Analytic torsion for cubes of vector bundles and Gillet's Riemann-Roch theorem

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Abstract

We present an analytic proof of Gillet's Riemann-Roch theorem for the Beilinson regulator in the case of compact fibrations, thereby extending to higher K-theory the analytic approach to the Grothendieck-Riemann-Roch theorem. Our proof depends essentially on Burgos-Wang's description of the regulator and on the properties of Bismut-Köhler's higher analytic torsion forms. Moreover, our proof shows how to define analogs of these analytic torsion forms for cubes of vector bundles.

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1 Introduction

Let X, B be projective smooth complex varieties and let $f : X \to B$ be a smooth holomorphic map. A particular case of Gillet's Riemann-Roch theorem (see [11]) states that the diagram

$$\begin{array}{ccc} K_k(X) & \xrightarrow{Td(Tf).ch} & \bigoplus_{p\geq 0} H^{2p-k}(X,p) \\ \downarrow f_* & & \downarrow f_* \\ K_k(B) & \xrightarrow{ch} & \bigoplus_{p\geq 0} H^{2p-k}(B,p) \end{array}$$

commutes, where $f : X \to B$ is the natural morphism, $Td(Tf)$ is the Todd class of the tangent bundle, $K_k(\cdot)$ denotes the k-th higher K-theory group of vector bundles in the sense of Quillen, the groups $H^{2p-k}(\cdot, p)$ are part of the real Deligne-Beilinson cohomology of X and ch denotes the Chern character on higher K-theory (also called Beilinson's regulator). In [9], Burgos-Wang provide a description of a lift of the Chern character to the simplicial level. In fact, they provide a functorial simplicial set $S^H_\cdot(\cdot)$, a functorial complex of abelian groups $\bigoplus_{p\geq 0} D^-_T(\cdot,p)[2p-1]$ and a canonical simplicial map $\widetilde{ch} : S^H_\cdot(\cdot) \to \mathcal{K}(\bigoplus_{p\geq 0} D^-_T(\cdot,p)[2p-1])$, such that $\pi_{k+1}(S^H_\cdot(\cdot)) = K_k(\cdot)$, $\pi_{k+1}(\mathcal{K}(\bigoplus_{p\geq 0} D^-_T(\cdot,p)[2p-1])) = H^{k+1}(\bigoplus_{p\geq 0} D^-_T(\cdot,p)[2p-1]) = \bigoplus_{p\geq 0} H^{2p-k}(X,p)$ and such that $\pi_k(\widetilde{ch}) = ch$. Here $\pi_k(\cdot)$ is the functor taking the k-th homotopy group of a simplicial set and $\mathcal{K}(\cdot)$ is the Dold-Puppe functor associating a simplicial abelian group to a homology-type complex of abelian groups (see the next subsection). Now choose a representative in the cohomology class of $Td(Tf)$ (we shall do this via certain Weil connections later). By abuse of language, we shall also denote this choice $Td(Tf)$. Since by Gillet's Riemann-Roch theorem, the simplicial maps $\widetilde{ch} \circ f_*$ and $f_* \circ Td(Tf).\widetilde{ch}$ induce the same maps on the level of homotopy groups, it is natural to ask for a simplicial homotopy between these two maps. In fact, since $\widetilde{ch}$ is a map to a fibrant simplicial set, we see that there must be such a simplicial homotopy (see [6, 4.3, p. 245]). It is the purpose of this paper to describe one explicitly, simultaneously providing an alternative proof of Gillet's Riemann-Roch theorem, which is analytic and does not rely on the deformation to the normal cone technique. The description of the homotopy is given in Theorem 3.6; it appears first as a homotopy between the chain complex of cubes of acyclic hermitian bundles on X and the complex $\bigoplus_{p\geq 0} D^-_T(B,p)$. At the level of 0-cubes, this last homotopy is given by the Bismut-Köhler's analytic torsion. Our description of the homotopy thus gives an interpretation of the analytic torsion form as the first map in a family of maps, the maps Π_k, which define a homotopy of chain complexes and thus provides a natural generalisation of the analytic torsion to cubes of acyclic hermitian vector bundles.

Our initial impulse for the research related to this paper was the question of the functoriality of a yet to be defined higher arithmetic K-theory (see [10]).

The functoriality of the arithmetic K_0-group of Gillet and Soulé (see [13, II]) involves the analytic torsion and the possibility of a higher theory naturally leads to conjecture the existence of higher analogs of the analytic torsion. It is our hope that the above family of maps are precisely these analogs.
Acknowledgments. Many thanks to Jean-Michel Bismut for an enlightening conversation, which helped me a great deal in finding the right perspective on the analytic aspects of the present work. I also owe a lot to J.-I. Burgos, whose detailed comments were invaluable in the development of this paper. My thanks also to the referee, for quick and efficient work.

2 The simplicial description of the Beilinson regulator

In this section, we recall Burgos-Wang's description of a map of simplicial sets which induces Beilinson's regulator on the level of homotopy groups. We also prove various results related to covariant functoriality, which will be needed in the last section. The basic reference for this section is [9].

2.1 Exact k-cubes

For the material described in this subsection, see also [9, §3]. If $\mathcal{A}$ is an exact (resp. abelian) category, we shall denote by $SES(\mathcal{A})$ the exact (resp. abelian) category of short exact sequences in $\mathcal{A}$. For $k \geq 1$, the category $C_k\mathcal{A}$ of k-cubes in $\mathcal{A}$ is the category $SES^{\circ k}(\mathcal{A}) = SES(SES(\dots k\ times)(\mathcal{A})$. By convention, we set $\mathcal{C}_0(\mathcal{A}) := \mathcal{A}$. Another way to describe $C_k\mathcal{A}$ is the following. Let $\{-1,0,1\}^k$ be the k-th cartesian power of the set $\{-1,0,1\}$, where we use the convention $\{-1,0,1\}^0 = \{0\}$. This set is naturally partially ordered by the relation $\leq$, defined by the law $(i_1,\dots i_k) \leq (j_1,\dots j_k)$ iff $i_1 \leq j_1$, $i_2 \leq j_2$, $\dots i_k \leq j_k$ ($i_1,\dots i_k, j_1 \dots j_k \in \{-1,0,1\}$). We can view $\{-1,0,1\}^k$ as a category whose objects are the elements $(i_1,\dots i_k)$ and such that the set of morphisms from $(i_1,\dots i_k)$ to $(j_1,\dots j_k)$ contains one element if $(i_1,\dots i_k) \leq (j_1,\dots j_k)$ and is empty otherwise. The category of exact k-cubes is the category of functors $E : \{-1,0,1\}^k \to \mathcal{A}$ such that for any set of indices $i_1,\dots \widehat{i_l},\dots i_k$ (the sign $\widehat{(.)}$ means that the corresponding symbol has to be omitted) the sequence

$$E_{i_1,\dots i_{l-1},-1,i_{l+1},\dots i_k} \to E_{i_1,\dots i_{l-1},0,i_{l+1},\dots i_k} \to E_{i_1,\dots i_{l-1},1,i_{l+1},\dots i_k}$$

is a short exact sequence.
We shall denote by $\mathcal{S}_k$ the symmetric group on k letters. An element σ defines defines a map $\{-1,0,1\}^k \to \{-1,0,1\}^k$ by the rule $\sigma((i_1,\dots i_k)) := (i_{\sigma(1)}, i_{\sigma(2)},\dots i_{\sigma(k)})$. This map is by construction order preserving and is thus an equivalence of categories. We obtain an equivalence of categories $\mathcal{C}_k(\mathcal{A}) \to$

$\mathcal{C}_k(\mathcal{A})$ if we associate to each k-cube E the k-cube $E \circ \sigma$. We shall also call this equivalence σ.

Let l be an integer such that $1 \leq l \leq k$ and let $i \in \{-1, 0, 1\}$. There is a natural embedding of ordered sets and thus of categories $\{-1, 0, 1\}^{k-1} \to \{-1, 0, 1\}^k$ given by the rule $(i_1, \dots i_{k-1}) \mapsto (i_1, \dots i_{l-1}, i, i_l, \dots i_{k-1})$. Given an exact k-cube $E : \{-1, 0, 1\}^k \to \mathcal{A}$, we define $\partial_l^i E$ to be the $(k-1)$-cube obtained as the composition of E with the embedding $\{-1, 0, 1\}^{k-1} \to \{-1, 0, 1\}^k$. There is an equality $\partial_l^i \partial_m^j E = \partial_m^j \partial_{l+1}^i E$, for each l with $1 \leq l \leq k-1$.

We write $\mathbf{Z}C_k(\mathcal{A})$ for the free abelian group generated by all the k-cubes. Define a family of linear maps $\delta_k : \mathbf{Z}C_k(\mathcal{A}) \to \mathbf{Z}C_{k-1}(\mathcal{A})$ by the rule $E \mapsto \sum_{l=1}^k (-1)^{l-1}(\partial_l^0 E - \partial_l^{-1} E - \partial_l^1 E)$. The identity $\partial_l^i \partial_m^j = \partial_m^j \partial_{l+1}^i$ $(1 \leq l \leq k-1)$ shows that the groups $\mathbf{Z}C_k(\mathcal{A})$ together with the morphisms δ_k form a homology-type complex indexed by k. A k-cube E is called degenerate, if for some l with $1 \leq l \leq k$ one of the two following conditions hold: $\partial_l^{-1} E = \partial_l^0 E$ and the natural map $\partial_l^{-1} E \to \partial_l^0 E$ is the identity map or $\partial_l^0 E = \partial_l^1 E$ and the natural map $\partial_l^0 E \to \partial_l^1 E$ is the identity map. Let us call D_k the free abelian group generated by all the degenerate k-cubes. The family of the D_k form a subcomplex of $\mathbf{Z}C.(\mathcal{A})$ (see [9, (3.2)]).

If $F : \mathcal{A} \to \mathcal{B}$ is an exact functor between two abelian (resp. exact) categories, then F naturally sends exact k-cubes on exact k-cubes and F induces a map of complexes $F : \mathbf{Z}C.(\mathcal{A}) \to \mathbf{Z}C.(\mathcal{B})$.

