An Adams-Riemann-Roch theorem in Arakelov geometry

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Abstract

We prove an analog of the classical Riemann-Roch theorem for Adams operations acting on K-theory, in the context of Arakelov geometry.

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1 Introduction

In this paper, we shall investigate relative Riemann-Roch formulas for the λ-operations acting on Grothendieck groups "compactified" in the sense of Arakelov geometry. Let Y be a quasi-projective scheme over $\mathbf{Z}$, which is smooth over $\mathbf{Q}$. We shall call such a scheme an arithmetic variety. Following [21, II], one can associate to Y an arithmetic Grothendieck group $\widehat{K}_0(Y)$, whose generators are differential forms and vector bundles on Y equipped with hermitian metrics on the manifold $Y(\mathbf{C})$ of complex points of Y. The group $\widehat{K}_0(Y)$ is related to the Grothendieck group $K_0(Y)$ of vector bundles of Y via the sequence

$$K_1(Y) \to \tilde{A}(Y) \to \widehat{K}_0(Y) \to K_0(Y) \to 0$$

where $K_1(Y)$ is the first Quillen K-group of Y and $\tilde{A}(Y)$ is a space of differential forms on $Y(\mathbf{C})$. Recall that the exterior powers of vector bundles λ^k are well-defined on $K_0(Y)$ and give rise to a λ-ring structure. It is shown in [21, Th. 7.3.4, p. 235, II] that the exterior powers of hermitian bundles give rise to well-defined operations λ^k on $\widehat{K}_0(Y)$ as well, such that the morphism $\widehat{K}_0(Y) \to K_0(Y)$ is compatible with the operations. In [32] (see also [33]), we prove that they actually define a λ-ring structure on $\widehat{K}_0(Y)$; a different proof can be found in [30]. To such a structure is canonically associated a family of ring endomorphisms ψ^k, called Adams operations (they are universal polynomials in the λ-operations).
Let now B be another arithmetic variety and $g : Y \to B$ a morphism which is projective, flat and smooth over the rational numbers $\mathbf{Q}$ (abbreviated p.f.s.r.). We suppose that g is also a local complete intersection morphism and that $Y(\mathbf{C})$ is endowed with some Kähler metric (this is always possible, with the given assumptions on Y). Using the higher analytic torsion defined in [12], one can define a push-forward map $g_* : \widehat{K}_0(Y) \to \widehat{K}_0(B)$; its determinant is represented in $\widehat{K}_0(B)$ by the determinant of the cohomology, endowed with the Quillen metric. The main result of the following paper is to give a Riemann-Roch theorem for the Adams operations, relatively to the push-forward map. More precisely, for any $y \in \widehat{K}_0(Y) \otimes \mathbf{Z}[\frac{1}{k}]$, we have

$$\psi^k(g_*(y)) = g_*(\theta^k_A(\overline{Tg}^\vee)^{-1}.\psi^k(y)) \tag{1}$$

where $\theta^k_A(\overline{Tg}^\vee)^{-1}$ is an element of $\widehat{K}_0(Y) \otimes \mathbf{Z}[\frac{1}{k}]$, which depends on g only. An algebraic analog of this equation can be found in [19, Th. 7.6, p. 149] (see also [29, 16.6, p. 71]). The formula (1) is deduced from another Riemann-Roch theorem, describing the behaviour of Adams operations under immersions. To prove (1) for the natural projection $\mathbf{P}^n_{\mathbf{Z}} \to Spec\mathbf{Z}$ of a projective space of dimension n over $Spec\mathbf{Z}$, we combine an induction argument on n with the Riemann-Roch theorem for immersions, applied to the diagonal immersion $\mathbf{P}^n_{\mathbf{Z}} \to \mathbf{P}^n_{\mathbf{Z}} \times \mathbf{P}^n_{\mathbf{Z}}$.

([1]) Via a projection formula and a base change formula, we show that (1) holds for the projection from any relative projective space to its base. The existence of this method, which has an algebraic analog, shows that the Riemann-Roch theorem for local complete intersection p.f.s.r. morphisms can be derived from the Riemann-Roch theorem for immersions in an almost formal way. See also the remarks at the end of the section 7. To obtain (1) in general, we show that the Riemann-Roch theorem for immersions implies that (1) is itself compatible with immersions. To describe the Riemann-Roch theorem for immersions, let $i : Y \to X$ be a regular immersion into an arithmetic variety X and $f : X \to B$ a p.f.s.r. morphism to B, such that $g = f \circ i$. We suppose that X is endowed with a Kähler metric and that Y carries the induced metric. We endow the normal bundle N of Y in X with the quotient metric. Let η be a hermitian bundle on Y and

$$0 \to \xi_m \to \xi_{m-1} \to \ldots \to \xi_0 \to i_*\eta \to 0$$

a resolution of η by vector bundles on X. We suppose that the $\xi.$ and η are endowed with hermitian metrics. Furthermore, we suppose that these metrics satisfy Bismut's assumption (A) (see [10, Def. 1.1, p. 258]) with respect to the metric of N. The theorem reads

$$g_*(\theta^k(\overline{N}^\vee)\psi^k(\overline{\eta})i^*(x)) = \sum_{i=0}^{m}(-1)^i f_*(\psi^k(\overline{\xi}_i)x) +$$

$$\int_{Y/B} Td(Tg)ch(i^*(x))ch(\psi^k(\eta)\theta^k(N^\vee))R(N) + \int_{X/B} kTd(\overline{Tf})\phi^k(T(h^{\xi.}))ch(x) -$$

$$\int_{Y/B} k^{rg(N)}ch(i^*(x))ch(\psi^k(\overline{\eta}))\phi^k(Td^{-1}(\overline{N}))\widetilde{Td}(g/f) \qquad (2)$$

Here $T(h^\xi)$ is a current whose singular support is Y and $\widetilde{Td}(g/f)$ is the Bott-Chern secondary class of the normal sequence associated to i on Y. The class R is the R-genus of Gillet and Soulé, an additive real cohomological class which will be described below. For $k = 1$, this theorem follows immediately from Bismut's theorem describing the behaviour of analytic torsion under immersions (see [6]). To prove it in general, we use the deformation to the normal cone technique of [3]. Since both sides of (2) depend on the Kähler metric of P, we have to control the Kähler metrics of the fibers of the deformation; the "good" metrics on the deformation space appear to satisfy certain normality conditions; they are constructed via the Grassmannian graph construction.
All these analytical and geometric techniques also appear in the proof of the arithmetic Riemann-Roch theorem for the first Chern class with values in arithmetic Chow groups (see [23, 4.2.3]). Furthermore, the following weak connection between that theorem and the theorem (1) can be established. If X and B are

[1] In this part of the proof, we were helped by Nicusor Dan.

regular varieties, the arithmetic Chow groups can be defined and proceeding as in [23, 4.2.3], using [6] rather than [13] (which was the only formula available at that time), one can prove a Riemann-Roch theorem for the full Chern character with values in arithmetic Chow groups (this extension of [23, 4.2.3] is not yet published). Using this Riemann-Roch theorem and the fact that arithmetic K_0-theory and arithmetic Chow theory are isomorphic *modulo torsion* (see [21, 7. p. 219, II]), it is possible to derive the formula (1) in a purely formal manner, provided we consider that both sides are elements of $\widehat{K}_0(Y) \otimes_{\mathbf{Z}} \mathbf{Q}$. The formula (1) shows that denominators can be removed, up to powers of $\frac{1}{k}$. In the book [16], a method of proof of a Riemann-Roch theorem for the full Chern character with values in arithmetic Chow groups is outlined, which doesn't use the analytical results of Bismut; the result [16, Th. 6.1, p. 77] stated there could also be used to establish the logical connection mentioned above if one could identify (perhaps only compare) the definition of the direct image in arithmetic K_0-theory defined there (see [16, Lecture 5]) and the one used here, which makes use of the torsion forms of Bismut-Köhler.
In the last section of the paper, a Riemann-Roch theorem for a Chern character with values in a graded ring arising from the λ-structure of arithmetic K_0-theory is deduced from (1). It is formally similar to either of the Riemann-Roch theorems for the Chern character mentioned above (see also the end of section 8) and also implies arithmetic analogs of the Hilbert-Samuel theorem. The main results of this paper are announced in [34].
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2 The λ-structure of arithmetic K_0-theory

Let X be a scheme of finite type over $\mathbf{Z}$, with smooth generic fiber. We shall write $X(\mathbf{C})$ for the manifold of complex points associated to X. Complex conjugation induces an antiholomorphic automorphism F_∞ on $X(\mathbf{C})$. We define $A^{p,p}(X)$ as the set of differential forms ω of type p,p on $X(\mathbf{C})$, that satisfy the equation $F_\infty^*\omega = (-1)^p\omega$ and we write $Z^{p,p}(X) \subseteq A^{p,p}(X)$ for the kernel of the operation $d = \partial + \overline{\partial}$. We also define $\tilde{A}(X) := \bigoplus_{p\geq 0}(A^{p,p}(X)/(Im\partial + Im\overline{\partial}))$ and $Z(X) = \bigoplus_{p\geq 0} Z^{p,p}(X)$. A hermitian bundle $\overline{E} = (E,h)$ is a vector bundle E on X, endowed with a hermitian metric h, which is invariant under F_∞, on the holomorphic bundle $E_{\mathbf{C}}$ on $X(\mathbf{C})$, which is associated to E. We denote by $ch(\overline{E})$ the representative of the Chern character associated by the formulas of Chern-Weil to the hermitian holomorphic connection defined by h. Let $\mathcal{E} : 0 \to E' \to E \to E'' \to 0$ be an exact sequence of vector bundles on X. We shall write $\overline{\mathcal{E}}$ for the sequence $\mathcal{E}$ and hermitian metrics on $E'_{\mathbf{C}}$, $E_{\mathbf{C}}$ and $E''_{\mathbf{C}}$ (invariant under F_∞). To $\overline{\mathcal{E}}$ are associated three hermitian bundles $\overline{E}'$, $\overline{E}$ and $\overline{E}''$ as well as a secondary Bott-Chern class $\widetilde{ch}(\overline{\mathcal{E}}) \in \tilde{A}(X)$; for the definition, we refer to [9, Par. f)].

Definition 2.1 *The arithmetic Grothendieck group $\widehat{K}_0(X)$ associated to X is the group generated by $\tilde{A}(X)$ and the isometry classes of hermitian bundles on X, with the relations*

(a) *For every exact sequence $\overline{\mathcal{E}}$ as above, we have $\widetilde{ch}(\overline{\mathcal{E}}) = \overline{E}' - \overline{E} + \overline{E}''$*

(b) *If $\eta \in \tilde{A}(X)$ is the sum of two elements η' and η'', then $\eta = \eta' + \eta''$ in $\widehat{K}_0(X)$.*

Notice that there is an exact sequence of groups

$$\tilde{A}(X) \to \widehat{K}_0(X) \to K_0(X) \to 0 \tag{3}$$

where the second map sends element of $\tilde{A}(X)$ on 0 and hermitian vector bundles on the corresponding vector bundles. Let us consider the group $\Gamma(X) := Z(X) \oplus \tilde{A}(X)$. We equip it with the grading whose term of degree p is $Z^{p,p}(X) \oplus \tilde{A}^{p-1,p-1}(X)$ if $p \geq 1$ and $Z^{0,0}(X)$ if $p = 0$. We define a bilinear map $*$ from $\Gamma(X) \times \Gamma(X)$ to $\Gamma(X)$ via the formula

$$(\omega, \eta) * (\omega', \eta') = (\omega \wedge \omega', \omega \wedge \eta' + \eta \wedge \omega' + (dd^c\eta) \wedge \eta').$$

Recall that $d^c = \frac{1}{4\pi i}(\partial - \overline{\partial})$. This map endows $\Gamma(X)$ with the structure of a commutative graded $\mathbf{R}$-algebra (cf. [21, Lemma 7.3.1, p. 233]). There is thus a unique λ-ring structure on $\Gamma(X)$ such that the k-th associated Adams operation acts by the formula $\psi^k(x) = \sum_{i \geq 0} k^i x_i$, where x_i stands for the component of degree i of the element $x \in \Gamma(X)$ (see [24, 7.2, p. 361, Appendice]). For the definition of the term λ-ring (also called special λ-ring), see [24, Def. 2.1, p. 314].

