LAMBDA-STRUCTURE ON GROTHENDIECK GROUPS OF HERMITIAN VECTOR BUNDLES

BY

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ABSTRACT

We define a "compactification" of the representation ring of the linear group scheme over $\mathrm{Spec}\mathbb{Z}$, in the spirit of Arakelov geometry. We show that it is a λ-ring which is canonically isomorphic to a localized polynomial ring and that it plays a universal role with respect to natural operations on the K_0-theory of hermitian bundles defined by Gillet–Soulé. As a byproduct, we prove that the natural pre-λ-ring structure of the K_0-theory of hermitian bundles is a λ-ring structure. This last result plays a key role in the proof of the main results of [18] and [12].

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1. Introduction

In their paper [6, 6., II], Gillet and Soulé gave a definition of arithmetic Grothendieck groups, i.e. they showed how to associate Grothendieck groups to arithmetic varieties "compactified" in the sense of Arakelov geometry. Furthermore, they gave a description of a pre-λ-structure on these groups. As is the case for the traditional Grothendieck groups of vector bundles on schemes, this structure arises from the exterior power operations on vector bundles. Once this pre-λ-structure is given, one is naturally lead to ask if it is a λ-structure, i.e. if there exist universal polynomials computing products as well as compositions of λ-operations (see subsection 2.1 for a brief introduction to λ-rings). In the traditional setting of the Grothendieck groups of vector bundles mentioned above, such polynomials exist and there are two main ways to prove their existence: the first one relies on the geometric splitting of vector bundles when lifted to their own projectivized total space (see [9, VI]) and the second one relies on the existence of a universal ring for natural operations on Grothendieck groups of vector bundles (see [9, p. 393] and [19]), which is a λ-ring. This last ring is in fact the representation ring of the linear group scheme over $\mathbb{Z}$. Both methods extend to the Arakelovian setting and yield the existence of a λ-structure on arithmetic Grothendieck groups. We refer to the (unpublished) text [15] for a proof using the first method. It is the purpose of the following paper to prove the existence of a universal ring for natural operations on arithmetic Grothendieck groups, which is a λ-ring and thus give a proof of the existence of a λ-structure taking the second route. It will appear that this ring is a "compactification", in the sense of Arakelov geometry, of the representation ring of the linear group scheme over $\mathbb{Z}$. The existence of such a "compactification" was apparently already known to J.-F. Burnol (private conversations) but we don't know if his methods of proof coincide with ours. This second proof apparently also works for a much more general class of schemes (Meissner's result is only formulated for projective varieties and our result applies to all schemes X such that $X \times_{\mathbb{Z}} \mathbb{Q}$ is smooth over $\mathbb{Q}$) and differs in method from the proof of its algebraic analog (see 3.1) in that it doesn't rely on the theory of the Brauer homomorphisms for its main step. Furthermore, one might hope that the universal ring defined here could be used to define a λ-structure on a yet to be defined higher arithmetic $\widehat{K}$-theory (see [20]) since its traditional counterpart has found such an application in the paper [13]. The hermitian representations that we use to construct our "compactifications" also appear in [5], in the context of the theory of heights. The main results of this paper are announced in [16]. The existence of a special

λ-structure on arithmetic Grothendieck groups plays a crucial role in the proof of the main results of [18] and [12] (see the end of the paper).

2. Preliminaries

2.1. λ-STRUCTURES.   Here we gather some general facts about λ-rings. The reader will find [1] a good reference. A **pre-λ-ring** is a unital commutative ring R with operations $\lambda^k \colon R \to R$ ($k \geq 0$), such that
  (i) $\lambda^0 = 1, \lambda^1 = Id$;
  (ii) $\lambda^k(x + y) = \sum_{i=0}^{k} \lambda^i(x)\lambda^{k-i}(y)$
for all $k \geq 1$ and for all $x, y \in R$. Elements $x \in R$ are called **line elements** if $\lambda^i(x) = 0$ for all $i > 1$. Now let $k, j \geq 0$ and $x \in R$ and suppose that $x = l_1 + \cdots + l_r$, where $l_1, \ldots, l_r$ are line elements, and that $r > kj$. The expression $\lambda^k(\lambda^j(x))$ can then be computed via the rule (ii) and yields a polynomial with integer coefficients in the l_i, which is symmetric by construction. By the fundamental theorem on symmetric functions, this polynomial can be written as a polynomial in the symmetric functions $\sigma_1(l_1, \ldots, l_r), \ldots, \sigma_{kj}(l_1, \ldots, l_r)$, which turns out to be independent of r; we denote it by $P_{k,l}$. Similarly, let $k \geq 0$ and let $x \in R$ (resp. $x' \in R$) and suppose that $x = l_1 + \cdots + l_r$ (resp. $x' = l_1' + \cdots + l_{r'}'$), where $l_1, \ldots, l_r$ (resp. $l_1', \ldots, l_{r'}'$) are line elements, and that $r > k$, $r' > k$. The expression $\lambda^k(x + x')$ can then be computed via the rule (ii) and yields a polynomial with integer coefficients in the l_i and l_i', which is symmetric in the l_i, as well as symmetric in the l_i'. By the theorem on symmetric functions it can be written as a polynomial in the symmetric functions $\sigma_1(l_1, \ldots, l_r), \ldots, \sigma_k(l_1, \ldots, l_r), \sigma_1(l_1', \ldots, l_{r'}'), \ldots, \sigma_k(l_1', \ldots, l_{r'}')$. Again it can be proved that this polynomial is independent of r; we denote it by P_k.

A λ-**ring** is a pre-λ-ring satisfying the following additional conditions:
  (iii) $\lambda^k(xy) = P_k(\lambda^1(x), \ldots, \lambda^k(x), \lambda^1(y), \ldots, \lambda^k(y))$;
  (iv) $\lambda^k(\lambda^l(x)) = P_{k,l}(\lambda^1(x), \ldots, \lambda^{kl}(x))$.
An element x such that $\lambda^k(x) = 0$ for all $k > d$ is said to be λ-**finite** of λ-dimension d. A pre-λ-ring with the property that all its elements are equal to differences of elements of finite λ-dimension is also said to be λ-finite. In a λ-finite ring, the conditions (iii) and (iv) are by definition automatically satisfied.

This also holds if R has a λ-ring extension in which this is satisfied.

Let $\lambda_t(x)$: $R \to 1 + R[[t]]$ be defined as $\lambda_t(x) = 1 + \sum_{k=1}^{\infty} \lambda^k(x)t^k$, where $1 + R[[t]]$ is the multiplicative subgroup of the ring of formal power series $R[[t]]$ consisting of power series with constant coefficient 1. The first condition states that λ_t is a group homomorphism from R as an additive group into $1 + R[[t]]$ as a multiplicative group. A pre-λ-ring structure on R gives rise to a collection of group homomorphisms, called **Adams operations**.

Definition 2.1: The Adams operations ψ^k: $R \to R$ are defined by the equations

$$\psi_{-t}(x) := \frac{-t.d\lambda_t(x)/dt}{\lambda_t(x)}, \quad \psi_t(x) =: \sum_{k \geq 1} \psi^k(x)t^k.$$

The Adams operations are additive. They are multiplicative and satisfy the identities $\psi^k \circ \psi^l = \psi^{kl}$ $(k, l \geq 1)$, if the pre-λ-structure is a λ-structure. If R is a $\mathbb{Q}$-algebra and we are provided with a collection of ring endomorphisms ψ^k: $R \to R$, such that $\psi^k \circ \psi^l = \psi^{kl}$, we can define a λ-structure via the formula

$$(1) \qquad\qquad \lambda_t := \exp\left(\sum_{k=1}^{\infty}(-1)^k \psi^k . \frac{t^k}{k}\right)$$

which inverts the formula given above for ψ_t. Given any commutative graded unitary $\mathbb{Q}$-algebra $R = \bigoplus_{i=0}^{\infty} R_i$, one can define Adams operations by the formula $\psi^k(x) = \sum_{i=0}^{\infty} k^i . x_i$, where x_i is the component of degree i. Any $\mathbb{Q}$-graded ring thus carries a canonical λ-ring structure. See [9, V, Appendice].

2.2. ARITHMETIC GROTHENDIECK GROUPS. In this subsection, we recall the definition of the arithmetic Grothendieck group $\widehat{K}_0(X)$ of a scheme X such that $X \times_{\mathbb{Z}} \mathbb{Q}$ is smooth over $\mathbb{Q}$. We describe its canonical pre-λ-ring structure.

We shall denote the complex manifold associated to its complex points by $X(\mathbb{C})$.

The complex conjugation induces an anti-holomorphic automorphism F_{∞} of $X(\mathbb{C})$. A **hermitian bundle** $\overline{E} = (E, h)$ on X is a geometric vector bundle E on X, together with a conjugation invariant hermitian metric h on $E_{\mathbb{C}}$, the holomorphic vector bundle associated to E on $X(\mathbb{C})$. A metric h on $E_{\mathbb{C}}$ is conjugation invariant if $\overline{(F_{\infty}^*(h))(s,l)} = h(s,l)$, for all C^{∞} sections s, l of $E_{\mathbb{C}}$. We shall denote by $A^{p,p}(X)$ the set of real differential forms ω of type (p,p) on $X(\mathbb{C})$ that satisfy the equation $F_{\infty}^*\omega = (-1)^p\omega$ and by $\tilde{A}^{p,p}(X)$ the quotient $(A^{p,p}(M)/(\text{im}\,\partial + \text{im}\,\overline{\partial}))$. The direct sum $\bigoplus_{p \geq 0}(A^{p,p}(M)/(\text{im}\,\partial + \text{im}\,\overline{\partial}))$ will be referred to as $\tilde{A}(X)$. Finally, let $Z^{p,p}(X)$ be the kernel of the operator $d = \partial + \overline{\partial}$

in $A^{p,p}(X)$ and let $Z(X)$ be the direct sum sum $\bigoplus_{p\geq 0} Z^{p,p}(X)$. Let

$$(2) \qquad\qquad \mathcal{E}\colon 0 \to E' \to E \to E'' \to 0$$

be an exact sequence of vector bundles on X. Denote by $\overline{\mathcal{E}}$ the sequence $\mathcal{E}$ together with F_∞-invariant hermitian metrics h', h and h'' on E', E and E''. To these data is associated a **Bott–Chern** secondary class $\tilde{\mathrm{ch}}(\overline{\mathcal{E}}) \in \tilde{A}(X)$. It satisfies the equation

$$(3) \qquad\qquad dd^c\tilde{\mathrm{ch}}(\overline{\mathcal{E}}) = \mathrm{ch}(\overline{E}' \oplus \overline{E}'') - \mathrm{ch}(\overline{E})$$

(recall that $d^c = \frac{1}{4\pi i}(\partial - \overline{\partial})$, so that $dd^c = \frac{1}{2\pi i}\overline{\partial}\partial$) where $\mathrm{ch}(\overline{E}')$, $\mathrm{ch}(\overline{E})$ and $\mathrm{ch}(\overline{E}'')$ are the Chern character forms associated to the unique holomorphic hermitian connections of E', E and E''. It vanishes if the map $E' \to E$ is metric preserving and the map $E \to E''$ admits a metric preserving holomorphic left inverse. More generally, if ρ is any power series in the Chern forms associated to the holomorphic hermitian connection of a hermitian holomorphic vector bundle, one can associate a Bott–Chern secondary class $\tilde{\rho}(\overline{\mathcal{E}})$ to the same data as above, with similar properties. For a definition of the secondary classes, see [6, Th. 1.2.2, p. 167]. The following definition is taken from [6, II].

