Strongly semistable sheaves and the Mordell-Lang conjecture over function fields

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Abstract

We give a new proof of the Mordell-Lang conjecture in positive characteristic, in the situation where the variety under scrutiny is a smooth subvariety of an abelian variety. Our proof is based on the theory of semistable sheaves in positive characteristic, in particular on Langer's theorem that the Harder-Narasimhan filtration of sheaves becomes strongly semistable after a finite number of iterations of Frobenius pull-backs.

1 Introduction

Let B be an abelian variety over an algebraically closed field F of characteristic $p > 0$. Let Y be an irreducible reduced closed subscheme of B. Let $\Lambda \subseteq B(F)$ be a subgroup. Suppose that $\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is a finitely generated $\mathbb{Z}_{(p)}$-module (here, as is customary, we write $\mathbb{Z}_{(p)}$ for the localization of $\mathbb{Z}$ at the prime p).

The Mordell-Lang conjecture for Y and B is the following statement.

Theorem 1.1 (Mordell-Lang conjecture for abelian varieties; Hrushovski [3]). *Suppose that $Y \cap \Lambda$ is Zariski dense in Y. Then there is a projective variety Y' over a finite subfield $\mathbb{F}_{p^r} \subseteq F$ and a finite and surjective morphism $h : Y'_F \to Y/\mathrm{Stab}(Y)$.*

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Here $\mathrm{Stab}(Y) = \mathrm{Stab}_B(Y)$ is the translation stabilizer of Y. This is the closed subgroup scheme of B, which is characterized uniquely by the fact that for any scheme S and any morphism $b : S \to B$, translation by b on the product $B \times_F S$ maps the subscheme $Y \times S$ to itself if and only if b factors through $\mathrm{Stab}_B(Y)$. Its existence is proven in [2, exp. VIII, Ex. 6.5 (e)].

Theorem 1.1 was first proven by Hrushovski in [3] using model-theoretic methods and other proofs were given in [8], [12] and [1].

Remark. The formulation of the Mordell-Lang conjecture given in [3] (see also [8, Intro.]) is more involved than the formulation given here but the two formulations are equivalent (we leave the verification of this equivalence as an exercise for the reader).

In the following article, we shall give a new proof of Theorem 1.1, under the the supplementary assumption that Y is smooth over F and that F has transcendence degree 1 over $\mathbb{F}_p$.

Our method of proof is based on the theory of semistable sheaves in positive characteristic, in particular on Langer's theorem that the Harder-Narasimhan filtration of sheaves becomes strongly semistable after a finite number of iterations of Frobenius pull-backs (see Theorem 2.1 below).

Our method allows us to give an upper-bound for the generic degree of the morphism h in Theorem 1.1 in terms of the Frobenius-stabilised slopes of the cotangent bundle of Y (see Lemma 1.2 and Corollary 1.4 below). The possibility of obtaining such an upper-bound was the main motivation for looking for the partial proof of Theorem 1.1 given here.

To describe our results precisely, we now switch notation. Let k_0 be an algebraically closed field of characteristic $p > 0$ and let U be a smooth curve over k_0. Let $\mathcal{A}$ be an abelian scheme over U and let $\mathcal{X} \hookrightarrow \mathcal{A}$ be a closed subscheme. We let K_0 be the function field of U and let $A := \mathcal{A}_{K_0}$ (resp. $X := \mathcal{X}_{K_0}$) be the generic fibre of $\mathcal{A}$ (resp. $\mathcal{X}$).

For all $n \geqslant 0$, we define

$$\mathrm{Crit}^n(\mathcal{X}, \mathcal{A}) := [p^n]_*(J^n(\mathcal{A}/U)) \cap J^n(\mathcal{X}/U).$$

Here $J^n(\bullet/U)$ refers to the n-th jet scheme of $\bullet$ over U. See [8, par. 2] for this and some more explanations. The scheme $J^n(\mathcal{A}/U)$ is naturally a commutative group scheme over U and $[p^n]$ refers to the multiplication-by-p^n-morphism. The notation $[p^n]_*(J^n(\mathcal{A}/U))$ refers to the scheme-theoretic image of $J^n(\mathcal{A}/U)$ by $[p^n]$.

There are natural morphisms $\Lambda^{\mathcal{A}}_{n,n-1} : J^n(\mathcal{A}/U) \to J^{n-1}(\mathcal{A}/U)$ and these lead to a projective system of U-schemes

$$\cdots \to \mathrm{Crit}^2(\mathcal{X},\mathcal{A}) \to \mathrm{Crit}^1(\mathcal{X},\mathcal{A}) \to \mathcal{X}.$$

whose connecting morphisms are finite. See [8, par. 3.1] for all this. We let $\mathrm{Exc}^n(\mathcal{A},\mathcal{X}) \hookrightarrow \mathcal{X}$ be the scheme-theoretic image of $\mathrm{Crit}^n(\mathcal{A},\mathcal{X})$ in $\mathcal{X}$. We let $\mathrm{Crit}^n(A,X)$ (resp. $\mathrm{Exc}^n(A,X) \hookrightarrow X$) be the generic fibre of $\mathrm{Crit}^n(\mathcal{A},\mathcal{X})$ (resp. $\mathrm{Exc}^n(\mathcal{A},\mathcal{X}) \hookrightarrow \mathcal{X}$).
Now fix once a for all an ample line bundle M on $X_{\bar{K}_0}$.

