On the determinant bundles of abelian schemes

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Abstract

Let $\pi : \mathcal{A} \to S$ be an abelian scheme over a scheme S which is quasi-projective over an affine noetherian scheme and let $\mathcal{L}$ be a symmetric, rigidified, relatively ample line bundle on $\mathcal{A}$. We show that there is an isomorphism

$$\det(\pi_*\mathcal{L})^{\otimes 24} \simeq \left(\pi_*\omega_{\mathcal{A}}^{\vee}\right)^{\otimes 12d}$$

of line bundles on S, where d is the rank of the (locally free) sheaf $\pi_*\mathcal{L}$. We also show that the numbers 24 and $12d$ are sharp in the following sense: if $N > 1$ is a common divisor of 12 and 24, then there are data as above such that

$$\det(\pi_*\mathcal{L})^{\otimes(24/N)} \not\simeq \left(\pi_*\omega_{\mathcal{A}}^{\vee}\right)^{\otimes(12d/N)}.$$

1. Introduction

Let $\pi : \mathcal{A} \to S$ be a abelian scheme, where S is a scheme which is quasi-projective over an affine noetherian scheme. We denote as usual by $\omega_{\mathcal{A}}$ the determinant of the sheaf of differentials of π. Let $\mathcal{L}$ be a line bundle on $\mathcal{A}$. Let $\epsilon : S \to \mathcal{A}$ be the zero-section and suppose that the line bundle $\epsilon^*\mathcal{L}$ is the trivial line bundle. Suppose furthermore that there is an isomorphism $[-1]^*\mathcal{L} \simeq \mathcal{L}$ and that $\mathcal{L}$ is ample relatively to π. In this situation, Chai and Faltings prove the following result (see [CF90, Th. 5.1, p. 25]):

Theorem 1.1 (Chai-Faltings). *There is an isomorphism* $\det(\pi_*\mathcal{L})^{\otimes 8d^3} \simeq \left(\pi_*\omega_{\mathcal{A}}^{\vee}\right)^{\otimes 4d^4}$ *of line bundles on S.*

Here d is the rank of the (locally free) sheaf $\pi_*\mathcal{L}$. This is a refinement of a special case of the "formule clé" considered by Moret-Bailly in his monograph [MB85].

In [CF90, p. 27], Chai and Faltings state that it is nevertheless likely that the factor d^3 can be cancelled on both sides of the above isomorphism or in other words that it is likely that there is an isomorphism

$$\det(\pi_*\mathcal{L})^{\otimes 8} \simeq \left(\pi_*\omega_{\mathcal{A}}^{\vee}\right)^{\otimes 4d}. \tag{1}$$

Let us introduce the line bundle

$$\Delta(\mathcal{L}) := \det(\pi_*\mathcal{L})^{\otimes 2} \otimes \pi_*\omega_{\mathcal{A}}^{\otimes d}.$$

The existence of the isomorphism (1) is the statement that $\Delta(\mathcal{L})^{\otimes 4}$ is trivial.

The aim of this text is to present the proof of the following statements about $\Delta(\mathcal{L})$:

Theorem 1.2. (a) *There is an isomorphism* $\Delta(\mathcal{L})^{\otimes 12} \simeq \mathcal{O}_S$.

(b) *For every $g \geqslant 1$, there exist data $\pi : \mathcal{A} \to S$ and $\mathcal{L}$ as above such that $\dim(\mathcal{A}/S) = g$ and such that $\Delta(\mathcal{L})$ is of order 12 in the Picard group of S.*

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The following corollary follows immediately from Theorem 1.2 (a) and Theorem 1.1.

Corollary 1.3. *If* $(3, d) = 1$ *then* $\Delta(\mathcal{L})^{\otimes 4}$ *is trivial.*

Notice that Theorem 1.2 (b) in particular implies that the exponent 4 surmised by Chai and Faltings is not the right one (it has to be replaced by the exponent 12). The corollary says that the exponent 4 is nevertheless the right one when $(3, d) = 1$.