Let $\Delta^\circ Ab$ be the category of simplicial abelian groups and let $C_{\leq 0}$ be the category of homology-type chain complexes of abelian groups. There exists a pair of functors $\mathcal{N} : \Delta^\circ Ab \to C_{\leq 0}$ and $\mathcal{K} : C_{\leq 0} \to \Delta^\circ Ab$, which induces an equivalence of the associated homotopy categories (see [18, 5.32]). We shall describe the functor $\mathcal{N}$ explicitly. Let $G.$ be a simplicial abelian group. We denote the face maps by $\partial.$. For each $k \geq 0$, the object of $\mathcal{N}(G.)$ which is of index k is G_k and the differential map d_k at k is the map $\sum_{l=0}^k (-1)^l \partial_l$.

Let now X be a smooth quasi-projective complex variety. We shall denote the category of vector bundles on X by $\mathcal{C}(X)$ and the category of exact k-cubes of vector bundles by $C_k(X)$. We recall Burgos-Wang's modification of Gillet-Soulé's construction of a geometric splitting of a short exact sequence of vector bundles (see [9, (3.7)]). Let

$$F : 0 \to E' \xrightarrow{i} E \to E'' \to 0 \qquad (1)$$

be a short exact sequence of vector bundles on X. Let σ_∞ (resp. σ_0) be the canonical section of the tautological bundle $\mathcal{O}(1)$ on $\mathbf{P}^1_\mathbf{C}$ which vanishes only at ∞ (resp. at 0). We lift the sequence (1) to $X \times \mathbf{P}^1_\mathbf{C}$ and we define a map $E' \to E(1) \oplus E'(1)$ by the rule $e' \mapsto i(e') \otimes \sigma_0 \oplus e' \otimes \sigma_\infty$; we can form the quotient sheaf $\langle F \rangle := (E(1) \oplus E'(1))/E'$, which is again a vector bundle. By construction there are functorial isomorphisms $\langle F \rangle|_{X \times \{0\}} \simeq E$ and $\langle F \rangle|_{X \times \{\infty\}} \simeq E' \oplus E''$. Moreover, this construction yields an exact functor $\langle (.) \rangle$ from the category of short exact sequences on X to the category of vector bundles on $X \times \mathbf{P}^1_\mathbf{C}$. We now extend the functor $\langle(.)\rangle$ to the category $C_k\mathcal{C}(X)$. By convention the functor $\langle(.)\rangle$ is the identity functor if $k=0$. Let now $k\geq 1$. We shall give a definition which depends inductively on k. For $k=1$, the functor $\langle(.)\rangle$ coincides with the functor $\langle(.)\rangle$ defined above. Suppose it is defined and exact for $k\geq 1$. Let F be $k+1$-cube. This cube gives rise to a family of short exact sequences of vector bundles

$$0\to F_{-1,i_1,\dots i_k}\to F_{0,i_1,\dots i_k}\to F_{1,i_1,\dots i_k}\to 0$$

for $i_1,\dots i_k\in\{-1,0,1\}$. Applying the Burgos-Gillet-Soulé construction to each member of this family, we obtain a family of vector bundles on $X\times(\mathbf{P}^1_{\mathbf{C}})^1$. Since the Gillet-Soulé construction corresponds to an exact functor, this family corresponds to a k-cube; applying the functor $\langle(.)\rangle$ to this k-cube is the definition of $\langle F\rangle$.

Let now σ be an element of $\mathcal{S}_k$. In the next proposition, we denote by the same letter the automorphism of $(\mathbf{P}^1_{\mathbf{C}})^k$ which sends $(p_1,p_2,\dots p_k)$ on $(p_{\sigma(1)},p_{\sigma(2)},\dots p_{\sigma(k)})$.

Proposition 2.1 *Let E be a k-cube on X. Then there is a functorial isomorphism $\sigma^*\langle E\rangle\simeq\langle\sigma(E)\rangle$.*

Proof: Since $\mathcal{S}_k$ is generated by the permutations which swap successive elements in $\{1,\dots k\}$ and since the isomorphism $\sigma^*\langle E\rangle\simeq\langle\sigma(E)\rangle$ is stable under composition of permutations, we are reduced to the case $k=2$ and to the permutation which swaps 1 and 2. So consider the 2-cube F

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & E_{-1,-1} & \stackrel{i_{-1}}{\to} & E_{-1,0} & \to & E_{-1,1} & \to & 0\\
 & & \downarrow j_{-1} & & \downarrow j_0 & & \downarrow j_1 & & \\
0 & \to & E_{0,-1} & \stackrel{i_0}{\to} & E_{0,0} & \to & E_{0,1} & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & E_{1,-1} & \stackrel{i_1}{\to} & E_{1,0} & \to & E_{1,1} & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By construction, $\langle F\rangle$ is then the cokernel of the map Q

$$E_{0,-1}(1)_2\oplus E_{-1,-1}(1)_2\oplus E_{-1,0}(1)_1\oplus E_{-1,-1}(1)_1 \tag{2}$$

$$\longrightarrow$$

$$E_{0,0}(1)_1(1)_2\oplus E_{0,-1}(1)_1(1)_2\oplus E_{-1,0}(1)_1(1)_2\oplus E_{-1,-1}(1)_1(1)_2 \tag{3}$$

given by

$$a\otimes b\oplus c\otimes d\oplus e\otimes f\oplus g\otimes h \tag{4}$$

$$\mapsto$$

$$\Big(i_0(a)\otimes\sigma_0^1\otimes b + j_0(e)\otimes f\otimes\sigma_0^2\Big)\oplus\Big(a\otimes\sigma_\infty^1\otimes b + j_{-1}(g)\otimes h\otimes\sigma_0^2\Big)\oplus$$
$$\Big(e\otimes f\otimes\sigma_\infty^2 + i_{-1}(c)\otimes\sigma_0^1\otimes d\Big)\oplus\Big(g\otimes h\otimes\sigma_2^\infty + c\otimes\sigma_\infty^1\otimes d\Big)$$

where $(1)_1$ (resp. $(1)_2$) denotes the twist with the tautological bundle coming from the first factor of $(\mathbf{P}^1_{\mathbf{C}})^2$ (resp. the second one) and $\sigma_0^{\cdot}$ (resp. $\sigma_\infty^{\cdot}$) is the corresponding section at 0 (resp. at ∞). Let H be the direct sum in (2) and H' the direct sum in (3). To obtain the bundle $\sigma^* < \sigma(F) >$, we have to swap $E_{i,j}$ and $E_{j,i}$ as well as $(1)_1$ and $(1)_2$ in the expressions for H and H' and we have to swap $\sigma_{\cdot}^1$ and $\sigma_{\cdot}^2$ as well as $i_{\cdot}$ and $j_{\cdot}$ in the expression for Q. One can check directly that up to a permutation of the factors we are left again with the original expressions for H, H' and Q once these swappings have been performed. Thus $\sigma^* < \sigma(F) > \simeq < F >$ and since σ is its own inverse as a permutation we have $\sigma^* < F > \simeq < \sigma(F) >$. This concludes the proof. **Q.E.D.**

The following corollary, without the explicit determination of the isomorphisms, can be found in [9, Prop. 3.9]. For $1 \leq l \leq k$, let τ_{kl} be the permutation on k letters defined by the following rules: $\tau_{kl}(i) := i+1$ if $1 \leq i \leq l-1$, $\tau_{kl}(l) := 1$, $\tau_{kl}(i) := i$ if $l+1 \leq i \leq k$.

Corollary 2.2 *Let E be a k-cube of vector bundles. There are isomorphisms*

(a) $\big\langle E\big\rangle|_{X\times\mathbf{P}^1_{\mathbf{C}}\times\ldots\{0\}\ldots\times\mathbf{P}^1_{\mathbf{C}}} \simeq \tau_{kl}^*\big\langle E\big\rangle|_{X\times\{0\}\times\ldots\times\mathbf{P}^1_{\mathbf{C}}} \simeq \big\langle\partial_1^0\tau_{kl}(E)\big\rangle \simeq \big\langle\partial_l^0(E)\big\rangle$

(b) $\big\langle E\big\rangle|_{X\times\mathbf{P}^1_{\mathbf{C}}\times\ldots\{\infty\}\ldots\times\mathbf{P}^1_{\mathbf{C}}} \simeq \tau_{kl}^*\big\langle E\big\rangle|_{X\times\{\infty\}\times\ldots\times\mathbf{P}^1_{\mathbf{C}}} \simeq \big\langle\partial_1^{-1}\tau_{kl}(E)\big\rangle\oplus\big\langle\partial_1^1\tau_{kl}(E)\big\rangle \simeq \big\langle\partial_l^{-1}(E)\big\rangle\oplus\big\langle\partial_l^1(E)\big\rangle$

where 0 and ∞ stand at the l-th place in the product.

Proof: For $l = 1$, the isomorphisms in (a) and (b) follow from the properties of the Burgos-Wang construction at the last step of the recursion that appears in the definition of the functor $\big\langle(.)\big\rangle$. For $l > 0$, they follow from the case $l = 1$ and the last proposition. **Q.E.D.**

We now adress the question of the covariant functoriality of the Burgos-Wang construction. Until the end of the paragraph, let $f : X \to B$ be a proper and flat morphism of smooth quasi-projective complex varieties and let E be a k-cube all of whose vector bundle components are f-acyclic (recall that a vector bundle V on X is f-acyclic if the relative cohomology sheaves R^if_*V vanish for $i > 0$). We shall also call such a k-cube acyclic. Define $\widetilde{f}_k$ to be the morphism $X \times (\mathbf{P}^1_{\mathbf{C}})^k \to B \times (\mathbf{P}^1_{\mathbf{C}})^k$ naturally induced by f.

