be a local complete regular equicharacteristic ring of dimension n with $\mathfrak{m} = (u_1, ..., u_n)$. Let ν be a valuation of $K = Frac(R)$, centered on R, of value group Γ and Γ_1 the smallest non-zero isolated subgroup of Γ. Define:

$$H = \{f \in R \mid \nu(f) \notin \Gamma_1\}.$$

H is a prime ideal of R (see the proof of Theorem 6.4). Moreover assume that:

$$n = e(R, \nu) = emb.dim\,(R/H)\,,$$

that is to say:

$$H \subset \mathfrak{m}^2.$$

Denote $r = r(R, u, \nu) = \dim_{\mathbb{Q}}\left(\sum_{i=1}^{n} \mathbb{Q}\nu(u_i)\right)$.

The valuation ν is unique if $ht(H) = 1$; it is the composition of the valuation $\mu : L^* \to \Gamma_1$ of rank 1 centered on R/H, where $L = Frac(R/H)$, with the valuation $\theta : K^* \to \Gamma/\Gamma_1$, centered on R_H, such that $k_\theta \simeq \kappa(H)$.

By abuse of notation, for $f \in R$, we denote $\mu(f)$ instead of $\mu(f \mod H)$. By the Cohen's theorem, we can assume that R is of the following form:

$$R = k\,[[u_1, ..., u_n]]\,.$$

For $j \in \{r+1, ..., n\}$, we denote by $\{Q_{j,i}\}_{i\in\Lambda_j}$ the set of key polynomials of the extension $k((u_1, ..., u_{j-1})) \hookrightarrow k((u_1, ..., u_{j-1}))(u_j)$, $\mathbf{Q}_{j,i} = \{Q_{j,i'} | i' \in \Lambda_j, i' < i\}$, $\Gamma^{(j)}$ the value group of $\nu_{|k((u_1,...,u_j))}$ and $\nu_{j,i}$ the i-truncation of ν for this extension.

For the definition of local framed sequences, see Definition 7.1 and sections 4.1 and 4.2 of [9].

Theorem 6.1. *Assume that $R_{n-1} = k[[u_1, ..., u_{n-1}]]$. Then:*

1. *One of this two following alternatives holds:*
   (a) *$H \cap R_{n-1} \neq (0)$ and there exists a local framed sequence $(R_{n-1}, u) \to (R', u')$ such that:*
   $$e(R', \nu) < e(R_{n-1}, \nu);$$
   (b) *$H \cap R_{n-1} = (0)$ and for all $f \in R_{n-1}$, there exists a local framed sequence $(R_{n-1}, u) \to (R', u')$ such that f is a monomial in u' times a unit of R'.*
2. *The local framed sequence $(R_{n-1}, u) \to (R', u')$ of (1) can be chosen defined over T.*

Furthermore assume that $k((u_1, ..., u_{n-1})) \hookrightarrow k((u_1, ..., u_n))/H$ is defectless. Then assumptions 1. and 2. are true with R instead of R_{n-1}.

Proof. The proof is the same as those of Theorem 5.1 and Theorem 7.2 of [9]. With the assumptions of Theorem 6.1, we can use the Proposition 5.2 of [9]. Then H is generated by a irreducible monic polynomial in u_n. Since $k((u_1, ..., u_{n-1})) \hookrightarrow k((u_1, ..., u_n))/H$ is defectless, by Proposition 5.1, the set of key polynomials $\{Q_{j,i}\}_{i\in\Lambda_j}$ has not limit key polynomial. To conclude it is sufficient to apply Theorem 7.2 of [9]. □

As a consequence, we obtain the local uniformization of a valuation of rank 1 centered on a local quasi-excellent equicharacteristic domain, satisfying some assumptions of lack of defect. The proof uses the notion of implicit prime ideal, for more details see [5] or section 4.3 of [9].