The fact that $\Delta(\mathcal{L})$ is a torsion line bundle is a consequence of the Grothendieck-Riemann-Roch theorem. This was shown by Moret-Bailly and Szpiro in the Appendix 2 to Moret-Bailly's monograph [MB85] and also by Chai in his thesis (see [Chai85, Chap. V, par. 3, th. 3.1, p. 209]). The link between the Grothendieck-Riemann-Roch theorem and the fact that $\Delta(\mathcal{L})$ is a torsion line bundle was already known to Mumford in the early sixties (private communication between Chai and the authors). If S is a smooth quasi-projective scheme over $\mathbb{C}$, then 1.2 (a) is contained in a theorem of Kouvidakis (see [K, Th. A]). The method of proof of the theorem of Kouvidakis is analytic and is based on the study of the transformation formulae of theta functions. It extends earlier work by Moret-Bailly (see [MB90]), who considered the case where $d = 1$. The result of Kouvidakis was extended by Polishchuk to more general bases S in [P]. Polishchuk's proof is a refinement of Chai and Faltings proof of Theorem 1.1; this last proof is not based on the Riemann-Roch theorem. The Theorem 0.1 in [P] shows in particular that there exists a constant $N(g)$, which depends only on the relative dimension g of $\mathcal{A}$ over S, such that $\Delta(\mathcal{L})^{\otimes N(g)}$ is trivial. The Theorems 0.1, 0.2, 0.3 of [P] give various bounds for $N(g)$, which depend on d, g and on the residue characteristics of S. In this context, the content of Theorem 1.2 is that $N(g) = 12$ is a possible choice and that for each $g \geqslant 1$, it is the best possible choice.

A key input in Polishchuk's refinement of the proof of Chai and Faltings is a formula describing the behaviour of $\Delta(\mathcal{L})$ under isogenies of abelian schemes ([P, Th. 1.1]; see also the end of section 2), which generalises an earlier formula by Moret-Bailly (see [MB85, VIII, 1.1.3, p. 188]), who considered the case $d = 1$. The proof of Theorem 1.2 (a) presented in this paper combines Polishchuk's isogeny formula and a refinement of the Grothendieck-Riemann-Roch theorem, called the Adams-Riemann-Roch theorem (see section 2). In spirit, it is close to Mumford's original approach. Our method can also be related to Moret-Bailly's proof of the "formule clé" in positive characteristic; see the first remark at the end of the text. Our proof of Theorem 1.2 (b) is based on a lemma of Polishchuk and on two constructions of Mumford.

The plan of the article is as follows. The second section contains some preliminaries to the proof; these preliminaries are the Adams-Riemann-Roch theorem and the two results of Polishchuk mentionned in the last paragraph. The proof itself is contained in the third section.

Notation and conventions. Suppose that $\mathcal{M}$ is a line bundle on a group scheme $\mathcal{C}$ over a base B, with zero-section $\epsilon : B \to \mathcal{C}$. We shall say that $\mathcal{M}$ is rigidified if $\epsilon^*\mathcal{M}$ is the trivial line bundle. We shall say that $\mathcal{M}$ is symmetric, if $[-1]^*\mathcal{M} \simeq \mathcal{M}$. Suppose that x is an element of an abelian group G and that k is a positive integer; we shall say that x is k^∞-torsion element of G if there exists an integer $n \geqslant 0$ such that $k^n \cdot x = 0$ in G. If G is a group or a group functor and k is a strictly positive integer, we shall write $[k]$ for the map $G \to G$ such that $[k](x) = x + \cdots + x$ (k-times) for every $x \in G$.

2. Preliminaries

In this section, "scheme" will be short for "noetherian scheme".

2.1 The Adams-Riemann-Roch theorem

In this section, we first describe the special case of the Adams-Riemann-Roch theorem that we shall need. We then go on to describe Polishchuk's isogeny formula.