Proposition 2.3 *The bundle $\big\langle E\big\rangle$ is $\widetilde{f}_k$-acyclic and there is a functorial isomorphism $\widetilde{f}_{k,*}\big\langle E\big\rangle \simeq \big\langle f_*E\big\rangle$. Moreover the isomorphisms in 2.2 are natural under f_*.*

Proof: In view of the recursive definition of the functor $\langle(.)\rangle$, we can without loss of generality restrict ourselves to the case $k = 1$. So let us suppose that E is an f-acyclic vector bundle and consider the exact sequence of vector bundles

$$\mathcal{E} : 0 \to E' \to E(1) \oplus E'(1) \to \langle E\rangle \to 0$$

defining $\langle E\rangle$. Using the theorem on cohomology and flat base change, we see that E and E' considered on $X \times (\mathbf{P}^1_{\mathbf{C}})^k$ are $\widetilde{f}_k$-acyclic. Using the projection formula and the properties of the long exact cohomology sequence, we see that E', $E(1) \oplus E'(1)$ are $\widetilde{f}_k$-acyclic as well. Thus, applying the projection formula to the long cohomology sequence of $\mathcal{E}$, we obtain the sequence

$$0 \to f_{k,*}E' \to f_{k,*}E(1) \oplus f_{k,*}E'(1) \to f_{k,*}\langle E\rangle \to 0.$$

The proof of the second statement follows from the first statement, the fact that f_* commutes with τ^*_{kl}, the functor $\tau_{kl}(.)$ on k-cubes and the explicit description of the isomorphisms given in 2.2. **Q.E.D**.

2.2 Hermitian k-cubes

Let now $\mathcal{C}^H(X)$ be the abelian category whose elements are vector bundles on X endowed with hermitian metrics and whose morphisms are the vector bundle morphisms. Notice that $\mathcal{C}^H(X)$ and $\mathcal{C}(X)$ are equivalent categories, an equivalence being described by the functor $\mathcal{C}^H(X) \to \mathcal{C}(X)$ forgetting the metric and a functor $\mathcal{C}(X) \to \mathcal{C}^H(X)$ given by a choice of a hermitian metric for each vector bundle. We shall call this category the category of hermitian exact k-cubes and we shall write $C^H_k(X)$ for $C_k(\mathcal{C}^H(X))$. Let $\overline{E}$ be a hermitian k-cube. Associated to E is a set of short exact sequences

$$0 \to E_{i_1,\ldots i_{l-1},-1,i_{l+1},\ldots i_k} \to E_{i_1,\ldots i_{l-1},0,i_{l+1},\ldots i_k} \to E_{i_1,\ldots i_{l-1},1,i_{l+1},\ldots i_k} \to 0 \quad (5)$$

$(1 \leq l \leq k)$. The three vector bundles appearing in this sequence carry a hermitian structure by definition. The following definition is [9, Def. 3.5].

Definition 2.4 (Burgos-Wang) *The hermitian k-cube $\overline{E}$ is called an emi k-cube, if for all l the vector bundle $E_{i_1,\ldots i_{l-1},1,i_{l+1},\ldots i_k}$ in the sequence (5) carries the metric induced by $E_{i_1,\ldots i_{l-1},0,i_{l+1},\ldots i_k}$.*

From the definitions, one can see that if $\overline{E}$ is an *emi* k-cube then $\partial^i_l\overline{E}$ is also an *emi* k-cube, for $i \in \{-1, 0, 1\}$ and $0 \leq l \leq k$. Notice that if we endow the tautological bundle $\mathcal{O}(1)$ with the Fubini-Study metric, the construction of the vector bundle $\langle E\rangle$ endows it with a natural hermitian structure. We shall denote by $\langle\overline{E}\rangle$ the bundle $\langle E\rangle$ endowed with this hermitian structure. Notice that by construction there is an isometric isomorphism $\sigma^*\langle\overline{E}\rangle \simeq \langle\sigma(\overline{E})\rangle$. For the proof of the following proposition, see [9, Prop. 3.9].

Proposition 2.5 (Burgos-Wang) *If $\overline{E}$ is an emi k-cube, the isomorphisms (a) and (b) in 2.2 are isometries, if all the bundles are endowed with their natural metrics. Moreover, in that case, the direct sum in (b) is orthogonal.*

We denote by $\lambda_l^1(\overline{E})$ the hermitian cube obtained if one sets the metric induced by $E_{i_1,\dots i_{l-1},0,i_{l+1},\dots i_k}$ on $E_{i_1,\dots i_{l-1},1,i_{l+1},\dots i_k}$, for all $(i_1,\dots,\widehat{i_l},\dots i_k)$. We denote by $\lambda_l^2(\overline{E})$ the hermitian k-cube such that $\partial_l^{-1}(\lambda_l^2(\overline{E})) = \partial_l^1\overline{E}$, $\partial_l^0(\lambda_l^2(\overline{E})) = \partial_l^1(\lambda_l^1(\overline{E}))$ and $\partial_l^1(\lambda_l^2(\overline{E})) = 0$. Let now $\lambda_k : \mathbf{Z}C_k^H(X) \to \mathbf{Z}C_k^H(X)$ be given by the formula

$$\lambda_k(\overline{E}) := \sum_{(r_1,\dots r_k)\in\{1,2\}^k} \lambda_k^{r_k}\circ\lambda_{k-1}^{r_{k-1}}\circ\dots\lambda_1^{r_1}(\overline{E})$$

for each $k\geq 0$. For the proof of following proposition, see [9, (3.5)].

Proposition 2.6 (Burgos-Wang) *The maps λ_k induce a map of complexes $\lambda : \mathbf{Z}C_\cdot^H(X) \to \mathbf{Z}C_\cdot^H(X)$. Moreover, for each k, the image of λ_k consists of emi cubes.*

Let now $f : X \to B$ be a smooth proper map of smooth quasi-projective varieties. Let $\overline{V}$ be a an f-acyclic vector bundle V on X, endowed with a hermitian metric h. Let ω be the Kähler form of some Kähler metric on X. By definition, elements $U, W \in f_*V|_p$ of a fiber of f_*V at a point $p\in B$ correspond to sections of $V|_{f^{-1}p}$. Let $d = dim(X) - dim(B)$; we define a pairing $<.,.>$ on $f_*V|_p$ by the formula

$$<U,W>:=\frac{1}{(2\pi)^d}\int_{f^{-1}p} h(U,W)\omega^d/d!.$$

This pairing defines a hermitian metric on f_*V, which shall be denoted by the symbol f_*h (see also [5, p. 666]). We shall say that the metric f_*h is obtained by integration along the fibers and we shall write $f_*\overline{V}$ for the hermitian bundle (f_*V, f_*h). This definition gives an exact functor from the exact category $C^H_{f-ac}(X)$ of the f-acyclic hermitian vector bundles on X to the abelian category of the hermitian vector bundles on B.

2.3 Computation of the Deligne-Beilinson cohomology

In this subsection, we recall the definition of the real Deligne-Beilinson cohomology of a compact complex manifold and recall Burgos-Wang's description of it (this description is suggested in [10]). We suppose for the time of this subsection that X is a projective complex manifold. Let $p\geq 0$. We shall denote by $\mathbf{R}(p)$ the subgroup $(2i\pi)^p.\mathbf{R}\subset\mathbf{C}$. The i-th real Deligne-Beilinson cohomology group of X is the i-th hypercohomology group of the complex of abelian sheaves

$$\mathbf{R}(p)_{\mathcal{D}} : 0\to\mathbf{R}(p)\to\Omega_X^0\xrightarrow{d}\Omega_X^1\to\dots\Omega_X^{p-1}\to 0.$$

where Ω^j_X is the sheaf of holomorphic differential forms of degree j on X (see *Deligne-Beilinson cohomology* in [17]).
For $p, q \geq 0$, a multiplicative structure is defined by the map of complexes

$$\cup : \mathbf{R}(p)_{\mathcal{D}} \otimes \mathbf{R}(q)_{\mathcal{D}} \to \mathbf{R}(p+q)_{\mathcal{D}}$$

which for elements $x \in (\mathbf{R}(p)_{\mathcal{D}})_n$ and $y \in (\mathbf{R}(q)_{\mathcal{D}})_m$ is defined by the rules $x \cup y = x \wedge y$ if $n = 0$, $x \cup y = x \wedge dy$ if $n > 0$ and $m = q$, 0 otherwise.
Let now $A^{i,j}(X)$ denote the differential forms of type i, j on X and let $A^n(X)$ denote the differential forms of degree n. Let $A^n_{\mathbf{R}}(X)$ denote the real differential forms of degree n. The subgroup $(2i\pi)^p.A^n_{\mathbf{R}}(X) \subseteq A^n(X)$ will be written $A^n_{\mathbf{R}}(X)(p)$. The set of all differential forms is denoted by $A(X)$. The i, j-component of $x \in A(X)$ will be written $x^{i,j}$. For $k \geq 0$, define also an operator $F^{k,k} : A(X) \to A(X)$ by the rule $F^{k,k}(x) := \sum_{l \geq k, l' \geq k} x^{l,l'}$ and an operator $F^k : A(X) \to A(X)$ by the rule $F^k(x) := \sum_{l \geq k, l' \geq 0} x^{l,l'}$. The operator $\pi_p : A(X) \to A_{\mathbf{R}}(X)(p)$ is defined as $\frac{1}{2}(x + (-1)^p \overline{x})$.
Let $D^*(X, p)$ be the cohomological complex whose group at $n \geq 0$ is

$$A^{n-1}_{\mathbf{R}}(X)(p-1) \cap \bigoplus\nolimits_{p'+q'=n-1, p'<p, q'<p} A^{p',q'}(X) \text{ if } n \leq 2p-1,$$

$$A^n_{\mathbf{R}}(X)(p) \cap \bigoplus\nolimits_{p'+q'=n, p' \geq p, q' \geq p} A^{p',q'}(X) \text{ if } n \geq 2p$$

and whose differential $d_{\mathcal{D}}$ is given by the formula dx if $n \geq 2p$, the formula $-\pi_{p-1} F^{deg(x)-p+1, deg(x)-p+1} dx$ if $n < 2p-1$ and the formula $-2\partial\overline{\partial}x$ if $n = 2p-1$. Notice that in view of the local definition of $D^*(X, p)$, there is a natural complex of (fine) sheaves $\mathbf{D}^*(X, p)$, such that $\Gamma(\mathbf{D}^*(X, p)) = D^*(X, p)$. Notice also that the complex $D^*(X, p)$ is naturally defined on any complex manifold, although we only consider the quasi-projective case in this paragraph. For the proof of the following proposition, see [8, Th. 2.6].

