On abelian varieties with an infinite group of separable p^∞-torsion points

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Abstract

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We would like to present the proof of the following proposition.

If $n \in \mathbb{N}$, we write $[n]$ for the multiplication by n endomorphism on an abelian variety. If h is a finite endomorphism of an abelian variety A over a field L, we write

$$A(L)[h^\ell] := \{x \in A(L) \,|\, h^{\circ\ell}(x) = 0\}$$

and

$$A(L)[h^\infty] := \{x \in A(L) \,|\, \exists n \in \mathbb{N} : h^{\circ n}(x) = 0\}.$$

Here $h^{\circ n}(x) := h(h(h(\cdots(x)\cdots)))$, where there are n pairs of brackets. The notation $A(L)[n^\ell]$ (resp. $A(L)[n^\infty]$) will be a shorthand for $A(L)[[n]^\ell]$ (resp. $A(L)[[n]^\infty]$).

Let now K_0 be the function field of a smooth and proper curve U over a finite field $\mathbb{F}$ of characteristic $p > 0$. Let B be an abelian variety over K_0. Suppose that for some $n > 3$ prime to p, the group scheme $B[n]$ is constant and that the Néron model of B over U has a semiabelian connected component.

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Proposition 0.1. *There exists*

- *an abelian variety C over K_0;*
- *an étale K_0-isogeny $\phi : B \to C$;*
- *an étale K_0-isogeny $f : C \to C$;*
- *a K_0-isogeny $g : C \to C$;*
- *a natural number $r \geq 0$*

such that

(a) $g \circ f = [p^r]$ *and* $g \circ f = f \circ g$;

(b) $C(K_0^{\text{sep}})[p^\infty] = C(K_0^{\text{sep}})[f^\infty] = C(\bar{K}_0)[f^\infty]$;

(c) $C(K_0^{\text{sep}})[g^\infty] = 0$.

Proof. For $\ell \geq 0$, define inductively

$$B_0 := B$$

and

$$B_{\ell+1} := B_\ell/(B_\ell(K_0^{\text{sep}})[p]).$$

For $\ell_2 \geq \ell_1$, let $\phi_{\ell_1,\ell_2} : B_{\ell_1} \to B_{\ell_2}$ be the (étale !) morphism obtained by composing the natural morphisms $B_{\ell_1} \to B_{\ell_1+1} \to \cdots \to B_{\ell_2}$. We first claim that

$$(\ker \phi_{\ell_1,\ell_2})(K_0^{\text{sep}}) = B_{\ell_1}(K_0^{\text{sep}})[p^{\ell_2-\ell_1}] \tag{1}$$

We prove the claim by induction on $\ell_2 - \ell_1$. For $\ell_2 - \ell_1 \leq 1$, the claim is true by definition. Suppose that $\ell_2 - \ell_1 \geq 1$. Let $x \in B(K_0^{\text{sep}})[p^{\ell_2-\ell_1}]$. Then $[p^{\ell_2-\ell_1-1}](x) \in B(K_0^{\text{sep}})[p]$ and thus

$$\phi_{\ell_1,\ell_1+1}([p^{\ell_2-\ell_1-1}](x)) = [p^{\ell_2-\ell_1-1}](\phi_{\ell_1,\ell_1+1}(x)) = 0.$$

Applying the inductive assumption to $\phi_{\ell_1,\ell_1+1}(x)$, we see that $\phi_{\ell_1+1,\ell_2}(\phi_{\ell_1,\ell_1+1}(x)) = \phi_{\ell_1,\ell_2}(x) = 0$. This proves that $(\ker \phi_{\ell_1,\ell_2})(K_0^{\text{sep}}) \supseteq B_{\ell_1}(K_0^{\text{sep}})[p^{\ell_2-\ell_1}]$. To prove the opposite inclusion, let $x \in (\ker \phi_{\ell_1,\ell_2})(K_0^{\text{sep}})$. We compute

$$\phi_{\ell_1,\ell_2}(x) = \phi_{\ell_1+1,\ell_2}(\phi_{\ell_1,\ell_1+1}(x)) = 0,$$

which implies (by the inductive hypothesis) that

$$[p^{\ell_2-\ell_1-1}](\phi_{\ell_1,\ell_1+1}(x)) = \phi_{\ell_1,\ell_1+1}([p^{\ell_2-\ell_1-1}](x)) = 0,$$

which in turn implies that $[p]([p^{\ell_2-\ell_1-1}](x)) = [p^{\ell_2-\ell_1}](x) = 0$. This proves that $(\ker \phi_{\ell_1,\ell_2})(K_0^{\rm sep}) \subseteq B_{\ell_1}(K_0^{\rm sep})[p^{\ell_2-\ell_1}]$ and completes the proof of the claim.

Now we know that by the reasoning made in the last page of my article *Infinitely p-divisible points on abelian varieties defined over function fields of characteristic* $p > 0$ (see my homepage), that there are only finitely many isomorphism classes of abelian varieties over K_0 in the sequence $\{B_\ell\}_{\ell\in\mathbb{N}}$. Let C be an abelian variety over K_0, which appears at least twice in the sequence $\{B_\ell\}_{\ell\in\mathbb{N}}$. Let $n_2 > n_1$ be such that $C \simeq B_{n_1} \simeq B_{n_2}$. Then by construction (under the identification $C = B_{n_1}$)

$$\phi_{n_1,n_2}^{\circ\ell} = \phi_{n_1,n_1+\ell\cdot(n_2-n_1)} \tag{2}$$

for any $\ell \geq 1$ and thus

$$C(K_0^{\rm sep})[p^\infty] = C(K_0^{\rm sep})[\phi_{n_1,n_2}^\infty] \tag{3}$$

Now define $f := \phi_{n_1,n_2}$ and $r := n_2 - n_1$. Define g as the only K_0-isogeny such that $g \circ f = [p^r]$.

Notice then that the identity $g \circ f = [p^r]$ implies the identity $f \circ g = [p^r]$. To see this last fact directly, recall first that there are natural injection of rings

$$\mathrm{End}_{K_0}(C) \hookrightarrow \mathrm{End}_{\bar{K}_0}(C_{\bar{K}_0}) \hookrightarrow \mathrm{End}_{\mathbb{Z}_t}(T_t(C(\bar{K}_0))) \hookrightarrow \mathrm{End}_{\mathbb{Q}_t}(T_t(C(\bar{K}_0)) \otimes \mathbb{Q}_t)$$

where $T_t(C(\bar{K}_0))$ is the classical Tate module of $C_{\bar{K}_0}$ and $t > 0$ is some prime number $\neq p$. Now if M and N are two square matrices of the same size with coefficients in a field of characteristic 0, such that $M \cdot N = p^r$, then $p^{-r}N$ is the inverse matrix of M and thus $N \cdot M = p^r$. This fact combined with the existence of the above injections implies that $f \circ g = [p^r]$ if $g \circ f = [p^r]$.

We have already proven (a). Point (b) is contained in equation (3).

We now prove (c). Suppose that for some $\ell \geq 0$ and some $x \in C(K_0^{\rm sep})$, we have $g^{\circ l}(x) = 0$. Let $y \in (f^{\circ \ell})^{-1}(x) \subseteq C(K_0^{\rm sep})$. Then $g^{\circ l}(f^{\circ l}(y)) = [p^{r\ell}](y) = 0$. Hence $f^{\circ l}(y) = 0 = x$ by (1) and (2). $\square$