If Y is a scheme, we shall write as usual $K_0(Y)$ for the Grothendieck group of coherent locally free sheaves. The tensor product of locally free sheaves descends to a bilinear pairing on $K_0(Y)$, which makes it into a commutative ring. If $f : X \to Y$ is a morphism of schemes, the pull-back of $\mathcal{O}_Y$-modules induces a ring morphism $f^* : K_0(Y) \to K_0(X)$. As a ring, $K_0(Y)$ is endowed with a family $(\psi^k)_{k\in\mathbb{N}^*}$ of (ring) endomorphisms, called the Adams operations. They have the property that $\psi^k(\mathcal{M}) = \mathcal{M}^{\otimes k}$ in $K_0(Y)$ for every line bundle $\mathcal{M}$ on Y. Furthermore, if $f : X \to Y$ is a scheme morphism as before, then $f^* \circ \psi^k = \psi^k \circ f^*$. The Adams operations are uniquely determined by these two last properties and by the fact that they are ring endomorphisms.

We shall also need Bott's "cannibalistic" classes. We shall denote thus a family of operations $(\theta^k)_{k\in\mathbb{N}^*}$, each of which associates elements of $K_0(Y)$ to coherent locally free sheaves on Y. They have the following three properties, which determine them uniquely. For every line bundle $\mathcal{M}$ on Y, we have

$$\theta^k(\mathcal{M}) = 1 + \mathcal{M} + \mathcal{M}^{\otimes 2} + \cdots + \mathcal{M}^{\otimes(k-1)}$$

in $K_0(Y)$. If

$$0 \to E' \to E \to E'' \to 0$$

is an exact sequence of coherent locally free sheaves on Y, then $\theta^k(E')\theta^k(E'') = \theta^k(E)$. And finally, if $f : X \to Y$ is a morphism of schemes, then $f^*(\theta^k(E)) = \theta^k(f^*E)$, for every coherent locally free sheaf on Y. About the operations θ^k, the following lemma holds. Suppose for the time of the lemma that Y is quasi-projective over an affine scheme.

Lemma 2.1. *For any coherent locally free sheaf E on Y, the element $\theta^k(E)$ is invertible in the ring $K_0(Y)[\frac{1}{k}]$.*

Proof. See [R, Par. 4, Prop. 4.2] (for lack of a standard reference). □

Let $f : X \to Y$ be a flat and projective morphism of schemes. We may consider the Grothendieck group $K_0^{\mathrm{ac}}(X)$ of f-acyclic coherent locally free sheaves on X, i.e. coherent locally free sheaves E such that $R^i f_* E = 0$ for every $i > 0$. There is a unique morphism of groups $f_* : K_0^{\mathrm{ac}}(X) \to K_0(Y)$ such that $f_*(E) = R^0 f_* E$ for every coherent locally free sheaf on X. A theorem of Quillen (see [Q, Par. 4, Th. 3, p. 108]) now implies that the natural map $K_0^{\mathrm{ac}}(X) \to K_0(X)$ is an isomorphism. Hence we obtain a morphism $f_* : K_0(X) \to K_0(Y)$. This morphism satisfies the projection formula: for all $y \in K_0(Y)$ and all $x \in K_0(X)$, the identity $f_*(f^*(y) \otimes x) = y \otimes f_*(x)$ holds.

Let us now consider a smooth and projective morphism of schemes $f : X \to Y$, where Y is quasi-projective over an affine scheme. Let Ω be the sheaf of differentials associated to f; it is a locally free sheaf on X. In this situation, the Adams-Riemann-Roch theorem is the following statement:

Theorem 2.2 (Grothendieck et al.). *For any $x \in K_0(X)[\frac{1}{k}]$, the equality*

$$\psi^k(f_*(x)) = f_*(\theta^k(\Omega)^{-1}\psi^k(x))$$

holds in $K_0(Y)[\frac{1}{k}]$.

For a proof of the Adams-Riemann-Roch theorem, see [FL, V, par. 7, Th. 7.6, p. 149].