Proposition 2.7 (Burgos-Wang) *Define a map of complexes of sheaves $\rho : \mathbf{R}(p)_{\mathcal{D}} \to \mathbf{D}(X, p)$ by the rule $\rho(\omega) = \pi_{p-1}(\omega)$. Then this map is a quasi-isomorphism. Moreover, the multiplicative structure*

$$\bullet : D(X, p) \otimes D(X, q) \to D(X, p+q)$$

given for $x \in D^n(X, p)$ and $y \in D^m(X, q)$ by

$$x \bullet y = \begin{cases} (-1)^n 2\pi_p(F^p dx) \wedge y + x \wedge 2\pi_q(F^q y) & if\ n < 2p \text{ and } m < 2q, \\ \pi_{p+q-1}(F^{n+m-p-q, n+m-p-q}(x \wedge y)) & if\ n < 2p,\ m \geq 2q \text{ and } l < 2r, \\ F^{r,r}(2\pi_p(F^p(dx)) \wedge y) + 2\pi_r \partial((x \wedge dy)^{r-1, l-r}) & if\ n < 2p,\ m \geq 2q, l \geq 2r, \\ x \wedge y & if\ n \geq 2p \text{ and } m \geq 2q \end{cases}$$

where $l = n + m$, $r = p + q$, gives the product in Deligne-Beilinson cohomology under this quasi-isomorphism. Moreover if $n = 2p$, $d_{\mathcal{D}} x = 0$, $s \geq 0$ and $z \in D(X, s)$, then $x \bullet y = y \bullet x$, $x \bullet (y \bullet z) = (x \bullet y) \bullet z$.

Let now $f : X \to B$ be a smooth map of projective complex smooth varieties.

Proposition 2.8 *Let $d = dim(X) - dim(B)$. The map $f_* : D(X,p)[2p] \to D(B, p-d)[2(p-d)]$ given by $\frac{1}{(2i\pi)^d}\int_{X/B}$ is a map of complexes. The map $f^* : D(B,p)[2p] \to D(X,p)[2p]$ given by the pull-back of differential forms is a map of complexes. Moreover, if $x \in D(X,p)[2p]$ and $y \in D(B,q)[2q]$, then the projection formula $f_*(x \bullet f^*(y)) = f_*(x) \bullet y$ holds.*

Proof: The fact that f_* gives a map of complexes with degree shift follows from the fact that the fiber integral commutes with ∂, $\overline{\partial}$, with the conjugation operator $\overline{(.)}$ and because

$$\int_{X/B} F^{deg(x)-p+1,deg(x)-p+1}x = F^{deg(x)-p-d+1,deg(x)-p-d+1}\int_{X/B} x =$$

$$F^{deg(\int_{X/B} x)-(p-d)+1,deg(\int_{X/B} x)-(p-d)+1}\int_{X/B} x \tag{6}$$

if $deg(x) < 2p-1$. The fact that the degree shift is 0 follows from the fact that the fiber integral reduces the total degree of a differential form by $2d$.
The second statement follows readily from the definitions. To prove the projection formula, notice first that

$$\int_{X/B} F^{deg(x)-p+1}x = F^{deg(x)-p-d+1}\int_{X/B} x = F^{deg(\int_{X/B} x)-(p-d)+1}\int_{X/B} x$$

if $deg(x) < 2p$. From this, (6), the commutation relations mentioned at the beginning of the proof and the projection formula for the fiber integral, the projection formula follows. **Q.E.D.**

2.4 Secondary classes for hermitian k-cubes

In this subsection, we shall recall Burgos-Wang's definition of secondary classes for hermitian k-cubes, which generalize the secondary classes of Gillet-Soulé (see [4, I, Par. f)]). Let $\mathcal{S}_k$ be the symmetric group on k letters. Let $u_1, \dots u_k$ be elements of $\bigoplus_{p\geq 0} D^{2p-1}(X,p)$. We shall write $D_T^-(X,p)[l]$ for the homology-type complex whose k-th object is $D^{l-k}(X,p)$ if $k \geq 0$ (and is 0 otherwise). Define an element of $\bigoplus_{p\geq 0} D_T^-(X,p)[2p]$ by the formula

$$C_k(u_1, \dots u_k) = (1/2)^{k-1}\sum_{\sigma\in\mathcal{S}_k}(-1)^{sgn(\sigma)}u_{\sigma(1)} \bullet (u_{\sigma(2)} \bullet (\dots u_{\sigma(k)} \dots)$$

The following lemma is a slight generalisation of an unpublished result of Burgos thesis (see [7, Prop. 2.5, p. 121]).

Lemma 2.9 *The following identity holds*

$$d_{\mathcal{D}}C_k(u_1,\dots u_k) = (-1/2)k.\sum_{j=1}^{k}(-1)^{j-1}d_{\mathcal{D}}(u_j)\bullet C_{k-1}(u_1,\dots\widehat{u_j},\dots u_k) =$$

$$(-1/2)k.\sum_{j=1}^{k}(-1)^{j-1}d_{\mathcal{D}}(u_j)\wedge C_{k-1}(u_1,\dots\widehat{u_j},\dots u_k)$$

Proof: We first prove the second equality. Let first $v \in D^{2q}(X,q)$ and $u \in D^{2p-k}(X,p)$. We compute

$$v\bullet u = \pi_{p+q-1}F^{2q+(2p-k)-(p+q),2q+(2p-k)-(p+q)}v\wedge u = \pi_{p+q-1}F^{p+q-k,p+q-k}v\wedge u =$$

$$\pi_{p+q-1}\sum_{l\geq p+q-k,l'\geq p+q-k}(v\wedge u)^{l,l'} = \pi_{p+q-1}\sum_{l=p+q-k}^{p+q-1}\sum_{l'=p+q-k}^{p+q-1}(v\wedge u)^{l,l'}$$

Since the total degree of $v\wedge u$ is $2p-k-1+2q$ and since $2p-k-1+2q \geq (p+q-k)+(p+q-1) = 2p+2q-k-1$, the last expression equals

$$\pi_{p+q-1}v\wedge u = v\wedge u$$

and thus $v\bullet u = v\wedge u$. This settles the second equality. To prove the first one, we compute

$$d_{\mathcal{D}}C_k(u_1,\dots u_k) = (1/2)^{k-1}\sum_{\sigma\in\mathcal{S}_k}(-1)^{sgn(\sigma)}d_{\mathcal{D}}(u_{\sigma(1)}\bullet(u_{\sigma(2)}\bullet(\dots u_{\sigma(k)}\dots)) =$$

$$(1/2)^{k-1}\sum_{\sigma\in\mathcal{S}_k}(-1)^{sgn(\sigma)}\sum_{i=1}^{k}(-1)^{i-1}u_{\sigma(1)}\bullet(u_{\sigma(2)}\bullet(\dots(d_{\mathcal{D}}u_{\sigma(i)}\dots u_{\sigma(k)}\dots)) =$$

$$(1/2)^{k-1}\sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{\sigma\in\mathcal{S}_k,\sigma(i)=j}(-1)^{sgn(\sigma)}(-1)^{i-1}u_{\sigma(1)}\bullet(u_{\sigma(2)}\bullet(\dots(d_{\mathcal{D}}u_{\sigma(i)}\dots u_{\sigma(k)}\dots)) =$$

$$(1/2)^{k-1}\sum_{i=1}^{k}\sum_{j=1}^{k}d_{\mathcal{D}}u_{\sigma(i)}\bullet\sum_{\sigma\in\mathcal{S}_k,\sigma(i)=j}(-1)^{sgn(\sigma)}(-1)^{i-1}u_{\sigma(1)}\bullet(u_{\sigma(2)}\bullet(\dots(\widehat{u_{\sigma(i)}}\dots u_{\sigma(k)}\dots)) =$$

$$(1/2)\sum_{i=1}^{k}(-1)^{i-1}\sum_{j=1}^{k}d_{\mathcal{D}}u_j(-1)^{i-j}\bullet C_{k-1}(u_1,\dots\widehat{u_j},\dots u_k)$$

which is the first equality. From the second to the third line of the last string of equalities, we used the distributivity of the operator $d_{\mathcal{D}}$, which is a consequence of the fact that the $\bullet$ product arises as a map of complexes of abelian groups (see the beginning of 2.7) and from the third to the fourth we used the commutativity statement at the end of 2.7. **Q.E.D**.

Let now $\overline{E}$ be a hermitian k-cube on a complex projective smooth variety X. Consider the canonical coordinate system of $\mathbf{P}^1_{\mathbf{C}}$ defined by the map $\mathbf{C} \to \mathbf{P}^1_{\mathbf{C}}$ which sends z on $[z, 1]$. The function $\log|z|$ defines an L^1 function on $\mathbf{P}^1_{\mathbf{C}}$, which we consider as a current. We shall denote by $\log|z_1|, \dots \log|z_k|$ the corresponding currents on $(\mathbf{P}^1_{\mathbf{C}})^k$. In the next definition, we shall consider that the $\log|z_i|$ are formally elements of $D^1((\mathbf{P}^1_{\mathbf{C}})^k, 1)$. They satisfy the equation of currents

$$d_{\mathcal{D}} \log|z_i| = -2\partial\overline{\partial}\log|z_i| = -4i\pi.(\delta_{\mathbf{P}^1_{\mathbf{C}}\times\mathbf{P}^1_{\mathbf{C}}\times\dots\{\infty\}\times\dots\mathbf{P}^1_{\mathbf{C}}} - \delta_{\mathbf{P}^1_{\mathbf{C}}\times\mathbf{P}^1_{\mathbf{C}}\times\dots\{0\}\times\dots\mathbf{P}^1_{\mathbf{C}}}) \tag{7}$$

where $\delta_{\cdot}$ takes the Dirac current associated to a closed submanifold and ∞ (resp. 0) stands at the i-th place. The following definition is taken from [9, Def. 6.12].