Definition 2.2 *If $\overline{E} + \eta, \overline{E}' + \eta'$ are two generators of $\widehat{K}_0(X)$, the product $\otimes$ is given by the formula*

$$(\overline{E} + \eta) \otimes (\overline{E}' + \eta') = \overline{E} \otimes \overline{E}' + [(ch(\overline{E}), \eta) * (ch(\overline{E}'), \eta')]$$

where $[.]$ refers to the projection on the second component of $\Gamma(X)$. If $k \geq 0$, set

$$\lambda^k(\overline{E} + \eta) = \lambda^k(\overline{E}) + [\lambda^k(ch(\overline{E}), \eta)]$$

where $\lambda^k(\overline{E})$ is the k-th exterior power of $\overline{E}$ and $\lambda^k(ch(\overline{E}), \eta)$ stands for the image of $(ch(\overline{E}), \eta)$ under the k-th λ-operation of $\Gamma(X)$.

H. Gillet and C. Soulé have shown in [21, Th. 7.3.4, p. 235] that $\otimes$ and λ^k are compatible with the defing relations of $\widehat{K}_0(X)$ and that it endows it with the structure of a pre-λ-ring. In [32], we show that $\widehat{K}_0(X)$ is actually a λ-ring.

3 The statement

An **arithmetic variety** will denote a quasi-projective scheme over $\mathbf{Z}$, with smooth generic fiber. Let $g : Y \to B$ be a projective, flat morphism of arithmetic varieties, which is smooth over the rational numbers $\mathbf{Q}$ (abbreviated p.f.s.r.). Fix a conjugation invariant Kähler metric h_Y on Y. Let η be an element of $\tilde{A}(Y)$ and (E, h) a hermitian bundle on Y, acyclic relatively to g. The sheaf of modules g_*E, which is the direct image of E, is then locally free and we write g_*h for the smooth metric it inherits from E by integration on the fibers (see [5, p. 278] or below). We write $T(h_Y, h^E)$ for the higher analytic torsion of (E, h) relatively to the Kähler fibration defined by g and h_Y. We shall recall its definition in paragraph 5.1. We write $\overline{Tg_{\mathbf{C}}}$ for the tangent bundle relative to $g_{\mathbf{C}}$, endowed with the induced metric and $Td(\overline{Tg_{\mathbf{C}}})$ for the Todd form associated to the holomorphic hermitian connection of $Tg_{\mathbf{C}}$.

Proposition 3.1 *There is a unique group morphism $g_* : \widehat{K}_0(Y) \to \widehat{K}_0(B)$ such that $g_*((E, h)+\eta) = (g_*E, g_*h) - T(h_Y, h^E) + \int_{Y/B} Td(\overline{Tg_{\mathbf{C}}})\eta$ for all (E, h) and η as above.*

The proof of 3.1 will be given below after the Theorem 5.16. Proposition 3.1 and its proof are similar to [22, Th. 3.2, p. 46] and its proof. See also [16, Lecture 5]. The group morphism of the last Proposition will be called the **push-forward** map associated to g and h_Y. To state the Riemann-Roch theorem, we need to define a characteristic class. The following definition is taken from [22, 1.2.3, p. 25].

Definition 3.2 *The R genus is the unique additive characteristic class defined for a line bundle L by the formula*

$$R(L) = \sum_{m\ odd, \geq 1} (2\zeta'(-m) + \zeta(-m)(1 + \frac{1}{2} + \ldots + \frac{1}{m}))c_1(L)^m/m!$$

where $\zeta(s)$ is the Riemann zeta function.

For any λ-ring A, denote by A_{fin} its subset of elements of finite λ-dimension (an element a is of finite λ-dimension if $\lambda^k(a) = 0$ for all $k >> 0$). For each $k \geq 1$, the Bott cannibalistic class θ^k (see [2, Prop. 7.2, p. 268]) is uniquely determined by the following properties:

(a) For every λ-ring A, θ^k maps A_{fin} into A_{fin} and the equation $\theta^k(a+b) = \theta^k(a)\theta^k(b)$ holds for all $a, b \in A_{fin}$;

(b) The map θ^k is functorial with respect to λ-ring morphisms;

(c) If e is an element of λ-dimension 1, then $\theta^k(e) = \sum_{i=0}^{k-1} e^i$.

If $H = \oplus_{i=0}^{\infty} H_i$ is a graded commutative group, we define $\phi^k(h) = \sum_{i=0}^{\infty} k^i h_i$, where h_i is the component of degree i of $h \in H$. If H is also a commutative graded ring, the ϕ^k coincide with the Adams operations canonically associated to H. Let now $H = \tilde{A}(Y)$ be endowed with the grading giving degree p to differential forms of type p, p. If $\omega \in \tilde{A}(Y)$, then one computes that $\psi^k(\omega) = k.\phi^k(\omega)$, where on the left side ω is viewed as an element of $\Gamma(Y)$ (endowed with the λ-structure described in section 2) and on the right side ω is viewed as an element of the graded group $\tilde{A}(Y)$. Thus we shall often write $k.\phi^k$ for ψ^k in that case. Consider now the form $k^{-rg(E)} Td^{-1}(\overline{E})\phi^k(Td(\overline{E}))$, where $\overline{E}$ is a hermitian bundle and $Td(\overline{E})$ is viewed as an element of the group $Z(X)$ endowed with its natural grading. This form is by construction a universal polynomial in the Chern forms $c_i(\overline{E})$ and we shall denote the associated symmetric polynomial in $r = rg(E)$ variables by CT^k. One can compute from the definitions that $CT^k = k^r \prod_{i=1}^{r} \frac{e^{T_i}-1}{T_i e^{T_i}} \frac{k.T_i e^{k.T_i}}{e^{k.T_i}-1}$ where $T_1, \dots T_r$ are the variables.

Definition 3.3 *Let $\mathcal{E} : 0 \to E' \to E \to E'' \to 0$ be an exact sequence of hermitian holomorphic bundles on a complex manifold. The Bott-Chern secondary class associated to $\mathcal{E}$ and to CT^k will be denoted by $\tilde{\theta}^k(\overline{\mathcal{E}})$.*

Let $g : Y \to B$ be a local complete intersection p.f.s.r. morphism of arithmetic varieties. Suppose that Y is endowed with a Kähler metric. Let $i : Y \to X$ be a regular closed immersion into an arithmetic variety X and $f : X \to B$ a smooth map, such that $g = f \circ i$. Endow X with a Kähler metric and the normal bundle $N_{Y/X}$ with some hermitian metric. Let $\overline{\mathcal{N}_{\mathbf{C}}}$ be the sequence $0 \to Tg_{\mathbf{C}} \to Tf_{\mathbf{C}} \to N_{X(\mathbf{C})/Y(\mathbf{C})} \to 0$, endowed with the the induced metrics on $Tg_{\mathbf{C}}$ and $Tf_{\mathbf{C}}$. In the next definition, the notation $\mathbf{Z}[\frac{1}{k}]$ refers to the localization of $\mathbf{Z}$ at the multiplicative subset generated by the integer k.

Definition 3.4 *The arithmetic Bott class $\theta^k(\overline{Tg}^{\vee})^{-1}$ (or $\theta^k(\overline{T}^{\vee}_{Y/B})^{-1}$) of g is the element $\theta^k(\overline{N}^{\vee}_{Y/X})\tilde{\theta}^k(\overline{\mathcal{N}_{\mathbf{C}}}) + \theta^k(\overline{N}^{\vee}_{Y/X})\theta^k(i^*\overline{Tf}^{\vee})^{-1}$ in $\widehat{K}_0(Y) \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{k}]$.*

We shall prove later (see after 4.5) that the Bott class θ^k of every hermitian bundle has an inverse in $\widehat{K}_0(Y) \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{k}]$, so that the above definition is meaningful.

Lemma 3.5 *The arithmetic Bott class of g doesn't depend on i nor on the metrics on X and N.*

We shall prove this after 7.3. We shall also show later (see 7.2) that when g is smooth, the arithmetic Bott class of g is simply the inverse of the Bott element of the dual of the relative tangent bundle Tg, endowed with the induced metric (as the notation suggests).

Let A be a λ-ring and let $\lambda_t(x) : A \to 1 + t.A[[t]]$ be defined as $\lambda_t(x) = 1 + \sum_{k=1}^{\infty} \lambda^k(x)t^k$, where $1 + t.A[[t]]$ is the multiplicative subgroup of the ring of formal power series $A[[t]]$ consisting of power series with constant coefficient 1. We recall the relationship between the Adams operations ψ^k and the λ-operations (cf. [24, V, Appendice]): define a formal power series ψ_t by the formula

$$\psi_t(x) := \frac{t.d\lambda_{-t}(x)/dt}{\lambda_{-t}(x)}.$$

The Adams operations are then given by the identity $\psi_t(x) =: \sum_{k\geq 1} \psi^k(x)t^k$. The Adams operations are ring endomorphisms of A and satisfy the identities $\psi^k \circ \psi^l = \psi^{kl}$ $(k, l \geq 1)$. We are now ready for the statement of the Riemann-Roch theorem for Adams operations and local complete intersection p.f.s.r. morphisms:

Theorem 3.6 *Let $g : Y \to B$ be a p.f.s.r. local complete intersection morphism of arithmetic varieties. For each $k \geq 0$, let $\theta^k_A(\overline{Tg}^\vee)^{-1} = \theta^k(\overline{Tg}^\vee)^{-1}.(1 + R(Tg_{\mathbf{C}}) - k.\phi^k(R(Tg_{\mathbf{C}})))$. Then for the map $g_* : \widehat{K}_0(Y) \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{k}] \to \widehat{K}_0(B) \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{k}]$, the equality*

$$\psi^k(g_*(y)) = g_*(\theta^k_A(\overline{Tg}^\vee)^{-1}.\psi^k(y))$$

holds in $\widehat{K}_0(B) \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{k}]$ for all $k \geq 1$ and $y \in \widehat{K}_0(Y) \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{k}]$.

The sections 4 to 7 will be devoted to a proof of this statement.

4 The γ-filtration of arithmetic K_0-theory

In this subsection, we shall prove that on any arithmetic variety, the ring $\widehat{K}_0(Y)$ has a locally nilpotent γ-filtration. For the definition of these terms, see [24, V, 3.10, p. 331] or below.

Let R be any λ-ring endowed with an augmentation homomorphism $rk : R \to \mathbf{Z}$. The γ **operations** are defined by the formula

$$\gamma_t(x) = \sum_{i\geq 0} \gamma^i(x)t^i := \lambda_{\frac{t}{1-t}}(x).$$

By construction, the γ-operations also define a pre-λ-ring structure, i.e. the equalities $\gamma_t(x + y) = \gamma_t(x).\gamma_t(y)$, $\gamma_0 = 1$ and $\gamma_1 = Id$ are satisfied. We use them to construct the γ-filtration F^nR $(n \in \mathbf{Z})$ of R. Define $F^nR = R$ for $n \leq 0$

and $F^1 R := ker\ rk$. Further, define $F^n R$ to be the additive subgroup generated by the elements $\gamma^{r_1}(x_1)\dots\gamma^{r_k}(x_k)$, where $x_1 \dots x_k \in F^1 R$, $\sum_{i=1}^k r_i \geq n$. By construction, $F^1 R \supseteq F^2 R \supseteq F^3 R \supseteq \dots$ and it is easily checked that the $F^n R$ are ideals that form a ring filtration. The γ-filtration of R is said to be **locally nilpotent**, if for each $y \in F^1 R$, there is a natural number $n(y)$, depending on y, such that $\gamma^{r_1}(y)\gamma^{r_2}(y)\dots\gamma^{r_d}(y) = 0$, if $r_1 + \dots + r_d > n(y)$. If this condition is fulfilled for a particular $y \in R$, we shall say that the γ-filtration is **nilpotent at** y. Until the end of the text, we shall use the the notation $G_{\mathbf{Q}} = G \otimes_{\mathbf{Z}} \mathbf{Q}$, for any commutative ring G.

