On the group of purely inseparable points of an abelian variety defined over a function field of positive characteristic

Damian RÖSSLER*

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Abstract

Let K be the function field of a smooth and proper curve S over an algebraically closed field k of characteristic $p > 0$. Let A be an ordinary abelian variety over K. Suppose that the Néron model $\mathcal{A}$ of A over S has some closed fibre $\mathcal{A}_s$, which is an abelian variety of p-rank 0.

We show that in this situation the group $A(K^{\mathrm{perf}})$ is finitely generated (thus generalizing a special case of the Lang-Néron theorem). Here $K^{\mathrm{perf}} = K^{p^{-\infty}}$ is the maximal purely inseparable extension of K. This result implies in particular that the "full" Mordell-Lang conjecture is verified in the situation described above. The proof relies on the theory of semistability (of vector bundles) in positive characteristic and on the existence of the compactification of the universal abelian scheme constructed by Faltings-Chai.

1 Introduction

Let k be an algebraically closed field and let S be a connected, smooth and proper curve over k. Let $K := \kappa(S)$ be its function field.

*Institut de Mathématiques, Equipe Emile Picard, Université Paul Sabatier, 118 Route de Narbonne, 31062 Toulouse cedex 9, FRANCE, E-mail: rossler@math.univ-toulouse.fr

If V/S is a locally free coherent sheaf on S, we denote by

$$0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{\mathrm{hn}(V)} = V$$

the Harder-Narasimhan filtration of V. We write as usual

$$\deg(*) := \deg(\mathrm{c}_1(*)),\ \mu(*) := \deg(*)/\mathrm{rk}(*)$$

and

$$\mu_{\min}(V) := \mu(V/V_{\mathrm{hn}(V)-1}),\ \mu_{\max}(V) := \mu(V_1).$$

See [3, chap. 5] (for instance) for the definition of the Harder-Narashimha filtration and for the notion of semistable sheaf, which underlies it.

Suppose from now on that k has characteristic $p > 0$.

If X is a scheme of characteristic p, we denote the absolute Frobenius endomorphism of X by F_X. If $f : X \to Y$ is a morphism between two schemes of characteristic p and $\ell > 0$, we denote by $Y^{(p^\ell)}$ the source of the fibre product of f and $F_X^{\circ\ell}$, where $F_X^{\circ\ell}$ is the ℓ-th power of the Frobenius endomorphism F_X of X.

A locally free sheaf V on S is said to be strongly semistable if $F_S^{r,*}(V)$ is semistable for all $r \in \mathbb{N}$. A. Langer proved in [17, Th. 2.7, p. 259] that there is an $n_0 = n_0(V) \in \mathbb{N}$ such that the quotients of the Harder-Narasimhan filtration of $F_S^{n_0,*}(V)$ are all strongly semistable. This shows in particular that the following definitions :

$$\bar{\mu}_{\min}(V) := \lim_{\ell\to\infty} \mu_{\min}(F_S^{\ell *}(V))/p^\ell$$

and

$$\bar{\mu}_{\max}(V) := \lim_{\ell\to\infty} \mu_{\max}(F_S^{\ell *}(V))/p^\ell$$

make sense.

With these definitions in hand, we are now in a position to formulate the results that we are going to prove in the present text.

Let $\pi : \mathcal{A} \to S$ be a smooth commutative group scheme and let $A := \mathcal{A}_K$ be the generic fibre of $\mathcal{A}$. Let $\epsilon : S \to \mathcal{A}$ be the zero-section and let $\omega := \epsilon^*(\Omega^1_{\mathcal{A}/S})$ be the Hodge bundle of $\mathcal{A}$ over S.

Fix an algebraic closure $\bar{K}$ of K. For any $\ell \in \mathbb{N}$, let

$$K^{p^{-\ell}} := \{x \in \bar{K} | x^{p^l} \in K\},$$

which is a field. We may then define the field

$$K^{\mathrm{perf}} = K^{p^{-\infty}} = \cup_{\ell\in\mathbb{N}} K^{p^{-\ell}},$$

which is often called the *perfection* of K.