Definition 2.2: The **arithmetic Grothendieck group** $\widehat{K}_0(X)$ of X is the free abelian group generated by the hermitian bundles on X and $\tilde{A}(X)$, quotiented by the relations:
  (a) $\overline{E} + \tilde{\mathrm{ch}}(\overline{\mathcal{E}}) = \overline{E}' + \overline{E}''$ for exact sequences of hermitian bundles $\overline{\mathcal{E}}$ as above;
  (b) if η', η, η'' are elements of $\tilde{A}(X)$ and η is the sum in $\tilde{A}(X)$ of η' and η'', then $\eta = \eta' + \eta''$ in $\widehat{K}_0(X)$.

Note that there is a natural exact sequence of groups

$$(4) \qquad\qquad \tilde{A}(X) \overset{a}{\to} \widehat{K}_0(X) \to K_0(X) \to 0$$

where the second map sends elements of $\tilde{A}(X)$ on 0 and hermitian vector bundles on the corresponding vector bundles.

One can define a multiplication and λ-operations on $\widehat{K}_0(X)$ as follows. Let $\Gamma(X)$ be the $\mathbb{R}$-vector space $Z(X) \oplus \tilde{A}(X)$ with the grading giving degree p to the subspace $Z^{p,p} \oplus \tilde{A}^{p-1,p-1}$ when $p > 0$ and degree 0 to the subspace $Z^{0,0} \oplus 0$. Denote by $*$ the bilinear pairing defined on $\Gamma(X)$ by the formula

$$(\omega, \eta) * (\omega', \eta') := (\omega \wedge \omega', \omega \wedge \eta' + \eta \wedge \omega' + (dd^c\eta) \wedge \eta').$$

This pairing defines a commutative, unitary, $\mathbb{R}$-graded ring structure on $\Gamma(X)$. For a proof, see [6, 7.3.2, p. 233]. The space $\Gamma(X)$ is thus endowed with a canonical λ-ring structure.

PROPOSITION-DEFINITION 2.3: *Let* $\overline{E}, \eta, \overline{E}', \eta'$ *be generators of* $\widehat{K}_0(X)$. *Define a product by the equations*

$$\overline{E}.\overline{E}' := \overline{E} \otimes \overline{E}', \quad \eta.\eta' = [(0, \eta) * (0, \eta')], \quad \overline{E}.\eta := \mathrm{ch}(\overline{E}) \wedge \eta$$

where [.] *is the projection on the second component in* $\Gamma(X)$. *Then this product is well-defined on* $\widehat{K}_0(X)$ *and turns it into a commutative unital ring, where the unit is represented by the trivial line bundle, endowed with the trivial metric.*

See [6, Th. 7.3.4, p. 235] for the proof. It follows from the definitions that $\tilde{A}(X)$ is an ideal of $\Gamma(X)$ closed under the λ^k-operations for $k \geq 1$.

Let now V be a finite dimensional complex vector space and let h be a hermitian metric on V; let $v_1, \ldots, v_n$ be a basis of V, which is orthonormal for h. The k-th exterior power $\lambda^k(V)$ of V is then also endowed with a natural metric, the unique one making the standard basis $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}$ $(1 \leq i_1 < i_2 < \cdots < i_k \leq n)$ orthonormal. One can show that the k-th exterior power metric is independent of the basis chosen for its construction. The basis $v_1, \ldots v_n$ also determines a representation of $s: \mathrm{GL}_n(\mathbb{C}) \to \mathrm{End}(V)$ of $\mathrm{GL}_n(\mathbb{C})$ in V; if h' is any metric on V, with matrix H' in the basis $v_1, \ldots, v_n$, one can show from the definition that, in the standard basis consisting of the $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}$, the matrix H' of the k-th exterior power metric of h' on $\lambda^k(V)$ is $(\lambda^k(s))(H')$. From this, we see that the construction of the exterior power metric also carries over to vector bundles. For $k \geq 1$, let us define

$$\lambda^k(\overline{E}) = \lambda^k(E, h) := (\lambda^k(E), \lambda^k(h))$$

which is the k-th exterior power of E endowed with the k-th exterior power metric $\lambda^k(h)$. We thus obtain operations λ^k $(k \geq 1)$ acting on all the generators of $\widehat{K}_0(X)$. The operation λ^0 will by convention send all the generators on the trivial line bundle. It is proved in [6, II] that these operations are well-defined on $\widehat{K}_0(X)$ and give it the structure of a pre-λ-ring.

3. Universal relations

In this section, we shall show that there is a "compactification" in the sense of Arakelov geometry of the representation ring of the linear group scheme over $\mathrm{Spec}\mathbb{Z}$, the group $\widehat{R}_{\mathbb{Z}}(\mathbf{GL}_n \times \mathbf{GL}_m)$, which is a λ-ring. The statement of the

main result is in Corollary 3.22, where the exact structure of that group is also given. In the last section, we shall prove that $\widehat{R}_{\mathbb{Z}}(\mathbf{GL}_n \times \mathbf{GL}_m)$ plays a universal role with respect to arithmetic K_0-theory (Corollary 4.4) and we shall use this universality to prove that the pre-λ-ring structure defined above on the arithmetic Grothendieck group is a λ-ring structure.

3.1. G-MODULES.   In this subsection, we collect some general facts about representation theory (cf. [19]). Let A be a commutative unital ring and $\mathbf{G}$ an affine group scheme over A. Let E be an A-module. For any A-algebra A', let $\mathbf{G}(A')$ refer to the group of sections of $\mathbf{G}_{A'}$ over A'. This group is also called the group of points of $\mathbf{G}$ over A'. Let $\mathrm{Aut}_{A'}(E)$ refer to the group of all A' automorphisms of $E \otimes_A A'$ as an A'-module. A $\mathbf{G}$-**module** structure on E is a collection of group morphisms $d_{A'}\colon \mathbf{G}(A') \to \mathrm{Aut}_{A'}(E)$, varying functorially with A'. A map of $\mathbf{G}$-modules is a map of A-modules $f\colon E \to E'$ such that $(d_{A'}(P))((f \otimes_A \mathrm{Id})(v)) = (f \otimes_A \mathrm{Id})((d_{A'}(P))(v))$ for all A-algebras A', all $v \in E \otimes_A A'$ and all $P \in \mathbf{G}(A')$. Furthermore, the $\mathbf{G}$-modules form an abelian category. We denote the associated Grothendieck group by $R_A(\mathbf{G})$. If $E = A^{\oplus n}$, a $\mathbf{G}$-module structure on E is equivalent to a map of group schemes from $\mathbf{G}$ to $\mathbf{GL}_n$, the n-dimensional linear group scheme over A. If $\mathbf{G}$ is the n-dimensional torus $\mathbf{T}^n$ over A, it can be shown (see [3, Expose I, 4.7.3]) that a $\mathbf{T}^n$-module structure on an A-module is equivalent to a $\mathbb{Z}^n$-grading on this module. There is a natural ring morphism $i^*\colon R_A(\mathbf{GL}_n) \to R_A(\mathbf{T}^n)$, which is the pull-back map arising from the immersion $i\colon \mathbf{T}^n \to \mathbf{GL}_n$. The following is a particular case of a theorem of Serre (see [19, p. 49, Th. 4 and p. 52, 3.8]). If R is a ring and S a subset of R, we shall write R_S for the localization of R at the multiplicative set generated by S.

THEOREM 3.1 (Serre): *Let i denote the natural map $\mathbf{T}^n \times \mathbf{T}^m \to \mathbf{GL}_n \times \mathbf{GL}_m$. If A is a field, then the induced map $i^*\colon R_A(\mathbf{GL}_n \times \mathbf{GL}_m) \to R_A(\mathbf{T}^n \times \mathbf{T}^m)$ is injective and the ring $R_A(\mathbf{GL}_n \times \mathbf{GL}_m)$ is isomorphic to the (localized) polynomial ring*

$$\mathbb{Z}[\lambda^1(\mathrm{Id}_n \times 1), \ldots, \lambda^n(\mathrm{Id}_n \times 1), \lambda^1(1 \times \mathrm{Id}_m), \ldots, \lambda^m(1 \times \mathrm{Id}_m)]_{\lambda^n(\mathrm{Id}_n \times 1), \lambda^m(1 \times \mathrm{Id}_m)}.$$

Moreover, if A is a principal ideal domain with fraction field F then $R_A(\mathbf{GL}_n) \simeq R_F(\mathbf{GL}_n)$, where the isomorphism is induced by the functor $(.) \otimes_A F$.

Recall that the Weyl group of $\mathbf{T}^{n+m}$ in $\mathbf{GL}_n \times \mathbf{GL}_m$ is a finite group W (see [3]), given by the quotient of the normalizer of $\mathbf{T}^{n+m}$ in $\mathbf{GL}_n \times \mathbf{GL}_m$ by $\mathbf{T}^{n+m}$. This group acts on $\mathbf{T}^{n+m}$ by automorphisms and is isomorphic to the

symmetric group on $n + m$ elements; given a permutation σ and a $\mathbb{Z}^{n+m}$-graded A-module $V = \bigoplus_{\vec{i} \in \mathbb{Z}^{n+m}} V_{\vec{i}}$, the image $\sigma^* V$ of V under σ is the graded module $\bigoplus_{\vec{i} \in \mathbb{Z}^{n+m}} V_{\sigma(\vec{i})}$, where $\sigma(\vec{i})$ denotes the vector $\vec{i}$ with components permuted by σ.

3.2. HERMITIAN **G**-MODULES. In the following, D is $\mathbb{Z}$ or $\mathbb{Q}$. Let **G** be a closed group subscheme of $(\mathbf{GL}_n \times \mathbf{GL}_m)_D$, with the property that the associated closed holomorphic group $\mathbf{G}(\mathbb{C}) \subseteq (\mathbf{GL}_n \times \mathbf{GL}_m)(\mathbb{C})$ is connected for the ordinary topology and invariant under the operation of taking the matrix adjoint. This hypothesis on **G** will hold until the end of the paper. We shall construct a Grothendieck group of **G**-modules with metrics. This construction will turn out to yield the universal ring we seek, when applied to $\mathbf{G} = \mathbf{GL}_n \times \mathbf{GL}_m$.

3.2.1. First definitions and results. For any finitely generated projective **G**-module V over D, we let $V_{\mathbb{C}}$ be $V \otimes_D \mathbb{C}$ endowed with the induced $\mathbf{G}(\mathbb{C})$-module structure. Denote the corresponding group map $\mathbf{G}(\mathbb{C}) \to \mathrm{Aut}_{\mathbb{C}}(V_{\mathbb{C}})$ by r.