Proposition 4.1 *Let $A = \oplus_{i=0}^d A_i$ be a graded ring with finite grading, such that $A_0 = \mathbf{Z}$. Endow it with the λ-structure associated to the grading and with the augmentation arising from the projection on A_0. Then the filtration induced on $A_{\mathbf{Q}}$ by the γ-filtration of A coincides with the filtration arising from the grading of $A_{\mathbf{Q}}$.*

Proof: See [24, Cor. 6.6.7., p. 352] **Q.E.D**.

Proposition 4.2 *Suppose that R is an augmented, locally γ-nilpotent λ-ring. Then, for every λ-finite element $e \in R$, the Bott element $\theta^k(e)$ is invertible in $R \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{k}]$.*

Proof: Suppose first that $e = u_1 + \dots u_r$, where the u_i are line elements (i.e. of λ-dimension 1). We can write

$$\theta^k(e) = \prod_{i=1}^{r} \sum_{j=0}^{k-1} u_i^j = \prod_{i=1}^{r} \sum_{j=0}^{k-1} (1 + (u_i - 1))^j.$$

The last expression is a symmetric polynomial in the $u_j - 1$ with constant coefficient j^r. The k-th symmetric function of the $u_j - 1$ is by definition $\gamma^k((u_1 + \dots u_r) - r) = \gamma^k(e - r)$. Thus $\theta^k(e) = j^r + P(\gamma^1(e-r), \dots \gamma^{m(k,r)}(e-r))$, where P is a polynomial with $m(k,r)$ variables, with vanishing constant coefficient, for some $m(k,r) \geq 1$.
Returning to the case where e is any λ-finite element, consider that by [2, p. 266], there exists a λ-ring R' containing R, in which e is a sum of line elements. This implies that the formula $\theta^k(e) = j^r + P(\gamma^1(e-r), \dots \gamma^{m(k,r)}(e-r))$ holds for any λ-finite element e. Now consider the element

$$j^{-r} \sum_{l=0}^{\infty} (\ -j^{-r} P(\gamma^1(e-r), \dots \gamma^{m(k,r)}(e-r))\ )^l$$

in $R \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{k}]$, which is the geometric series applied to the element $-j^{-r}P(\gamma^1(e-r), \dots \gamma^{m(k,r)}(e-r))$. The sum on l is finite, since $P(\gamma^1(e-r), \dots \gamma^{m(k,r)}(e-r)) \in$

F^1R and R is locally nilpotent. By construction, it yields an inverse of $\theta^k(e)$. **Q.E.D.**

If $g = e - e'$ where e, e' are λ-finite elements in R, then the element $\theta^k(g)$ can be defined by the formula $\theta^k(e)\theta^k(e')^{-1}$ in $R \otimes \mathbf{Z}[\frac{1}{k}]$. It is independent of the choice of e and e'.
We define an augmentation on $rk : \widehat{K}_0(Y) \to \mathbf{Z}$ by the formula $rk(\overline{E} + \eta) = rank(E)$, for all hermitian bundles $\overline{E}$ and differential forms $\eta \in \tilde{A}(Y)$. To prove the statement mentioned at the beginning of the section, we first consider a particular case. Let $G_{i,j}$ be the Grassmannian representing the functor assigning to each $\mathbf{Z}$-scheme T the set of locally free quotients of $\mathcal{O}_T^{i+j}$ of rank j (see [15, Th. I.9.7.4]). It is a model over $\mathbf{Z}$ of the usual complex Grassmannian. Denote its universal bundle by $Q_{i,j}$. We endow $Q_{i,j}$ with the standard quotient metric.

Lemma 4.3 *Let $i = q.j$, for some positive integer q. The ring $\widehat{K}_0(G_{i,j})$ is locally γ-nilpotent at $\overline{Q}_{i,j} - j$.*

Proof: Let q be the element $\overline{Q}_{i,j} - j$ of $\widehat{K}_0(G_{i,j})$. Let m and $r_1, \dots r_d$ be natural numbers such that $r_1 + \dots + r_d > m$. Let m be greater than $dim(G_{i,j})$. Notice the following facts:
(a) $\gamma^{r_1}(q)\gamma^{r_2}(q)\dots\gamma^{r_d}(q) \in \tilde{A}(G_{i,j})$.
This follows from the fact that the forgetful map $\widehat{K}_0(G_{i,j}) \to K_0(G_{i,j})$ is a map of augmented λ-rings and from the fact that the γ-filtration of $K_0(G_{i,j})$ vanishes in degree greater than $dim(G_{i,j})$ (see [24, Th. 6.9, p. 413]).
(b) $ch(\gamma^{r_1}(q)\gamma^{r_2}(q)\dots\gamma^{r_d}(q)) = 0$.
From [21, Lemma 7.3.3, p. 235] we can deduce that

$$ch(\gamma^{r_1}(q)\gamma^{r_2}(q)\dots\gamma^{r_d}(q)) = \gamma^{r_1}(ch(q))\gamma^{r_2}(ch(q))\dots\gamma^{r_d}(ch(q))).$$

Thus we can deduce (b) from 4.1 and the fact that $Z(G_{i,j})$ vanishes in degrees greater than $dim(G_{i,j}) - 1$.
Therefore $\gamma^{r_1}(q)\gamma^{r_2}(q)\dots\gamma^{r_d}(q)$ lies in the image in $\widehat{K}_0(G_{i,j})$ of the even de Rahm cohomology $H(G_{i,j}(\mathbf{C}))$, which consists of the kernel of the operator dd^c acting on $\tilde{A}(G_{i,j})$. Now consider that there is a morphism of schemes $\mu : G_{q,1}^{\oplus j} \to G_{i,j}$, such that $\mu^*\overline{Q}_{i,j}$ is isometrically isomorphic to an orthogonal sum of line bundles $\overline{L}_1 \oplus \overline{L}_2 \oplus \dots \overline{L}_j$ (see [35, 4.2, p. 84]). Moreover the map μ induces an injection on cohomology $\mu^* : H(G_{i,j}(\mathbf{C})) \to H(G_{q,1}^{\oplus j}(\mathbf{C}))$ (see [21, Lemma 3.1.5, p. 182]). It follows from the definitions that for a line bundle $\overline{L}_i$, we have $\gamma^n(\overline{L}_i - 1) = 0$ for $n > 1$ and thus $\widehat{K}_0(G_{q,1}^{\oplus j})$ is locally γ- nilpotent at $\overline{L}_i - 1$. Now we can compute

$$\mu^*(\gamma^{r_1}(q)\gamma^{r_2}(q)\dots\gamma^{r_d}(q)) =$$

$$\gamma^{r_1}(\overline{L}_1 + \dots + \overline{L}_j - j)\dots\gamma^{r_d}(\overline{L}_1 + \dots + \overline{L}_j - j) =$$

$\gamma^{r_1}((\overline{L}_1 - 1) + (\overline{L}_2 - 1) + \ldots (\overline{L}_j - 1)) \ldots \gamma^{r_d}((\overline{L}_1 - 1) + (\overline{L}_2 - 1) + \ldots (\overline{L}_j - 1))$.

By the preceding remark and 4.4, the last expression vanishes for $m >> 0$. Therefore $\gamma^{r_1}(q)\gamma^{r_2}(q)\ldots\gamma^{r_d}(q)$ vanishes also for such m, since it lies in the even de Rahm cohomology and μ^* is injective there. This completes the proof. **Q.E.D.**

For the next Proposition, we shall need the

Lemma 4.4 *Let $y_1, \ldots y_r$ be elements of an augmented λ-ring R. Suppose that R is locally γ-nilpotent at each of the $y_1, \ldots y_d$. Then it is locally γ-nilpotent at the sum $y_1 + \ldots + y_d$.*

Proof: Since we can apply induction on d, we can assume without loss of generality that $d = 2$. Let m and $r_1, \ldots + r_d$ be natural numbers such that $r_1 + \ldots + r_d > m$. Using the fact that γ_t is a homomorphism, we can compute

$$\gamma^{r_1}(y_1 + y_2)\gamma^{r_2}(y_1 + y_2)\ldots\gamma^{r_d}(y_1 + y_2) = \prod_{i=1}^{d}(\sum_{j=0}^{r_i}\gamma^j(y_1)\gamma^{r_i - j}(y_2)).$$

The last expression is a sum of terms of the form

$$\gamma^{r'_1}(y_1)\gamma^{r'_2}(y_1)\ldots\gamma^{r'_l}(y_1)\gamma^{r'_{l+1}}(y_2)\ldots\gamma^{r'_d}(y_2)$$

where $1 \leq l \leq d$ and $r'_1 + \ldots + r'_d > m$. Now choose m such that $m > 2.n(y_1)$ and $m > 2.n(y_2)$. Then either $\gamma^{r'_1}(y_1)\gamma^{r'_2}(y_1)\ldots\gamma^{r'_l}(y_1) = 0$ or $\gamma^{r'_{l+1}}(y_2)\ldots\gamma^{r'_d}(y_2) = 0$, since either $r'_1 + \ldots + r'_l > m/2$ or $r'_{l+1} + \ldots + r'_{d'} > m/2$. This shows that we can choose $n(y_1 + y_2) = m$ and ends the proof. **Q.E.D.**

Notice that if any morphism $g : Y \to B$ of arithmetic varieties is given, there is a natural pull-back map $g^* : \widehat{K}_0(B) \to \widehat{K}_0(Y)$, given by the formula $g^*((E, h) + \eta) := (g^*E, g^*h) + g^*\eta$. The pull-back map is a ring morphism, which preserves the λ-operations.

Proposition 4.5 *Let Y be any arithmetic variety. The γ-filtration of $\widehat{K}_0(Y)$ is locally nilpotent.*

Proof: In $\widehat{K}_0(Y)$, for all $y \in F^1\widehat{K}_0(Y)$, we have $y = \kappa + \overline{E} - \overline{F}$, where $\kappa \in \tilde{A}(Y)$ is a differential form and $\overline{E}$, $\overline{F}$ are hermitian vector bundles of same rank. Notice the following:

Lemma 4.6 *The Grothendieck group of vector bundles $K_0(Y)$ of Y is generated as a group by globally generated vector bundles.*

Proof of 4.6: since Y is quasi-projective, there is an immersion $Y \to \mathbf{P}^r_{\mathbf{Z}}$. Recall that there is an isomorphism $\mathbf{Z}[T]/((1 - T)^{r+1}) \simeq K_0(\mathbf{P}^r_{\mathbf{Z}})$ given by

$T \mapsto \mathcal{O}(1)$. This implies that if E is any vector bundle on Y, we can write $E = E(1-(1-\mathcal{O}(1))^{r+1})^k$, for any $k \geq 1$. But $E(1-(1-\mathcal{O}(1))^{r+1})^k$ is a linear combination of elements $E(i)$, for $i \geq k$. If we let k be sufficiently big, all the $E(i)$ will thus be globally generated (see [25, Th. 8.8, p. 252, III]), which finishes the proof. **Q.E.D**.