Lemma-Definition 1.2. *Suppose that X is smooth and connected over K_0 and that* $\mathrm{Stab}(X_{\bar{K}_0}) = 0$. *Then* $\bar{\mu}_{\min}(\Omega_{X_{\bar{K}_0}}) > 0$ *and*

$$\mathfrak{DB}(X) := p^{\sup\{n\in\mathbb{N}\ |\ H^0(X, F_X^{n,*}\Omega^\vee_{X/K_0}\otimes\Omega_{X/K_0})\neq 0\}} \leqslant \frac{\bar{\mu}_{\max}(\Omega_{X_{\bar{K}_0}})}{\bar{\mu}_{\min}(\Omega_{X_{\bar{K}_0}})}$$

Here $\bar{\mu}_{\min}(\cdot) = \bar{\mu}_{\min,M}(\cdot)$ (resp. $\bar{\mu}_{\max}(\cdot) = \bar{\mu}_{\max,M}(\cdot)$) refers to the Frobenius-stabilised minimal (resp. maximal slope) with respect to M. See section 2 below for the definition.

Theorem 1.3. *Suppose that $\mathcal{X}$ is smooth over U with geometrically connected fibres and suppose that* $\mathrm{Stab}(X) = 0$. *Consider the statements:*

(a) *For any $n \geqslant 0$ there is a $Q = Q(n) \in \Gamma_0$ such that* $\mathrm{Exc}^n(A, X^{+Q}) \hookrightarrow X$ *is an isomorphism.*

(b) *For any closed point $u_0 \in U$, there is an $n_0 = n_0(u_0)$ such that $p^{n_0} \leqslant \mathfrak{DB}(X)$ and a finite and surjective morphism of $\widehat{\mathcal{O}}_{u_0}$-schemes*

$$\iota = \iota_{u_0} : \mathcal{X}_{u_0}^{p^{-n_0}} \times_{k_0} \widehat{\mathcal{O}}_{u_0} \to \mathcal{X}_{\widehat{\mathcal{O}}_{u_0}}$$

of degree equal to $p^{\dim(X)n_0}$.

Then (a) *implies* (b).

Here U_{u_0} is the spectrum of the local ring of U at u_0 and $\widehat{U}_{u_0}$ is its completion. The notation X^{+Q} refers to the pushforward by the addition-by-Q morphism of the subscheme X of A. The scheme $\mathcal{X}_{u_0}$ is the k_0-scheme, which is the fibre of $\mathcal{X}$ at u_0. The symbol $\mathcal{X}_{u_0}^{p^{-r}}$ refers to the scheme obtained from $\mathcal{X}_{u_0}$ by composing the structure map of $\mathcal{X}_{u_0}$ with the n-th power $\mathrm{Frob}_{k_0}^{-1,\circ n}$ of the inverse of the absolute Frobenius morphism Frob_{k_0} of Spec k_0 (recall that Frob_{k_0} is an automorphism because k_0 is perfect).

Notice that the morphism ι must be flat by "miracle flatness" (see [7, Th. 23.1]), since both source and target of ι are regular schemes. By the degree of ι, we mean as usual

$$\deg(\iota) := \mathrm{rk}(\iota_*(\mathcal{O}_{\mathcal{X}_{u_0}^{p^{-n_0}}\times_{k_0}\widehat{\mathcal{O}}_{u_0}})),$$

noting that $\iota_*(\mathcal{O}_{\mathcal{X}_{u_0}^{p^{-n_0}}\times_{k_0}\widehat{\mathcal{O}}_{u_0}})$ is a locally free sheaf, since ι is flat.

Let now Γ be a subgroup of $A(\bar{K}_0)$. Suppose that

$$\Gamma = \mathrm{Div}^p(\Gamma_0) := \{\gamma \in A(\bar{K}_0) \mid \exists n \in \mathbb{N}^* : (n,p) = 1 \ \&\ n\cdot\gamma \in \Gamma_0\}$$

where Γ_0 is a finitely generated subgroup of $A(K_0)$. In particular, $\Gamma\otimes\mathbb{Z}_{(p)}$ is a finitely generated $\mathbb{Z}_{(p)}$-module.