Definition 3.2: An admissible hermitian metric on V is a conjugation invariant hermitian metric h on $V_{\mathbb{C}}$ such for all $M \in \mathbf{G}(\mathbb{C})$, $r(M^*)$ is the adjoint with respect to h of the endomorphism $r(M)$. The pair $\overline{V} := (V, h)$ is called a **hermitian G-module**.

Recall that the conjugation $V_{\mathbb{C}} \to V_{\mathbb{C}}$ is an $\mathbb{R}$-linear map given by the formula $v \otimes z \mapsto v \otimes \overline{z}$. A metric h on V is conjugation invariant if $h(\overline{x}, \overline{y})$ is the conjugate of the complex number $h(x, y)$, for all $x, y \in V_{\mathbb{C}}$.

LEMMA 3.3: *The representation r is determined uniquely by its restriction to $\mathbf{G}(\mathbb{C}) \cap U_n \times U_m$. The metric h is admissible if and only if for every $M \in \mathbf{G}(\mathbb{C}) \cap U_n \times U_m$, the automorphism $r(M)$ is unitary with respect to h.*

Here U_i denotes the group of unitary $i \times i$ matrices.

Proof of Lemma 3.3: Recall that the underlying complex vector space of the (holomorphic) Lie algebra of $\mathbf{GL}_n(\mathbb{C}) \times \mathbf{GL}_m(\mathbb{C})$ is the complex vector space $M_n \times M_m$, where M_i is the space of complex $i \times i$ matrices. The matrix adjunction $(\cdot)^*$, viewed as a map from $\mathbf{GL}_n(\mathbb{C}) \times \mathbf{GL}_m(\mathbb{C})$ to itself, is a C^∞ map and its differential at the origin is the matrix adjunction viewed as a map from $M_n \times M_m$ to itself. Thus, the Lie algebra $T\mathbf{G}(\mathbb{C})$ of $\mathbf{G}(\mathbb{C})$ is invariant under matrix adjunction. Let now $M_0 \in T\mathbf{G}(\mathbb{C})$. The elements $A := \frac{1}{2}i(M_0 + M_0^*)$ and $B := \frac{1}{2}(M_0 - M_0^*)$ lie in $T\mathbf{G}(\mathbb{C})$ and by construction $M_0 = -i.A + B$. Recall now that the (real) Lie algebra of $U_n \times U_m$ consists of the matrices $M \in M_n \times M_m$ such that $M^* = -M$; by construction again A and B lie in $T(U_n \times U_m)$ and as the representation r is holomorphic we can write $r(M_0) = -i.r(A) + r(B)$.

This settles the first statement in the lemma. To see that the second one is true suppose that for every $M \in \mathbf{G}(\mathbb{C}) \cap U_n \times U_m$, the automorphism $r(M)$ is unitary with respect to h and let M_0, A, B be as before. Consider the string of equations

$$r(\exp(M_0)^*) = \exp(dr(M_0^*)) = \exp(i.dr(A^*) + dr(B^*))$$
$$= \exp(i.dr(A)^* + dr(B)^*) = \exp(dr(M_0)^*) = \exp(dr(M_0))^*$$
$$= r(\exp(M_0))^*$$

These equations show that the admissibility condition is satisfied for all the elements in a small neighborhood of the identity in $T\mathbf{G}(C)$. Using the fact that matrix adjunction reverses the order of products and the fact that any neighborhood of the identity generates $T\mathbf{G}(\mathbb{C})$, we see that it is satisfied for all the elements of $T\mathbf{G}(C)$. This proves one direction of the equivalence; the other direction follows immediately from the definitions, so we are done.   ∎

COROLLARY 3.4: *There exists an admissible metric on every finitely generated* $\mathbf{G}$-*module.*

Proof of Corollary 3.4:   The group $\mathbf{G}(\mathbb{C}) \cap U_n \times U_m$ is compact and one can thus obtain a $\mathbf{G}(\mathbb{C}) \cap U_n \times U_m$-invariant metric on a representation of $\mathbf{G}(\mathbb{C})$ by integration. Using the second statement in the lemma, the result follows.   ∎

Next, we define a Grothendieck group of hermitian $\mathbf{G}$-modules.

Definition 3.5: Let $\widehat{R}_D(\mathbf{G})$ be the free group generated by all the hermitian $\mathbf{G}$-modules, with the relations $(V, h) = (V', h') + (V'', h'')$ for exact sequences

$$(5) \qquad\qquad\qquad 0 \to V' \to V \to V'' \to 0$$

of $\mathbf{G}$-modules such that in the induced sequence

$$(6) \qquad\qquad\qquad 0 \to V'_{\mathbb{C}} \to V_{\mathbb{C}} \to V''_{\mathbb{C}} \to 0$$

the first map preserves the metric and the second map induces an isometry from the orthogonal complement of $(V'_{\mathbb{C}}, h')$ in $(V_{\mathbb{C}}, h)$ onto $(V''_{\mathbb{C}}, h'')$.

A sequence as above will be said to be **orthogonal**.

If $\mathbf{G}$ is the trivial group scheme $\mathrm{Spec} D$, then one might wonder if the groups $\widehat{K}_0(D)$ and $\widehat{R}_D(\mathbf{G})$ are isomorphic. The answer is affirmative; for the proof, compare [6, Th. 6.2. (i)]GS3 and [7]. We shall from now on identify $\widehat{K}_0(D)$ with $\widehat{R}_D(\mathbf{G})$ in that case.

LEMMA 3.6: *The orthogonal complement* $(V'_{\mathbb{C}})^{\perp}$ *of* $V'_{\mathbb{C}}$ *in* $V_{\mathbb{C}}$ *is invariant under conjugation and is a sub-$\mathbf{G}(\mathbb{C})$-module. In particular, in the sequence* (6), *the map* $V_{\mathbb{C}} \to V''_{\mathbb{C}}$ *has a left inverse which preserves the metric of* $V''_{\mathbb{C}}$ *and is a morphism of $\mathbf{G}(\mathbb{C})$-modules.*

Proof of Lemma 3.6: We first prove conjugation invariance. We have to check that $\overline{x}$ is orthogonal to $V_{\mathbb{C}}$, if x is orthogonal to $V_{\mathbb{C}}$. Let $y \in V_{\mathbb{C}}$; we compute $h(x, y) = 0 = \overline{h(\overline{x}, \overline{y})} = h(\overline{x}, \overline{y})$. Since conjugation induces an ($\mathbb{R}$-linear) automorphism $V_{\mathbb{C}} \to V_{\mathbb{C}}$, this implies that $h(\overline{x}, y) = 0$ for all $y \in V_{\mathbb{C}}$ and thus proves our first claim. To prove the second claim, let $r \colon \mathbf{G}(\mathbb{C}) \to \mathrm{Aut}_{\mathbb{C}}(V_{\mathbb{C}})$ be the map arising from the $\mathbf{G}(\mathbb{C})$-module structure. The fact that $V'_{\mathbb{C}}$ is a sub-$\mathbf{G}(\mathbb{C})$-module is expressed by the relation $r(M)x \in V'_{\mathbb{C}}$ for all $x \in V'_{\mathbb{C}}$ and for all $M \in G(\mathbb{C})$. For $(V_{\mathbb{C}})^{\perp}$ to be a sub-$\mathbf{G}(\mathbb{C})$-module, the equation $h(x, r(M)y) = 0$ has to hold for all $x \in V'_{\mathbb{C}}$ and all $y \in (V'_{\mathbb{C}})^{\perp}$. Now using the admissibility of h, we can compute $h(x, r(M)y) = h(r(M^*)x, y)$ and $h(r(M^*)x, y)$ vanishes by hypothesis. Thus we are done.   ∎

Remark: The last lemma entails that the properties of $\mathbf{G}(\mathbb{C})$ that are given at the beginning of the paragraph (connectedness and invariance under the operation of taking the matrix adjoint) imply that $\mathbf{G}(\mathbb{C})$ is a reductive holomorphic group.

The two last lemmata immediately imply the following facts.

Let V' be a sub-$\mathbf{G}$-module of the projective $\mathbf{G}$-module V. Let h be an admissible metric on V and h' its restriction to V'. Suppose that V/V' is projective as well. When V/V' is endowed with the quotient metric, it is a hermitian $\mathbf{G}$-module.

Let (V, h), (V', h') be hermitian $\mathbf{G}$-modules; the tensor product metric on the $\mathbf{G}$-module $V \otimes V'$ is admissible; the exterior power metric $\lambda^k(h)$ is admissible on the $\mathbf{G}$-module $\lambda^k(V)$.

From these facts, one deduces

PROPOSITION 3.7: *The tensor product endows the group $\widehat{R}_D(\mathbf{G})$ with a commutative unital ring structure.*

We shall show that exterior powers are well-defined on $\widehat{R}_D(\mathbf{G})$.

PROPOSITION 3.8: *The equality*

$$\lambda^r(\overline{V}) = \sum_{i=0}^{r} \lambda^i(\overline{V}') . \lambda^{r-i}(\overline{V}'')$$

holds in $\widehat{R}_D(\mathbf{G})$.

Proof of Proposition 3.8: Let V', V, V'' be as in Definition 3.5. The Proposition will follow from the next lemma.

LEMMA 3.9: *For $i > 0$, let $\lambda_i^r(V)$ be the D-submodule of $\lambda^r(V)$ generated by the elements $x_1' \wedge \cdots \wedge x_i' \wedge y_{i+1} \wedge \cdots \wedge y_r$, where $x_1', \ldots, x_i'$ are elements of V'. For $i = 0$, let $\lambda_i^r(V)$ be $\lambda^r(V)$. Then $\lambda_i^r(V)$ is actually a sub-$\mathbf{G}$-module and there is a natural isometric isomorphism of $\mathbf{G}$-modules*

$$\lambda^i(V') \otimes \lambda^{r-i}(V'') \simeq \lambda_i^r(V)/\lambda_{i+1}^r(V).$$

Proof of Lemma 3.9: For $i = 0$, the statement is obvious. For $i > 0$, the map is given by

$$(7) \qquad x_1' \wedge \cdots \wedge x_i' \otimes x_1'' \wedge \cdots \wedge x_{r-i}'' \to x_1' \wedge \cdots \wedge x_i' \wedge x_1'' \wedge \cdots \wedge x_{r-i}''$$

where $x_1', \ldots, x_i' \in V'$ and $x_1'', \ldots, x_{r-i}''$ are inverse images of elements of V''.