To prove Proposition 4.5, consider that in view of the preceding lemma, we can assume that E and F are globally generated. We can also assume that E and F are endowed with some metrics of our choice, since a modification of the metrics is equivalent to the addition of an element of $\tilde{A}(Y)$, by the definition of $\widehat{K}_0(Y)$. By definition, there are natural numbers N and M and morphisms $f_E : Y \to G_{N,rg(E)}$ and $f_F : Y \to G_{M,rg(F)}$ such that the isomorphisms $f_E^*(Q_{N,rg(E)}) \simeq E$ and $f_F^*(Q_{M,rg(F)}) \simeq F$ hold. Clearly, we may assume that N is a multiple of $rg(E)$ and M a multiple of $rg(F)$. Endow the universal bundles $Q_{N,rg(E)}$ and $Q_{M,rg(F)}$ with their canonical quotient metrics. Endow E and F with the metrics arising from the isomorphisms. By the last Proposition, $\widehat{K}_0(Y)$ is locally γ-nilpotent at $f_E^*(\overline{Q}_{N,rg(E)} - rg(E)) = \overline{E} - rg(E)$ and at $f_F^*(\overline{Q}_{M,rg(F)} - rg(F)) = \overline{F} - rg(F)$. By 4.1, it is also locally γ-nilpotent at κ. By 4.4, it is thus locally nilpotent at $(\overline{E} - rg(E)) - (\overline{F} - rg(F)) + \kappa = \kappa + \overline{E} - \overline{F}$, which completes the proof. **Q.E.D**.

Notice that in view of 4.2, the Bott element of every hermitian bundle on Y is invertible in $\widehat{K}_0(Y) \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{k}]$.

Open questions. Is the group $\widehat{K}_0(Y)$ generated by λ-finite elements ? Does the γ-filtration $F^i\widehat{K}_0(Y)$ actually vanish for $i > dim(Y)$?

5 Analytical preliminaries

5.1 The higher analytic torsion

In this subsection, we shall recall the definition of the higher analytic torsion, as it is needed to define the push-forward map of arithmetic K_0-theory. The higher analytic torsion can be viewed as a sort of relative version of the Bott-Chern secondary classes and was defined in [12, Def. 3.8, p. 668]. In [22, Th. 3.1, p. 41] and [16, Lecture 5], one finds different attempts to define an object with properties similar to the object defined in [12, Def. 3.8, p. 668]. A good reference for the background material needed for this section is [5]. Let $f : M \to S$ be a proper smooth holomorphic map of complex manifolds. Denote by J^{Tf} the almost complex structure on the real tangent bundle underlying the relative complex tangent bundle Tf. Suppose that Tf is endowed with some hermitian metric h. Let T^HM be a (differentiable) complex subbundle of TM, such that there is a direct sum decomposition $TM = T^HM \oplus Tf$. In the following, we shall identify real differential forms with complex conjugation invariant differential forms. The following definition is taken from [12, Def. 1.1, p. 650].

Definition 5.1 *The map f together with the bundle $T^H M$ and the hermitian metric h define a Kähler fibration if there is a real closed $(1,1)$ form ω on M such that $T^H M$ and Tf are orthogonal with respect to ω and such that the equation $\omega(X,Y) = h(X, J^{Tf}Y)$ holds for all $X, Y \in Tf_m$ and all $m \in M$.*

We shall suppose that the triple $f, T^H M, h$ form a Kähler fibration and fix an associated differential form ω with the above properties. It is shown in [9, II,1.] that for a given Kähler fibration, the form ω is unique up to addition of a form $f^*\eta$, where η is a real closed $(1,1)$-form on S. Moreover, for given f, a Kähler metric on M defines a Kähler fibration, if we choose $T^H M$ to be the orthogonal complement of Tf in TM, ω to be the Kähler form associated to the metric and h to be the metric obtained by restriction.

We shall from now on use the subscript $_{\mathbf{R}}$ to denote the underlying real bundle of a complex bundle (e.g. $T^H_{\mathbf{R}} M$ etc.). The subscript $_{\mathbf{C}}$ will denote the complexification of the underlying real bundle of a complex bundle (e.g. $T^H_{\mathbf{C}} M = T^H_{\mathbf{R}} M \otimes_{\mathbf{R}} \mathbf{C}$ etc.). Fix a Riemannian metric on $T_{\mathbf{R}}S$. Let $\nabla^{T_{\mathbf{R}}S}$ be the Levi-Civita connection on S, which is the unique metric torsion free connection on $T_{\mathbf{R}}S$. Let $\nabla^{T_{\mathbf{R}}f}$ be the real connection induced on $T_{\mathbf{R}}f$ by the canonical holomorphic hermitian connection on Tf. The natural identification of C^∞ bundles $f^*T_{\mathbf{R}}S \simeq T^H_{\mathbf{R}}M$ yields a connection $\nabla^{T^H_{\mathbf{R}}M}$ on $T^H_{\mathbf{R}}M$. Via the direct sum decomposition $TM = T^H M \oplus Tf$, we thus get a connection on $T_{\mathbf{R}}M$. Denote its torsion by T; this is a (real) 2-form with values in $T_{\mathbf{R}}M$. It is shown in [9, II] that its values are in $T_{\mathbf{R}}f \subseteq T_{\mathbf{R}}M$ and that T doesn't depend on the metric chosen on $T_{\mathbf{R}}S$. The torsion T measures the extent to which the horizontal bundle is not integrable.

The bundle $T_{\mathbf{C}}f$ carries a natural hermitian metric and thus yields a bundle of Clifford algebras $C(T_{\mathbf{C}}f)$ (for the definition of a Clifford algebra, see (see [28, Th. 8.1, p. 512]).

Now let ξ be a holomorphic bundle on M. Denote by $T^{(0,1)}f$ the differentiable bundle of $-i$ eigenspaces of the endomorphism $J^{Tf} \otimes_{\mathbf{R}} \mathbf{C}$ of $T_{\mathbf{C}}f$ and let $T^{*(0,1)}f$ its complex dual. Let $T^{(1,0)}$ be the differentiable bundle of i eigenspaces of $J^{Tf} \otimes_{\mathbf{R}} \mathbf{C}$. Denote by $\Lambda(T^{*(0,1)}f)$ the associated bundle of exterior algebras. There is a fibrewise $C(T_{\mathbf{C}}f)$-module structure on the bundle $\Lambda(T^{*(0,1)}f) \otimes \xi$. By the universal property of Clifford algebras, to define the module structure, it is sufficient to describe the action of elements $W \in T_{\mathbf{C}}f_m$ on $(\Lambda(T^{*(0,1)}f) \otimes \xi)_m$ ($m \in M$). Let $W = U + V$, where $U \in T^{(1,0)}f_m$ and $V \in T^{(0,1)}f_m$. Let U' be element of $T^{*(0,1)}f_m$ defined by the formula $U'(Y) = h(U, Y)$ (where we view h as extended to $T_{\mathbf{C}}f$). We define the complex endomorphism $c(W)$ by the formula $c(W)(.) = \sqrt{2}U' \wedge (.) - \sqrt{2}\iota_{(.)}$, where ι is the contraction operator (see [5, Def. 1.6., p. 18]). In the following $\widehat{\otimes}$ refers to the $\mathbf{Z}_2$-graded tensor product. Recall that every $\mathbf{Z}$-graded vector space carries a natural $\mathbf{Z}_2$-grading.

The following definition is taken from [12, Def. 1.6, p. 653].

Definition 5.2 *For each point $p \in S$, let $f_1, \dots f_{2n}$ be a basis of $T_{\mathbf{R}} S_p \subseteq T_{\mathbf{C}} S_p$ and $f^1, \dots f^{2n}$ be its dual basis in $T^*_{\mathbf{R}} S_p$. The element*

$$c(T) \in (f^* \Lambda(T^*_{\mathbf{C}} S) \widehat{\otimes} (End(\Lambda(T^{*(0,1)} f) \otimes \xi)))^{odd}$$

is defined by the formula

$$c(T) = \frac{1}{2} \sum_{\substack{1 \le \alpha \le 2n \\ 1 \le \beta \le 2n}} f^\alpha \wedge f^\beta \widehat{\otimes} c(T(f^H_\alpha, f^H_\beta))$$

The upperscript $(.)^H$ refers to the horizontal lift, obtained via the natural isomorphism $f^* T_{\mathbf{R}} S \simeq T^H_{\mathbf{R}} M$. It can be shown that the definition 5.2 doesn't depend on the choice of the basis. Notice now that the bundle $\Lambda(T^{*(0,1)} f)$ carries a natural connection, induced by the holomorphic hermitian connection on Tf. Suppose that ξ is equipped with a hermitian metric h^ξ. The bundle $\Lambda(T^{*(0,1)} f) \otimes \xi$ is then also endowed with a natural connection, which is the tensor product of the connection on $\Lambda(T^{*(0,1)} f)$ with the hermitian holomorphic connection on ξ. Both of these connections are by construction hermitian. We now let E be the infinite dimensional bundle on S whose fiber at each point $p \in S$ consists of the C^∞ sections of $(\Lambda(T^{*(0,1)} f)) \otimes \xi|_{f^{-1} p}$. The following definition is taken from [12, (b), p. 651]:

Definition 5.3 *Let $u > 0$. The Bismut (or Levi-Civita) superconnection on E is the differential operator*

$$B_u = \nabla^E + \sqrt{u}(\overline{\partial}^Z + \overline{\partial}^{Z*}) - \frac{1}{2\sqrt{2u}} c(T)$$

on $f^*(\Lambda(T^*_{\mathbf{C}} S)) \widehat{\otimes} (\Lambda(T^{*(0,1)} f) \otimes \xi)$.

The operator ∇^E is the superconnection on E associated to the hermitian connection on $\Lambda(T^{*(0,1)} f) \otimes \xi$ and the horizontal bundle $T^H_{\mathbf{C}} M$; see [5, Prop. 9.13, p. 283] for the definition. The operator $\overline{\partial}^Z$ is the Dolbeaut operator along the fibers of f and we let $\overline{\partial}^{Z*}$ denote its formal adjoint. Both are differential operators on $\Lambda(T^{*(0,1)} f) \otimes \xi$.

Definition 5.4 *The operator N_V is the endomorphism of $\Lambda(T^{*(0,1)} f) \otimes \xi$ acting on $\Lambda^p(T^{*(0,1)} f) \otimes \xi$ as multiplication by p. The element $\omega^{H\overline{H}}$ is the section of $f^*(\Lambda^2(T^*_{\mathbf{R}} S)) \subseteq f^*(\Lambda^2(T^*_{\mathbf{C}} S))$ defined by the formula $\omega^{H\overline{H}}(U, V) = \omega(U^H, V^H)$, where U, V are in some fiber of $T_{\mathbf{C}} S$. For $u > 0$, let N_u be the section of $f^*(\Lambda(T^*_{\mathbf{C}} S)) \widehat{\otimes} End(\Lambda(T^{*(0,1)} f) \otimes \xi)$ defined by the formula $N_u := N_V + \frac{i}{u} \omega^{H\overline{H}}$.*

Following [12, Def. 3.8, p. 668], we now proceed to define the higher analytic torsion. From now on we make the hypothesis that ξ is f-acyclic, i.e. its non-zero relative cohomology groups vanish. Let ϕ be the endomorphism of $\Lambda(T^*_{\mathbf{C}}S)$ which acts as multiplication by $(2i\pi)^{-q/2}$ on $\Lambda^q(T^*_{\mathbf{C}}S)$ (we fix an arbitrary square root of i). Do not confuse ϕ with the operator ϕ^k defined before 3.3! Notice that since B_u is a superconnection, its square B_u^2 is a family of differential operators acting on the fibers of f, with differential form coefficients. Furthermore, the restriction of B_u^2 to each fiber of f is the sum of a nilpotent operator and a generalized Laplacian on E; we can thus associate to B_u^2 a (smooth) family of kernels (see [5, Th. 9.51, p. 315]), which is written $exp(-B_u^2)$. The family $exp(-B_u^2)$ can be viewed as a section of $f^*(\Lambda(T^*_{\mathbf{C}}S))\widehat{\otimes}End(\Lambda(T^{*(0,1)}f)\otimes\xi)$. The bundle $f_*\xi$ can be endowed with a metric built from the metric of ξ and the form ω associated to the fibration. By definition, elements $U, V \in f_*\xi|_p$ of a fiber of $f_*\xi$ at a point $p \in S$ correspond to holomorphic sections of $\xi|_{f^{-1}p}$. Let $d = dim(M) - dim(S)$; we define a pairing $< ., . >$ on $f_*\xi|_p$ by the formula

$$< U, V >:= \frac{1}{(2\pi)^d}\int_{f^{-1}p} h^\xi(U, V)\omega^d/d!.$$

This pairing defines a hermitian metric on $f_*\xi$, which shall be denoted by the symbol f_*h^ξ (see also [12, p. 666]). For each section l of $f^*(\Lambda(T^*_{\mathbf{C}}S))\widehat{\otimes}End(\Lambda(T^{*(0,1)}f)\otimes\xi)$, we can form the pointwise supertrace $Tr_s(l) \in f^*(\Lambda(T^*_{\mathbf{C}}S))$; if we take the mean of $Tr_s(l)$ over the fibers of $f^*(\Lambda(T^*_{\mathbf{C}}S))$, with the volume form $\omega^d/d!$, we obtain an element of $\Lambda(T^*_{\mathbf{C}}S)$, which we also call $Tr_s(l)$ (see [5, p. 285]). The symbol $-(\nabla^{f_*\xi})^2$ will refer to the square of the hermitian holomorphic connection on $f_*\xi$ endowed with the metric f_*h^ξ and with the trivial $\mathbf{Z}_2$-grading. It is an element of $\Lambda(T^*_{\mathbf{C}}S)\widehat{\otimes}End(f_*\xi)$. In the coming definition, Γ will be Euler's Gamma function.