Definition 6.2. For a local noetherian ring $(R, \mathfrak{m})$, with $\mathfrak{m} = (u) = (u_1, ..., u_n)$ and $f_1, ..., f_s \in \mathfrak{m}$, we call the **monomial property for R and $\boldsymbol{f_1, ..., f_s}$** the three following assertions:

1. R is regular;
2. For $1 \leqslant j \leqslant s$, f_j is a monomial in u times a unit of R;
3. For $1 \leqslant j \leqslant s$, f_1 divides f_j in R.

Definition 6.3. Let $(S, \mathfrak{m}, k)$ be a local domain of quotient field L and μ a valuation of L of rank 1 and of value group Γ_1, centered on S.

Denote by $u = (u_1, ..., u_n)$ a minimal set of generators of $\mathfrak{m}$ and by $\overline{H}$ the implicit prime ideal of $\widehat{S}$.

Denote $u = (y, x)$ with $x = (x_1, ..., x_l)$, $l = emb.dim\left(\widehat{S}/\overline{H}\right)$ and such that the images of $x_1, ..., x_l$ in $\widehat{S}/\overline{H}$ induce a minimal set of generators of $(\mathfrak{m}\widehat{S})/\overline{H}$.

Let $f_1, ..., f_s \in \mathfrak{m}$ such that $\mu(f_1) = \min_{1 \leqslant j \leqslant s}\{\mu(f_j)\}$.

1. We say that **S and $f_1, ..., f_s$ have the quotient local uniformization property of dimension l** if there exists a local framed sequence:

$$(S, u, k) = \left(S^{(0)}, u^{(0)}, k^{(0)}\right) \xrightarrow{\rho_0} \left(S^{(1)}, u^{(1)}, k^{(1)}\right) \xrightarrow{\rho_1} \dots \xrightarrow{\rho_{i-1}} \left(S^{(i)}, u^{(i)}, k^{(i)}\right),$$

   such that $\widehat{S^{(i)}}/\overline{H}^{(i)}$ and $\overline{f}_1, ..., \overline{f}_s$ have the monomial property, where $\overline{H}^{(i)}$ is the implicit prime ideal of $\widehat{S^{(i)}}$ and $\overline{f_j}$ are the images of $f_j \mod \overline{H}^{(i)}$, $1 \leqslant j \leqslant s$.
2. We say that **S and $f_1, ..., f_s$ have the local uniformization property of dimension l** if there exists a local framed sequence:

$$(S, u, k) = \left(S^{(0)}, u^{(0)}, k^{(0)}\right) \xrightarrow{\rho_0} \left(S^{(1)}, u^{(1)}, k^{(1)}\right) \xrightarrow{\rho_1} \dots \xrightarrow{\rho_{i-1}} \left(S^{(i)}, u^{(i)}, k^{(i)}\right),$$

   such that $S^{(i)}$ and $\overline{f}_1, ..., \overline{f}_s$ have the monomial property.

Theorem 6.4. *Let $(S, \mathfrak{m}, k)$ be a local noetherian equicharacteristic domain of quotient field L and μ be a valuation of L of rank 1 with value group Γ_1, centered on S.*

Denote by $u = (u_1, ..., u_n)$ a minimal set of generators of $\mathfrak{m}$ and by $\overline{H}$ the implicit prime ideal of $\widehat{S}$.

Denote $u = (y, x)$ with $x = (x_1, ..., x_l)$, $l = emb.dim\left(\widehat{S}/\overline{H}\right)$ and such that the images of $x_1, ..., x_l$ in $\widehat{S}/\overline{H}$ induce a minimal set of generators of $(\mathfrak{m}\widehat{S})/\overline{H}$.

Assume that the quotient local uniformization property of dimension $l - 1$ holds for all local domain, and that $k((x_1, ..., x_{l-1})) \hookrightarrow k((x_1, ..., x_{l-1}))[x_l]/\overline{H}$ is defectless.