Corollary 1.4. *Suppose that $X_{\bar{K}_0}\cap\Gamma$ is dense in $X_{\bar{K}_0}$. Suppose also that X is smooth over K_0.*

Then there exists a smooth projective variety X' over k_0 and a finite and surjective K_0^{sep}-morphism

$$h : X'_{K_0^{\mathrm{sep}}} \to (X/\mathrm{Stab}(X))_{K_0^{\mathrm{sep}}}$$

such that

$$\deg(h) \leqslant \mathfrak{DB}(X/\mathrm{Stab}(X))^{\dim(X/\mathrm{Stab}(X))}.$$

In particular, $(X/\mathrm{Stab}(X))_{K_0^{\mathrm{sep}}}$ has a model over k_0 if $\Omega_{(X/\mathrm{Stab}(X))_{\bar{K}_0}}$ is strongly semistable.

Remark. It seems likely that there are "many" varieties with strongly semistable ample cotangent bundle. Indeed, recall that the cotangent bundle Ω_S of a smooth and projective variety S over $\mathbb{C}$ is semistable with respect to $\det(\Omega_S)$, if $\det(\Omega_S)$ is ample. This is a consequence of the main result of [11]. On the other hand, there is speculation (see for example [9] and the references therein) that in many situations the reduction modulo a prime number p of a semistable sheaf is strongly semistable for "most" prime numbers p.

Notations and conventions. If Y is a scheme of characteristic p, we write $F_Y : Y \to Y$ for the absolute Frobenius endomorphism of Y. The short-hand wrog refers to "without restriction of generality".

2 The geometry of vector bundles in positive characteristic

Let L be an ample line bundle on a smooth and projective variety Y over an algebraically closed field l_0. If V is torsion free coherent sheaf on Y, we shall write

$$\mu(V) = \mu_L(V) = \deg_L(V)/\mathrm{rk}(V)$$

for the slope of V (with respect to L). Here $\mathrm{rk}(V)$ is the rank of V, which is the dimension the stalk of V at the generic point of Y. Furthermore,

$$\deg_L(V) := \int_X \mathrm{c}_1(V) \cdot \mathrm{c}_1(L)^{\dim(Y)-1}.$$

Here $c_1(\cdot)$ refers to the first Chern class with values in an arbitrary Weil cohomology theory and the integral sign $\int_X$ is a short-hand for the push-forward morphism to Spec l_0 in that theory. Recall that V is called semistable (with respect to L) if for every coherent subsheaf W of V, we have $\mu(W) \leqslant \mu(V)$. The torsion free sheaf V is called strongly semistable if $\mathrm{char}(l_0) > 0$ and $F_X^{*,n}V$ is semistable for all $n \geqslant 0$.
In general, there exists a filtration

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{r-1} \subseteq V_r = V$$

of V by subsheaves, such that the quotients V_i/V_{i-1} are all semistable and such that the slopes $\mu(V_i/V_{i-1})$ are strictly decreasing for $i \geqslant 1$. This filtration is unique and is called the Harder-Narasimhan (HN) filtration of V. We shall write

$$\mu_{\min}(V) := \inf\{\mu(V_i/V_{i-1})\}_{i\geqslant 1}$$

and

$$\mu_{\max}(V) := \sup\{\mu(V_i/V_{i-1})\}_{i\geqslant 1}$$

An important consequence of the definitions is the following fact: if V and W are two torsion free sheaves on Y and $\mu_{\min}(V) > \mu_{\max}(W)$, then $\mathrm{Hom}_Y(V, W) = 0$.
For more on the theory of semistable sheaves, see the monograph [4].
The following theorem will be a key input in our proof of Theorem 1.3. For the proof see [5, Th. 2.7].

Theorem 2.1 (Langer). *If V is torsion free coherent sheaf on Y and* $\mathrm{char}(l_0) > 0$*, then there exists $n_0 \geqslant 0$ such that $F_X^{\circ n,*}V$ has a strongly semistable HN filtration for all $n \geqslant n_0$.*

If V is a torsion free sheaf on Y and $\mathrm{char}(l_0) > 0$, we now define

$$\bar{\mu}_{\min}(V) := \lim_{r\to\infty} \mu_{\min}(F_Y^{\circ r,*}V)/\mathrm{char}(l_0)^r$$

and

$$\bar{\mu}_{\max}(V) := \lim_{r\to\infty} \mu_{\max}(F_Y^{\circ r,*}V)/\mathrm{char}(l_0)^r.$$

Note that Theorem 2.1 implies that the sequences $\mu_{\min}(F_Y^{\circ r,*}V)/\mathrm{char}(l_0)^r$ (resp. $\mu_{\max}(F_Y^{\circ r,*}V)/\mathrm{char}(l_0)^r$) become constant when r is sufficiently large, so the above definitions of $\bar{\mu}_{\min}$ and $\bar{\mu}_{\max}$ make sense.