Definition 5.5 *For $s \in \mathbf{C}$ with $Re(s) > 1$ let*

$$\zeta^1(s) := -\frac{1}{\Gamma(s)}\int_0^1 u^{s-1}\{\phi(Tr_s(N_u.exp(-B_u^2)) - Tr_s(N_V.exp(-(\nabla^{f_*\xi})^2)))\}du$$

and similarly for $s \in \mathbf{C}$ with $Re(s) < 1/2$ let

$$\zeta^2(s) := -\frac{1}{\Gamma(s)}\int_1^\infty u^{s-1}\{\phi(Tr_s(N_u.exp(-B_u^2)) - Tr_s(N_V.exp(-(\nabla^{f_*\xi})^2)))\}du$$

It it shown in [12, p. 668] that ζ^1 extends to a meromorphic function of s, holomorphic for $|Re(s)| < 1/2$ and that $\zeta^2(s)$ is holomorphic for $|Re(s)| < 1/2$.

Definition 5.6 *The higher analytic torsion $T(\omega, h^\xi)$ of ξ is the differential form $\frac{\partial}{\partial s}(\zeta^1 + \zeta^2)(0)$.*

When the fibration arises from a Kähler metric h_M on M, we shall also use the notation $T(h_M, h^\xi)$ in place of $T(\omega, h^\xi)$. The higher analytic torsion satisfies the following equality, which establishes the link with the Bott-Chern secondary classes appearing in the definition of arithmetic K_0-theory.

Proposition 5.7 *The form $T(\omega, h^\xi)$ is real (conjugation invariant) and a sum of forms of type (p,p) $(p>0)$. It satisfies the equation of currents*

$$dd^c T(\omega, h^\xi) = ch(f_*\xi, f_*h^\xi) - \int_{M/S} Td(Tf, h^{Tf}) ch(\xi, h^\xi).$$

Its component in degree 0 is the Ray-Singer analytic torsion of ξ in each fiber of M over S.

For the proof, whose essential ingredient is the local index theorem, we refer to [12, Th. 3.9, p. 669]. Notice that the last Proposition can be viewed as a "double transgressed" version of the Riemann-Roch theorem with values in real de Rahm cohomology. The following theorem studies the dependence of T on ω:

Theorem 5.8 *Let ω' be the form associated to another Kähler fibration for $f: M \to S$. Let $g^{'Tf}$ be the metric on Tf in this new fibration. The following identity holds in $\tilde{A}(S) = \oplus_{p\geq 0}(A^{p,p}(S)/(Im\partial + Im\overline{\partial}))$:*

$$T(\omega', h^{'\xi}) - T(\omega, h^\xi) = -\int_{M/S} \widetilde{Td}(Tf, g^{Tf}, g^{'Tf}) ch(\xi, h^\xi) + \widetilde{ch}(g_*^\omega h^E, g_*^{\omega'} h^E).$$

Here $\widetilde{Td}(Tf, g^{Tf}, g^{'Tf})$ refers to the Todd secondary class of the sequence

$$0 \to Tf \to Tf \to 0 \to 0,$$

where the second term is endowed with the metric g^{Tf} and the third term with the metric $g^{'Tf}$. The term $\widetilde{ch}(g_*^\omega h^\xi, g_*^{\omega'} h^\xi)$ is the Chern secondary class of the sequence

$$0 \to g_*\xi \to g_*\xi \to 0 \to 0,$$

where the second term carries the metric obtain by integration along the fibers with the volume form coming from ω' and the third one the metric obtain by integration along the fibers with the volume form coming from ω. For the proof, we refer to [12, Th. 3.10, p. 670].

5.2 The singular Bott-Chern current

The singular Bott-Chern current is a generalisation of the usual Bott-Chern form to sequences involving coherent sheaves supported on regular closed subvarieties.

In the sequel, let $M' \xrightarrow{i} M$ be an embedding of complex manifolds, with normal bundle N. Recall that the space of currents $D^{p,q}(M)$ is the topological dual of the space of differential forms $A^{n-p,n-q}(M)$ ($n = dim(M)$) equipped with the Schwartz topology. Furthermore, to each current γ on M, one may associate a closed conical subset $WF(\gamma)$ of $T^*_{\mathbf{R}}M$, called the **wave front set** of γ; if two currents have disjoint wave front sets, their exterior products can be defined. See [27] for more details.

Definition 5.9 *The set $P^M_{M'}$ is the vector space of real currents ω on M such that*

(a) *ω is a sum of currents of type p,p ($p \geq 0$);*

(b) *The wave front set of ω is contained in $N^*_{\mathbf{R}} \subseteq T^*_{\mathbf{R}}M$.*

Definition 5.10 *The set $P^{M,0}_{M'}$ is the subset of $P^M_{M'}$ consisting of currents of the form $\partial\alpha + \overline{\partial}\beta$, where α and β are currents whose wave front set is included in $N^*_{\mathbf{R}}$. The sets P^M and $P^{M,0}$ are defined similarly, omtting condition (b).*

Let

$$\Xi : \ 0 \to \xi_m \to \xi_{m-1} \to \ldots \to \xi_0 \to i_*\eta \to 0$$

be a resolution in M by holomorphic vector bundles ξ_i of the coherent analytic sheaf $i_*\eta$, where η is a vector bundle on M'. Let $F = \oplus_{i=0}^m H^i(\Xi)$ be the direct sum of the homology sheaves of Ξ. There is a natural identification of graded bundles $i^*F \simeq \oplus_{i=0}^{rk(N)} \Lambda^i(p_N^*(N^\vee)) \otimes \eta$ (see [24, Prop. 2.5, p. 431]). Now fix hermitian metrics on N and η and hermitian metrics on the ξ_i. Homology sheaves carry the quotient metrics and direct sums, duals, exterior powers and tensor products of bundles carry the orthogonal sum, dual, exterior power and tensor product metrics; thus we see that both of the just described graded bundles carry natural metrics.

Definition 5.11 *We say that the hermitian metrics on the bundles ξ_i satisfy Bismut's assumption (A) with respect to the metrics on N and η if the isomorphism $i^*F \simeq \oplus_{i=0}^{rk(N)} \Lambda^i(p_N^*(N^\vee)) \otimes \eta$ also identifies the metrics.*

It is proved in [8, Prop. 1.6] that there always exist metrics on the ξ_i such that this assumption is satisfied. For more details see [10, p. 259]. Let us suppose now that the bundles ξ_i are equipped with hermitian metrics on M and that the bundle η is equipped with a hermitian metric on M', which satisfy Bismut's assumption (A) with respect to N. The **singular Bott-Chern current** of $\overline{\Xi}$ is an element $T(h^{\xi_\cdot})$ of $P^M_{M'}$ satisfying the equation

$$dd^c T(h^{\xi_\cdot}) = i_*(Td^{-1}(\overline{N})ch(\overline{\eta})) - \sum_{i=0}^{m} (-1)^i ch(\overline{\xi}_i)$$

(see [10, Th. 2.5, p. 266]). Here i_* refers to the pushforward of currents. If i is the identity, Ξ becomes an exact sequence of bundles on M and the singular Bott-Chern current a differential form, which coincides with the Bott-Chern secondary class of Ξ defined in [9, Par. f)].
If $f : F \to M$ is a holomorphic map tranversal to M', the equation $T(h^{f^*\xi_\cdot}) = f^*T(h^{\xi_\cdot})$ holds for the holomorphic resolution $f^*\xi_\cdot$ of $(f|_{f^{-1}(M')})^*\eta$ (endowed with the pull-back metric). Furthermore, the following result holds:

Proposition 5.12 *Let*

$$\begin{array}{ccccccccc} & 0 & & 0 & & & 0 & & 0 \\ & \uparrow & & \uparrow & & & \uparrow & & \uparrow \\ 0 \to & \xi^0_m & \to & \xi^0_{m-1} & \to \dots \to & & \xi^0_0 & \to & i_*\eta^0 \to 0 \\ & \uparrow & & \uparrow & & & \uparrow & & \uparrow \\ 0 \to & \xi^1_m & \to & \xi^1_{m-1} & \to \dots \to & & \xi^1_0 & \to & i_*\eta^1 \to 0 \\ & \uparrow & & \uparrow & & & \uparrow & & \uparrow \\ & \dots & & \dots & & & \dots & & \dots \\ & \uparrow & & \uparrow & & & \uparrow & & \uparrow \\ 0 \to & \xi^n_m & \to & \xi^n_{m-1} & \to \dots \to & & \xi^n_0 & \to & i_*\eta^n \to 0 \\ & \uparrow & & \uparrow & & & \uparrow & & \uparrow \\ & 0 & & 0 & & & 0 & & 0 \end{array}$$

be the elements of an exact sequence of complexes resolving an exact sequence of bundles η_j on M', for $0 \leq j \leq n$. Fix a hermitian metric on the normal bundle $N_{M/M'}$ and suppose that the rows are endowed with metrics satisfying Bismut's assumption (A). Then the following formula holds:

$$\sum_{j=0}^{n} (-1)^j T(h^{\xi^j_\cdot}) = i_*(Td^{-1}(\overline{N})\widetilde{ch}(\overline{\eta}^\cdot)) - (\sum_{i=0}^{m} (-1)^i \widetilde{ch}(\overline{\xi}^\cdot_i))$$

in $P^M_{M'}/P^{M,0}_{M'}$.

Proof: See [11, Th. 2.9, p. 279]. **Q.E.D.**

We shall not recall the definition of $T(h^{\xi_\cdot})$ here since we shall only need its above mentioned properties and since it doesn't appear in the final result of the paper; see [10] for the definition.

Proposition 5.13 *Let $\overline{\xi}$ be a hermitian holomorphic vector bundle on M and let s be a regular section of ξ. Let*

$$0 \to \Lambda^{rank(\xi)}(\xi^\vee) \to \dots \xi^\vee \to \mathcal{O}_{Z(s)} \to 0$$

be the Koszul resolution it induces on M, where $Z(s)$ is the zero-scheme of s. Endow the elements of this resolution with the exterior power metrics, the normal bundle with the metric induced by ξ and $\mathcal{O}_{Z(s)}$ with the trivial metric. Then *these metrics satisfy Bismut's assumption (A) and the current* $Td(\overline{\xi})T(h^{\Lambda^{\cdot}\xi})$ *is of type* $rank(\xi)-1, rank(\xi)-1$ *in* $P^M_{M'}/P^{M,0}_{M'}$.