It is known (see [14, Prop. 9.3, p. 592]) that this map is well-defined and an isomorphism of $\mathbf{G}$-modules. Let $b_1', \ldots, b_{n'}' \in V'_{\mathbb{C}}$ and $b_1'', \ldots, b_{n''}'' \in V''_{\mathbb{C}}$ be orthonormal bases. Considering $V''_{\mathbb{C}}$ as included in $V_{\mathbb{C}}$ by the map given in Lemma 3.6, we can consider the union $b_1', \ldots b_{n'}' \in V'_{\mathbb{C}}, b_1'', \ldots b_{n''}'' \in V''_{\mathbb{C}}$ as an orthogonal base of $V_{\mathbb{C}}$. To prove that the map (7) is an isometry amounts to showing that the elements

$$b_{j_1}' \wedge \cdots \wedge b_{j_i}' \wedge b_{k_1}'' \wedge \cdots \wedge b_{k_{r-i}}''$$

where $1 \leq j_1 < \cdots < j_i \leq n'$, $1 \leq k_1 < \cdots < k_{r-i} \leq n''$, are orthonormal and in the orthogonal complement of $\lambda_{i+1}^r(V_{\mathbb{C}})$. They are clearly of norm 1. Consider now that the vector space $\lambda_{i+1}^r(V_{\mathbb{C}})$ is generated by elements of the type

$$b_{j_1}' \wedge \cdots \wedge b_{j_{i+p}}' \wedge b_{k_1}'' \wedge \cdots \wedge b_{k_{r-i-p}}''$$

for $p \geq 1$. These elements are all orthogonal to the elements $b_{j_1}' \wedge \cdots \wedge b_{j_i}' \wedge b_{k_1}'' \wedge \cdots \wedge b_{k_{r-i}}''$ mentioned above, since they are both distinct members of the canonical orthonormal basis of $\lambda^r(V_{\mathbb{C}})$, arising from $b_1', \ldots, b_{n'}', b_1'', \ldots, b_{n''}''$, so we are done.   ∎

Now we can compute

$$\lambda^r(\overline{V}) = \sum_{i=0}^{r} \lambda_i^r(V)/\lambda_{i+1}^r(V) = \sum_{i=0}^{r} \lambda^i(V') \otimes \lambda^{r-i}(V'')$$

and we are done.   ∎

THEOREM 3.10: *The exterior powers are well-defined on $\widehat{R}_D(\mathbf{G})$ and endow it with the structure of a pre-λ-ring.*

Proof of Theorem 3.10: Let $\mathcal{I}$ be the set of all isometric isomorphism classes of hermitian $\mathbf{G}$-modules. Define a map $\lambda_t \colon I \to 1 + \widehat{R}_D(C)^+[[t]]$ by the formula $\lambda_t(\overline{V}) = 1 + \sum_{i=1}^{\infty} \lambda^i(\overline{V})$. Extend this map by linearity to obtain a homomorphism $\mathbb{Z}[\mathcal{I}] \to 1 + \widehat{R}_D(C)^+[[t]]$. Proposition 3.8 shows that the kernel of λ_t contains the elements $\overline{V} - \overline{V}' - \overline{V}''$ as above. Thus λ_t and therefore all the λ^k are well-defined on $\widehat{R}_D(\mathbf{G})$ and the fact that λ_t is a homomorphism corresponds to the second part of the statement of the theorem. ∎

There is a natural embedding of $\widehat{K}_0(D)$ in $\widehat{R}_D(\mathbf{G})$, which associates to every hermitian D-module the corresponding $\mathbf{G}$-module endowed with the trivial structure. We shall denote by $\widehat{\mathrm{Pic}}(D) \subseteq \widehat{K}_0(D)$ the isometric isomorphism classes of hermitian projective D-modules of rank 1.

3.2.2. Tori. In this subsubsection, we shall determine the structure of $\widehat{R}_D(\mathbf{T}^n)$. See [19] for the corresponding algebraic statements.

LEMMA 3.11: *A (conjugation invariant) metric h on a hermitian $\mathbf{T}^n$-module V is admissible if and only if the pieces of the $\mathbb{Z}^n$-grading of $\overline{V}$ are pairwise orthogonal.*

Proof of Lemma 3.11: Each $\mathbf{T}^n$-module structure on V induces a map $S^n(\mathbb{C}) \xrightarrow{f} \mathrm{Aut}_{\mathbb{C}}(V_{\mathbb{C}})$, where $S^n(\mathbb{C})$ is the set of diagonal matrices in $\mathrm{GL}_n(\mathbb{C})$. The natural grading of $V_{\mathbb{C}}$ is connected to f in the following way. Let S be a diagonal matrix with diagonal elements $X_{11}, X_{22}, \ldots, X_{n,n}$; the piece $V_{\mathbb{C}(a_1,\ldots,a_n)}$ is an eigenspace of $f(S)$ with eigenvalue $X_{11}^{a_1} X_{22}^{a_2} \cdots X_{nn}^{a_n}$. Now suppose that V is endowed with an admissible metric. The admissibility property 3.2 implies that the images of elements of $S^n(\mathbb{C})$ by f are normal (i.e. commute with their adjoints) for the hermitian metric of $V_{\mathbb{C}}$. For two distinct non-zero pieces of the grading $(a_1, \ldots, a_n)$ and $(a'_1, a'_2, \ldots, a'_n)$ there clearly exists an element $S \in S^n(\mathbb{C})$, such that $V_{\mathbb{C}(a_1,\ldots,a_n)}$, $V_{\mathbb{C}(a'_1,a'_2,\ldots,a'_n)}$ lie in distinct eigenspaces of $f(S)$. Since $f(S)$ is normal, its distinct eigenspaces are orthogonal. This implies that the pieces of the $\mathbb{Z}^n$-grading of V are pairwise orthogonal.

To prove the converse, choose an orthonormal base on each piece of the grading and join all these bases to obtain an orthonormal base of $V_{\mathbb{C}}$, thus defining a hermitian metric on V. Let S be a diagonal matrix as above. If we identify $f(S)$ with its matrix in this basis, $f(S)$ is diagonal and the elements on the diagonal are the polynomials $X_{11}^{a_1} X_{22}^{a_2} \cdots X_{nn}^{a_n}$; thus we deduce $f(S^*) = f(S)^*$. Since the

adjoint of $f(S)$ for the metric h can be identified with the matrix $f(S)^*$, this concludes the proof. ∎

We can determine the structure of $\widehat{R}_D(\mathbf{T}^n)$ exactly. In the next Proposition, the elements $(e^{\vec{m}})_{\vec{m} \in \mathbb{Z}^n}$ are the basis elements of the group algebra $\widehat{K}_0(D)[\mathbb{Z}^n]$.

PROPOSITION 3.12: *There is an isomorphism $\widehat{R}_D(\mathbf{T}^n) \overset{\alpha}{\simeq} \widehat{K}_0(D)[\mathbb{Z}^n]$, with α defined by the formula $\alpha(\overline{V}) := \sum_{\vec{m} \in D^n} \overline{V}_{\vec{m}} e^{\vec{m}}$, where the $\overline{V}_{\vec{m}}$ are considered equipped with the induced metric.*

Proof of Proposition 3.12:   We shall describe an inverse that is a homomorphism of rings. For $\vec{m} \in \mathbb{Z}^n$, define a map $f_{\vec{m}}: \widehat{K}_0(D) \mapsto \widehat{R}_D(\mathbf{T}^n)$ which sends hermitian D-modules $\overline{V}$ to $\mathbf{T}^n$-modules with their $\vec{m}$-component equal to $\overline{V}$ and all the others 0. This map is well-defined by definition and the family of all $f_{\vec{m}}$ defines a map $f: \widehat{K}_0(D)[\mathbb{Z}^n] \mapsto \widehat{R}_D(\mathbf{T}^n)$, which is seen to be a two-sided inverse of α. ∎

PROPOSITION 3.13: *All elements of $\widehat{R}_D(\mathbf{T}^n)$ are sums of line elements.*

Proof of Proposition 3.13:   Let $\overline{V}$ be a hermitian $\mathbf{T}^n$-module. Since the pieces of the grading are orthogonal by Lemma 3.11, we can write $\overline{V} = \sum_{\vec{m} \in \mathbb{Z}^n} \overline{V}_{\vec{m}}$ in $\widehat{R}_D(\mathbf{T}^n)$, where the $\overline{V}_{\vec{m}} = (V_{\vec{m}}, h_{\vec{m}})$ are considered endowed with the induced metrics and with the grading that is non-zero on the $\vec{m}$-th piece of the grading only. It will therefore be sufficient to show that any such $\overline{V}_{\vec{m}}$ can be written as sums of hermitian 1-dimensional $\mathbb{Z}^n$-graded modules in $\widehat{R}_D(\mathbf{T}^n)$; to prove this, remember that the structure theorem for finitely generated projective modules over a Dedekind domain (see [14, Par. 2, p. 532]) shows that there exists a fractional ideal J in the quotient field of D and an exact sequence of D-modules

$$0 \to J \to V_{\vec{m}} \to V_{\vec{m}}/J \to 0$$

where $V_{\vec{m}}/J$ is projective as well. Endow J with the metric induced by h and with the grading induced by $V_{\vec{m}}$. Endow $V_{\vec{m}}/J$ with the quotient grading and metric. One can then apply induction on the dimension to obtain the last claim and complete the proof. ∎

3.2.3. Linear group schemes over $\mathbb{Q}$.   In this subsubsection, we shall prove a structure theorem for $\widehat{R}_{\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_m)$. The proof will also show that the pre-λ-ring structure of this ring is a λ-ring structure. We let I:

$$R_{\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_m) \simeq K_0(\mathbb{Q})[\lambda^1(\mathrm{Id}_n \times 1), \ldots, \lambda^n(\mathrm{Id}_n \times 1), \lambda^1(1 \times \mathrm{Id}_m),$$
$$\ldots, \lambda^m(1 \times \mathrm{Id}_m)]_{\lambda^n(\mathrm{Id}_n \times 1), \lambda^m(1 \times \mathrm{Id}_m)}$$

denote the standard isomorphism. We shall use the same letter for the standard isomorphism $I: R_{\mathbb{Q}}(\mathbf{T}^{n+m}) \simeq K_0(\mathbb{Q})[\mathbb{Z}^{n+m}]$.

First notice that from Lemmas 3.3 and 3.6 and the fact that the group $U_n \times U_m$ is a maximal compact subgroup of $(\mathbf{GL}_n \times \mathbf{GL}_m)(\mathbb{C})$ we can deduce the following facts.

Let $\overline{S}$ be a simple hermitian $(\mathbf{GL}_n \times \mathbf{GL}_m)_{\mathbb{Q}}$-module. Then its metric is unique up to multiplication by a constant (real, positive) factor.

Let $\overline{V}$ be a hermitian $(\mathbf{GL}_n \times \mathbf{GL}_m)_{\mathbb{Q}}$-module. Then there are simple $(\mathbf{GL}_n \times \mathbf{GL}_m)_{\mathbb{Q}}$-modules with admissible metrics $\overline{V}_1, \ldots, \overline{V}_r$, such that

$$\overline{V} = \overline{V}_1 + \cdots + \overline{V}_r$$

in $\widehat{R}_{\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_m)$.

PROPOSITION 3.14: *There is an exact sequence*

$$\widehat{\mathrm{Pic}}(\mathbb{Q}) \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_m) \xrightarrow{\psi} \widehat{R}_{\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_m) \xrightarrow{\phi} R_{\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_m) \to 0$$

where ϕ is the map forgetting the metric. The map ψ is defined by

$$\psi(a \otimes b) = (a-1).\kappa(I(b)),$$

where κ is the ring morphism sending $\lambda^i(\mathrm{Id}_n \times 1)$ to $\lambda^i(\overline{\mathrm{Id}}_n \times 1)$ and $\lambda^i(1 \times \mathrm{Id}_m)$ to $\lambda^i(1 \times \overline{\mathrm{Id}}_m)$.