Lemma 2.2. *Let V be a torsion free sheaf on Y. Suppose that V is globally generated and of degree 0 with respect to L. Then there exists an isomorphism $V \simeq \mathcal{O}_Y^{\oplus \mathrm{rk}(V)}$.*

Proof. Let $\phi : \mathcal{O}_Y^{\oplus l} \to V$ be a surjection, where l is chosen as small as possible. Suppose that $\ker\ \phi \neq 0$ (otherwise the Lemma is proven). Let $V_0 = \ker\ \phi$. Then $\mu(V_0) = 0$ and furthermore, since $\mathcal{O}_Y^{\oplus l}$ is semistable, every semistable subsheaf of V_0 has slope $\leqslant 0$ and thus V_0 is also semistable. Now for any $i \in \{1, \ldots, l\}$, let $\pi_i : V_0 \to \mathcal{O}_Y$ be the projection on the i-th coordinate. Choose $i_0 \in \{1, \ldots, l\}$ so that π_{i_0} is non-vanishing. Then π_{i_0} is surjective in codimension 2, because otherwise, the degree of the image of π_{i_0} would be < 0, which would contradict the semistability of V_0. Now replace V_0 be a non-zero semistable subsheaf of $\ker \pi_{i_0}$ and repeat the above reasoning, unless π_{i_0} is an isomorphism outside a closed subset of codimension at least 2. Continuing in the same way, we end up with a semistable torsion free sheaf $M_0 \subseteq \ker\ \phi \subseteq \mathcal{O}^{\oplus l}$ of rank 1, endowed with an arrow $M_0 \to \mathcal{O}_Y$, which is an isomorphism outside a closed subset of codimension at least 2. We thus obtain a complex

$$\mathcal{O}_Y|_{Y\setminus Y_0} \to \mathcal{O}_Y^{\oplus l}|_{Y\setminus Y_0} \to V|_{Y\setminus Y_0},$$

where Y_0 is a closed subscheme of Y, which is of codimension at least 2. Since Y is normal, the arrow $\mathcal{O}_Y|_{Y\setminus Y_0} \to \mathcal{O}_Y^{\oplus l}|_{Y\setminus Y_0}$ extends uniquely to all of Y. We thus obtain a surjection $\mathcal{O}_Y^{\oplus l}/\mathcal{O}_Y \simeq \mathcal{O}_Y^{\oplus l-1} \to V$. This contradicts the minimality of l and proves the lemma. $\square$

Corollary 2.3. *Let V be a torsion free sheaf. Suppose that V is globally generated. Then $V \simeq V_0 \oplus \mathcal{O}_Y^l$ for some $l \geqslant 0$ and for some torsion sheaf V_0 such that $\mu(V_0) > 0$.*

Corollary 2.4. *Let V be a vector bundle over Y. Suppose that*
- for any surjective finite morphism $\phi : Y_0 \to Y$, we have $H^0(Y_0, \phi^*V) = 0$;
- $V^\vee$ is globally generated.
Then for any surjective finite morphism $\phi : Y_0 \to Y$, such that Y_0 is smooth over l_0, we have $\mu_{\min}(\phi^* V^\vee) > 0$. In particular, if $\mathrm{char}(l_0) > 0$ then $\bar{\mu}_{\min}(V^\vee) > 0$.

Proof. The bundle $V^\vee$ is globally generated so $\mu_{\min}(\phi^* V^\vee) \geqslant 0$. Now to obtain a contradiction, suppose that $\phi^* V^\vee$ has a non-zero semistable quotient Q of degree 0. Then we have $\phi^* V^\vee \simeq Q_0 \oplus \mathcal{O}_{Y_0}^{\oplus l}$ for some $l > 0$ by Corollary 2.3. This implies that $\phi^* V$ has a non-vanishing section, which contradicts the assumptions. □

The following elementary lemma is crucial to this article. The assumption that Y is smooth over l_0 is not used in the next lemma.

Lemma 2.5. *Let*

$$0 \to V \to W \to N \to 0$$

be an exact sequence of vector bundles on Y.
Suppose that $W \simeq \mathcal{O}_Y^l$ for some $l > 0$.
Then for any dominant proper morphism $\phi : Y_0 \to Y$, where Y_0 is integral, the morphism

$$\phi^* : H^0(Y, V) \to H^0(Y_0, \phi^* V)$$

is an isomorphism.

Proof. We have a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & H^0(Y,V) & \longrightarrow & H^0(Y,W) & \longrightarrow & H^0(Y,N) \\ & \downarrow{\phi^*} & & \downarrow{\phi^*} & & \downarrow{\phi^*} \\ 0 \longrightarrow & H^0(Y_0,\phi^*V) & \longrightarrow & H^0(Y_0,\phi^*W) & \longrightarrow & H^0(Y_0,\phi^*N) \end{array}$$

In this diagram, all three vertical arrows are injective by construction. Furthermore, the middle vertical arrow is an isomorphism, also by construction. The five lemma now implies that the left vertical arrow is surjective. □

In the following lemma, the smoothness assumption on Y is not used either. The proof of the following lemma is extracted from [6, p. 49, before Prop. 3], where the argument is attributed to Moret-Bailly.