Proof: See [11, Th. 3.17, p. 301]. **Q.E.D.**

Recall that a section s as above is regular iff it is transverse to the zero section. The current $g = Td(\overline{\xi})T(h^{\Lambda^{\cdot}\xi})$ will be called *the Green current associated to s.*

Proposition 5.14 *Suppose that* $\tilde{M}' \xrightarrow{\tilde{i}} M$ *and* $M' \xrightarrow{i} M$ *are closed analytic subvarieties meeting transversally. Suppose that the normal bundles* $\tilde{N}$ *of* $\tilde{M}'$ *and* N *of* M' *are endowed with hermitian metrics. Let*

$$\Xi : 0 \to \xi_m \to \xi_{m-1} \to \dots \xi_0 \to i_*\eta \to 0$$

be a resolution by hermitian bundles in M *of the hermitian bundle* η *in* M' *and let*

$$\tilde{\Xi} : 0 \to \tilde{\xi}'_{\tilde{m}} \to \tilde{\xi}'_{\tilde{m}-1} \to \dots \tilde{\xi}'_0 \to \tilde{i}_*\tilde{\eta} \to 0$$

be a resolution by hermitian bundles in M *of the hermitian bundle* $\tilde{\eta}$ *in* $\tilde{M}'$. *Let*

$$\Xi'' : 0 \to \xi''_{m+\tilde{m}} \to \xi''_{m+\tilde{m}-1} \to \dots \xi''_0 \to \eta|_{\tilde{M}'} \otimes \tilde{\eta}|_{M'} \to 0$$

be the tensor product resolution $\Xi \otimes \Xi'$, *which resolves the bundle* $\eta|_{\tilde{M}'} \otimes \tilde{\eta}|_{M'}$ *on* $M' \cap \tilde{M}'$. *Suppose that the resolutions* Ξ *and* $\tilde{\Xi}$ *both satisfy Bismut's assumption (A) with respect to the metrics on the normal bundles. Endow the normal bundle of* $M' \cap \tilde{M}'$ *in* M *with the metric arising from its canonical identification with* $N|_{\tilde{M}'} \oplus \tilde{N}|_{M'}$. *Then the formula*

$$T(h^{\xi''_{\cdot}}) = \{\sum_{i=0}^{m}(-1)^i ch(\overline{\xi}_i))T(h^{\tilde{\xi}_{\cdot}}\} + \tilde{i}_*\{Td^{-1}(\overline{\tilde{N}})ch(\overline{\tilde{\eta}})\tilde{i}^*(T(h^{\xi_{\cdot}}))\}$$

holds in $P^M_{M'\cup\tilde{M}'}/P^{M,0}_{M'\cup\tilde{M}'}$.

Here the space $P^M_{M'\cup\tilde{M}'}/P^{M,0}_{M'\cup\tilde{M}'}$ is defined similarly to the space $P^M_{M'}/P^{M,0}_{M'}$, by requiring all the involved currents to have their wave front sets included in $N^*_{\mathbf{R}} + \tilde{N}^*_{\mathbf{R}}$. For the proof of 5.14, we refer to [10, Th. 2.7, p. 271].

Corollary 5.15 *The singular Bott-Chern current of the resolution*

$$\Xi \otimes \alpha : 0 \to \xi_m \otimes \alpha \to \xi_{m-1} \otimes \alpha \to \dots \xi_0 \otimes \alpha \to i_*(\eta \otimes \alpha) \to 0$$

where $\overline{\alpha}$ *is a hermitian bundle on* M, *is equal to* $ch(\overline{\alpha})T(h^{\xi_{\cdot}})$ *in* $P^M_{M'}/P^{M,0}_{M'}$.

5.3 Bismut's theorem

We shall now state the fundamental theorem of Bismut describing the behaviour of the relative analytic torsion under immersions (see [6]). Let $i : M' \to M$ be closed immersion of complex manifolds and let $g : M' \to S$, $f : M \to S$ be smooth proper holomorphic maps such that $g = f \circ i$. Let

$$\Xi : 0 \to \xi_m \to \dots \xi_0 \to i_*\eta \to 0$$

be a resolution with metrics as at the beginning of 5.2. Suppose that Tf is endowed with a hermitian metric h and that a horizontal tangent bundle $T^H M$ is given, such that h, $T^H M$ and f define a Kähler fibration. Let ω be a real $(1,1)$-form associated to this fibration. We endow M' with the fibration structure, which is the restriction of the fibration structure on M and with the associated form $\omega' = i^*\omega$. We shall write $\widetilde{Td}(g/f)$ for $\widetilde{Td}(\overline{\mathcal{N}})$, where $\overline{\mathcal{N}}$ is the sequence

$$0 \to Tg \to Tf \to N \to 0$$

where N is the normal bundle of the immersion, endowed with the quotient metric. Recall that $\widetilde{Td}(\overline{\mathcal{N}})$ is a Bott-Chern secondary class and satisfies the equation

$$dd^c\widetilde{Td}(\overline{\mathcal{N}}) = Td(\overline{Tf}) - Td(\overline{Tg} \oplus \overline{N}).$$

We also suppose in this subsection that the ξ_i are f-acyclic and that η is g-acylic. Let $f_*\Xi$ denote the sequence

$$0 \to f_*(\xi_m) \to f_*(\xi_{m-1}) \to \dots \to f_*(\xi_0) \to g_*\eta \to 0.$$

It is exact, by the properties of long exact cohomology sequences associated to the functor f_*. By the semi-continuity of the Euler characteristic, all the elements of $f_*(\Xi)$ are vector bundles and we can thus endow them with the metrics f_*h^{ξ} and g_*h^{η} obtained by integration on the fibers.

Theorem 5.16 *The equality*

$$\sum_{i=0}^{m}(-1)^i T(\omega, h^{\xi_i}) - T(\omega', h^{\eta}) + \widetilde{ch}(f_*\Xi) =$$

$$\int_{M'/S} ch(\eta)R(N)Td(Tg) + \int_{M/S} T(h^{\xi_\cdot})Td(\overline{Tg}) - \int_{M'/S} ch(\overline{\eta})\widetilde{Td}(g/f)Td^{-1}(\overline{N})$$

holds in $\tilde{A}(S)$.

For an announcement of the proof, see [6]; for the proof itself, which is very long and technical, see [7]. A proof in the case that S is a point, as well as an overview of the involved techniques is contained in [13].

Proof of 3.1: Consider the group $\widehat{K}_0^{ac}(Y)$, whose generators are the g-acyclic hermitian bundles on Y and the elements of $\tilde{A}(Y)$, with same relations as the group $\widehat{K}_0(Y)$. A theorem of Quillen (see [31, Cor. 3., p. 111]) for the algebraic analogs of these groups implies that the natural map $\widehat{K}_0^{ac}(Y) \to \widehat{K}_0(Y)$ is an isomorphism. Consider now an exact sequence

$$\mathcal{E} : 0 \to E' \to E \to E'' \to 0$$

of g-acyclic bundles on Y, endowed with (conjugation invariant) hermitian metrics. Using the just mentioned isomorphism, the definition of g_* and the defining relations of $\widehat{K}_0(Y)$, we see that to prove our claim, it will be sufficient to prove that

$$(g_*E, g_*h^E) - T(h_Y, h^E) + \int_{Y/B} Td(\overline{Tg_{\mathbf{C}}})\widetilde{ch}(\overline{\mathcal{E}}) - (g_*E', g_*h^{E'}) +$$

$$T(h_Y, h^{E'}) - (g_*E'', g_*h^{E''}) + T(h_Y, h^{E''}) = 0 \qquad (4)$$

in $\widehat{K}_0(B)$. According to 5.16 (applied with the identity as immersion) and the remarks made before 5.14, the equation

$$T(h_Y, h^{E'}) - T(h_Y, h^E) + T(h_Y, h^{E''}) + \widetilde{ch}(g_*\mathcal{E}) = -\int_{Y/B} Td(\overline{Tg_{\mathbf{C}}})\widetilde{ch}(\overline{\mathcal{E}}) \quad (5)$$

holds in $\tilde{A}(B)$. By the defining relations of $\widehat{K}_0(Y)$, we have

$$(g_*E, g_*h^E) + \widetilde{ch}(g_*\mathcal{E}) - (g_*E', g_*h^{E'}) - (g_*E'', g_*h^{E''}) = 0 \qquad (6)$$

in $\widehat{K}_0(B)$. Combining (5) and (6), we see that (4) holds. This ends the proof. **Q.E.D.**

Remarks. (a) The case of 5.16 used in the above proof can be deduced from 5.7 by a simple geometric deformation argument (see [22, Th. 3.2, p. 46] for such an argument).
(b) On any arithmetic variety, one can define a K_0-theory of hermitian coherent sheaves (see [23, Def. 25, p. 499]); if the variety is regular, it can be proved to coincide with the K_0-theory of hermitian bundles (see [22, Lemma 13, p. 499]). If B is regular, one can use this isomorphism to obtain a push-forward map even if g is not flat. The argument is similar to the above argument, with hermitian bundles replaced by hermitian coherent sheaves. We shall stick to the definition of the push-forward map for the flat case, however, in view of the relative nature of the notion of flatness of a morphism and because arithmetic varieties are often assumed to be flat over $\mathbf{Z}$.

6 An Adams-Riemann-Roch formula for closed immersions

This section is devoted to the proof of the Adams-Riemann-Roch formula (2) mentioned in the introduction. The exact statement can be found in 6.22.

6.1 Geometric preliminaries

In this subsection, we shall define the geometric objects that will be needed for the proof.

6.1.1 The deformation to the normal cone

The strategy of proof of the Adams-Riemann-Roch theorem for closed immersions consists in studying the behaviour of the Adams operations along the fibres of a deformation parameterized by $\mathbf{P}^1_{\mathbf{Z}}$. Let $Y \xrightarrow{i} X$ be a regular closed immersion of schemes over a Dedekind domain D. Let N denote the normal bundle of i. Since we want to consider the arithmetic as well as the complex case, let D be $\mathbf{Z}$ or $\mathbf{C}$ in this subsection. In the sequel, the notation $\mathbf{P}(E)$, where E is a vector bundle on any scheme, will refer to the space $Proj(Sym(E^\vee))$. Note that P can naturally be considered as a **covariant** functor.

Definition 6.1 *The deformation to the normal cone of the immersion i is the blow up W of $X \times \mathbf{P}^1_D$ along $Y \times \{\infty\}$.*

We define p_X to be the projection $X \times \mathbf{P}^1_D \to X$, p_Y the projection $Y \times \mathbf{P}^1_D \to Y$ and π the blow-down map $W \to X \times \mathbf{P}^1_D$. Let also q be the projection $X \times \mathbf{P}^1_D \to \mathbf{P}^1_D$ and q_W the map $q \circ \pi$. From the universality of the blow-up construction, we know that there is a canonical closed immersion $Y \times \mathbf{P}^1_D \xrightarrow{j} W$ such that $\pi \circ j = i \times Id$. We shall denote the map $\pi^{-1}|_{X \times \{0\}}$ by i_X. The following is known about the structure of W:

Proposition 6.2 *The closed subscheme $q^{-1}(\infty)$ is a Cartier divisor with two irreducible components P and $\tilde{X}$, that meet regularly. The component P is isomorphic to $\mathbf{P}(N \oplus 1)$ and the component $\tilde{X}$ is isomorphic to the blow-up of X along Y. The component $\tilde{X}$ does not meet $j(Y \times \mathbf{P}^1_D)$ and $j(Y \times \mathbf{P}^1_D) \cap P$ (scheme-theoretic intersection) is the image of the canonical section of $\mathbf{P}(N \oplus 1) \to Y$.*

Proof: See [18, Ch. 5]. **Q.E.D.**

The canonical section $i_\infty : Y \to \mathbf{P}(N \oplus 1)$ arises from the morphism of vector bundles $\mathcal{O}_Y \to N \oplus \mathcal{O}_Y$.