Theorem 1.1. *Suppose that $\mathcal{A}/S$ is semiabelian and that A is a principally polarized abelian variety. Suppose that the vector bundle ω is ample. Then there exists $\ell_0 \in \mathbb{N}$ such the natural injection $A(K^{p^{-\ell_0}}) \hookrightarrow A(K^{\mathrm{perf}})$ is surjective (and hence a bijection).*

For the notion of ampleness, see [13, par. 2]. A smooth commutative S-group scheme $\mathcal{A}$ as above is called semiabelian if each fibre of $\mathcal{A}$ is an extension of an abelian variety by a torus (see [7, I, def. 2.3] for more details).

We recall the following fact, which is proven in [1]: *a vector bundle V on S is ample if and only if $\bar{\mu}_{\min}(V) > 0$.*

Theorem 1.2. *Suppose that A is an ordinary abelian variety. Then*

(a) $\bar{\mu}_{\min}(\omega) \geqslant 0$*;*

(b) *if there is a closed point $s \in S$ such that $\mathcal{A}_s$ is an abelian variety of p-rank 0, then $\bar{\mu}_{\min}(\omega) > 0$.*

Remark. Fix the following notations for the time of this remark. Let C be a smooth, proper and connected curve over the complex numbers $\mathbb{C}$. Let $\mathcal{G} \to C$ be a smooth group scheme over C and let $\epsilon_{\mathcal{G}} : C \to \mathcal{G}$ be its zero-section.

Suppose that the fibre of $\mathcal{G}$ over the generic point of C is an abelian variety.

$$\text{The inequality } \mu_{\min}(\epsilon_{\mathcal{G}}^{*}\Omega^1_{\mathcal{G}/C}) \geqslant 0 \text{ then holds.} \tag{1}$$

(in other words: $\epsilon_{\mathcal{G}}^{*}\Omega^1_{\mathcal{G}/C}$ is semiample). Inequality (1) follows from a theorem of P. Griffiths. A more algebraic proof of (1) was given by J.-B. Bost in [4], where the link with Griffiths's theorem is explained and further references are given.

Theorem 1.2 (a) may thus be viewed as an analog of (1) in positive characteristic. If A is not ordinary, it seems difficult to give general criteria of (semi)ampleness for ω in positive characteristic. In this context, notice that the strict analog of (1) in positive characteristic is false; see [22, chap. 8] for a counterexample by L. Moret-Bailly, in which the generic fibre of the group scheme has p-rank 0.

Corollary 1.3. *Suppose that A is ordinary and that there is a closed point $s \in S$ such that $\mathcal{A}_s$ is an abelian variety of p-rank 0. Then*

(a) *there exists $\ell_0 \in \mathbb{N}$ such that the natural injection $A(K^{p^{-\ell_0}}) \hookrightarrow A(K^{\mathrm{perf}})$ is surjective;*

(b) *the group $A(K^{\mathrm{perf}})$ is finitely generated.*

Here is an application of Corollary 1.3. Suppose until the end of the sentence that A is an elliptic curve over K and that $j(A) \notin k$ (here $j(\cdot)$ is the modular j-invariant); then A is ordinary and there is a closed point $s \in S$ such that $\mathcal{A}_s$ is an elliptic curve of p-rank 0 (i.e. a supersingular elliptic curve); thus $A(K^{\mathrm{perf}})$ is finitely generated. This was also proven by D. Ghioca (see [8]) using a different method.

We list two further applications of Theorems 1.1 and 1.2.

Let Y be an integral closed subscheme of $B := A_{\bar{K}}$.

The following proposition is a special case of the (unproven) "full" Mordell-Lang conjecture, first formulated by Abramovich and Voloch. See [9] and [21, Conj. 4.2] for a formulation of the conjecture and further references.

Proposition 1.4. *Suppose that A is an ordinary abelian variety. Suppose that there is a closed point $s \in S$ such that $\mathcal{A}_s$ is an abelian variety of p-rank 0. If $Y \cap A(K^{\mathrm{perf}})$ is Zariski dense in Y then Y is the translate of an abelian subvariety of B by an point in $B(\bar{K})$.*

Proof (of Proposition 1.4). This is a direct consequence of Corollary 1.3 and of the Mordell-Lang conjecture over function fields of positive characteristic; see [14] for the latter. □

Our second application is to a question of a A. Langer and H. Esnault. See [6, Remark 6.3] for the latter. The following proposition gives a positive answer to their question in a specific situation.