Lemma 2.6 (Moret-Bailly)**.** *Suppose given a vector bundle V on Y with the following property: if $\phi : Y_0 \to Y$ is a surjective and finite morphism and Y_0 is irreducible, then we have $H^0(Y_0, \phi^*V) = 0$.*
Let $f : T \to Y$ be a torsor under V and let $Z \hookrightarrow T$ be a closed immersion. Suppose that $f|_Z : Z \to Y$ is finite and surjective and that Z is irreducible.
Then $f|_Z$ is generically purely inseparable.

Proof. Let $f : T \times_Y T \to Y$. We consider the scheme $T \times_Y (T \times_Y T)$. Via the projection on the second factor $T \times_Y T$, this scheme is naturally a torsor under the vector bundle f^*V . This torsor has two sections:
- the section σ_1 defined by the formula $t_1 \times t_2 \mapsto t_1 \times (t_1 \times t_2)$;
- the section σ_2 defined by the formula $t_1 \times t_2 \mapsto t_2 \times (t_1 \times t_2)$.
Since $T \times_Y (T \times_Y T)$ is a torsor under f^*V, there is a section $s \in H^0(T \times_Y T, f^*V)$ such that $\sigma_1 + s = \sigma_2$ and by construction $s(t_1 \times t_2) = 0$ iff $t_1 = t_2$. In other words, s vanishes precisely on the the diagonal of $T \times_Y T$.
Consider now the closed immersion $Z \times_Y Z \hookrightarrow T \times_Y T$. Suppose for contradiction that $f|_Z$ is not generically purely inseparable. Then there is an irreducible component C of $Z \times_Y Z$, which is not contained in the diagonal and such that $f|_C : C \to Y$ is dominant and hence surjective. Indeed, if $f|_Z$ is not generically purely inseparable, then there is by constructibility an open subset $U \subseteq Y$, such that for any closed point $u \in U$, there is a point $P(u) \in Z_u \times_u Z_u$ such that $P(u)$ is not contained in the diagonal of $Z_u \times_u Z_u$. Hence there is an irreducible component of $Z \times_Y Z$, which does not coincide with the diagonal and furthermore there is one, which dominates U for otherwise not every $P(u)$ would be contained in an irreducible component of $Z \times_Y Z$.
Now consider $f|_C^*V$. By construction the section $s|_C \in H^0(C, f|_C^*V)$ does not vanish. This contradicts the assumption on V. $\square$

We now quote a result proved in [10, exp. 2, Prop. 1].

Proposition 2.7 (Szpiro, Lewin-Ménégaux). *Suppose that* $\mathrm{char}(l_0) > 0$. *If* $H^0(Y, F_Y^*(V) \otimes \Omega_{Y/l_0}) = 0$ *then the natural map of abelian groups*

$$H^1(Y, V) \to H^1(Y, F_Y^* V)$$

is injective.

Corollary 2.8. *Suppose that* $\mathrm{char}(l_0) > 0$. *Let* V *be a vector bundle over* Y. *Suppose that*
- for any surjective finite morphism $\phi : Y_0 \to Y$, *we have* $H^0(Y_0, \phi^* V) = 0$;
- $V^\vee$ *is globally generated.*
Then there is an $n_0 \in \mathbb{N}$ *such that* $H^0(S, F_Y^{n,*}(V) \otimes \Omega_{Y/l_0}) = 0$ *for all* $n > n_0$.
Furthermore, let $T \to Y$ *be a torsor under* $F_Y^{n_0,*}(V)$. *Let* $\phi : Y' \to Y$ *be a proper surjective morphism and suppose that* Y' *is irreducible. Then the map*

$$H^1(Y, F_Y^{n_0,*}(V)) \to H^1(Y', \phi^*(F_Y^{n_0,*}(V)))$$

is injective.

Proof. (of Corollary 2.8). The existence of n_0 is a consequence of Corollary 2.4 and Theorem 2.1.
For the second assertion, by Lemma 2.6, we may assume wrog that ϕ is generically purely inseparable. Let H be the function field of Y and let $H'|H$ be the (purely inseparable) function field extension given by ϕ. Let $\ell_0 > 0$ be sufficiently large so that the extension $H'|H$ factors through the extension $H^{p^{-\ell_0}}|H$. We may suppose wrog that Y' is a normal scheme, since we may replace Y' by its normalization without restriction of generality. On the other hand the morphism $F_Y^{\ell_0} : Y \to Y$ gives a presentation of Y as its own normalization in $H^{p^{-\ell}}$. Thus there is a natural factorization $Y \to Y' \stackrel{\phi}{\to} Y$, where the composition of the two arrows is given by $F_Y^{\ell_0}$. Now by Proposition 2.7 there is a natural injection $H^1(Y, F_Y^{n_0,*}(V)) \hookrightarrow H^1(Y, F_Y^{\ell_0,*}(F_Y^{n_0,*}(V)))$. Hence the torsor T is not trivialized by $F_Y^{\ell_0}$ and thus cannot be trivialized by ϕ. □

3 Proof of Lemma 1.2, Theorem 1.3 and Corollary 1.4

First notice the important fact that we have $H^0(X, \Omega_X^\vee) = 0$. The follows from the fact that $\mathrm{Stab}(X) = 0$.