The embeddings of P and $\tilde{X}$ in W will be denoted by i_P and $i_{\tilde{X}}$. Let $p : P \to Y$ be the projection and $\phi := p_X \circ \pi : W \to X$.
The interest of W comes from the possibility to control the rational equivalence classe of the fibres $q^{-1}(p)$ ($p \in \mathbf{P}^1_D$). In the language of line bundles, this is expressed by the fact that $\mathcal{O}(X) \simeq \mathcal{O}(P + \tilde{X}) \simeq \mathcal{O}(P) \otimes \mathcal{O}(\tilde{X})$, which is an immediate consequence of the isomorphism $\mathcal{O}(\infty) \simeq \mathcal{O}(0)$ on $\mathbf{P}^1_D$.
This equality will enable us to reduce certain computations on X to computations on P, which is often much easier to handle. Indeed on P, the canonical quotient bundle Q has a canonical regular section s, which vanishes exactly on Y. Thus, the section s determines a global Koszul resolution

$$\mathcal{K} : 0 \to \Lambda^{dim(Q)}(Q^\vee) \to \ldots \to Q^\vee \to \mathcal{O}_P \to i_{\infty *}\mathcal{O}_Y \to 0.$$

Also, the immersion i_P (resp. i_X) is Tor independent of the immersion j. If X and Y are integral, we have the following alternative description of W via the Grassmannian graph construction of MacPherson.

Proposition 6.3 *Suppose that X and Y are integral. Let s be a section of a vector bundle E on X such that $Z(s) = Y$. Let f be the morphism $X \times \mathbf{A}^1_D \to \mathbf{P}(E \oplus 1) \times \mathbf{P}^1_D$ given by $f(x, a) = [a.s(x), 1] \times a$, where $[., .]$ denotes homogeneous coordinates and $\mathbf{A}^1_D \subseteq \mathbf{P}^1_D$ is the affine line over D. There is an isomorphism between W and the Zariski closure of $Im\ f$.*

Proof: [18, Ch. 5]. **Q.E.D.**

Lemma 6.4 *Suppose that X and Y are integral. The composition of the inclusion of W in $\mathbf{P}(E \oplus 1) \times \mathbf{P}^1_D$ with the morphism j is given by the formula $y \times a \mapsto [0, 1] \times a$.*

Proof: We have to show that the image of the map given by the formula $y \times a \mapsto [0, 1] \times a|_{y \in Y, a \in \mathbf{P}^1_D}$ is the closure of the image of the map given by the formula $y \times a \mapsto [0, 1] \times a|_{y \in Y, a \in \mathbf{A}^1_D}$. That is, we have to show that every Zariski closed subset of $\mathbf{P}(E \oplus 1) \times \mathbf{P}^1_D$ containing the image of the second map contains the image of the first map. Now the closure of $Y \times \mathbf{A}^1_D$ in $X \times \mathbf{P}^1_D$ is $Y \times \mathbf{P}^1_D$ and the map $p : \mathbf{P}(E \oplus 1) \times \mathbf{P}^1_D \to X \times \mathbf{P}^1_D$ is proper. Thus p_* sends every closed subset of $\mathbf{P}(E \oplus 1) \times \mathbf{P}^1_D$ containing the image of $y \times a \mapsto [0, 1] \times a|_{y \in Y, a \in \mathbf{A}^1_D}$ onto a set containing $Y \times \mathbf{P}^1_D$, which proves our claim.
Q.E.D.

6.1.2 Deformation of resolutions

One of the difficulties of a Riemann-Roch formula for embeddings in $\widehat{K}_0$-theory comes from the impossibility to represent explicitly coherent sheaves, in particular images of locally free sheaves by the embedding. One has to stick to certain explicit resolutions of these sheaves by locally free ones. Let η be a vector bundle on Y and

$$\Xi : 0 \to \xi_m \to \dots \xi_0 \to i_*\eta \to 0$$

a resolution of $i_*\eta$ in X. We shall make use of a particular extension of Ξ to W, whose existence is ensured by the following result:

Proposition 6.5 *There exists a resolution*

$$\tilde{\Xi} : 0 \to \tilde{\xi}_m \to \dots \tilde{\xi}_0 \to j_* p_Y^*(\eta) \to 0$$

on W extending Ξ and such that
(1) The restriction $\tilde{\Xi}|_{\tilde{X}}$ is split acyclic;
(2) There is an exact sequence of complexes on P

$$0 \to \mathcal{S} \to i_P^*(\tilde{\Xi}) \to \mathcal{K} \otimes p^*(\eta) \to 0$$

where $\mathcal{S}$ is split acyclic.

For the proof, we refer to [11, Th. 4.8, p. 318]. Recall that $\mathcal{K}$ is a Koszul resolution. We shall denote the complex $i_P^*(\tilde{\Xi})$ by $\xi_.^\infty$.

6.2 Proof of the Adams-Riemann-Roch theorem for closed immersions

In the next paragraphs, we shall very often use the following key fact. Let M be a complex manifold and let ζ, κ be two elements of P^M, with disjoint wave front sets. Then the equation

$$(dd^c\zeta) \wedge \kappa = \zeta \wedge (dd^c\kappa) \tag{7}$$

holds in $P^M/P^{M,0}$. The proof follows from the equalities $\partial(\zeta \wedge \overline{\partial}\kappa) = \partial\zeta \wedge \overline{\partial}\kappa + \zeta \wedge \partial\overline{\partial}\kappa$ and $-\overline{\partial}(\partial\zeta \wedge \kappa) = \partial\zeta \wedge \overline{\partial}\kappa + \partial\overline{\partial}\zeta \wedge \kappa$.

6.2.1 The case $k = 1$

In this subsection, we shall use Bismut's theorem to derive a formula comparing the push-forwards of $\overline{\eta}$ and of $\sum_{i=0}^m (-1)^i \overline{\xi}_i$ to a base B. This formula can be considered as a Riemann-Roch theorem for the immersion i and the Adams operation $\psi^1 = Id$.

Proposition 6.6 *Let $i : Y \to X$ be a regular closed immersion of arithmetic varieties and $g : Y \to B$, $f : X \to B$ be p.f.s.r. morphisms to an arithmetic variety B such that $g = f \circ i$. Let*

$$\Xi : \ 0 \to \xi_m \to \xi_{m-1} \to \dots \to \xi_0 \to i_*\eta \to 0$$

be a resolution by f-acyclic vector bundles on X of a g-acylic vector bundle η on Y. Suppose that X is endowed with a Kähler metric h_X, that Y carries the induced metric h_Y and that the normal bundle N of i carries the quotient metric. Suppose that η and the ξ_i are endowed with hermitian metrics satisfying Bismut's assumption (A) with respect to the metric of N. Then the equality

$$g_*(\overline{\eta}) - \sum_{i=0}^{m}(-1)^i f_*(\overline{\xi}_i) =$$

$$\int_{Y/B} ch(\eta)R(N)Td(Tg) + \int_{X/B} T(h^{\xi_\cdot})Td(\overline{Tf}) - \int_{Y/B} ch(\overline{\eta})\widetilde{Td}(g/f)Td^{-1}(\overline{N})$$

holds in $\widehat{K}_0(B)$.

Proof: Using the defining relations of arithmetic K_0-theory, we compute

$$g_*(\overline{\eta})-\sum_{i=0}^{m}(-1)^i f_*(\overline{\xi}_i) = (g_*\eta, g_*h_\eta)-T(h_Y,\eta)-(\sum_{i=0}^{m}(-1)^i((f_*\xi_i, g_*h_{\xi_i})-T(h_X,\xi_i))) =$$

$$\widetilde{ch}(f_*\Xi) - T(h_Y,\eta) + \sum_{i=0}^{m}(-1)^i T(h_X,\xi_i).$$

Comparing the last expression with the formula in 5.16 yields the proof. **Q.E.D.**

Theorem 6.7 *The Proposition 6.6 holds without acyclicity conditions on η and the ξ_i.*

Proof: Since f is projective, there is a closed embedding $k : X \to \mathbf{P}^r_B$ in a relative projective space over B, such that $f = p \circ k$, where $p : \mathbf{P}^r_B \to B$ is the natural projection. On $P = \mathbf{P}^r_B$, we have a canonical exact sequence

$$\mathcal{K}_P : 0 \to \mathcal{O}_P \to p^*(\mathcal{E}^\vee)(1) \to \ldots \to p^*(\Lambda^{r+1}\mathcal{E}^\vee)(r+1) \to 0$$

(see [19, p. 107]) where $\mathcal{E} = \mathcal{O}_B^{\oplus r+1}$. Restricting this sequence to X, we obtain an exact sequence of exact sequences (a double complex):

$$0 \to \Xi \to \Xi \otimes f^*(\mathcal{E}^\vee)(1) \to \ldots \to \Xi \otimes f^*(\Lambda^{r+1}\mathcal{E}^\vee)(r+1) \to 0.$$

Endow $\mathcal{E}$ with the trivial metric. Let us make the assumption that 6.6 holds for $\Xi \otimes f^*(\Lambda^n\overline{\mathcal{E}}^\vee)(n)$, $n \geq 1$. We show that this implies that it holds for Ξ. We compute

$$g_*(\overline{\eta}) = g_*(-\sum_{j=1}^{r+1}(-1)^j\overline{\eta} \otimes g^*(\Lambda^j\overline{\mathcal{E}}^\vee)(j)) + \int_{Y/B} Td(\overline{Tg})ch(\overline{\eta})\widetilde{ch}(\overline{\mathcal{K}}_P)$$

and

$$\sum_{i=0}^{m}(-1)^i f_*(\overline{\xi}_i) = \sum_{i=0}^{m}(-1)^i f_*(-\sum_{j=1}^{r+1}(-1)^j \overline{\xi}_i \otimes f^*(\Lambda^j \overline{\mathcal{E}}^\vee)(j)) +$$

$$\sum_{i=0}^{m}(-1)^i \int_{X/B} Td(\overline{Tf})ch(\overline{\xi}_i)\widetilde{ch}(\overline{\mathcal{K}}_P)$$

by the relations of arithmetic K_0-theory. Moreover

$$\int_{Y/B} ch(\eta)R(N)Td(Tg) = \int_{Y/B} -\sum_{j=1}^{r+1}(-1)^j ch(\eta \otimes g^*(\Lambda^j \mathcal{E}^\vee)(j))R(N)Td(Tg)$$

and

$$\int_{X/B} Td(\overline{Tf})T(h^{\xi_\cdot}) = \int_{X/B} Td(\overline{Tf})\{\delta_Y Td^{-1}(\overline{N})ch(\overline{\eta})\widetilde{ch}(\overline{\mathcal{K}}_P) -$$

$$\sum_{i=0}^{m}(-1)^i ch(\overline{\xi}_i)\widetilde{ch}(\overline{\mathcal{K}}_P) - \sum_{j=1}^{r+1}(-1)^j T(h^{\xi_\cdot})ch(f^*(\Lambda^j \overline{\mathcal{E}}^\vee)(j))\}$$

by 5.12 and 5.15. We also have

$$\int_{Y/B} ch(\overline{\eta})Td^{-1}(\overline{N})\widetilde{Td}(g/f) =$$

$$\int_{Y/B} \{\ dd^c\widetilde{ch}(\overline{\mathcal{K}}_P)ch(\overline{\eta}) - \sum_{j=1}^{r+1}(-1)^j ch(\overline{\eta} \otimes g^*(\Lambda^j \overline{\mathcal{E}}^\vee)(j))\ \}Td^{-1}(\overline{N})\widetilde{Td}(g/f) =$$

$$\int_{Y/B} \widetilde{ch}(\overline{\mathcal{K}}_P)ch(\overline{\eta})(Td^{-1}(\overline{N})Td(\overline{Tf}) - Td(\overline{Tg})) -$$

$$\int_{Y/B} \sum_{j=1}^{r+1}(-1)^j ch(\overline{\eta} \otimes g^*(\Lambda^j \overline{\mathcal{E}}^\vee)(j))Td^{-1}(\overline{N})\widetilde{Td}(g/f)$$