Definition 2.10 (Burgos-Wang) *The Bott-Chern secondary class $\widetilde{ch}(\overline{E})$ of $\overline{E}$ is the element of $\bigoplus_{p\geq 0} D^{2p-k}(X,p)$ given by*

$$\frac{(-1)^k}{2k!(2i\pi)^k}\sum_{(r_1,\dots r_k)\in\{1,2\}^k}\int_{X\times(\mathbf{P}^1_{\mathbf{C}})^k/X} ch(\langle\lambda_k^{r_k}\circ\lambda_{k-1}^{r_{k-1}}\circ\dots\lambda_1^{r_1}(\overline{E})\rangle)\bullet C_k(\log|z_1|,\log|z_2|,\dots\log|z_k|).$$

In this definition, $ch(\cdot)$ is the representative of the Chern character class associated by the Chern-Weil formulae to the unique connection of a hermitian bundle (without the $2i\pi$ factor), which is of type $(1,0)$ and is compatible with the hermitian structure (see [14, Lemma, p. 73]). For the proof of the following proposition, see [9, §6].

Proposition 2.11 (Burgos-Wang) *The equation*

$$d_{\mathcal{D}}\widetilde{ch}(\overline{E}) = \sum_{l=1}^{k}(-1)^{l-1}\Big(\widetilde{ch}(\partial_l^0(\overline{E})) - \widetilde{ch}(\partial_l^{-1}(\overline{E})) - \widetilde{ch}(\partial_l^1(\overline{E}))\Big)$$

holds.

In the next corollary, we write $\widetilde{\mathbf{Z}}C^H_{\cdot}(X)$ for $\mathbf{Z}C^H_{\cdot}(X)/D.(\mathcal{C}^H(X))$.

Corollary 2.12 (Burgos-Wang) *The secondary class $\widetilde{ch}$ induces a map of complexes $\widetilde{\mathbf{Z}}C^H_{\cdot}(X) \to \bigoplus_{p\geq 0} D_T^-(X,p)[2p]$.*

2.5 The description of the regulator

We recall Waldhausen's construction of a simplicial set whose homotopy groups are canonically isomorphic to the higher K-theory groups of Quillen. Let $\mathcal{A}$ be a small exact category with a fixed 0 object. Let $\mathbf{M}_n$ be the category whose objects are the elements (i,j) of the Cartesian product $\{0,1,\dots n\}\times\{0,1\dots n\}$ such that $i\leq j$ and whose sets of morphisms $Mor((i,j),(k,l))$ contain one element if $i\leq k$ and $j\leq l$ and are empty otherwise. Let π_i be the functor from the category of simplicial sets to category of abelian groups which takes the i-th homotopy group of the geometric realisation of a simplicial set. In [19, p. 182] a simplicial set $S.(\mathcal{A})$ is defined, whose set of n-simplices is the set of functors $\tau:\mathbf{M}_n\to\mathcal{A}$ such that for all i with $0\leq i\leq n$, $\tau(i,i)=0$ and for i,j,k with $1\leq i\leq j\leq k\leq n$, the sequence $\tau(i,j)\to\tau(i,k)\to\tau(j,k)$ is a short exact sequence. Call ∂_m its face maps and s_m its degeneracy maps ($0\leq m\leq n$). For all $i\geq 0$, there is a canonical isomorphism $\pi_{i+1}(S.(\mathcal{A}))\simeq K_i(\mathcal{A})$. Notice that the set of k-simplices carries in a natural way the structure of an exact category. The face and degeneracy maps are exact functors for these categorical structures. We shall define inductively an exact functor Cub from the category of k-simplices to the category of $k-1$-cubes (see [9, (4.4)]). If $k=1$ and if $\tau:\mathbf{M}_k\to\mathcal{A}$ is a 1-simplex, we define $Cub(\tau):=\tau(0,1)$. Suppose now the functor Cub is defined for $k-1$ ($k>1$) and let E be a k-simplex; there is a natural exact sequence of $k-1$-simplices

$$0\to s_{k-1}s_{k-2}\dots s_2(\tau(0,1))\to\partial_1\tau\to\partial_1\tau/(s_{k-1}s_{k-2}\dots s_2(\tau(0,1)))\to 0$$

Applying the functor Cub to this sequence, we obtain an exact sequence of $k-1$-cubes, i.e. a k-cube. Using the exactness properties of the face and degeneracy maps, we see that this also gives an exact functor from the category of k-simplices to the category of $k-1$-cubes and so we are done. We shall write $\mathbf{Z}S.(\mathcal{A})$ for the free simplicial abelian group generated by $S.(\mathcal{A})$. Recall that the k-th object of the homology-type complex $\mathcal{N}(\mathbf{Z}S.(\mathcal{A}))$ coincides with the set $\mathbf{Z}S_k(\mathcal{A})$ (see the first subsection). Extending the functor Cub by linearity, we obtain a map $\mathbf{Z}S_k(\mathcal{A})\to\mathbf{Z}C_{k-1}(\mathcal{A})$. For the proof of the following proposition, see [9, Cor. 4.8].

Proposition 2.13 (Burgos-Wang) *The map Cub induces a mapping of complexes $\mathcal{N}(\mathbf{Z}S.(\mathcal{A}))\to(\mathbf{Z}C.(\mathcal{A})/D.(\mathcal{A}))[-1]$.*

Let again X be a smooth projective complex variety. We specialize the above discussion to $\mathcal{C}^H(X)$. We shall write $S^H_.(X)$ for $S.(\mathcal{C}^H(X))$. Let H_i be the functor on the category of homology-type complexes which takes the i-th homology group of a complex. For the proof of the following theorem, see [9, §5].

Theorem 2.14 (Burgos-Wang) *Let Hu be the Hurewicz map $S^H_.(X)\to\mathbf{Z}S^H_.(X)$. Let zch be the composition of maps of homology-type complexes*

$$\mathcal{N}(\mathbf{Z}S^H_.(X))\stackrel{Cub}{\to}\widetilde{\mathbf{Z}}C^H_.(X)[-1]\stackrel{\widetilde{ch}}{\to}\bigoplus_{p\geq 0}D^-_T(X,p)[2p-1].$$

Then the composition $H_i(zch)\circ\pi_i(Hu)$ is Beilinson's regulator on $K_i(X)$.

3 Analytic torsion for hermitian k-cubes

Before giving the main statement of this section, we recall some properties of Bismut-Koehler's analytic torsion form.

3.1 The higher analytic torsion

The higher analytic torsion can be viewed as a sort of relative version of the Bott-Chern secondary classes and was defined in [5, Def. 3.8, p. 668]. Let $f : M \to S$ be a proper smooth holomorphic map of complex manifolds. We suppose that M can be endowed with a Kähler metric and we let ω be the Kähler form of a Kähler metric on M. The pair f, ω is a special case of Kähler fibration (see [5, (a), p. 649]). Furthermore, we let ξ be an f-acyclic holomorphic bundle on M and we denote by h^ξ a hermitian metric on ξ. The higher analytic torsion $T(f, \omega, h^\xi)$ is an element of $\bigoplus_{p \geq 0} D^{2p-1}(S, p)$, which depends on f, ω, ξ and h^ξ and satisfies the equality

$$(-1/2).d_{\mathcal{D}}T(f, \omega, h^\xi) = ch(f_*\xi, f_*h^\xi) - \int_{M/S} Td(Tf, h^{Tf})ch(\xi, h^\xi). \quad (8)$$

In this formula, $ch(\cdot)$ (resp. $Td(\cdot)$) is the representative of the Chern character (resp. Todd) class associated by the Chern-Weil formulae to the unique connection of a hermitian bundle, which is of type $(1, 0)$ and is compatible with the hermitian structure.

Warning. The definition of the analytic torsion form we use here coincides with Bismut-Köhler's only up to a rescaling. In [5, Rem. 3.3, p. 667], one defines an operator ϕ which acts on differential forms. If we denote by $T'(f, \omega, h^\xi)$ Bismut-Köhler's torsion, then the equality $\phi(T'(f, \omega, h^\xi)) = T(f, \omega, h^\xi)$ holds.

The equality (8) refines the Grothendieck-Riemann-Roch theorem with values in $\partial\overline{\partial}$-cohomology on the level of differential forms. We shall sometimes write $T(\omega, h^\xi)$ or $T(h^\xi)$ for $T(f, \omega, h^\xi)$, when there is no ambiguity about the underlying map or Kähler form. Consider now the following setting. Let Z be a compact Kählerian complex manifold and let Z' be a closed submanifold of Z. Choose a Kähler metric on Z and endow Z' with the restricted metric. Let ω be the Kähler form of the product metric on $M \times Z$ and let $f_Z : M \times Z \to S \times Z$ be the induced map. Similarly, let ω' be the Kähler form of the product metric on $M \times Z'$ and let $f_{Z'} : M \times Z' \to S \times Z'$ be the induced map. Call j (resp. i) the natural embedding $M \times Z' \to M \times Z$ (resp. $S \times Z' \to S \times Z$). Let $\tilde{\xi}$ be an f_Z-acyclic bundle on $M \times Z$, which is equipped with a hermitian metric $h^{\tilde{\xi}}$.

Lemma 3.1 *The equality $i^*(T(f_Z, \omega, h^{\tilde{\xi}})) = T(f_{Z'}, \omega', j^*h^{\tilde{\xi}})$ holds.*

For the proof (which follows readily from the definition of the torsion) see [5, p. 683] and also [12, p. 47]. Suppose now that ξ' and ξ'' are two vector bundles on

M which are f-acyclic and that $h^{\xi'}$ is a hermitian metric on ξ', $h^{\xi''}$ a hermitian metric on ξ''. Let $\xi := \xi' \oplus \xi''$ be the direct sum and let h^{ξ} be the hermitian metric arising as the orthogonal direct sum of $h^{\xi'}$ and $h^{\xi''}$.