Proof of Lemma 1.2. Notice that

$$H^0(X, F_X^{\circ n,*}\Omega^\vee_{X/K_0}\otimes\Omega_{X/K_0})\simeq \mathrm{Hom}_X(F_X^{\circ n,*}\Omega_{X/K_0},\Omega_{X/K_0})$$

and furthermore, for any $r\geqslant 0$, there is a natural inclusion

$$\mathrm{Hom}_X(F_X^{\circ n,*}\Omega_{X/K_0},\Omega_{X/K_0})\subseteq \mathrm{Hom}_X(F_X^{\circ(n+r),*}\Omega_{X/K_0},F_X^{\circ r,*}\Omega_{X/K_0})$$

given by pulling back morphisms of vector bundles by $F_X^{\circ r,*}$. Now choose r sufficiently large so that $F_X^{\circ r,*}\Omega_{X/K_0}$ has a Harder-Narasimhan filtration with strongly semistable quotients. This is possible by Theorem 2.1. Then we have

$$\mu_{\min}(F_X^{\circ(n+r),*}\Omega_{X/K_0})=p^n\cdot\mu_{\min}(F_X^{\circ r,*}\Omega_{X/K_0})$$

and thus $\mathrm{Hom}_X(F_X^{\circ(n+r),*}\Omega_{X/K_0},F_X^{\circ r,*}\Omega_{X/K_0})=0$ if

$$p^n\cdot\mu_{\min}(F_X^{\circ r,*}\Omega_{X/K_0})>\mu_{\max}(F_X^{\circ r,*}\Omega_{X/K_0}).$$

Furthermore, by Corollary 2.4 and Lemma 2.5, we have $\mu_{\min}(F_X^{\circ r,*}\Omega_{X/K_0})>0$. Hence we will have

$$\sup\{p^n\mid n\in\mathbb{N}\ \&\ H^0(X,F_X^{n,*}\Omega^\vee_{X/K_0}\otimes\Omega_{X/K_0})\neq 0\}\leqslant\frac{\mu_{\max}(F_X^{\circ r,*}\Omega_{X/K_0})}{\mu_{\min}(F_X^{\circ r,*}\Omega_{X/K_0})}=\frac{\bar\mu_{\max}(\Omega_{X/K_0})}{\bar\mu_{\min}(\Omega_{X/K_0})}.$$

Proof of Theorem 1.3. Let $Q\in\mathcal{A}(U)$. Consider the infinite commutative diagram of $\mathcal{X}$-schemes

$$\begin{array}{ccccccc}
\dots\longrightarrow & \mathrm{Crit}^2(\mathcal{X}^{+Q},\mathcal{A}) & \longrightarrow & \mathrm{Crit}^1(\mathcal{X}^{+Q},\mathcal{A}) & \longrightarrow & \mathcal{X}\\
 & \downarrow & & \downarrow & & \downarrow{=}\\
\dots\longrightarrow & J^2(\mathcal{X}/U) & \longrightarrow & J^1(\mathcal{X}/U) & \longrightarrow & \mathcal{X}
\end{array}$$

For any $n\geqslant 0$, we shall write

$$\begin{array}{ccccccc}
\dots\longrightarrow & \mathrm{Crit}^2(\mathcal{X}^{+Q},\mathcal{A})^{(p^n)} & \longrightarrow & \mathrm{Crit}^1(\mathcal{X}^{+Q},\mathcal{A})^{(p^n)} & \longrightarrow & \mathcal{X}\\
 & \downarrow & & \downarrow & & \downarrow{=}\\
\dots\longrightarrow & J^2(\mathcal{X}/U)^{(p^n)} & \longrightarrow & J^1(\mathcal{X}/U)^{(p^n)} & \longrightarrow & \mathcal{X}
\end{array}$$

for the diagram obtained by pulling back the original diagram by $F_{\mathcal{X}}^{n,*}$. Let

$$n_0:=\sup\{n\in\mathbb{N}^*\mid H^0(X,F_X^{n,*}\Omega^\vee_{X/K_0}\otimes\Omega_{X/K_0})\neq 0\}.$$

Suppose that (a) in Theorem 1.3 is satisfied. We shall study diagram (3) in the case where $n = n_0$. Now fix any $m \geqslant 1$ and choose some $Q \in \mathcal{A}(U)$ such that $\mathrm{Exc}^m(A, X^{+Q}) \hookrightarrow X$ is an isomorphism. This is possible by assumption. By construction, the morphism

$$\mathrm{Crit}^m(\mathcal{X}^{+Q}, \mathcal{A})^{(p^{n_0})} \to \mathcal{X}$$

is then surjective. Choose an irreducible component $\mathrm{Crit}^m(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})} \hookrightarrow \mathrm{Crit}^m(\mathcal{X}^{+Q}, \mathcal{A})^{(p^{n_0})}$, which dominates $\mathcal{X}$. Endow $\mathrm{Crit}^m(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})}$ with its induced reduced scheme structure and for any $l < m$, let $\mathrm{Crit}^l(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})} \hookrightarrow \mathrm{Crit}^l(\mathcal{X}^{+Q}, \mathcal{A})^{(p^{n_0})}$ be the irreducible component obtained by direct image from $\mathrm{Crit}^m(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})}$.