2.2 Some results of Polishchuk on $\Delta(\mathcal{L})$

Let T be any base scheme and let $\kappa : \mathcal{B} \to T$ be an abelian scheme. Let $\alpha : \mathcal{B} \to \mathcal{B}$ be a finite and flat T-homomorphism of group schemes. Let $\mathcal{M}$ be a symmetric and rigidified line bundle on $\mathcal{B}$, which is ample relatively to κ. Suppose that the sheaf $\kappa_*\mathcal{M}$ has strictly positive rank. The following special case of Polishchuk's isogeny formula [P, Th. 1.1] plays a crucial role in the proof of Theorem 1.2 a):

Theorem 2.3 (Polishchuk). (a) *Let* $n := (12, \deg(\alpha))$. *There is an isomorphism*

$$\det(\kappa_*(\alpha^*\mathcal{L}))^{\otimes 2n} \simeq \det(\kappa_*(\mathcal{L}))^{\otimes(2n\cdot\deg(\alpha))}.$$

(b) *Let* $m := (3, \deg(\alpha))$. *Suppose that* $\deg(\alpha)$ *is odd and that* $\mathrm{rk}(\kappa_*\mathcal{L})$ *is even. There is then an isomorphism*

$$\det(\kappa_*(\alpha^*\mathcal{L}))^{\otimes m} \simeq \det(\kappa_*(\mathcal{L}))^{\otimes(m\cdot\deg(\alpha))}.$$

The two following lemmata are needed in the proof of Theorem 1.2 (b).

Lemma 2.4 (Polishchuk). *Suppose that* $\dim(\mathcal{B}/T) = 1$ *and that* $\mathcal{M} = \mathcal{O}(O_{\mathcal{B}}) \otimes \omega_{\mathcal{B}}$. *Then for every* $r \geqslant 1$, *there is an isomorphism*

$$\Delta(\mathcal{M}^{\otimes r}) \simeq \omega_{\mathcal{B}}^{\otimes(r^2+2)}.$$

Here $O_{\mathcal{B}}$ is the image of the unit section of $\mathcal{B}/T$. Notice that the image of the unit section is a Cartier divisor and that its normal bundle is isomorphic to the restriction of $\omega_{\mathcal{B}}$ via the unit section (this is a consequence of the fact that κ is smooth; see for instance [FL, IV, par. 3, Lemma 3.8]). This implies that the pull-back of $\mathcal{M}$ via the unit section is the trivial line bundle (see [FL, IV, par. 3, Prop. 3.2 (b)]).

For the proof of the Lemma 2.4, see [P, Prop. 5.1].

Let now $\kappa' : \mathcal{B}' \to T$ be an abelian scheme and let $\mathcal{M}'$ be a symmetric and rigidified line bundle on $\mathcal{B}'$, which is ample relatively to κ'. Let p (resp. p') be the natural projection $\mathcal{B} \times_B \mathcal{B}' \to \mathcal{B}$ (resp. $\mathcal{B} \times_B \mathcal{B}' \to \mathcal{B}'$). Let m (resp. m') be the rank of $\kappa_*\mathcal{M}$ (resp. $\kappa'_*\mathcal{M}'$).

Lemma 2.5. *There is an isomorphism*

$$\Delta(p^*\mathcal{M} \otimes p'^*\mathcal{M}') \simeq \Delta(\mathcal{M})^{\otimes m'} \otimes \Delta(\mathcal{M}')^{\otimes m}$$

Proof. Left to the reader (use the Künneth formula). □

3. The proof

3.1 The isomorphism $\Delta(\mathcal{L})^{\otimes 12} \simeq \mathcal{O}_S$

In this subsection, we shall prove assertion (a) in Theorem 1.2. We shall now apply the Adams-Riemann-Roch theorem 2.2 to abelian schemes. We work in the situation of the introduction. We may also suppose without restriction of generality that $d \geqslant 1$. Let $k \geqslant 2$. Let g be the relative dimension of $\mathcal{A}$ over S. Recall that the theorem of the cube (see [MB85, Par. 5.5, p. 29]) implies that $[k]^*\mathcal{L} \simeq \mathcal{L}^{\otimes k^2}$. Write Ω for the sheaf of differentials of π. We compute in $K_0(S)[\frac{1}{k}]$:

$$\begin{aligned}\psi^{k^2}(\pi_*\mathcal{L}) &= \pi_*(\theta^{k^2}(\Omega)^{-1}\psi^{k^2}(\mathcal{L})) = R\pi_*(\theta^{k^2}(\Omega)^{-1}\mathcal{L}^{\otimes k^2}) \\ &= \pi_*(\theta^{k^2}(\Omega)^{-1}[k]^*(\mathcal{L})) = \pi_*([k]^*\mathcal{L})\theta^k(\pi_*\Omega)^{-1}\end{aligned}$$