Lemma 3.2 *The equality $T(\omega, h^{\xi}) = T(\omega, h^{\xi'}) + T(\omega, h^{\xi''})$ holds.*

For the proof (which again is not difficult), see [5, Th. 3.10, p. 670].

3.2 Simplicial refinement of Gillet's Riemann-Roch theorem

Let $f : X \to B$ be a smooth map of smooth projective complex varieties. Fix a Kähler metric on X. By [16, 2.7, p.117] the embedding of categories $\mathcal{C}_{f-ac}(X) \to \mathcal{C}(X)$ induces an isomorphism on the level of K-theory. In view of this fact, we shall identify the category $\mathcal{C}(X)$ with the category $\mathcal{C}_{f-ac}(X)$ until the end of the paper and work everywhere with acyclic bundles. Consider now the diagram of simplicial complexes

$$\begin{array}{ccccccc}
S^H_.(X) & \stackrel{Hu}{\to} & \mathbf{Z}S^H_.(X) & \stackrel{Cub}{\to} & \mathcal{K}(\widetilde{\mathbf{Z}}C^H_.(X)[-1]) & \stackrel{\mathcal{K}(\widetilde{ch})}{\to} & \mathcal{K}(\oplus_{p\geq 0}D^-_T(X,p)[2p-1]) \\
\downarrow\ f_* & & \downarrow\ f_* & & \downarrow\ \mathcal{K}(f_*) & & \downarrow\ \mathcal{K}(f_*) \\
S^H_.(B) & \stackrel{Hu}{\to} & \mathbf{Z}S^H_.(B) & \stackrel{Cub}{\to} & \mathcal{K}(\widetilde{\mathbf{Z}}C^H_.(B)[-1]) & \stackrel{\mathcal{K}(\widetilde{ch})}{\to} & \mathcal{K}(\oplus_{p\geq 0}D^-_T(B,p)[2p-1])
\end{array}$$

By construction, the first square and the second square on the left of the diagram commute. The third square however does not commute; we shall see that it commutes up to a natural simplicial homotopy, once a correction factor $Td(\overline{Tf})$. has been inserted on the left side of its top row. The description of this homotopy will be the content of 3.6. The next lemmata and propositions are prolegomena to 3.6.

Let Y be some smooth quasi-projective complex variety. Let $\overline{V} := (V, h)$ and $\overline{W} := (V, h')$ be two hermitian bundles on $Y \times (\mathbf{P}^1_{\mathbf{C}})^k$ with same underlying bundle. Let $p_1 : Y \times (\mathbf{P}^1_{\mathbf{C}})^{k+1} \to Y \times (\mathbf{P}^1_{\mathbf{C}})^k$ be the map defined by the rule $(y, p_1, \dots p_{k+1}) \mapsto (y, p_2, p_3, \dots p_{k+1})$. We shall write $H\{\overline{V} \Rightarrow \overline{W}\}$ for the bundle $p_1^* V$ endowed with the metric

$$g := \frac{|w_1|^2 h + |w_2|^2 h'}{|w_1|^2 + |w_2|^2}. \tag{9}$$

where w_1, w_2 are homogeneous coordinates for the first $\mathbf{P}^1_{\mathbf{C}}$ factor in $Y \times (\mathbf{P}^1_{\mathbf{C}})^{k+1}$. The hermitian bundles $\widetilde{f}_k(\langle \lambda_k^{r_k} \circ \lambda_{k-1}^{r_{k-1}} \dots \lambda_1^{r_1}(\overline{E})\rangle)$ and $\langle \lambda_k^{r_k} \circ \lambda_{k-1}^{r_{k-1}} \dots \lambda_1^{r_1}(f_*(\overline{E}))\rangle$ $(1 \leq l \leq k,\ r_l = 1, 2)$ are canonically isomorphic as bundles, but carry in general different metrics. In the following, we shall write $\overline{H}^{r_1,\dots r_k}_{1,2,\dots k}$ instead of

$H\{\widetilde{f}_k(\langle\lambda_k^{r_k}\circ\lambda_{k-1}^{r_{k-1}}\dots\lambda_1^{r_1}(\overline{E})\rangle)\Rightarrow\langle\lambda_k^{r_k}\circ\lambda_{k-1}^{r_{k-1}}\dots\lambda_1^{r_1}(f_*(\overline{E}))\rangle\}$. Now we define

$$\Pi'_k(\overline{E}) := \frac{(-1)^k}{2(k+1)!(2i\pi)^{k+1}}\sum_{(r_1,\dots r_k)\in\{1,2\}^k}\int_{(\mathbf{P}^1_{\mathbf{C}})^{k+1}} ch(\overline{H}^{r_1,\dots r_k}_{1,2,\dots k})\bullet C_{k+1}(\log|z_1|,\dots\log|z_{k+1}|).$$

By linear extension, we obtain a map $\Pi'_k:\mathbf{Z}C^H_k(X)\to\bigoplus_{p\geq 0}D^{2p-k-1}(B,p)$.

Lemma 3.3 *The map Π'_k vanishes on degenerate k-cubes.*

Proof: The proof will be deduced from the following fact: if $\overline{E}$ is a degenerate hermitian k-cube on a quasi-projective smooth variety Y, then for some l with $1\leq l\leq k$, $\langle\overline{E}\rangle$ is invariant under the automorphism of $Y\times(\mathbf{P}^1_{\mathbf{C}})^k$ given by $z_l\mapsto\frac{1}{z_l}$. To see this, suppose first that $\overline{E}$ is an exact sequence

$$0\to\overline{E}'\xrightarrow{Id.}\overline{E}'\to 0\to 0$$

where $\overline{E}'$ is a hermitian $k-1$-cube. The definition of $\langle\cdot\rangle$ then tells us that $\langle\overline{E}\rangle$ is isomorphic to the cokernel of the map $\langle\overline{E}'\rangle\to\langle\overline{E}'\rangle(1)\oplus\langle\overline{E}'\rangle(1)$ given by $e\mapsto e\otimes\sigma_0\oplus e\otimes\sigma_\infty$, where $\langle\overline{E}'\rangle$ is twisted with the tautological bundle coming from the first $\mathbf{P}^1_{\mathbf{C}}$ factor of $Y\times(\mathbf{P}^1_{\mathbf{C}})^k$. Since the automorphism $z_1\mapsto 1/z_1$ exchanges σ_0 and σ_∞, this cokernel is naturally invariant under that automorphism and we are done in that case. The general case follows directly from this one, under use of the isometry mentioned before 2.5.
Returning to the hypotheses of the lemma, suppose that $\overline{E}$ is degenerate for the index l; by construction $\lambda_k^{r_k}\circ\lambda_{k-1}^{r_{k-1}}\circ\dots\lambda_1^{r_1}(\overline{E})$ and $\lambda_k^{r_k}\circ\lambda_{k-1}^{r_{k-1}}\circ\dots\lambda_1^{r_1}(f_*\overline{E})$ are then also degenerate for the index l and thus we see that $ch(\overline{H}^{r_1,\dots r_k}_{1,2,\dots k})$ is invariant under $z_l\mapsto 1/z_l$. Since $C_{k+1}(\log|z_1|,\dots\log|z_{k+1}|)$ changes sign under that automorphism, $\int_{(\mathbf{P}^1_{\mathbf{C}})^{k+1}}ch(\overline{H}^{r_1,\dots r_k}_{1,2,\dots k})\bullet C_{k+1}(\log|z_1|,\dots\log|z_{k+1}|)$ vanishes. This completes the proof. **Q.E.D**.

In the next proposition and its proof, we write

$$\int_{(\mathbf{P}^1_{\mathbf{C}})^k} ch(\widetilde{f}_k(\langle\lambda(\overline{E})\rangle))\bullet C_{k+1}(\log|z_1|,\log|z_2|,\dots\log|z_{k+1}|)$$

for

$$\sum_{(r_1,\dots r_k)\in\{1,2\}^k}\int_{(\mathbf{P}^1_{\mathbf{C}})^k} ch(\widetilde{f}_k(\langle\lambda_k^{r_k}\circ\lambda_{k-1}^{r_{k-1}}\circ\dots\lambda_1^{r_1}(\overline{E})\rangle))\bullet C_{k+1}(\log|z_1|,\log|z_2|,\dots\log|z_{k+1}|)$$

Proposition 3.4 *The equality*

$$d_{\mathcal{D}}\circ\Pi'_k(\overline{E})+\Pi'_{k-1}\circ\delta(\overline{E})=\widetilde{ch}(f_*(\overline{E}))-$$

$$\frac{(-1)^k}{2k!(2i\pi)^k}\int_{(\mathbf{P}^1_{\mathbf{C}})^k} ch(\widetilde{f}_k(\langle\lambda(\overline{E})\rangle))\bullet C_{k+1}(\log|z_1|,\log|z_2|,\dots\log|z_{k+1}|)$$

holds.