Proof of Proposition 3.14: We have to show that ψ is well-defined. We only have to check that it is additive in the first factor, i.e. that $(ab-1) = (a-1) + (b-1)$. Since the image of the inclusion $\widehat{\mathrm{Pic}}(\mathbb{Q}) \subseteq \widehat{R}_{\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_m)$ lies in $\widehat{K}_0(\mathbb{Q}) \subseteq \widehat{R}_{\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_m)$, this equation has only to be checked in $\widehat{K}_0(\mathbb{Q})$. Now, in view of the isomorphism $K_0(\mathbb{Q}) \overset{\mathrm{rank}}{\simeq} \mathbb{Z}$ and the sequence (4), we have $(a-1) \in \tilde{A}(\mathrm{Spec}\,\mathbb{Q})$, $(b-1) \in a(\tilde{A}(\mathrm{Spec}\,\mathbb{Q}))$ and by the definition of the product in $\tilde{A}$ we have $(a-1).(b-1) = 0$ and thus $(ab-1) = (a-1) + (b-1)$.

Next we have to show that ψ is surjective onto $\ker \phi$. Since $\mathbb{Q}$ is a field, the Jordan–Hölder theorem implies that $R_{\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_m)$ is a free abelian group with generators the simple $(\mathbf{GL}_n \times \mathbf{GL}_m)_{\mathbb{Q}}$-modules. Let

$$(8) \qquad\qquad n_1\overline{V}_1 + \cdots + n_r\overline{V}_r$$

be in $\ker \phi$. Suppose, without loss of generality, that $n_1, \ldots, n_{r_0} \geq 0, n_{r_0+1}, \ldots, n_r \leq 0$ and write $\overline{V}_i = \overline{V}_{i1} + \cdots + \overline{V}_{is_i}$ $(1 \leq i \leq r)$ where the $\overline{V}_{i1}, \ldots$ are simple, as in 3.2.3. Under this decomposition, the expression (8) becomes

$$(9) \qquad n_1\overline{V}_{11} + n_1\overline{V}_{12} + \cdots + -|n_{r_0+1}|\overline{V}_{r_0+1,1} - \cdots - |n_r|\overline{V}_{rs_r}$$

and the remark preceding (8) shows that the same simple modules (with multi-plicities) must appear in the positive part of (9) as in its negative part. Therefore ker ϕ is generated by expressions $(S, h) - (S, h')$, where S is simple. The image of ψ is generated by expressions $(S, h) - (S, t.h)$ $(t > 0)$, so that we shall have proved surjectivity if we can show that all the metrics of a simple module are proportional. This is the content of 3.2.3.

Finally, to prove surjectivity on the right, it will be sufficient to show that for every $(\mathbf{GL}_n \times \mathbf{GL}_m)_{\mathbb{C}}$-module, there exists an admissible metric. This follows from the structure theorem 3.1 for the case $A = \mathbb{C}$, Lemma 3.3, the Jordan–Hölder Theorem and Lemma 3.6.   ∎

On Spec$\mathbb{Z}$ all the line bundles are isomorphic in $\mathbb{Z}$. Thus every element of $\widehat{\mathrm{Pic}}(\mathbb{Z})$ can be represented by a hermitian bundle of the type $(\mathbb{Z}, h)$. Moreover, it is easy to see that there is an isomorphism $\widehat{\mathrm{Pic}}(\mathbb{Z}) \simeq \mathbb{R}_*^+$ sending $(\mathbb{Z}, h)$ on $\sqrt{h(1, 1)}$. Similarly, there is an isomorphism $\widehat{\mathrm{Pic}}(\mathbb{Q}) \simeq \mathbb{R}_*^+ / \mathbb{Q}_*^+$.

LEMMA 3.15: *There are isomorphisms $\widehat{K}_0(\mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{R}_*^+$ and $\widehat{K}_0(\mathbb{Q}) \simeq \mathbb{Z} \oplus \mathbb{R}_*^+ / \mathbb{Q}_*^+$ given by $\overline{V} \mapsto \mathrm{rank}(V) \oplus \det(\overline{V})$.*

Proof of Lemma 3.15:   See [6, Prop. 2.5, p. 177 and Th. 7.2.1, p. 225].   ∎

LEMMA 3.16: *There is a commutative diagram with exact rows*
(10)

$$0 \longrightarrow \widehat{\mathrm{Pic}}(\mathbb{Q}) \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(\mathbf{T}^{n+m}) \overset{\phi}{\longrightarrow} \widehat{R}_{\mathbb{Q}}(\mathbf{T}^{n+m}) \longrightarrow R_{\mathbb{Q}}(\mathbf{T}^{n+m}) \longrightarrow 0$$

$$\uparrow \mathrm{Id} \otimes i^* \qquad\qquad \uparrow i^* \qquad\qquad \uparrow i^*$$

$$\widehat{\mathrm{Pic}}(\mathbb{Q}) \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_m) \overset{\psi}{\succ} \widehat{R}_{\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_m) \succ R_{\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_m) \succ 0$$

where i is the natural map $\mathbf{T}^{n+m} \mapsto \mathbf{GL}_n \times \mathbf{GL}_m$ and ϕ is defined by

$$\phi(a \otimes b) = (a - 1).\mu(I(b));$$

here, μ sends $r.\vec{z} \in \mathbb{Z}[\mathbb{Z}^{n+m}] \simeq R_{\mathbb{Q}}(\mathbf{T}^{n+m})$ on $\mathbb{Z}^r$ endowed with the standard metric and with trivial grading except at $\vec{z}$.

Proof of Lemma 3.16:   Only the exactness of the first row requires proof. Under the isomorphisms given in Proposition 3.12 and Lemma 3.15, it becomes

$$\widehat{\mathrm{Pic}}(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}^{n+m}] \to (\mathbb{Z} \oplus \widehat{\mathrm{Pic}}(\mathbb{Q}))[\mathbb{Z}^{n+m}] \to \mathbb{Z}[\mathbb{Z}^{n+m}] \to 0$$

where the first map is given by $a \otimes r.\vec{z} \mapsto (0 \oplus a^r).\vec{z}$ and the second one by $(r \oplus a).\vec{z} \mapsto r.\vec{z}$. From this description, the surjectivity of the second map and the exactness in the middle follows. We must prove the injectivity of the first map. To prove this, notice that ϕ is a direct sum of morphisms $\phi_{\vec{z}} : \widehat{\mathrm{Pic}}(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Z} \to \widehat{\mathrm{Pic}}(\mathbb{Q})$, sending $a \otimes r$ on a^r, for all $\vec{z} \in \mathbb{Z}^n$. One checks that a double-sided inverse to $\phi_{\vec{z}}$ is given by the map sending a on $a \otimes 1$. Taking the direct sum over $\mathbb{Z}^n$ of these inverses yields an inverse of ϕ and ends the proof.   ∎

In the next corollary we shall denote by W the Weyl group associated to the immersion of $\mathbf{T}^{n+m}$ in $\mathbf{GL}_n \times \mathbf{GL}_m$. If S is a set on which W acts, we shall write S^W for the elements of S which are fixed under each element of W.

COROLLARY 3.17: *The pre-λ-structure of the ring $\widehat{R}_{\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_m)$ is a λ-structure. There is natural action of the Weyl group W on $\widehat{R}_{\mathbb{Q}}(\mathbf{T}^{n+m})$ and there are isomorphisms*

$$\widehat{R}_{\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_m) \overset{i^*}{\simeq} (\widehat{R}_{\mathbb{Q}}(\mathbf{T}^{n+m}))^W$$
$$\simeq \widehat{K}_0(\mathbb{Q})[\lambda^1(\overline{\mathrm{Id}}_n \times 1), \ldots, \lambda^n(\overline{\mathrm{Id}}_n \times 1), \lambda^1(1 \times \overline{\mathrm{Id}}_m),$$
$$\ldots, \lambda^m(1 \times \overline{\mathrm{Id}}_m)]_{\lambda^n(\overline{\mathrm{Id}}_n \times 1), \lambda^m(1 \times \overline{\mathrm{Id}}_m)}.$$

Proof of Corollary 3.17: The commutativity of the first square in (10) and the injectivity of ϕ show that ψ is injective too. Since the first and last vertical maps are injective, we can apply the 5-lemma and conclude that the middle vertical map is injective. This proves that $\widehat{R}_{\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_m)$ is a λ-ring, by the discussion before the definition of the arithmetic Grothendieck group. There is a natural action on $\widehat{\mathrm{Pic}}(\mathbb{Q}) \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(\mathbf{T}^{n+m})$ of the elements w of W, defined by the equation $w^*(a \otimes b) = a \otimes w^*(b)$. Note that an element w of W only modifies the $\mathbb{Z}^{n+m}$-index of the pieces of the $\mathbb{Z}^{n+m}$-gradings of the hermitian $\mathbf{T}^{n+m}$-modules; therefore they remain pairwise orthogonal after the action of w, which means that w sends hermitian $\mathbf{T}^{n+m}$-modules into hermitian $\mathbf{T}^{n+m}$-modules; therefore there is also a natural action of W on $\widehat{R}_{\mathbb{Q}}(\mathbf{T}^{n+m})$. Moreover, this action is compatible with the action of W on $\widehat{\mathrm{Pic}} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(T^{n+m})$; to show this, notice that the image of $\widehat{\mathrm{Pic}}(\mathbb{Q})$ in $\widehat{R}_{\mathbb{Q}}(\mathbf{T}^{n+m})$ is invariant under W, since it consists of modules with trivial grading except at 0, whose grading is invariant under permutation of the components of the index vectors in $\mathbb{Z}^{n+m}$. We can thus compute $w^*(\phi(a \otimes b)) = w^*((a-1).\mu(I(b))) = (a-1).\mu(I(w^*(b)))$, which is the compatibility. Since the isomorphisms $\widehat{\mathrm{Pic}}(\mathbb{Q}) \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_m) \overset{i^*}{\simeq}$ $(\widehat{\mathrm{Pic}}(\mathbb{Q}) \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(\mathbf{T}^{n+m}))^W$ and $R_{\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_m) \overset{i^*}{\simeq} (R_{\mathbb{Q}}(\mathbf{T}^{n+m}))^W$ hold, we deduce that the isomorphism $(\widehat{R}_{\mathbb{Q}}(\mathbf{T}^{n+m}))^W \overset{i^*}{\simeq} \widehat{R}_{\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_m)$ holds, which

ends the proof.   ∎

3.2.4. Linear group schemes over $\mathbb{Z}$. In this subsubsection, we shall prove a structure theorem for $\widehat{R}_{\mathbb{Z}}(\mathbf{GL}_n \times \mathbf{GL}_m)$; as in the previous section the proof will show that $\widehat{R}_{\mathbb{Z}}(\mathbf{GL}_n \times \mathbf{GL}_m)$ is a λ-ring. Let now G satisfy the properties stated at the beginning of section 3.2, with D taken to be $\mathbb{Z}$. We denote by $R_t(\mathbf{G})$ the Grothendieck group of the finitely generated torsion $\mathbf{G}$-modules.