where we have used the theorem of the cube, the projection formula and the fact that $\pi^*\pi_*\Omega = \Omega$. In other words, we have the identity

$$\theta^{k^2}(\pi_*\Omega)\psi^{k^2}(\pi_*\mathcal{L}) = \pi_*([k]^*\mathcal{L}) \qquad (2)$$

in $K_0(S)[\frac{1}{k}]$. Let us now introduce the (truncated) Chern character

$$\mathrm{ch} : K_0(S)[\frac{1}{k}] \to \mathbb{Z}[\frac{1}{k}] \oplus \mathrm{Pic}(S)[\frac{1}{k}],$$

which is defined by the formula

$$\mathrm{ch}(s/k^t) := \mathrm{rank}(s)/k^t \oplus \det(s)^{1/k^t}$$

for every $s \in K_0(S)$ and $t \in \mathbb{N}$. Let us introduce the pairing

$$(r/k^t, m/k^l) \bullet ((r')^{1/k^{t'}}, (m')^{1/k^{l'}}) := r \cdot r'/k^{t+t'} \ \oplus \ (m')^{r/k^{t+l'}} \otimes m^{r'/k^{t'+l}}$$

in the group $\mathbb{Z}[\frac{1}{k}] \oplus \mathrm{Pic}(S)[\frac{1}{k}]$. The pairing $\bullet$ makes this group into a commutative ring. The properties of the determinant show that the Chern character is a ring morphism. We now apply the Chern character to the identity (2). As we shall compute in the ring $\mathbb{Z}[\frac{1}{k}] \oplus \mathrm{Pic}(S)[\frac{1}{k}]$, we switch from multiplicative notation ("$\otimes$") to additive notation ("$+$") in the group $\mathrm{Pic}(S)$. For the purposes of computation, we may suppose without loss of generality that $\pi_*\Omega = \omega_1 + \dots \omega_g$ in $K_0(S)$, where $\omega_1, \dots, \omega_g$ are line bundles. We compute

$$\begin{aligned}\mathrm{ch}(\theta^{k^2}(\pi_*\Omega)) &= \big(k^2 + \frac{k^2(k^2-1)}{2}\det(\omega_1)\big) \bullet \dots \bullet \big(k^2 + \frac{k^2(k^2-1)}{2}\det(\omega_g)\big) \\ &= k^{2g} + \frac{k^2(k^2-1)k^{2g-2}}{2}\det(\pi_*\Omega)\end{aligned}$$

and

$$\begin{aligned}\mathrm{ch}(\theta^{k^2}(\pi_*\Omega))\mathrm{ch}(\psi^{k^2}(\pi_*\mathcal{L})) &= \big(k^{2g} + \frac{k^2(k^2-1)k^{2g-2}}{2}\det(\pi_*\Omega)\big) \bullet \big(d + k^2\det(\pi_*\mathcal{L})\big) \\ &= k^{2g}d + k^{2g+2}\det(\pi_*\mathcal{L}) + \frac{dk^2(k^2-1)k^{2g-2}}{2}\det(\pi_*\Omega).\end{aligned}$$

On the other hand, we have

$$\mathrm{ch}(\pi_*([k]^*\mathcal{L})) = dk^{2g} + \det(\pi_*[k]^*\mathcal{L}).$$

Here we have used the fact that the degree of the isogeny given by multiplication by k on $\mathcal{A}$ is k^{2g} and the fact that the rank of $\pi_*[k]^*\mathcal{L}$ is dk^{2g} (see [M70, III, par. 12, Th. 2, p. 121]). Thus, (2) leads to the equality

$$k^{2g}d + k^{2g+2}\det(\pi_*\mathcal{L}) + \frac{dk^2(k^2-1)k^{2g-2}}{2}\det(\pi_*\Omega) = dk^{2g} + \det(\pi_*[k]^*\mathcal{L})$$