Proof: We compute

$$d_{\mathcal{D}}\circ\Pi'_k(\overline{E})=$$

$$\frac{(-1)^k}{2(k+1)!(2i\pi)^{k+1}}\sum_{(r_1,\dots r_k)\in\{1,2\}^k}\int_{(\mathbf{P}^1_{\mathbf{C}})^{k+1}} ch(\overline{H}^{r_1,\dots r_k}_{1,2,\dots k})\bullet d_{\mathcal{D}}C_{k+1}(\log|z_1|,\dots\log|z_{k+1}|)=$$

$$\frac{(-1)^k}{2(k+1)!(2i\pi)^{k+1}}\sum_{(r_1,\dots r_k)\in\{1,2\}^k}\int_{(\mathbf{P}^1_{\mathbf{C}})^{k+1}} ch(\overline{H}^{r_1,\dots r_k}_{1,2,\dots k})\bullet\Big((-1/2)(k+1).$$

$$\sum_{j=1}^{k+1}(-1)^{j-1}(-4i\pi).(\delta_{z_j=\infty}-\delta_{z_j=0})\bullet C_k(\log|z_1|,\dots\widehat{\log|z_j|},\dots\log|z_{k+1}|)\Big)=$$

$$\frac{(-1)^k}{2(k+1)!(2i\pi)^{k+1}}\sum_{(r_1,\dots r_k\in\{1,2\}^k}\int_{(\mathbf{P}^1_{\mathbf{C}})^{k+1}} ch(\overline{H}^{r_1,\dots r_k}_{1,2,\dots k})\bullet\Big((-1/2)(k+1). \quad (10)$$

$$\sum_{j=2}^{k+1}(-1)^{j-1}(-4i\pi).(\delta_{z_j=\infty}-\delta_{z_j=0})\bullet C_k(\log|z_1|,\dots\widehat{\log|z_j|},\dots\log|z_{k+1}|)\Big)$$

$$+$$

$$\frac{(-1)^k}{2k!(2i\pi)^k}\int_{(\mathbf{P}^1_{\mathbf{C}})^k} ch(\widetilde{f}_k(\langle\lambda(\overline{E})\rangle))\bullet C_k(\log|z_1|,\log|z_2|,\dots\log|z_k|)$$

$$-$$

$$\frac{(-1)^k}{2k!(2i\pi)^k}\int_{(\mathbf{P}^1_{\mathbf{C}})^k} ch(\langle\lambda(f_*\overline{E})\rangle))\bullet C_k(\log|z_1|,\log|z_2|,\dots\log|z_k|).$$

Now we compute the expression (10):

$$\frac{(-1)^k}{2(k+1)!(2i\pi)^{k+1}}\sum_{(r_1,\dots r_k\in\{1,2\}^k}\int_{(\mathbf{P}^1_{\mathbf{C}})^{k+1}} ch(\overline{H}^{r_1,\dots r_k}_{1,2,\dots k})\bullet\Big((-1/2)(k+1).$$

$$\sum_{j=2}^{k+1}(-1)^{j-1}(-4i\pi).(\delta_{z_j=\infty}-\delta_{z_j=0})\bullet C_k(\log|z_1|,\dots\widehat{\log|z_j|},\dots\log|z_{k+1}|)\Big)=$$

$$\frac{(-1)^k}{2(k+1)!(2i\pi)^{k+1}}\sum_{(r_1,\dots r_k)\in\{1,2\}^k}\int_{(\mathbf{P}^1_{\mathbf{C}})^k}\Big((-1/2)(k+1).$$

$$\sum_{j=2}^{k+1}(-1)^{j-1}(-4i\pi).\Big(-ch(H\{\widetilde{f}_{k-1,*}(\big\langle\partial^0_{j-1}(\lambda_k^{r_k}\circ\ldots\lambda_1^{r_1}(\overline{E}))\big\rangle)\Rightarrow\big\langle\partial^0_{j-1}(\lambda_k^{r_k}\circ\ldots\lambda_1^{r_1}(f_*(\overline{E})))\big\rangle\})+$$

$$ch(H\{\widetilde{f}_{k-1,*}(\big\langle\partial^{-1}_{j-1}\lambda_k^{r_k}\circ\ldots\lambda_1^{r_1}(\overline{E})\big\rangle\oplus\big\langle\partial^{1}_{j-1}\lambda_k^{r_k}\circ\ldots\lambda_1^{r_1}(\overline{E})\big\rangle)$$

$$\Rightarrow\big\langle\partial^{-1}_{j-1}\lambda_k^{r_k}\circ\ldots\lambda_1^{r_1}(f_*(\overline{E}))\big\rangle\oplus\big\langle\partial^{1}_{j-1}\lambda_k^{r_k}\circ\ldots\lambda_1^{r_1}(f_*(\overline{E}))\big\rangle\})\Big)\bullet C_k(\log|z_1|,\ldots\log|z_k|)\Big)=$$

$$\frac{(-1)^{k-1}}{2k!(2i\pi)^k}\sum_{(r_1,\ldots r_{k-1})\in\{1,2\}^{k-1}}\int_{(\mathbf{P}^1_{\mathbf{C}})^k}$$

$$\Big(\sum_{j=1}^{k}(-1)^{j-1}\Big(-ch(H\{\widetilde{f}_{k-1,*}(\big\langle\lambda_{k-1}^{r_{k-1}}\circ\ldots\lambda_1^{r_1}(\partial^0_j(\overline{E}))\big\rangle)$$

$$\Rightarrow\big\langle\lambda_{k-1}^{r_{k-1}}\circ\ldots\lambda_1^{r_1}(\partial^0_j(f_*(\overline{E})))\big\rangle\})+ch(H\{\widetilde{f}_{k-1,*}(\big\langle\lambda_{k-1}^{r_{k-1}}\circ\ldots\lambda_1^{r_1}(\partial^{-1}_j(\overline{E}))\big\rangle\oplus\big\langle\lambda_{k-1}^{r_{k-1}}\circ\ldots\lambda_1^{r_1}(\partial^1_j(\overline{E}))\big\rangle)$$

$$\Rightarrow\big\langle\lambda_{k-1}^{r_{k-1}}\circ\ldots\lambda_1^{r_1}(\partial^{-1}_j(f_*(\overline{E})))\big\rangle\oplus\big\langle\lambda_{k-1}^{r_{k-1}}\circ\ldots\lambda_1^{r_1}(\partial^1_j(f_*(\overline{E})))\big\rangle\})\Big)\bullet C_k(\log|z_1|,\ldots\log|z_k|)\Big)=$$

$$-\Pi'_{k-1}\circ\delta_k(\overline{E}).$$

For the first equality, we used the first statement of 2.8; for the second equality, we used 2.9 and (7); for the third and fourth equality, we use the definition of the metric g in (9) and the second statement in 2.3; for the fifth equality, we use 2.6, the additivity of ch and the additivity of H; for the sixth one, we use the definition of δ. **Q**.**E**.**D**.

Consider now the map $\widetilde{f}_k : X\times(\mathbf{P}^1_{\mathbf{C}})^k\to(\mathbf{P}^1_{\mathbf{C}})^k$. We equip $X\times(\mathbf{P}^1_{\mathbf{C}})^k$ with the product metric. For a hermitian k-cube $\overline{E}$, define

$$\Pi''_k(\overline{E}):=\frac{(-1)^k}{2(k+1)!(2i\pi)^k}\sum_{(r_1,\ldots r_k)\in\{1,2\}^k}\int_{(\mathbf{P}^1_{\mathbf{C}})^k}C_{k+1}(T(\big\langle\lambda_k^{r_k}\lambda_{k-1}^{r_{k-1}}\ldots\lambda_1^{r_1}(\overline{E})\big\rangle),\log|z_1|,\log|z_2|,\ldots\log|z_k|)$$

where $T(\big\langle\lambda_k^{r_k}\lambda_{k-1}^{r_{k-1}}\ldots\lambda_1^{r_1}\overline{E}\big\rangle)$ is the higher analytic torsion of the hermitian bundle $\big\langle\lambda_k^{r_k}\lambda_{k-1}^{r_{k-1}}\ldots\lambda_1^{r_1}\overline{E}\big\rangle$ relative to the just defined Kähler fibration.

Lemma 3.5 *The map Π''_k vanishes on degenerate k-cubes.*

Proof: Suppose that $\overline{E}$ is degenerate for the index l. Then $\lambda_k^{r_k}\lambda_{k-1}^{r_{k-1}}\ldots\lambda_1^{r_1}\overline{E}$ is degenerate for the index l and the fact mentioned at the beginning of the proof of 3.3 shows that $\big\langle\lambda_k^{r_k}\lambda_{k-1}^{r_{k-1}}\ldots\lambda_1^{r_1}\overline{E}\big\rangle$ is invariant under the automorphism $A : z_l\mapsto 1/z_l$ of $X\times(\mathbf{P}^1_{\mathbf{C}})^k$. Thus the equation $A^*T(\big\langle\lambda_k^{r_k}\lambda_{k-1}^{r_{k-1}}\ldots\lambda_1^{r_1}(\overline{E})\big\rangle)=T(\big\langle\lambda_k^{r_k}\lambda_{k-1}^{r_{k-1}}\ldots\lambda_1^{r_1}(\overline{E})\big\rangle)$ holds, because A leaves the Fubini-Study Kähler form invariant. Thus $A^*C_{k+1}(T(\big\langle\lambda_k^{r_k}\lambda_{k-1}^{r_{k-1}}\ldots\lambda_1^{r_1}(\overline{E})\big\rangle)),\log|z_1|,\log|z_2|,\ldots\log|z_k|)=-C_{k+1}(T(\big\langle\lambda_k^{r_k}\lambda_{k-1}^{r_{k-1}}\ldots\lambda_1^{r_1}(\overline{E})\big\rangle)),\log|z_1|,\log|z_2|,\ldots\log|z_k|)$ and we are done. **Q**.**E**.**D**.