PROPOSITION 3.18: *Let E be a $\mathbf{G}_{\mathbb{Q}}$-module and L_0 a finitely generated $\mathbb{Z}$-module in E. There exists a finitely generated $\mathbf{G}$-module L in E, containing L_0.*

Proof of Proposition 3.18: See [19, p. 40, Prop. 2].   ∎

We shall define a map from $R_t(\mathbf{G})$ into $\widehat{R}_{\mathbb{Z}}(\mathbf{G})$ by taking projective resolutions of torsion $\mathbf{G}$-modules. For any torsion $\mathbf{G}$-module T, let $0 \to E \to P \xrightarrow{f} T \to 0$ be a resolution of T by finitely generated projective $\mathbf{G}$-modules. Such a resolution always exists by [19, Prop. 4, p. 42]; since such a resolution determines an isomorphism $E_{\mathbb{C}} \simeq P_{\mathbb{C}}$, we can choose any admissible metric on $E_{\mathbb{C}}$ and induce it on $P_{\mathbb{C}}$, to obtain hermitian $\mathbf{G}$-modules $\overline{E}$ and $\overline{P}$. We define $j(T) = \overline{P} - \overline{E}$.

PROPOSITION 3.19: *The map j is a well-defined group homomorphism. The sequence*

$$R_t(\mathbf{G}) \xrightarrow{j} \widehat{R}_{\mathbb{Z}}(\mathbf{G}) \to \widehat{R}_{\mathbb{Q}}(\mathbf{G}) \to 0,$$

where the second map arises from the functor $\otimes_{\mathbb{Z}}\mathbb{Q}$, is exact.

Proof of Proposition 3.19: We shall first prove that j is well-defined. Let g be the morphism $E \to P$. Consider another orthogonal projective resolution $0 \to E' \xrightarrow{g'} P' \xrightarrow{f'} T \to 0$. Let Q be the kernel of the morphism $P \oplus P' \xrightarrow{f+f'} T$. The projection $Q \to P$ is clearly surjective and its kernel by construction arises as the image of the morphism $E' \xrightarrow{0 \oplus g'} P \oplus P'$. Thus we get an orthogonal resolution $0 \to E' \to Q \to P \to 0$, where Q is considered endowed with the metric induced by the inclusion $Q \subseteq P \oplus P'$. Applying precisely the same reasoning to the projection $Q \to P'$, we get an orthogonal resolution $0 \to E \to Q \to P' \to 0$. Thus in $\widehat{R}_{\mathbb{Z}}(\mathbf{G})$ we can compute $\overline{P} + \overline{E}' = \overline{Q} = \overline{P}' + \overline{E}$, which implies that $\overline{E} - \overline{P} = \overline{E}' - \overline{P}'$. This proves our first claim. We shall now prove that j is a group homomorphism. Let

$$0 \to T' \to T \xrightarrow{\alpha} T'' \to 0$$

be an exact sequence of torsion G-modules, where T is the same as before. We get another orthogonal resolution

$$0 \to E'' \to P \xrightarrow{f''} T'' \to 0$$

of T'', if we define $f'' = \alpha \circ f$, $\overline{E}'' = \overline{\ker f''}$, where $\overline{\ker f''}$ is endowed with the metric induced by $\overline{P}$. The snake lemma applied to the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E & \longrightarrow & P & \xrightarrow{f} & T & \longrightarrow & 0 \\
  &   & \downarrow & & \downarrow \text{Id} & & \downarrow \alpha & & \\
0 & \longrightarrow & E'' & \longrightarrow & P & \xrightarrow{f''} & T'' & \longrightarrow & 0
\end{array}
$$

yields an isomorphism $T' \simeq E''/E$ such that the resolution

$$0 \to E \to E'' \to T' \to 0$$

is orthogonal. Now we can compute

$$j(T') - j(T) + j(T'') = \overline{E}'' - \overline{E} - (\overline{P} - \overline{E}) + \overline{P} - \overline{E}'' = 0$$

which yields our last claim. To prove the exactness, we shall construct a map $\psi\colon \widehat{R}_{\mathbb{Q}}(\mathbf{G}) \to \widehat{R}_{\mathbb{Z}}(\mathbf{G})/\operatorname{Im}(j)$ inverting the map $\widehat{R}_{\mathbb{Z}}(\mathbf{G})/\operatorname{Im}(j) \to \widehat{R}_{\mathbb{Q}}(\mathbf{G})$. Let $\overline{V}$ be a hermitian $\mathbf{G}_{\mathbb{Q}}$-module and use Proposition 3.18 to choose a lattice (i.e. a finitely generated $\mathbb{Z}$-module that generates E as a $\mathbb{Q}$-vector space) K in V that is a $\mathbf{G}$-module and induce the metric from V on K to obtain a hermitian $\mathbf{G}$-module $\overline{K}$. Define $\psi(\overline{V}) = \overline{K}$. To prove that it is well-defined, consider that if K' is another such lattice, then $K'/(K' \cap K)$ is a torsion $\mathbb{Z}$-module and by symmetry $K/(K' \cap K)$ is too. Therefore, the sequences

$$0 \to K' \cap K \to K' \to K'/(K' \cap K) \to 0$$

and

$$0 \to K' \cap K \to K \to K/(K' \cap K) \to 0$$

show that $\overline{K}' = \overline{K}(\operatorname{modulo} \operatorname{Im}(j))$. Finally, if

$$(11) \qquad\qquad 0 \to V' \to V \xrightarrow{f} V'' \to 0$$

is an orthogonal sequence of hermitian $\mathbf{G}_{\mathbb{Q}}$-modules, we choose lattices $K' \subseteq V'$, $K'' \subseteq V''$ as above, define $K := f^{-1}(K'')$ and we endow them with the induced metrics. We get an orthogonal sequence

$$0 \to K' \to K \to K'' \to 0$$

where the maps are induced from the sequence (11), so that $\psi(\overline{V}) = \psi(\overline{V}') + \psi(\overline{V}'')$. This proves that ψ is a group homomorphism and ends the whole proof. ∎

In the next lemma, we consider the sequence of the previous Proposition, with **G** the torus of rank $n + m$ over $\mathbb{Z}$.

LEMMA 3.20: *The sequence of groups*

$$0 \to R_t(\mathbf{T}^{n+m}) \xrightarrow{j} \widehat{R}_{\mathbb{Z}}(\mathbf{T}^{n+m}) \to \widehat{R}_{\mathbb{Q}}(\mathbf{T}^{n+m}) \to 0$$

is exact.

Proof of Lemma 3.20: Only the injectivity of j requires proof. The category of finitely generated torsion $\mathbb{Z}$-modules has a natural decomposition as a direct sum over the set of all primes p of the categories of p-primary torsion $\mathbb{Z}$-modules. The Grothendieck group of p-primary torsion $\mathbb{Z}$-modules coincides with the Grothendieck group of p-torsion $\mathbb{Z}$-modules (modules over $\mathbb{F}_p$), because every p-primary $\mathbb{Z}$-module has a canonical filtration by p-torsion $\mathbb{Z}$-modules. Moreover, the category of finitely generated torsion $\mathbf{T}_{\mathbb{Z}}^{n+m}$-modules is canonically the direct sum over $\mathbb{Z}^{n+m}$ of the (repeated) category of finitely generated torsion $\mathbb{Z}$-modules. Using this fact and Proposition 3.12, we see that the sequence of Lemma 3.20 can be written

$$(12) \qquad 0 \to (\bigoplus_p K_0(\mathbb{F}_p))[\mathbb{Z}^{n+m}] \xrightarrow{j} \widehat{K}_0(\mathbb{Z})[\mathbb{Z}^{n+m}] \xrightarrow{l} \widehat{K}_0(\mathbb{Q})[\mathbb{Z}^{n+m}] \to 0.$$

Since we can form projective two-term resolutions for each piece of the grading of a $\mathbf{T}^{n+m}$-module separately, this is a direct sum over $\mathbb{Z}^{n+m}$ whose elements are the unique sequence

$$(13) \qquad 0 \to \bigoplus_p K_0(\mathbb{F}_p) \xrightarrow{j} \widehat{K}_0(\mathbb{Z}) \xrightarrow{l} \widehat{K}_0(\mathbb{Q}) \to 0,$$

which is the sequence (12) for $n + m = 0$. Under the isomorphism of Lemma 3.15, this sequence translates to

$$0 \to \bigoplus_p \mathbb{Z} \xrightarrow{j} \mathbb{Z} \oplus \mathbb{R}_*^+ \xrightarrow{l} \mathbb{Z} \oplus \mathbb{R}_*^+ / \mathbb{Q}_*^+ \to 0.$$

The map l is then the quotient map $\mathbb{Z} \oplus \mathbb{R}_*^+ \to \mathbb{Z} \oplus \mathbb{R}_*^+ / \mathbb{Q}_*^+$, since the functor $\otimes_{\mathbb{Z}} \mathbb{Q}$ commutes with the operation of taking the rank and taking the determinant. To determine the map j, we note that for any finitely generated p-torsion $\mathbb{Z}$-module

J there exists a free $\mathbb{Z}$-module W so that $J \simeq W/pW$ (see [14, XV, Par. 2]). Therefore, we can write down the resolution

$$0 \to (W, p^2.h) \to (W, h) \to J \to 0$$

where the first map is multiplication by p and h is any metric; from this resolution we can compute

$$(\mathrm{rk} \oplus \det)(J) = (\mathrm{rk} \oplus \det)((W, h.p^2) - (W, h))$$
$$= 0 \oplus \det(W, h).p^{\mathrm{rk}(W)} \det(W, h)^{-1} = 0 \oplus p^{\mathrm{rk}(W)}.$$

Therefore j is given by the formula $a_{p_1} \oplus a_{p_2} \oplus \cdots \mapsto p_1^{a_{p_1}}.p_2^{a_{p_2}} \cdots$. Since all the positive rational numbers can be written uniquely as products of integer powers of primes, we deduce that j is injective, which completes the proof. ∎

PROPOSITION 3.21: *The diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R_t(\mathbf{T}^{n+m}) & \longrightarrow & \widehat{R}_{\mathbb{Z}}(\mathbf{T}^{n+m}) & \longrightarrow & \widehat{R}_{\mathbb{Q}}(\mathbf{T}^{n+m}) & \longrightarrow & 0 \\
& & \big\uparrow{\scriptstyle i^*} & & \big\uparrow{\scriptstyle i^*} & & \big\uparrow{\scriptstyle i^*} & & \\
& & R_t(\mathbf{GL}_n \times \mathbf{GL}_m) & \longrightarrow & \widehat{R}_{\mathbb{Z}}(\mathbf{GL}_n \times \mathbf{GL}_m) & \longrightarrow & \widehat{R}_{\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_m) & \longrightarrow & 0
\end{array}$$

is commutative and the first vertical map is injective.