Now notice that by Corollary 2.8 and Lemma 2.5 the $F_X^{n_0,*}\Omega^\vee_{X/K_0}$-torsor $J^1(X/K_0)^{(p^{n_0})} \to X$ must be trivial. Let $\sigma : X \to J^1(X/K_0)^{(p^{n_0})}$ be a section. The datum of the composed morphism $\mathrm{Crit}^1(X^{+Q}, A)_0^{(p^{n_0})} \to X \stackrel{\sigma}{\to} J^1(X/K_0)^{(p^{n_0})}$ is equivalent to the datum of a section of the pull-back of $\Omega^\vee_{X/k_0}$ to $\mathrm{Crit}^1(X^{+Q}, A)_0^{(p^{n_0})}$, which must vanish by Lemma 2.5. Hence the morphism $\mathrm{Crit}^1(X^{+Q}, A)_0^{(p^{n_0})} \to X$ is an isomorphism and thus by Zariski's main theorem, the morphism $\mathrm{Crit}^1(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})} \to \mathcal{X}$ is an isomorphism. We now repeat this reasoning for $\mathrm{Crit}^2(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})} \to \mathrm{Crit}^1(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})} \simeq \mathcal{X}$ and we conclude that

$$\mathrm{Crit}^2(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})} \to \mathrm{Crit}^1(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})}$$

is an isomorphism. Continuing this way, we see that in the whole tower

$$\mathrm{Crit}^m(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})} \to \mathrm{Crit}^{m-1}(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})} \to \cdots \to \mathrm{Crit}^1(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})} \to \mathrm{Crit}^1(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})} \to \mathcal{X}$$

the connecting morphisms are all isomorphisms. Using König's lemma, we may even choose the irreducible components $\mathrm{Crit}^m(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})}$ in such a way as to obtain an infinite chain

$$\cdots \to \mathrm{Crit}^m(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})} \to \mathrm{Crit}^{m-1}(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})} \to \cdots \to \mathrm{Crit}^1(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})} \to \mathrm{Crit}^1(\mathcal{X}^{+Q}, \mathcal{A})_0^{(p^{n_0})} \to \mathcal{X} \tag{1}$$

where all the connecting morphisms are isomorphisms.

Now choose a closed point $u_0 \in U$. View u_0 as a closed subscheme of U. For any $i \geqslant 0$, let u_i be the i-th infinitesimal neighborhood of $u_0 \simeq \mathrm{Spec}\, k_0$ in U (so that there is no ambiguity of notation for u_0). Notice that u_i has a natural structure of k_0-scheme. Recall that by the definition of the jet scheme (see [8, sec. 2]), the scheme $J^m(\mathcal{X}/U)_{u_0}$ represents the functor on k_0-schemes

$$T \mapsto \mathrm{Mor}_{u_m}(T \times_{k_0} u_m, \mathcal{X}_{u_m}).$$

Thus the infinite chain (1) gives rise to morphisms

$$\mathcal{X}_{u_0}^{(p^{-n_0})} \times_{k_0} u_m \to \mathcal{X}_{u_m} \tag{2}$$

compatible with each other under base-change. In particular, base-change to u_0 gives $F^{n_0}_{\mathcal{X}_{u_0}}$. View the $\widehat{U}_{u_0}$-schemes $\mathcal{X}_{u_0}^{(p^{-n_0})} \times_k \widehat{U}_{u_0}$ and $\mathcal{X}_{\widehat{U}_{u_0}}$ as formal schemes over $\widehat{U}_{u_0}$ in the next sentence. The family of morphisms (2) provides us with a morphism of formal schemes

$$\mathcal{X}_{u_0}^{(p^{-n_0})} \times_k \widehat{U}_{u_0} \to \mathcal{X}_{\widehat{U}_{u_0}}$$

and since both schemes are projective over $\widehat{U}_{u_0}$, Grothendieck's GAGA theorem shows that this morphism of formal schemes comes from a unique morphism of schemes

$$\iota : \mathcal{X}_{u_0}^{(p^{-n_0})} \times_k \widehat{U}_{u_0} \to \mathcal{X}_{\widehat{U}_{u_0}}.$$

By construction the morphism ι specializes to $F^{n_0}_{\mathcal{X}_{u_0}}$ at the closed point u_0 of $\widehat{U}_{u_0}$. Thus

$$\deg(\iota) = p^{\dim(X)n_0}.$$

Finally, $p^{n_0} \leqslant \mathfrak{DB}(X)$ by Lemma 1.2.