Proposition 1.5. *Suppose that A is an ordinary abelian variety and that there is a closed point $s \in S$ such that $\mathcal{A}_s$ is an abelian variety of p-rank 0. Suppose that for all $\ell \geqslant 0$ we are given a point $P_\ell \in A^{(p^\ell)}(K)$ and suppose that for all $\ell \geqslant 1$, we have $\mathrm{Ver}^{(p^\ell)}_{A/K}(P_\ell) = P_{\ell-1}$. Then P_0 is a torsion point.*

The morphism $\mathrm{Ver}_{A/K}^{(p^\ell)} : A^{(p^\ell)} \to A^{(p^{\ell-1})}$ is the Verschiebung morphism. See [10, $\mathrm{VII_A}$, 4.3] for the definition.

Proof (of Proposition 1.5). By assumption the point P_0 is p^∞-divisible in $A(K^{\mathrm{perf}})$, because $[p]_{A/K} = \mathrm{Ver}_{A/K} \circ \mathrm{Frob}_{A/K}$. Here $\mathrm{Frob}_{A/K}$ is the relative Frobenius morphism and $[p]_{A/K}$ is the multiplication by p morphism on A. Thus P_0 is a torsion point, because $A(K^{\mathrm{perf}})$ is finitely generated by Corollary 1.3. □

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2 Proof of 1.1, 1.2 & 1.3

2.1 Proof of Theorem 1.1

The idea behind the proof of Theorem 1.1 comes from an article of M. Kim (see [16]).

In this subsection, the assumptions of Theorem 1.1 hold. So we suppose that $\mathcal{A}/S$ is semiabelian, that A is a principally polarized abelian variety and that ω is ample.

If $Z \to W$ is a W-scheme and W is a scheme of characteristic p, then for any $n \geqslant 0$ we shall write $Z^{[n]} \to W$ for the W-scheme given by the composition of arrows

$$Z \to W \stackrel{F_W^n}{\to} W.$$

Now fix $n \geqslant 1$ and suppose that $A(K^{p^{-n}}) \backslash A(K^{p^{-n+1}}) \neq \emptyset$.

Fix $P \in A^{(p^n)}(K) \backslash A^{(p^{n-1})}(K) = A(K^{p^{-n}}) \backslash A(K^{p^{-n+1}})$. The point P corresponds to a commutative diagram of k-schemes

$$\begin{array}{ccc} & \overset{P}{\nearrow} & A \\ & & \downarrow \\ \mathrm{Spec}\ K^{[n]} & \xrightarrow[F_K^n]{} & \mathrm{Spec}\ K \end{array}$$

such that the residue field extension $K|\kappa(P(\mathrm{Spec}\ K^{[n]}))$ is of degree 1 (in other words P is birational onto its image). In particular, the map of K-vector spaces

$P^*\Omega^1_{A/k} \to \Omega^1_{K^{[n]}/k}$ arising from the diagram is non zero.

Now recall that there is a canonical exact sequence

$$0 \to \pi_K^*\Omega^1_{K/k} \to \Omega^1_{A/k} \to \Omega^1_{A/K} \to 0.$$

Furthermore the map $F_K^{n,*}\Omega^1_{K/k} \overset{F_K^{n,*}}{\to} \Omega^1_{K^{[n]}/k}$ vanishes. Also, we have a canonical identification $\Omega^1_{A/K} = \pi_K^*\omega_K$ (see [2, chap. 4., Prop. 2]). Thus the natural surjection $P^*\Omega^1_{A/k} \to \Omega^1_{K^{[n]}/k}$ gives rise to a non-zero map

$$\phi_n : F_K^{n,*}\omega_K \to \Omega^1_{K^{[n]}/k}.$$

The next lemma examines the poles of the morphism ϕ_n.