Proof of Proposition 3.21: The commutativity follows from the definitions. As to the second statement, notice that by construction i^* is a direct sum over all primes p of the maps $R_{\mathbb{F}_p}(\mathbf{GL}_n \times \mathbf{GL}_m) \xrightarrow{i^*} R_{\mathbb{F}_p}(\mathbf{T}^{n+m})$, under the isomorphisms $R_t(\mathbf{T}^{n+m}) \simeq \bigoplus_p R_{\mathbb{F}_p}(\mathbf{T}^{n+m})$ and $R_t(\mathbf{GL}_n \times \mathbf{GL}_m) \simeq \bigoplus_p R_{\mathbb{F}_p}(\mathbf{GL}_n \times \mathbf{GL}_m)$ described at the beginning of the last proof. We are therefore reduced to proving the injectivity of the maps $R_{\mathbb{F}_p}(\mathbf{GL}_n \times \mathbf{GL}_m) \xrightarrow{i^*} R_{\mathbb{F}_p}(\mathbf{T}^{n+m})$. Since $\mathbb{F}_p$ is a field, this is a consequence of Theorem 3.1. ∎

In the next corollary, W refers to the Weyl group associated to the immersion $\mathbf{T}_n \times \mathbf{T}_m \to \mathbf{GL}_n \times \mathbf{GL}_m$.

COROLLARY 3.22: *The pre-λ-structure of the ring $\widehat{R}_{\mathbb{Z}}(\mathbf{GL}_n \times \mathbf{GL}_m)$ is a λ-structure. There is a natural action of W on $\widehat{R}_{\mathbb{Z}}(\mathbf{T}_n \times \mathbf{T}_m)$ and there are isomorphisms*

$$\widehat{R}_{\mathbb{Z}}(\mathbf{GL}_n \times \mathbf{GL}_m) \simeq \widehat{R}_{\mathbb{Z}}(\mathbf{T}_n \times \mathbf{T}_m)^W$$
$$\simeq \widehat{K}_0(\mathbb{Z})[\lambda^1(\overline{\mathrm{Id}}_n \times 1), \ldots, \lambda^n(\overline{\mathrm{Id}}_n \times 1), \lambda^1(1 \times \overline{\mathrm{Id}}_m),$$
$$\ldots, \lambda^m(1 \times \overline{\mathrm{Id}}_m)]_{\lambda^n(\overline{\mathrm{Id}}_n \times 1), \lambda^m(1 \times \overline{\mathrm{Id}}_m)}.$$

Proof of Corollary 3.22:   Consider again the diagram of Proposition 3.21. First, the injectivity of the first vertical map and of the first upper horizontal map yields the injectivity of the first lower horizontal map of the last diagram. Therefore, it follows from the 5-lemma that the middle vertical map is injective; this entails that the pre-λ-structure $\widehat{R}_{\mathbb{Z}}(\mathbf{GL}_n \times \mathbf{GL}_m)$ is a λ-structure. Secondly, it follows from the description of the action of the Weyl group given at the end of subsection 3.1 and the remark following (12) that the action of the Weyl group W on the groups of the first row is compatible with the maps in the sequence. Now, since W consists of automorphisms of $\mathbf{T}_{\mathbb{Z}}^{n+m}$, its action on the fibers of $\mathbf{T}_{\mathbb{Z}}^{n+m}$ on Spec$\mathbb{Z}$ is well-defined. It corresponds in each case to the Weyl group of $\mathbf{T}_{\mathbb{F}_p}^{n+m}$ in the fiber $(\mathbf{GL}_n \times \mathbf{GL}_m)_{\mathbb{F}_p}$. The correspondance is a consequence of the fact that in both cases the Weyl group is the entire symmetric group on $n+m$ elements, exchanging the components of the indices of the $\mathbb{Z}^{n+m}$ grading. Therefore, the isomorphism $R_t(\mathbf{T}^{n+m})^W = i^*(R_t(\mathbf{GL}_n \times \mathbf{GL}_m))$ holds. From this and Corollary 3.17, we deduce that the isomorphism $\widehat{R}_{\mathbb{Z}}(\mathbf{GL}_n \times \mathbf{GL}_m) \simeq \widehat{R}_{\mathbb{Z}}(\mathbf{T}^{n+m})^W$ also holds. This concludes the proof.   ∎

OPEN QUESTION: Is the last corollary true for a ring of integers in a number field in place of $\mathbb{Z}$? To prove it, one might have to extend on the left the sequence appearing in Proposition 3.19. This would probably require the definition of higher K-groups for the category of hermitian $\mathbf{G}$-modules, since the sequence of Proposition 3.19 is a hermitian analog of the right end of Quillen's localization sequence for the torsion $\mathbf{G}$-modules viewed as a (Serre) subcategory of the finitely generated $\mathbf{G}$-modules.

4. A universal action

In this subsection, we show that $\widehat{R}_{\mathbb{Z}}(\mathbf{GL}_n \times \mathbf{GL}_m)$ is a universal object with respect to natural operations on hermitian vector bundles. This is made precise by the next theorem.

THEOREM 4.1: *Let X be a scheme such that $X \times_{\mathbb{Z}} \mathbb{Q}$ is smooth over $\mathbb{Q}$ and $\overline{E}$ (resp. $\overline{E}'$) be a hermitian vector bundle of rank n (resp. m) on X. Then there exists a (unique) pre-λ-ring morphism $r_{\overline{E},\overline{E}'}: \widehat{R}_{\mathbb{Z}}(\mathbf{GL}_n \times \mathbf{GL}_m) \mapsto \widehat{K}_0(X)$ such that $r_{\overline{E},\overline{E}'}(\overline{\mathrm{Id}}_n \times 1) = \overline{E}$ and $r_{\overline{E},\overline{E}'}(1 \times \overline{\mathrm{Id}}_m) = \overline{E}'$.*

Proof of Theorem 4.1:   Let h_E and $h_{E'}$ denote the metrics of E and E'. The local frames of the vector bundles E and E' give rise to principal bundles $\mathbf{GL}(E)$ and $\mathbf{GL}(E')$, called their frame bundles. By definition, $\mathbf{GL}(E)$ carries a group

scheme action of $\mathbf{GL}_n$ and $\mathbf{GL}(E')$ carries a group scheme action of $\mathbf{GL}_m$, so that the fibre product $\mathbf{GL}(E) \times_X \mathbf{GL}(E')$ carries an action of $\mathbf{GL}_n \times \mathbf{GL}_m$. For a finitely generated projective $\mathbf{GL}_n \times \mathbf{GL}_m$-module V, we can form the geometric vector bundle $V(\mathbf{GL}(E) \times_X \mathbf{GL}(E'))$ associated to V over $\mathbf{GL}(E) \times_X \mathbf{GL}(E')$ (cf. [9, VI, p. 396]). This construction yields an exact functor from the category of finitely generated projective $\mathbf{GL}_n \times \mathbf{GL}_m$-modules to the category of geometric vector bundles over X.

The metrics on the bundle E and E' gives rise to C^∞ principal bundles $U(E_\mathbb{C})$ and $U(E'_\mathbb{C})$ over $X(\mathbb{C})$, the bundles of local orthonormal frames, with groups $U(n)$ and $U(m)$, the unitary groups of dimensions n and m (cf. [10, 7.4, p. 68]). Endow V with an admissible metric h and let $f\colon U(n) \times U(m) \to \mathrm{Aut}_\mathbb{C}(V_\mathbb{C})$ be the representation induced on the product of unitary groups. Write $f(U(E_\mathbb{C}) \times U(E'_\mathbb{C}))$ for the associated vector bundle. Again, this construction yields an exact functor from the category of complex linear representations of $U(n) \times U(m)$ to the category of C^∞ vector bundles over $X(\mathbb{C})$ (cf. [10, II, 12., 5.]).

Let $N \times M \in U(n) \times U(m)$ be a product of unitary matrices. Using the admissibility of h we compute $f((N \times M)^*) = f((N \times M)^{-1}) = (f(N \times M))^{-1} = (f(N \times M))^*$, where $(f(N \times M))^*$ is the adjoint of $f(N \times M)$ for h. Thus f sends $U(n) \times U(m)$ into the unitary subgroup of $\mathrm{Aut}_\mathbb{C}(V_\mathbb{C})$ determined by h. Let $r = \dim(V_\mathbb{C})$ and let us choose an arbitrary orthonormal basis of $V_\mathbb{C}$ to identify f with a morphism $f\colon U(n) \times U(m) \to U(r)$. We can choose a trivialising cover of $X(\mathbb{C})$ for $E_\mathbb{C} \times E'_\mathbb{C}$ whose transition functions have image in $U(n) \times U(m)$. Now, by construction, the bundle $f(U(E_\mathbb{C}) \times U(E'_\mathbb{C})) \simeq V(\mathbf{GL}(E) \times_X \mathbf{GL}(E'))_\mathbb{C}$ is (canonically isometrically isomorphic to) a bundle given by transition functions whose image lies in $U(r)$. It is thus canonically endowed with a metric, which we shall call $V(h_E \oplus h'_E)$. Finally, we define $r_{\overline{E}, \overline{E}'}(V, h) := (V(\mathbf{GL}(E) \times_X \mathbf{GL}(E')), V(h_E \oplus h'_E))$. The proof now follows from the next two lemmata:

LEMMA 4.2: *The metric $V(h_E \oplus h'_E)$ is conjugation invariant.*

Proof of Lemma 4.2: The construction of the bundle

$$(V(\mathbf{GL}(E) \times_X \mathbf{GL}(E')), V(h_E \oplus h'_E))$$

is clearly local on X, so we may assume that X is an affine scheme where E and E' are free. Since it is of finite type over $\mathbb{Z}$, we have

$$X \simeq \mathrm{Spec}\mathbb{Z}[X_1, X_2, \ldots, X_i]/(f_1, \ldots, f_j).$$

By definition, the manifold $X(\mathbb{C})$ is then given by the zero set in $\mathbb{C}^i$ of the polynomials $f_1, \ldots, f_j$ and the total space of $E_{\mathbb{C}} \times E'_{\mathbb{C}}$ is then $X(\mathbb{C}) \times \mathbb{C}^{n+m}$. For every element $v \times w \in X(\mathbb{C}) \times \mathbb{C}^{n+m}$, we have $F_{\infty}^*(v \times w) = \overline{v} \times \overline{w}$, where the overbar $\overline{(.)}$ refers to complex conjugation of the coefficients. Let $H \colon X(\mathbb{C}) \to$ $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C})$ be the function giving the matrix of the metric of $E_{\mathbb{C}} \times E'_{\mathbb{C}}$ in the standard basis of $\mathbb{C}^{n+m}$. The total space of $V(\mathbf{GL}(E) \times_X \mathbf{GL}(E'))_{\mathbb{C}} \simeq$ $V_{\mathbb{C}}(\mathbf{GL}(E_{\mathbb{C}}) \times_{X(\mathbb{C})} \mathbf{GL}(E'_{\mathbb{C}}))$ is then $X(\mathbb{C}) \times V_{\mathbb{C}}$ and its metric is given by the formula $h((.), r(H)(.))$, where r is the associated complex representation as in 3.2. Now notice that

(a) $H(\overline{v}) = \overline{H}(v)$, where $\overline{H}(v)$ is the complex conjugate of the matrix $H(v)$.

(b) $r(\overline{M})(x) = \overline{r(M)(\overline{x})}$ for all $M \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C})$ and all $x \in V_{\mathbb{C}}$, where $\overline{x}$ is the conjugate of x for the natural conjugation on $V_{\mathbb{C}}$.