Let $f_1, ..., f_s \in \mathfrak{m}$ be such that $\mu(f_1) = \min_{1 \leqslant j \leqslant s}\{\mu(f_j)\}$. Then S and $f_1, ..., f_s$ have the quotient local uniformization property of dimension l.

Proof. By Theorem 2.1 of [5], μ extends uniquely in a valuation $\widehat{\mu}$ centered on $\widehat{S}/\overline{H}$.

By the Cohen's structure theorem, we know that there exists a complete regular local ring of characteristic zero R and a surjective morphism φ:

$$\varphi : R \twoheadrightarrow \widehat{S}/\overline{H}.$$

Denote $H = \ker \varphi$. Since $\overline{H}$ is a prime ideal (Theorem 2.1 of [5]), H is a prime ideal of R. Choose R such that $\dim(R) = l$. Denote K the quotient field of R. K is of the form $k((x_1, ..., x_l))$. Let θ be a valuation of K, centered on R_H, such that $k_\theta = \kappa(H)$. If we see $\widehat{\mu}$ as a valuation centered on R/H by the morphism φ, we can consider the valuation $\nu = \widehat{\mu} \circ \theta$ centered on R and of value group Γ. Then, Γ_1 is the non-zero smallest isolated subgroup of Γ and:

$$H = \{f \in R \mid \nu(f) \notin \Gamma_1\}.$$

Denote $T = \varphi^{-1}(\sigma(S))$. This is a local subring R of maximal ideal $\varphi^{-1}(\sigma(\mathfrak{m})) = \mathfrak{m} \cap T$. Then, T contains $x_1, ..., x_l$ and:

$$T/(\mathfrak{m} \cap T) \simeq k.$$

Since $k\left((x_1, ..., x_{l-1})\right) \hookrightarrow k\left((x_1, ..., x_{l-1})\right)[x_l]/H$, we can apply Theorem 6.1. We end the proof in the same way as the proof of Theorem 8.1 of [9]. □

Theorem 6.5. *Let $(S, \mathfrak{m}, k)$ be an equicharacteristic quasi-excellent local domain of quotient field L and μ be a valuation of L of rank 1 and of value group Γ_1, centered on S.*

Let $u = (u_1, ..., u_n)$ be a minimal set of generators of $\mathfrak{m}$ and $\overline{H}$ the implicit prime ideal of $\widehat{S}$.

Denote $u = (y, x)$ with $x = (x_1, ..., x_l)$, $l = emb.dim\left(\widehat{S}/\overline{H}\right)$ and such that the images of $x_1, ..., x_l$ in $\widehat{S}/\overline{H}$ induce a minimal set of generators of $(\mathfrak{m}\widehat{S})/\overline{H}$.

Assume that the local uniformization property of dimension $l-1$ is true for all local domain and $k\left((x_1, ..., x_{l-1})\right) \hookrightarrow k\left((x_1, ..., x_{l-1})\right)[x_l]/\overline{H}$ is defectless.

Let $f_1, ..., f_s \in \mathfrak{m}$, such that $\mu(f_1) = \min_{1 \leqslant j \leqslant s}\{\mu(f_j)\}$. Then S and $f_1, ..., f_s$ have the local uniformization property of dimension l.

In other words, μ admits an embedded local uniformization in the sense of Property 2.11 of [9].

Proof. With the notations of Theorem 6.4, we see that there exists a surjective morphism:

$$\psi : \widehat{S} \twoheadrightarrow \widehat{S}/\overline{H} \simeq R/H.$$

Since $k\left((x_1, ..., x_{l-1})\right) \hookrightarrow k\left((x_1, ..., x_{l-1})\right)[x_l]/\overline{H}$ is defectless, by Theorem 6.4, after an auxiliary local framed sequence, we can assume that $\widehat{S}/\overline{H}$ is regular, so that $R/H \simeq k[[x_1, ..., x_l]]$. The end of the proof is the same as the proof of Theorem 8.3 of [9]. □

Corollary 6.6. *Let $(S, \mathfrak{m}, k)$ be a quasi-excellent local domain of quotient field L and μ be a valuation of L of rank 1 centered on S, such that car $(k_\mu) = 0$.*

Then μ admits an embedded local uniformization in the sense of Property 2.11 of [9].