Proof of Corollary 1.4. We may replace X by $X/\mathrm{Stab}(X)$ without restriction of generality in the statement of Corollary 1.4. Thus we may (and do) assume that $\mathrm{Stab}(X) = 0$. Notice that by construction, for any $n \geqslant 1$, the natural homomorphism of groups

$$\Gamma_0/p^n\Gamma_0 \to \Gamma/p^n\Gamma$$

is a surjection. Furthermore, $\Gamma_0/p^n\Gamma_0$ is finite since Γ_0 is finitely generated. Hence, using the assumptions of Corollary 1.4, we see that for any $n \geqslant 1$, there exists $Q = Q(n) \in \Gamma_0$, such that $X^{+Q(n)} \cap p^n\Gamma$ is dense in X^{+Q}. This implies that $\mathrm{Exc}^n(A, X^{+Q(n)}) \hookrightarrow X$ is an isomorphism (see [8, par. 3.2] for more details or this). Now applying Theorem 1.3 (b), we obtain a surjective and finite morphism of $\widehat{\mathcal{O}}_{u_0}$-schemes

$$\mathcal{X}_{u_0}^{p^{-n_0}} \times_{k_0} \widehat{\mathcal{O}}_{u_0} \to \mathcal{X}_{\widehat{\mathcal{O}}_{u_0}}$$

for some closed point u_0 in U (in fact any will do) and some $n_0 \geqslant 0$ such that $p^{n_0} \leqslant \mathfrak{DB}(X)$. Let $\widehat{K}_0$ be the fraction field of $\mathcal{X}_{\widehat{\mathcal{O}}_{u_0}}$.

Since k_0 is an excellent field, we know that the field extension $\widehat{K}_0|K_0$ is separable. On the other hand the just constructed finite and surjective morphism $\mathcal{X}_{u_0} \times_{k_0} \widehat{K}_0 \to X_{\widehat{K}_0}$ is defined over a finitely generated (as a field over K) subfield K_0' of $\widehat{K}$. The field extension $K_0'|K$ is then still separable (because the extension $\widehat{K}_0|K_0$ is separable), so that by the theorem on separating transcendence bases, there exists a variety U'/K_0, which is smooth over K_0 and whose function field is K_0'. Furthermore, possibly replacing U' by one of its open subschemas, we may assume that the morphism $\mathcal{X}_{u_0} \times_k K_0' \to X_{K_0'}$ extends to a finite and surjective morphism

$$\alpha : \mathcal{X}_{u_0} \times_{k_0} U' \to X_{U'}.$$

Let $P \in U'(K_0^{\mathrm{sep}})$ be a K_0^{sep}-point over K (the set $U'(K_0^{\mathrm{sep}})$ is not empty because U' is smooth over K_0). The morphism α_P is the morphism h advertised in Theorem 1.3 (b). The inequality $\deg(h) \leqslant \mathfrak{DB}(X)^{\dim(X)}$ is verified by construction.

References

[1] Elisabeth Bouscaren Franck Benoist Anand Pillay, *On function field Mordell-Lang and Manin-Mumford*, Preprint arXiv:1404.6710.

[2] *Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 152, Springer-Verlag, Berlin, 1962/1964.

[3] Ehud Hrushovski, *The Mordell-Lang conjecture for function fields*, J. Amer. Math. Soc. **9** (1996), no. 3, 667–690.

[4] Daniel Huybrechts and Manfred Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics, E31, Friedr. Vieweg & Sohn, Braunschweig, 1997.

[5] Adrian Langer, *Semistable sheaves in positive characteristic*, Ann. of Math. (2) **159** (2004), no. 1, 251–276, DOI 10.4007/annals.2004.159.251.

[6] Mireille Martin-Deschamps, *Propriétés de descente des variétés à fibré cotangent ample*, Ann. Inst. Fourier (Grenoble) **34** (1984), no. 3, 39–64 (French, with English summary).

[7] Hideyuki Matsumura, *Commutative ring theory*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989. Translated from the Japanese by M. Reid.

[8] Damian Rössler, *On the Manin-Mumford and Mordell-Lang conjectures in positive characteristic*, Algebra Number Theory **7** (2013), no. 8, 2039–2057, DOI 10.2140/ant.2013.7.2039.

[9] N. I. Shepherd-Barron, *Semi-stability and reduction mod p*, Topology **37** (1998), no. 3, 659–664, DOI 10.1016/S0040-9383(97)00038-4.

[10] *Séminaire sur les Pinceaux de Courbes de Genre au Moins Deux*, Astérisque, vol. 86, Société Mathématique de France, Paris, 1981.

[11] Shing Tung Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), no. 5, 1798–1799.

[12] P. Ziegler, *Mordell-Lang in positive characteristic*, To appear in *Rendiconti del Seminario Matematico della Universit di Padova*.