in $\mathbb{Z}[\frac{1}{k}] \oplus \mathrm{Pic}(S)[\frac{1}{k}]$. Multiplying by k^{-2g} and specializing to $\mathrm{Pic}(S)[\frac{1}{k}]$, we get

$$k^2\det(\pi_*\mathcal{L}) + \frac{d(k^2-1)}{2}\det(\pi_*\Omega) = k^{-2g}\det(\pi_*[k]^*\mathcal{L})$$

in $\mathrm{Pic}(S)[\frac{1}{k}]$. Now Theorem 2.3 (a) shows that

$$2 \cdot k^{-2g}\det(\pi_*[k]^*\mathcal{L}) = 2 \cdot \det(\pi_*\mathcal{L}) \tag{3}$$

in $\mathrm{Pic}(S)[\frac{1}{k}]$. We deduce from the last two equalities that

$$(k^2-1) \cdot \big(2 \cdot \det(\pi_*\mathcal{L}) + d \cdot \det(\pi_*\Omega)\big) = 0. \tag{4}$$

in $\mathrm{Pic}(S)[\frac{1}{k}]$. In other words, $\Delta(\mathcal{L})^{\otimes(k^2-1)}$ is a k^∞-torsion line bundle. If we specialise to $k=2$, we see that $\Delta(\mathcal{L})^{\otimes 3}$ is a 2^∞-torsion line bundle. If we specialise to $k=3$, we see that $\Delta(\mathcal{L})^{\otimes 8}$ is a 3^∞-torsion line bundle. Hence $\Delta(\mathcal{L})^{\otimes 24}$ is a trivial line bundle.

Suppose now that d is odd. Theorem 1.1 says that $\Delta(\mathcal{L})^{\otimes 4d^3}$ is a trivial line bundle. Hence $\Delta(\mathcal{L})^{\otimes(24,4d^3)}$ is a trivial line bundle. Since $(24, 4d^3)$ divides 12, this implies that $\Delta(\mathcal{L})^{\otimes 12}$ is a trivial line bundle.

Suppose now that d is even. Theorem 2.3 (b) then shows that the equality

$$k^{-2g}\det(\pi_*[k]^*\mathcal{L}) = \det(\pi_*\mathcal{L})$$

holds in $\mathrm{Pic}(S)[\frac{1}{k}]$ (this equality refines (3)). Proceeding as we did after the equality (4), we obtain the equality

$$\frac{k^2-1}{2}\big(2\cdot\det(\pi_*\mathcal{L}) + d\cdot\det(\pi_*\Omega)\big) = 0.$$

in $\mathrm{Pic}(S)[\frac{1}{k}]$. In other words, $\Delta(\mathcal{L})^{\otimes(k^2-1)}$ is a k^∞-torsion line bundle if k is even and $\Delta(\mathcal{L})^{\otimes(k^2-1)/2}$ is a k^∞-torsion line bundle if k is odd. If we specialise to $k=3$, we see that $\Delta(\mathcal{L})^{\otimes 4}$ is a 3^∞-torsion line bundle. We saw above that $\Delta(\mathcal{L})^{\otimes 3}$ is a 2^∞-torsion line bundle and so we obtain again that $\Delta(\mathcal{L})^{\otimes 12}$ is a trivial line bundle.

This concludes the proof of the assertion (a) of Theorem 1.2.

Remark 1. Suppose that S is a scheme over $\mathbb{F}_p$, for some prime number p and that $d=1$. Moret-Bailly then proves that the line bundle $\Delta(\mathcal{L})^{\otimes(p^2-1)p^{2g+2}}$ is trivial (see [MB85, chap. VIII, par. 2, Th. 2.1, p. 193]). The equality (4) for $k=p$ is a variant of this. Notice furthermore that for any vector bundle E on S, we have $\psi^p(E) = F_S^*(E)$, where F_S is the absolute Frobenius endomorphism of S. Moret-Bailly's proof is based on the study of the behaviour of $\Delta(\mathcal{L})$ under base-change by F_S and on the case $d=1$ of the isogeny formula. In this sense, our proof of (2) over a general base can be considered as an extension of Moret-Bailly's proof of (2) in positive characteristic.