To prove (a), note that by the conjugation invariance of the metric, we have $w_1^* H(v) w_2 = \overline{\overline{w_1^*} H(\overline{v}) \overline{w_2}} = w_1^* \overline{H}(\overline{v}) w_2$ for all w_1, w_2 over v. To prove (b), let us choose a basis of V, to obtain an integral basis of $V_{\mathbb{C}}$ by base change. For the time of the next paragraph, identify r with a map $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C}) \to \mathrm{GL}_d(\mathbb{C})$, where d is the dimension of V. The matrix coefficients of r are then rational functions with integer coefficients in the components of the elements of $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C})$. Thus we see that $r(\overline{M}) = \overline{r(M)}$, where $\overline{r(M)}$ is the complex conjugate of $r(M)$ in $\mathrm{GL}_d(\mathbb{C})$. This proves the claim.

Using the conjugation invariance requirement for h, (a) and (b), we can now compute

$$h(x, r(H(v))y) = \overline{h(\overline{x}, \overline{r(H(v))}y)} = \overline{h(\overline{x}, r(\overline{H}(v))\overline{y})} = \overline{h(\overline{x}, r(H(\overline{v}))\overline{y})},$$

which proves that the metric on $V(\mathbf{GL}(E) \times_X \mathbf{GL}(E'))_{\mathbb{C}}$ is conjugation invariant and ends the proof. ∎

LEMMA 4.3: *The map $r_{\overline{E}, \overline{E}'}$ is well-defined as a map from $\widehat{R}_{\mathbb{Z}}(\mathbf{GL}_n \times \mathbf{GL}_m)$ and is a pre-λ-ring morphism.*

Proof of Lemma 4.3: Consider

$$0 \to V' \to V \to V'' \to 0,$$

an exact sequence of $\mathbf{GL}_n \times \mathbf{GL}_m$-modules, endowed with admissible metrics h', h and h'', which is orthogonal. To prove well-definedness, we have to prove that the sequence

(14) $0 \to V'(\mathbf{GL}(E) \times_X \mathbf{GL}(E')) \to V(\mathbf{GL}(E) \times_X \mathbf{GL}(E'))$

           $\to V''(\mathbf{GL}(E) \times_X \mathbf{GL}(E')) \to 0$

has a vanishing secondary class. Let f', f and f'' be the representations of $U(n) \times U(m)$ induced by V', V and V''. Until the end of the paragraph, consider $V'_{\mathbb{C}}$, $V_{\mathbb{C}}$ and $V''_{\mathbb{C}}$ as representation spaces of $U(n) \times U(m)$. Let $(V'_{\mathbb{C}})^*$, $(V_{\mathbb{C}})^*$ and $(V''_{\mathbb{C}})^*$ be the dual representations. The fact that the map $V' \to V$ preserves the metrics is expressed by the commutativity of the diagram

$$
\begin{array}{ccc}
(V'_{\mathbb{C}})^* & \longleftarrow & (V_{\mathbb{C}})^* \\
\uparrow & & \uparrow \\
V'_{\mathbb{C}} & \longrightarrow & V_{\mathbb{C}}
\end{array}
$$

where the two vertical arrows are the anti-linear isomorphisms of representations induced by the metrics and the upper horizontal arrow is the dual of the lower horizontal one. By functoriality, this diagram yields the commutative diagram of vector bundles

$$
\begin{array}{ccc}
(f'(U(E_{\mathbb{C}}) \times U(E'_{\mathbb{C}})))^* & \longleftarrow & (f(U(E_{\mathbb{C}}) \times U(E'_{\mathbb{C}})))^* \\
\uparrow & & \uparrow \\
f'(U(E_{\mathbb{C}}) \times U(E'_{\mathbb{C}})) & \longrightarrow & f(U(E_{\mathbb{C}}) \times U(E'_{\mathbb{C}}))
\end{array}
$$

which in turn implies that the map

$$(V'(\mathbf{GL}(E) \times_X \mathbf{GL}(E')))_{\mathbb{C}} \to (V(\mathbf{GL}(E) \times_X \mathbf{GL}(E')))_{\mathbb{C}}$$

preserves the hermitian metrics. By the same token, the left inverse $V''_{\mathbb{C}} \to V_{\mathbb{C}}$ appearing in Lemma 3.6 yields a metric preserving holomorphic map

$$(V''(\mathbf{GL}(E) \times_X \mathbf{GL}(E')))_{\mathbb{C}} \to (V(\mathbf{GL}(E) \times_X \mathbf{GL}(E')))_{\mathbb{C}},$$

which splits the sequence (14) orthogonally. By the remark preceding the definition of the arithmetic Grothendieck group, this shows that the secondary class of (14) vanishes. The map $r_{\overline{E}, \overline{E}'}$ is a pre-λ-ring morphism by construction, so we are done. ∎

COROLLARY 4.4: *Let X be a scheme such that $X \times_{\mathbb{Z}} \mathbb{Q}$ is smooth over $\mathbb{Q}$. Then $\widehat{K}_0(X)$ is a λ-ring.*

Proof of Corollary 4.4: We want to prove (iii) and (iv) in subsection 2.1. Assume $x = \overline{E}$ and $y = \overline{E}'$. Then Theorem 4.1 tells us that the image of $r_{\overline{E}, \overline{E}'}$ contains x and y. By Corollary 3.22, all the pairs of elements of $\widehat{R}_{\mathbb{Z}}(\mathbf{GL}_n \times \mathbf{GL}_m)$ satisfy

(iii) and (iv). This implies that they hold for x and y since $r_{\overline{E}, \overline{E}'}$ is a pre-λ-ring morphism. Now notice that if (iii) holds for $x = x'$ and $y = y'$ (resp. $x = x''$ and $y = y''$), then it holds for $x = x' + x'$ and $y = y' + y''$; this follows from the construction of the polynomials P_k and $P_{k,l}$ (see subsection 2.1) and the unicity statement in the fundamental theorem on symmetric functions; similarly, if (iv) holds for $x = x'$ (resp. $x = x''$), then it holds for $x = x' + x''$. We are thus reduced to the cases (a) $x = (0, \eta)$, $y = (0, \eta')$, where $\eta, \eta' \in \tilde{A}(X)$, and (b) $x = (0, \eta)$, $y = \overline{E}$, where $\eta \in \tilde{A}(X)$. The case (a) is settled by the remark following 2.3. To prove (b), we compute

$$
\begin{aligned}
\lambda^k(\overline{E}.\eta) &= \lambda^k((0, [(\mathrm{ch}(\overline{E}), 0) * (0, \eta)])) \\
&= (0, [\lambda^k((\mathrm{ch}(\overline{E}), 0) * (0, \eta))]) \\
&= (0, [P_k(\lambda^1((\mathrm{ch}(\overline{E}), 0)), \ldots, \lambda^k((\mathrm{ch}(\overline{E}), 0)), \lambda^1((0, \eta)), \ldots, \lambda^k((0, \eta)))]) \\
&= (0, [P_k((\mathrm{ch}(\lambda^1(\overline{E})), 0), \ldots, (\mathrm{ch}(\lambda^k(\overline{E})), 0), \lambda^1((0, \eta)), \ldots, \lambda^k((0, \eta)))]) \\
&= P_k(\lambda^1(\overline{E}), \ldots, \lambda^k(\overline{E}), \ldots, \lambda^1(\eta), \ldots, \lambda^k(\eta)),
\end{aligned}
$$

where we used the equality $\lambda^k((\mathrm{ch}(\overline{E}), 0)) = (\mathrm{ch}(\lambda^k(\overline{E})), 0)$ on the fourth line. This is the content of [6, Lemma 7.3.3, p. 235, II]. This concludes the proof. ∎

The next corollary follows from the previous one and the discussion at the end of subsection 2.1.

COROLLARY 4.5: *The Adams operations* ψ^k *($k \geq 1$) are ring endomorphisms of* $\widehat{K}_0(X)$ *and satisfy the equation* $\psi^k \circ \psi^l = \psi^{kl}$ *($k, l \geq 1$).*

The fact that the Adams operations are ring endomorphisms is essential to the proof of the arithmetic Adams–Riemann–Roch theorem stated in [17] and proved in [18]. Theorem 4.1 also plays a key role in the proof of the main result of [12] (which is announced in [11]), where it is used to prove the invertibility of the Todd element in a Lefschetz fixed point formula.

References

[1] M. Atiyah and D. O. Tall, *Group representations, λ-rings and the j-homomorphism*, Topology **8** (1969), 253–297.

[2] J.-M. Bismut, H. Gillet and C. Soulé, *Analytic torsion and holomorphic determinant bundles i, ii, iii*, Communications in Mathematical Physics **115** (1988), 49–78, 79–126, 301–351.

[3] M. Demazure and A. Grothendieck, *Schémas en groupes*, Lecture Notes in Mathematics **64**, Springer-Verlag, Berlin, 1970.

[4] W. Fulton and J. Harris, *Representation Theory*, Springer, Berlin, 1991.

[5] C. Gasbarri, *Hermitian vector bundles of rank two and adjoint systems on arithmetic surfaces*, Topology **38** (1999), 1161–1174.

[6] H. Gillet and C. Soulé, *Characteristic classes for algebraic vector bundles with hermitian metrics i, ii*, Annals of Mathematics **131** (1990), 163–203, 205–238.

[7] D. Grayson, *Modules projectifs munis d'une métrique sur les anneaux d'entiers de corps de nombres*, Comptes Rendus de l'Académie des Sciences, Paris **309** (1989), 573–575.

[8] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley, New York, 1978.

[9] A. Grothendieck, P. Berthelot and L. Illusie, *Théorie des intersections et théorème de Riemann-Roch*, Lecture Notes in Mathematics **225**, Springer-Verlag, Berlin, 1971.

[10] D. Husemoller, *Fibre Bundles*, Springer, Berlin, 1994.

[11] K. Köhler and D. Roessler, *Un théorème du point fixe de Lefschetz en géométrie d'Arakelov*, Comptes Rendus de l'Académie des Sciences, Paris, Série I **326** (1998), 719–722.

[12] K. Köhler and D. Roessler, *A fixed point formula of Lefschetz type in Arakelov geometry I: statement and proof*, IHES Preprint M/98/76 (Nov. 1998).

[13] C. Kratzer, *λ-structure en K-théorie algébrique.*, Commentarii Mathematics Helvetici **55** (1970), 233–254.

[14] S. Lang, *Algebra*, Addison-Wesley, Reading, MA, 1984.

[15] A. Meissner, *Arithmetische K-theorie*, Dissertation, University of Regensburg, 1993.

[16] D. Roessler, *Lambda structure en K-théorie des fibrés algébriques hermitiens*, Comptes Rendus de l'Académie des Sciences, Paris **322** (1996), 251–254.

[17] D. Roessler, *Un théorème d'Adams–Riemann–Roch en géométrie d'Arakelov*, Comptes Rendus de l'Académie des Sciences, Paris **322** (1996), 749–752.

[18] D. Roessler, *An Adams–Riemann–Roch theorem in Arakelov geometry*, Duke Mathematical Journal **96** (1999), 61–126.

[19] J.-P. Serre, *Groupes de Grothendieck des schémas en groupes réductifs déployés*, Publications mathématiques de l'IHES **34** (1968), 37–52.

[20] X. Wang, *Higher order characteristic classes in arithmetic geometry*, Ph. D. thesis, Harvard University, 1992.