We let E be the reduced closed subset, which is the union of the points $s \in S$, such that the fibre $\mathcal{A}_s$ is not complete.

Lemma 2.1. *The morphism ϕ_n extends to a morphism of vector bundles*

$$F_S^{n,*}\omega \to \Omega^1_{S^{[n]}/k}(E).$$

Proof (of 2.1). First notice that there is a natural identification $\Omega^1_{S^{[n]}/k}(\log E) = \Omega^1_{S^{[n]}/k}(E)$, because there is a sequence of coherent sheaves

$$0 \to \Omega_{S^{[n]}/k} \to \Omega^1_{S^{[n]}/k}(\log E) \to \mathcal{O}_E \to 0$$

where the morphism onto $\mathcal{O}_E$ is the residue morphism. Here the sheaf $\Omega^1_{S^{[n]}/k}(\log E)$ is the sheaf of differentials on $S^{[n]}\backslash E$ with logarithmic singularities along E. See [15, Intro.] for this result and more details on these notions.

Now notice that in our proof of Theorem 1.1, we may replace K by a finite extension field K' without restriction of generality. We may thus suppose that A is endowed with an m-level structure for some $m \geqslant 3$.

We now quote part of one the main results of the book [7]:

(1) there exists a regular moduli space $A_{g,m}$ for principally polarized abelian varieties over k endowed with an m-level structure;

(2) there exists an open immersion $A_{g,m} \hookrightarrow A^*_{g,m}$, such that the (reduced) complement $D := A^*_{g,m}\backslash A_{g,m}$ is a divisor with normal crossings and $A^*_{g,m}$ is regular and proper over k;

(3) the scheme $A^*_{g,m}$ carries a semiabelian scheme G extending the universal abelian scheme $f : Y \to A_{g,m}$;

(4) there exists a regular and proper $A^*_{g,m}$-scheme $\bar{f} : \bar{Y} \to A^*_{g,m}$, which extends Y and such that $F := \bar{Y} \backslash Y$ is a divisor with normal crossings (over k); furthermore

(5) on $\bar{Y}$ there is an exact sequence of locally free sheaves

$$0 \to \bar{f}^*\Omega^1_{A_{g,m}/k}(\log D) \to \Omega^1_{Y/k}(\log F) \to \Omega^1_{Y/A_{g,m}}(\log F/D) \to 0,$$

which extends the usual sequence of locally free sheaves

$$0 \to f^*\Omega^1_{A_{g,m}/k} \to \Omega^1_{Y/k} \to \Omega^1_{Y/A_{g,m}} \to 0$$

on $A_{g,m}$. Furthermore there is an isomorphism $\Omega^1_{Y/A_{g,m}}(\log F/D) \simeq \bar{f}^*\omega_G$. Here $\omega_G := \mathrm{Lie}(G)^\vee$ is the tangent bundle (relative to $A^*_{g,m}$) of G restricted to $A^*_{g,m}$ via the unit section.

See [7, chap. VI, th. 1.1] for the proof.

The datum of A/K and its level structure induces a morphism $\phi : K \to A_{g,m}$, such that $\phi^*Y \simeq A$, where the isomorphism respects the level structures. Call $\lambda : A \to Y$ the corresponding morphism over k. Let $\bar{\phi} : S \to A^*_{g,m}$ be the morphism obtained from ϕ via the valuative criterion of properness. By the unicity of semiabelian models (see [20, IX, Cor. 1.4, p. 130]), we have a natural isomorphism $\bar{\phi}^*G \simeq \mathcal{A}$ and thus we have a set-theoretic equality $\bar{\phi}^{-1}(D) = E$ and an isomorphism $\bar{\phi}^*\omega_G = \omega$. Let also $\bar{P}$ be the morphism $S^{[n]} \to \bar{Y}$ obtained from $\lambda \circ P$ via the valuative criterion of properness. By construction we now get an arrow

$$\bar{P}^*\Omega^1_{\bar{Y}/k}(\log F) \to \Omega^1_{S^{[n]}/k}(\log E)$$

and since the induced arrow

$$\bar{P}^*\bar{f}^*\Omega^1_{A_{g,m}/k}(\log D) = F_S^{n,*} \circ \bar{\phi}^*(\Omega^1_{A_{g,m}/k}(\log D)) \to \Omega^1_{S^{[n]}/k}(\log E)$$