3.2 Sharpness

In this subsection, we shall prove the assertion (b) in Theorem 1.2. We fix an affine noetherian base scheme B. All schemes and morphisms of schemes in this subsection will be relative to this base scheme. Furthermore, all schemes will be locally noetherian. We first recall a result of Mumford. Consider the following set of data:

- $\delta, g \in \mathbb{N}^*$;
- T a scheme;
- $\kappa : \mathcal{B} \to T$ a projective abelian scheme of relative dimension g;
- $\lambda : \mathcal{B} \to \mathcal{B}^\vee$ a polarisation over T of degree δ^2;
- a linear rigidification $\mathbb{P}(\kappa_*(L^\Delta(\lambda)^{\otimes 3})) \simeq \mathbb{P}_T^{6^g\cdot\delta-1}$.

Here $L^\Delta(\lambda)$ is the pull-back of the Poincaré line bundle on $\mathcal{B}\times_T\mathcal{B}^\vee$ via the map $\mathrm{Id}\times_T\lambda : \mathcal{B}\to\mathcal{B}\times_T B^\vee$. We shall call the scheme T the ground scheme of the set of data. If we are given two sets of data as above, there is an obvious notion of isomorphism between them. If we are given two sets of data with the same ground scheme T, an isomorphism between the two sets of data will be called a T-isomorphism if it restricts to the identity on T. For each scheme T, we shall write $\mathcal{H}_{g,\delta}(T)$ for the set of T-isomorphism classes of sets of data whose ground scheme is T. If $T'\to T$ is a morphism of schemes, the obvious base-change of sets of data from T to T' induces a map $\mathcal{H}_{g,\delta}(T)\to\mathcal{H}_{g,\delta}(T')$. One thus obtains a contravariant functor from the category of (locally noetherian) schemes to the category of sets. For more details, see [MFK, chap. 7, par. 2].

THEOREM 3.1 (Mumford). *The functor $\mathcal{H}_{g,\delta}$ is representable by a quasi-projective scheme over B.*

For the proof, see [MFK, Prop. 7.3, chap. 7, par. 2]. We shall refer to the scheme representing $\mathcal{H}_{g,\delta}$ as $H_{g,\delta}$.

Let now $\kappa_{1,1} : \mathcal{B}_{1,1} \to H_{1,1}$ be the universal abelian scheme over $H_{1,1}$. Let

$$\mathcal{L}_{\mathcal{B}_{1,1}} := \mathcal{O}(O_{\mathcal{B}_{1,1}})^{\otimes 3}\otimes\omega_{\mathcal{B}_{1,1}/H_{1,1}}^{\otimes 3}.$$

Here again $O_{\mathcal{B}_{1,1}}$ is the image of the unit section of $\mathcal{B}_{1,1} \to H_{1,1}$.

Proposition 3.2. *If* $B = \operatorname{Spec} \mathbb{C}$ *then the line bundle* $\Delta(\mathcal{L}_{\mathcal{B}_{1,1}})$ *is of order* 12 *in* $\operatorname{Pic}(H_{1,1})$.

Proof. Notice that there is a natural action of the group scheme PGL_6 on $H_{1,1}$, defined as follows. Consider a set of data $\mathfrak{D}$ of the type described at the beginning of the subsection; let $a \in \mathrm{PGL}_6(T)$; to a corresponds by construction an automorphism A of $\mathbb{P}^5_T$; we let a send $\mathfrak{D}$ on the set of data $\mathfrak{D}$ with its linear rigidification composed with A. This defines an action of the group functor PGL_6 on the functor $\mathcal{H}_{1,1}$ and hence an action of PGL_6 on $H_{1,1}$. Since $H_{1,1}$ is a fine moduli-space, this PGL_6-equivariant structure canonically lifts to a PGL_6-equivariant structure on $\mathcal{B}_{1,1}$, such that the morphism $\kappa_{1,1}$ is PGL_6-equivariant.