Theorem 3.6 *The diagram*

$$\begin{array}{ccc} \widetilde{\mathbf{Z}}C^H_\cdot(X) & \stackrel{\widetilde{ch}}{\to} & \oplus_{p\geq 0} D_T^-(X,p)[2p] \\ \downarrow f_* & & \downarrow f_* \circ Td(\overline{Tf}) \bullet (\cdot) \\ \widetilde{\mathbf{Z}}C^H_\cdot(B) & \stackrel{\widetilde{ch}}{\to} & \oplus_{p\geq 0} D_T^-(B,p)[2p] \end{array}$$

commutes up to homotopy of chain complexes. A homotopy between $\widetilde{ch} \circ f_*$ *and* $f_* \circ Td(\overline{Tf}) \bullet \widetilde{ch}$ *is given by the formula*

$$\Pi_k(\overline{E}) := \Pi'_k(\overline{E}) + \Pi''_k(\overline{E}) =$$

$$\frac{(-1)^{k+1}}{2(k+1)!(2i\pi)^{k+1}} \sum_{(r_1,\dots r_k \in \{1,2\}^k} \int_{(\mathbf{P}^1_{\mathbf{C}})^{k+1}} ch(\overline{H}^{r_1,\dots r_k}_{1,2,\dots k}) \bullet C_{k+1}(\log|z_1|, \dots \log|z_{k+1}|) +$$

$$\frac{(-1)^k}{2(k+1)!(2i\pi)^k} \sum_{(r_1,\dots r_k) \in \{1,2\}^k} \int_{(\mathbf{P}^1_{\mathbf{C}})^k} C_{k+1}(T(\langle \lambda_k^{r_k} \lambda_{k-1}^{r_{k-1}} \dots \lambda_1^{r_1} E \rangle), \log|z_1|, \log|z_2|, \dots \log|z_k|)$$

Proof: We compute

$$d_{\mathcal{D}} \Pi''_k(\overline{E}) = \tag{11}$$

$$\frac{(-1)^k}{2(k+1)!} \frac{1}{(2i\pi)^k} \int_{(\mathbf{P}^1_{\mathbf{C}})^k} d_{\mathcal{D}} C_{k+1}(T(\langle \lambda(\overline{E}) \rangle), \log|z_1|, \log|z_2|, \dots \log|z_k|) =$$

$$\frac{(-1)^k}{2(k+1)!} \frac{1}{(2i\pi)^k} \int_{(\mathbf{P}^1_{\mathbf{C}})^k} (-1/2).(k+1). \Big(\sum_{j=2}^{k+1} (-1)^{j-1} d_{\mathcal{D}}(\log|z_{j-1}|) \bullet$$

$$C_k(T(\langle \lambda(\overline{E}) \rangle), \log|z_1|, \dots \log\widehat{|z_{j-1}|}, \dots \log|z_k|) + d_{\mathcal{D}}(T(\langle \lambda(\overline{E}) \rangle)) \bullet C_k(\log|z_1|, \dots \log|z_k|) \Big) =$$

$$\frac{(-1)^k}{2(k+1)!} \frac{1}{(2i\pi)^k} .(-1/2).(k+1). \int_{(\mathbf{P}^1_{\mathbf{C}})^k} \Big(\sum_{j=2}^{k+1} (-1)^{j-1} d_{\mathcal{D}}(\log|z_{j-1}|) \wedge$$

$$C_k(T(\langle \lambda(\overline{E}) \rangle), \log|z_1|, \dots \log\widehat{|z_{j-1}|}, \dots \log|z_k|) + d_{\mathcal{D}}(T(\langle \lambda(\overline{E}) \rangle)) \bullet C_k(\log|z_1|, \dots \log|z_k|) \Big) =$$

$$(-1/2).(k+1). \frac{(-1)^k}{2(k+1)!} \frac{1}{(2i\pi)^k} \int_{(\mathbf{P}^1_{\mathbf{C}})^k} \Big(\sum_{j=2}^{k+1} (-1)^{j-1}. -4i\pi.(\delta_{z_{j-1}=\infty} - \delta_{z_{j-1}=0})$$

$$\wedge C_k(T(\langle \lambda(\overline{E}) \rangle), \log|z_1|, \dots \log\widehat{|z_{j-1}|}, \dots \log|z_k|) +$$

$$-2 \Big(ch(\widetilde{f}_{k*}(\langle \lambda(\overline{E}) \rangle) - \int_{X \times (\mathbf{P}^1_{\mathbf{C}})^k / (\mathbf{P}^1_{\mathbf{C}})^k} Td(\overline{T\widetilde{f}_k}) \bullet ch(\langle \lambda(\overline{E}) \rangle) \Big) \bullet C_k(\log|z_1|, \dots \log|z_k|) \Big) =$$

$$\frac{(-1)^k}{2(k+1)!}.(k{+}1).\frac{1}{(2i\pi)^{k-1}}\int_{(\mathbf{P}^1_{\mathbf{C}})^{k-1}}\sum_{j=2}^{k+1}(-1)^{j-1}(-1)(C_k(T(\langle\partial^0_{j-1}\lambda(\overline{E})\rangle),\log|z_1|,\dots\log|z_{k-1}|)-$$

$$C_k(T(\langle\partial^{-1}_{j-1}\lambda(\overline{E})\rangle),\log|z_1|,\dots\log|z_{k-1}|)-C_k(T(\langle\partial^{1}_{j-1}\lambda(\overline{E})\rangle),\log|z_1|,\dots\log|z_{k-1}|))+$$

$$\frac{(-1)^k}{2(k+1)!}(-1/2).(k+1)\frac{1}{(2i\pi)^k}\int_{(\mathbf{P}^1_{\mathbf{C}})^k}-2\Big(ch(\widetilde{f}_{k*}(\langle\lambda(\overline{E})\rangle)-$$

$$\int_{X\times(\mathbf{P}^1_{\mathbf{C}})^k/(\mathbf{P}^1_{\mathbf{C}})^k}Td(\overline{T\widetilde{f}_k})\bullet ch(\langle\lambda(\overline{E})\rangle)\Big)\bullet C_k(\log|z_1|,\dots\log|z_k|)=$$

$$-\Pi''_{k-1}(\delta_k(\lambda(\overline{E})))+\frac{(-1)^k}{2k!}\frac{1}{(2i\pi)^k}\Big(\int_{(\mathbf{P}^1_{\mathbf{C}})^k}ch(\widetilde{f}_{k*}(\langle\lambda(\overline{E})\rangle))\bullet C_k(\log|z_1|,\dots\log|z_k|)-$$

$$\int_{(\mathbf{P}^1_{\mathbf{C}})^k}\int_{X\times(\mathbf{P}^1_{\mathbf{C}})^k/(\mathbf{P}^1_{\mathbf{C}})^k}Td(\overline{T\widetilde{f}_k})\bullet ch(\langle\lambda(\overline{E})\rangle)\bullet C_k(\log|z_1|,\dots\log|z_k|)\Big)=$$

$$-\Pi''_{k}(\delta_k(\lambda(\overline{E})))+\frac{(-1)^k}{2k!}\frac{1}{(2i\pi)^k}\Big(\int_{(\mathbf{P}^1_{\mathbf{C}})^k}ch(\widetilde{f}_{k*}(\langle\lambda(\overline{E})\rangle))\bullet C_k(\log|z_1|,\dots\log|z_k|)$$

$$-\int_X\Big(Td(\overline{Tf})\bullet\int_{X\times(\mathbf{P}^1_{\mathbf{C}})^k/X}ch(\langle\lambda(\overline{E})\rangle)\bullet C_k(\log|z_1|,\dots\log|z_k|)\Big)\Big)=$$

$$-\Pi''_{k-1}(\delta_k(\lambda(\overline{E})))+\frac{(-1)^k}{2k!}\frac{1}{(2i\pi)^k}\int_{(\mathbf{P}^1_{\mathbf{C}})^k}ch(\widetilde{f}_{k*}(\langle\lambda(\overline{E})\rangle))\wedge C_k(\log|z_1|,\dots\log|z_k|)$$

$$-\int_X Td(\overline{Tf})\bullet\widetilde{ch}(\lambda(\overline{E})))\qquad(12)$$

For the first equality, we use the first statement of 2.8; for the second and third one, we use 2.9; for the fourth one, we use the equality (7) and the equation for the analytic torsion (8); for the fifth one, we use 3.2, 3.1 and 2.5; for the sixth one, we use the definition of δ; for the seventh one, we use the projection formula in 2.8; for the seventh one, we use the definition of the secondary classes 2.10. If we combine the equality between (11) and (12) with the equality in 3.4 and use 3.3 and 3.5 we get the result. **Q.E.D**.

If we apply the functor $\mathcal{K}$ to the homotopy of chain complexes defined in the last theorem and compose it with Cub, we get a simplicial homotopy between the maps $\mathcal{K}(\widetilde{ch})\circ\mathcal{K}(Cub)\circ Hu\circ f_*$ and $f_*\circ\mathcal{K}(Td(\overline{Tf})\bullet(\cdot))\circ\mathcal{K}(\widetilde{ch})\circ\mathcal{K}(Cub)\circ Hu$. Thus after application of the functor π_i to both maps, we get Gillet's Riemann-Roch theorem.
Let $u_1,\dots u_k\in\bigoplus_{p\geq 0}D^{2p-1}(X,p)$. Applying the definition of the $\bullet$ product, we obtain the following expression for C_k:

$$C_k(u_1,\dots u_k):=\sum_{i=1}^k\sum_{\sigma\in\mathcal{S}_k}(-1)^{i-1}(-1)^{sgn(\sigma)}u_{\sigma(1)}\wedge\partial u_{\sigma(2)}\wedge\dots\partial u_{\sigma(i)}\wedge\overline{\partial}u_{\sigma(i+1)}\wedge\dots\overline{\partial}u_{\sigma(k)}$$

(compare with [7, p. 120]). Using this equality, one can make the expression for Π_k completly explicit. The map Π_k might be considered as a generalisation of the Bismut-Köhler analytic torsion form to cubes of vector bundles. Notice also that if B is a point, then $\Pi_k(\overline{E})$ is a real number, which is equal to the Ray-Singer analytic torsion if $k = 1$ and is 0 if k is even.

Remark. If one tries to generalize the last theorem to quasi-projective varieties smooth over smooth quasi-projective bases which are not necesserally compact, one runs into serious analytic difficulties related to logarithmic singularities. If the variety X is non-compact, the complex computing the Deligne-Beilinson cohomology which was described in subsection 2.3 has to be restricted so as to contain only differential forms with logarithmic singularities along the boundary of some good compactification. Since Π_k has to lie in the corresponding complex with logarithmic singularities on B, one is lead to the question of the type of the singularities of the higher analytic torsion forms, if one compactifies X and B simultaneously. In his article [2] (see also [3]), Bismut investigates this question for the Ray-Singer torsion (the degree zero part of the analytic torsion form), in the case of the at most quadratic degeneration of f along the boundary of the compactification. Unfortunately, for degree reasons, the use of the full analytic torsion form is unavoidable in the definition of Π_k'' and one would need a result analogous to the main result of [2] for the full torsion form to be in a position to tackle the proof of the analog of our theorem in a relative context. Finally, notice that one could probably obtain the analog of the just discussed generalisation in the context of analytic Deligne cohomology, where logarithmic singularities do not play a role.

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