vanishes (because it vanishes generically), we get an arrow

$$\bar{P}^*\Omega^1_{\bar{Y}/A^*_{g,m}}(\log F/D) = F_S^{n,*} \circ \bar{\phi}^*\omega_G = F_S^{n,*}\omega \to \Omega^1_{S^{[n]}/k}(\log E) = \Omega^1_{S^{[n]}/k}(E),$$

which is what we sought. □

To conclude the proof of Proposition 1.1, choose l_0 large enough so that

$$\mu_{\min}(F_S^{l,*}(\omega)) > \mu(\Omega^1_{S/k}(E))$$

for all $l > l_0$. Such an l_0 exists because $\bar{\mu}_{\min}(\omega) > 0$. Now notice that since k is a perfect field, we have $\Omega^1_{S/k}(E) \simeq \Omega^1_{S^{[n]}/k}(E)$. We see that we thus have

$$\mathrm{Hom}(F_S^{l,*}(\omega), \Omega^1_{S^{[n]}/k}(E)) = 0$$

for all $l > l_0$ and thus by Lemma 2.1 we must have $n < l_0 + 1$. Thus we have

$$A(K^{(p^{-l})}) = A(K^{(p^{-l+1})})$$

for all $l \geqslant l_0$.

Remark. The fact that $\mathrm{Hom}(F_S^{l,*}(\omega), \Omega^1_{S^{[n]}/k}(E)) \simeq \mathrm{Hom}(F_S^{l,*}(\omega), \Omega^1_{S/k}(E))$ vanishes for large l can also be proven without appealing to the Harder-Narasimhan filtration. Indeed the vector bundle ω is also cohomologically p-ample (see [18, Rem. 6), p. 91]) and thus there is an $l_0 \in \mathbb{N}$ such that for all $l > l_0$

$$\begin{aligned}
\mathrm{Hom}(F_S^{l,*}(\omega), \Omega^1_{S/k}(E)) &= H^0(S, F_S^{l,*}(\omega)^\vee \otimes \Omega^1_{S/k}(E)) \\
&\overset{\text{Serre duality}}{=} H^1(S, F_S^{l,*}(\omega) \otimes \Omega^1_{S/k}(E)^\vee \otimes \Omega^1_{S/k})^\vee \\
&= H^1(S, F_S^{l,*}(\omega) \otimes \mathcal{O}(-E))^\vee = 0.
\end{aligned}$$

2.2 Proof of Theorem 1.2

In this subsection, we suppose that the assumptions of Theorem 1.2 hold. So we suppose that A is an ordinary abelian variety.

Notice first that for any $n \geqslant 0$, the Hodge bundle of $\mathcal{A}^{(p^n)}$ is $F_S^{n,*}\omega$. Hence, in proving Proposition 1.2, we may assume without restriction of generality that ω has a strongly semistable Harder-Narasimhan filtration.

Let $V := \omega/\omega_{\mathrm{hn}(\omega)-1}$. Notice that for any $n \geqslant 0$, we have a (composition of) Verschiebung(s) map(s) $\omega \rightarrow F_S^{n,*}\omega$. Composing this with the natural quotient map, we get a map

$$\phi : \omega \overset{\mathrm{Ver}_{\mathcal{A}}^{(p^n),*}}{\rightarrow} F_S^{n,*}V \tag{2}$$

The map ϕ is generically surjective, because by the assumption of ordinariness the map $\omega \overset{\mathrm{Ver}^{(p^n),*}}{\rightarrow} F_S^{n,*}\omega$ is generically an isomorphism.

We now prove (a). The proof is by contradiction. Suppose that $\bar{\mu}_{\min}(\omega) := \mu(V) < 0$. This implies that when $n \to \infty$, we have $\mu(F_S^{n,*}V) \to -\infty$. Hence if n is sufficiently large, we have $\mathrm{Hom}(\omega, F_S^{n,*}V) = 0$, which contradicts the surjectivity of the map in (2).