Let now $k \geqslant 1$ be the order of $\Delta(\mathcal{L}_{\mathcal{B}_{1,1}})$ in $\operatorname{Pic}(H_{1,1})$ (which is finite by Theorem 1.2 (a)). Notice that Lemma 2.4 shows that there is an isomorphism

$$\Delta(\mathcal{L}_{\mathcal{B}_{1,1}}) \simeq \kappa_{1,1*}\omega_{\mathcal{B}}^{11}.$$

We thus see that there is an isomorphism

$$\kappa_{1,1*}\omega_{\mathcal{B}_{1,1}}^{\otimes 11\cdot k} \simeq \mathcal{O}. \tag{5}$$

Fix such an isomorphism. This is tantamount to giving a trivialising section s of $\kappa_{1,1*}\omega_{\mathcal{B}_{1,1}}^{\otimes 11\cdot k}$. Notice now that the reduced closed subscheme $H_{1,1,\mathrm{red}}$ underlying $H_{1,1}$ carries a PGL_6-action such that the closed immersion $H_{1,1,\mathrm{red}} \hookrightarrow H_{1,1}$ is equivariant (this follows from the definition of a group-scheme action, from [EGA, I, par. 5, cor. 5.1.8] and from the fact that $\mathrm{PGL}_6 \times H_{1,1}$ is reduced, since PGL_6 is smooth over $\mathbb{C}$). Furthermore, there are no non-trivial characters $\mathrm{PGL}_6 \to \mathbb{G}_m$. Thus the restriction of the section s to $H_{1,1,\mathrm{red}}$ is PGL_6-invariant (for this, see [MFK, Prop. 1.4, chap. I, par. 3, p. 33]).

Now consider an elliptic curve $\kappa : E \to \operatorname{Spec} \mathbb{C}$, which has complex multiplication by $\mathbb{Z}[j]$, where $j = -1/2 + i\sqrt{3}/2$ is a primitive 3rd root of unity. The element j then acts on $\kappa_*\omega_E$ by multiplication by either j or $\overline{j}$ (this can be seen by considering the complex uniformisation of $E(\mathbb{C})$). Choose an arbitrary rigidification of $\mathbb{P}(\kappa_*(\mathcal{O}(O_E)^{\otimes 3}))$. The elliptic curve E together with its rigidification defines an element P of $H_{1,1,\mathrm{red}}(\mathbb{C})$, since $\operatorname{Spec} \mathbb{C}$ is reduced. Since the section s is PGL_6-invariant on $H_{1,1,\mathrm{red}}$, the element $s(P) \in \kappa_*\omega_E^{\otimes 11\cdot k}$ must satisfy the equation $s(P) = j^{11\cdot k}\cdot s(P)$ or the equation $s(P) = \overline{j}^{11\cdot k} \cdot s(P)$ and hence $3|k$. A similar reasoning with an elliptic curve with complex multiplication by the Gaussian integers $\mathbb{Z}[i]$ shows that $4|k$. Hence 12 divides k; on the other hand Theorem 1.2 (a) shows that k divides 12. Hence $k = 12$ and this concludes the proof. □

Remark 2. The idea to use rigidifications in the context of the Theorem 1.1 is due to Chai and Faltings; see [CF90, proof of th. 5.1]. The idea to use elliptic curves with complex multiplication to compute orders in Picard groups is due to Mumford; see [M63, par. 6].

We shall now prove Theorem 1.2 (b). Let g be a positive natural number. Let $B = \operatorname{Spec} \mathbb{C}$ and choose an elliptic curve E over $\mathbb{C}$. Let $\mathcal{L}_E := \mathcal{O}(O_E)$ be the line bundle associated to the zero-section. Consider E as an isotrivial abelian scheme over $H_{1,1}$. Consider the abelian scheme $\mathcal{A} := \mathcal{B}_{1,1} \times_{H_{1,1}} E^{g-1}$ over $S = H_{1,1}$ and the line bundle $\mathcal{L} := \mathcal{L}_{\mathcal{B}_{1,1}} \boxtimes \mathcal{L}_E^{\boxtimes(g-1)}$ on $\mathcal{A}$ (here $\boxtimes$ refers to the exterior tensor product). It is a consequence of Proposition 3.2 and Lemma 2.5 that $\Delta(\mathcal{L})$ is of exact order 12 in $\operatorname{Pic}(S)$.

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