Proof. Let $u = (u_1, ..., u_n)$ be a minimal set of generators of $\mathfrak{m}$, and $\overline{H}$ be the implicit prime ideal of $\widehat{S}$. Denote $u = (y, x)$ with $x = (x_1, ..., x_l)$, $l = emb.dim\left(\widehat{S}/\overline{H}\right)$ and such that the images of $x_1, ..., x_l$ in $\widehat{S}/\overline{H}$ induce a minimal set of generators of $(\mathfrak{m}\widehat{S})/\overline{H}$. By Theorem 2.1 of [5], μ extends uniquely in a valuation $\widehat{\mu}$ centered on $\widehat{S}/\overline{H}$. Since $car\,(k_\mu) = 0$, then $car(k_{\widehat{\mu}}) = 0$. We saw that there exists a surjective morphism:

$$\psi : \widehat{S} \twoheadrightarrow \widehat{S}/\overline{H} \simeq R/H,$$

where $H = \ker\psi$. The quotient field of R is of the form $k\,((x_1, ..., x_l))$. By Proposition 2.10, we deduce that the valued fields $\left(k\,((x_1, ..., x_{j-1}))\,, \widehat{\mu}_{|k((x_1,...,x_{j-1}))}\right)$ are defectless. In order to conclude, it is sufficient to apply recursively on $j \in \{1, ..., n\}$ the Theorem 6.5 for $\left(k\,((x_1, ..., x_{j-1}))\,, \widehat{\mu}_{|k((x_1,...,x_{j-1}))}\right)$. □

References

[1] Steven Dale Cutkosky, Ramification of valuations and counterexamples to local monomialization in positive characteristic, preprint, arXiv:1404.7459 [math.AG], 2014.
[2] Charles Favre, Mattias Jonsson, The Valuative Tree, Lecture Notes in Math., vol. 1853, Springer-Verlag, Berlin, 2004.
[3] F.J. Herrera Govantes, W. Mahboub, M.A. Olalla Acosta, M. Spivakovsky, Key polynomials for simple extensions of valued fields, preprint, arXiv:1406.0657 [math.AG], 2014.
[4] F.J. Herrera Govantes, M.A. Olalla Acosta, M. Spivakovsky, Valuations in algebraic field extensions, J. Algebra 312 (2) (2007) 1033–1074.
[5] F.J. Herrera Govantes, M.A. Olalla Acosta, M. Spivakovsky, B. Teissier, Extending a valuation centered in a local domain to the formal completion, Proc. Lond. Math. Soc. 105 (3) (2012) 571–621.
[6] Franz-Viktor Kuhlmann, Valuation Theory, http://math.usask.ca/~fvk/Fvkbook.htm, 2011.
[7] Wael Mahboub, Key polynomials, J. Pure Appl. Algebra 217 (6) (2013) 989–1006.
[8] Wael Mahboub, Une construction explicite de polynômes-clés pour des valuations de rang fini, Thèse de Doctorat, Institut de Mathématiques de Toulouse, 2013.
[9] Jean-Christophe San Saturnino, Uniformisation locale des schémas quasi-excellents de caractéristique nulle, preprint, arXiv:1311.3525 [math.AG], 2013.
[10] Michel Vaquié, Extension d'une valuation, Trans. Amer. Math. Soc. 359 (7) (2007) 3439–3481 (electronic).
[11] Michel Vaquié, Famille admissible de valuations et défaut d'une extension, J. Algebra 311 (2) (2007) 859–876.
[12] Michel Vaquié, Extensions de valuation et polygone de Newton, Ann. Inst. Fourier (Grenoble) 58 (7) (2008) 2503–2541.