We turn to the proof of (b). Again the proof is by contradiction. So suppose that $\bar{\mu}_{\min}(\omega) \leqslant 0$. By (a), we know that we then actually have $\bar{\mu}_{\min}(\omega) = 0 = \mu(V)$ and $V \neq 0$. If $\bar{\mu}_{\max}(\omega) > 0$ then the map $\omega_1 \to F_S^{n,*}V$ obtained by composing ϕ with the inclusion $\omega_1 \hookrightarrow \omega$ must vanish, because

$$\mu(\omega_1) > \mu(F_S^{n,*}V) = p^n \cdot \mu(V) = 0.$$

Hence we obtain a map $\omega/\omega_1 \to F_S^{n,*}V$. Repeating this reasoning for ω/ω_1 and applying induction we finally get a map

$$\lambda : V \to F_S^{n,*}V.$$

The map λ is generically surjective and thus globally injective, since its target and source are locally free sheaves of the same generic rank. Let T be the cokernel of λ (which is a torsion sheaf). We then have

$$\deg(V) + \deg(T) = 0 + \deg(T) = \deg(F_S^{n,*}V) = 0$$

and thus $T = 0$. This shows that λ is a (global) isomorphism. In particular, the map ϕ is surjective. Thus the map

$$\phi_s : \omega_s \overset{\mathrm{Ver}_{\mathcal{A}_s}^{(p^n),*}}{\to} F_s^{n,*}V_s$$

is surjective and thus non-vanishing. This contradicts the hypothesis on the p-rank at s.

2.3 Proof of Corollary 1.3

In this subsection, we suppose that the assumptions of Corollary 1.3 are satisfied. So we suppose that A is ordinary and that there is a closed point $s \in S$ such that $\mathcal{A}_s$ is an abelian variety of p-rank 0.

We first prove (a). First we may suppose without restriction of generality that A is principally polarized. This follows from the fact that the abelian variety $(A \times_K A^\vee)^4$ carries a principal polarization ("Zarhin's trick" - see [19, Rem. 16.12,

p. 136]) and from the fact that the abelian variety $(A \times_K A^\vee)^4$ also satisfies the assumptions of Corollary 1.3. Furthermore, we may without restriction of generality replace S by a finite extension S'. Thus, by Grothendieck's semiabelian reduction theorem (see [12, IX]) we may assume that $\mathcal{A}$ is semiabelian. Statement (a) then follows from Theorems 1.1 and 1.2.

We now turn to statement (b). Let ℓ_0 be as in (a). Let $\mathrm{Tr}_{K|k}(A^{(p^{\ell_0})})$ be the $K|k$-trace of $A^{(p^{\ell_0})}$. This is an abelian variety over k, which comes with a finite K-morphism $\tau_{K|k} : \mathrm{Tr}_{K|k}(A^{(p^{\ell_0})})_K \to A^{(p^{\ell_0})}$ (see [5] for the definition). The morphism $\tau_{K|k}$ uniquely extends to an S-morphism $T : \mathrm{Tr}_{K|k}(A^{(p^{\ell_0})})_S \to \mathcal{A}^{(p^{\ell_0})}$ by a theorem of Raynaud (see [20, IX, Cor. 1.4, p. 130]). The morphism T is finite on any complete fibre of $\mathcal{A}^{(p^{\ell_0})}$, as can be seen by looking at the image under T of the torsion points of order prime to p on such a fibre. Combining this with our assumptions, we see that $\mathrm{Tr}_{K|k}(A^{(p^{\ell_0})})$ is simultaneously ordinary and of p-rank 0, which in turn implies that $\mathrm{Tr}_{K|k}(A^{(p^{\ell_0})}) = 0$. Now by the Lang-Néron theorem, the group $A^{(p^{\ell_0})}(K)/\mathrm{Tr}_{K|k}(A^{(p^{\ell_0})})(k)$ is finitely generated and thus we see that

$$A^{(p^{\ell_0})}(K) = A(K^{p^{-\ell_0}}) = A(K^{\mathrm{perf}})$$

is finitely generated. This concludes the proof of (b).

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