by the definition of the Bott-Chern secondary class. We want to prove that

$$g_*(\overline{\eta}) - \sum_{i=0}^{m}(-1)^i f_*(\overline{\xi}_i) - \int_{Y/B} ch(\eta)R(N)Td(Tg) - \int_{X/B} T(h^{\xi_\cdot})Td(\overline{Tf})$$

$$+ \int_{Y/B} ch(\overline{\eta})\widetilde{Td}(g/f)Td^{-1}(\overline{N})$$

vanishes. Using our assumption and the previous computations, we see that the last expression equals

$$\int_{Y/B} Td(\overline{Tg})ch(\overline{\eta})\widetilde{ch}(\overline{\mathcal{K}}_P) -$$

$$\sum_{i=0}^{m}(-1)^i\int_{X/B}Td(\overline{Tf})ch(\overline{\xi}_i)\widetilde{ch}(\overline{\mathcal{K}}_P)-\int_{Y/B}\widetilde{ch}(\overline{\mathcal{K}}_P)ch(\overline{\eta})Td^{-1}(\overline{N})Td(\overline{Tf})+$$

$$\sum_{i=0}^{m}\int_{X/B}(-1)^iTd(\overline{Tf})ch(\overline{\xi}_i)\widetilde{ch}(\overline{\mathcal{K}}_P)+$$

$$\int_{Y/B}\widetilde{ch}(\overline{\mathcal{K}}_P)ch(\overline{\eta})(Td^{-1}(\overline{N})Td(\overline{Tf})-Td(\overline{Tg}))$$

vanishes (the sums $\sum_{j=1}^{r+1}(\ldots)$ cancel by the assumption). Therefore 6.6 holds for Ξ. Now, since $\mathcal{E}$ is free and endowed with the trivial metric, the formula of 6.6 holds for $\Xi\otimes f^*(\Lambda^n\overline{\mathcal{E}}^\vee)(n)$, if it holds for $\Xi(n)$, for $n\geq 1$. Applying descending induction on n, we see that the formula of 6.6 holds for Ξ, if it holds for $\Xi(n)$, for all $n>>0$. But this last condition is satisfied, since $\eta(n)$ is g-acyclic and the $\xi_i(n)$ are f-acyclic for $n>>0$ (see [25, Th. 12.11, p. 290, III] and [25, Th. 8.8(c), p. 252]). This ends the proof. **Q.E.D.**

6.2.2 A model for closed embeddings

Let Y be an arithmetic variety. In this subsection, we prove a Riemann-Roch formula for the closed immersion $i_\infty : Y\to\mathbf{P}(N\oplus 1)$ mentioned at the beginning of 6.1.1. The deformation theorem of the next subsection will then show that a Riemann-Roch formula for all regular immersions can be derived from that one. We suppose that $P=\mathbf{P}(N\oplus 1)$ is endowed with a Kähler metric, that Y carries the metric induced from P via i_∞ and we assume that the normal bundle N_∞ is endowed with the quotient metric. We fix an arithmetic variety B and a p.f.s.r. (i.e. projective and flat, smooth over $\mathbf{Q}$) map $g:Y\to B$. We fix a metric on Q (the universal quotient bundle on P) which yields the metric of N_∞, when restricted to Y. The resolution $\mathcal{K}$ carries the exterior product metrics of Q. We shall denote the elements of the resolution $\mathcal{K}\otimes p^*(\eta)$ by $\kappa.$, endowed with the tensor product metric. Moreover, for any arithmetic variety Y, we shall use the map $ch:\widehat{K}_0(Y)\to Z(Y)$, which is defined by $ch(\overline{E}+\eta)=ch(\overline{E})+dd^c\eta$. This map is well-defined by the definition of the $\widehat{K}_0$-groups. We shall also call an element of $\widehat{K}_0(Y)$ which lies in the subgroup generated by all the hermitian vector bundles a **virtual hermitian bundle**.

Proposition 6.8 *Let α be a virtual hermitian bundle on P. The equality*

$$g_*(\theta^k(\overline{N}_\infty^\vee)\psi^k(\overline{\eta})\alpha)=\sum_{i=0}^{rg(N)}(-1)^i(g\circ p)_*(\psi^k(\overline{\kappa}_i)\alpha)+$$

$$\int_{Y/B}ch(\alpha)Td(Tg)ch(\psi^k(\eta)\theta^k(N^\vee))R(N)+\int_{P/B}k.ch(\alpha)Td(\overline{T(g\circ p)})\phi^k(T(h^{\kappa_\cdot}))-$$

$$\int_{Y/B} k^{rg(N)} ch(\alpha) ch(\psi^k(\overline{\eta})) \phi^k (Td^{-1}(\overline{N}_\infty)) \widetilde{Td}(g/g \circ p)$$

holds in $\widehat{K}_0(B)$ *for all* $k \geq 1$.

Proof: We shall need a formula comparing restrictions by i_∞ and direct-images by p. This is the content of

Lemma 6.9 *The equality*

$$g_*(i_\infty^*(x\alpha)) = (g \circ p)_*(\lambda_{-1}(\overline{Q}^\vee) x\alpha) + \int_{Y/B} Td(Tg) i_\infty^*(ch(\alpha.x)) R(N) +$$

$$\int_{P/B} Td(\overline{T(g \circ p)}) T(h^{\mathcal{K}.}) ch(\alpha.x) - \int_{Y/B} i_\infty^*(ch(\alpha.x)) Td^{-1}(\overline{N}_\infty) \widetilde{Td}(g/g \circ p)$$

holds in $\widehat{K}_0(B)$ *for any virtual hermitian bundle* $x \in \widehat{K}_0(P)$.

Proof of 6.9: If $x = \overline{V}$ and $\alpha = \overline{V}'$ apply 5.15 and then 6.7 to the resolution $\overline{\mathcal{K}} \otimes \overline{V} \otimes \overline{V}'$. Since both sides of the formula are additive, this yields the result. **Q.E.D**.

As in the classical case the Riemann-Roch formula for the canonical model boils down to certain formal identies, contained in the next two lemmas:

Lemma 6.10 *Let* R *be any* λ*-ring and* $e \in R$ *an element of finite* λ*-dimension. The equality*

$$\psi^k(\lambda_{-1}(e)) = \lambda_{-1}(e)\theta^k(e)$$

holds.

The proof of 6.10 can be found in [2, Prop. 7.3, p. 269].

Lemma 6.11 *The identity* $ch(\theta^k(\overline{V})) = k^{dim(V)} Td(\overline{V}^\vee) \phi^k(Td^{-1}(\overline{V}^\vee))$ *holds for any hermitian bundle* $\overline{V}$.

Proof of 6.11: let $r = dim(V)$. Let Ω be the (local) curvature matrix of $V_{\mathbf{C}}$ associated to the canonical hermitian holomorphic connection. By construction $ch(\theta^k(\overline{V}))$ is a power series with real coefficients in the elements of the matrix Ω, which is invariant under conjugation. On the other hand the power series $Det(1 + e^{\Omega} + e^{2\Omega} + \ldots e^{(k-1)\Omega})$ has the same properties and coincides with $ch(\theta^k(\overline{V}))$ if Ω is diagonal. To verify that they coincide for all matrices of forms Ω, notice that it is sufficient to show that they coincide as (entire) functions defined on $M_{r \times r}$, the set of all matrices with complex coefficients. Since they coincide for all diagonalisable matrices (they are invariant under conjugation), they coincide on all matrices by continuity, since diagonalisable matrices are dense in $M_{r\times r}$. Thus, we are reduced to prove the identity

$$Det(1+e^{\Omega}+e^{2\Omega}+\dots e^{(k-1)\Omega})=k^r Det(\frac{-e^{-\Omega}\Omega}{e^{-\Omega}-I})Det(\frac{-ke^{-k\Omega}}{e^{-k\Omega}-I})$$

where we have used the multiplicativity of ϕ^k and the fact that $\phi^k(\Omega)=k.\Omega$. Both sides are power series with real coefficients in the coefficients of Ω and are invariant under conjugation, and so by the same density argument as above, we are reduced to verify that they coincide on diagonal matrices. Let $m_1, m_2, \dots m_r$ be the diagonal elements of a diagonal complex matrix M. On the left hand, we compute

$$Det(1+e^{M}+e^{2M}+\dots e^{(k-1)M})=\prod_{i=1}^{r}(1+e^{m_i}+\dots e^{(k-1)m_i})=\prod_{i=1}^{r}\frac{1-e^{km_i}}{1-e^{m_i}}$$

and on the right hand, we get

$$k^r\prod_{i=1}^{r}\frac{-e^{-m_i}m_i}{e^{-m_i}-1}\cdot\frac{e^{-km_i}-1}{-ke^{km_i}m_i}$$

The expressions for the left and right hand sides clearly coincide, so we are done. **Q.E.D.**

We now resume the proof of 6.8. Using the fact that the arithmetic K_0-groups are λ-rings (see [32, Cor. 3.30]) and 6.10, we compute

$$(g\circ p)_*(\psi^k(p^*(\overline{\eta}))\psi^k(\lambda_{-1}(\overline{Q}^{\vee}))\alpha)=$$

$$(g\circ p)_*(\psi^k(p^*(\overline{\eta}))\theta^k(\overline{Q}^{\vee})\lambda_{-1}(\overline{Q}^{\vee})\alpha)$$

By 6.9, the last expression equals

$$g_*(\psi^k(p^*(\overline{\eta}))\theta^k(\overline{Q}^{\vee})\alpha)-\int_{Y/B}ch(\alpha)Td(Tg)ch(\psi^k(p^*(\overline{\eta}))\theta^k(Q^{\vee}))R(N)-$$

$$\int_{P/B}ch(\alpha)Td(\overline{T(g\circ p)})T(h^{\mathcal{K}}_{\cdot})ch(\psi^k(p^*(\overline{\eta}))\theta^k(\overline{Q}^{\vee}))+$$

$$\int_{Y/B}ch(\alpha)ch(\psi^k(p^*(\overline{\eta}))\theta^k(\overline{Q}^{\vee}))Td^{-1}(\overline{N})\widetilde{Td}(g/g\circ p)$$

where we dropped the i^* and i^*_{∞} to make the expression less heavy. Using 5.13 to compute $T(h^{\mathcal{K}}_{\cdot})$ and 6.11, we see that the last expression equals

$$g_*(\psi^k(p^*(\overline{\eta}))\theta^k(\overline{Q}^{\vee})\alpha)-\int_{Y/B}ch(\alpha)Td(Tg)ch(\psi^k(\eta)\theta^k(N^{\vee}))R(N)-$$

$$\int_{P/B} ch(\alpha)Td(\overline{T(g\circ p)})ch(\psi^k(p^*(\overline{\eta}))\theta^k(\overline{Q}^\vee))Td^{-1}(\overline{Q})g+$$

$$\int_{Y/B} ch(\alpha)ch(\psi^k(\overline{\eta}))k^{rg(N)}Td(\overline{N})\phi^k(Td^{-1}(\overline{N}))Td^{-1}(\overline{N})\widetilde{Td}(g/g\circ p)$$

where g is the Green current of $\mathcal{K}$, described after 5.13 . Recall that it is of pure type $(rg(N)-1, rg(N)-1)$, so that $\phi^k(g) = k^{rg(N)-1}g$. To complete the proof of Proposition 6.8, we only have to compute the integral of the second line:

$$\int_{P/B} ch(\alpha)Td(\overline{T(g\circ p)})ch(\psi^k(p^*(\overline{\eta}))\theta^k(\overline{Q}^\vee))Td^{-1}(\overline{Q})g =$$

$$\int_{P/B} ch(\alpha)k.k^{-rg(Q)}\phi^k(g)Td(\overline{T(g\circ p)})\phi^k(ch(p^*(\overline{\eta}))\theta^k(\overline{Q}^\vee))Td^{-1}(\overline{Q}) =$$

$$\int_{P/B} k.ch(\alpha)Td(\overline{T(g\circ p)})\phi^k(ch(p^*(\overline{\eta})))\phi^k(Td^{-1}(\overline{Q})g) =$$

$$\int_{P/B} k.ch(\alpha)Td(\overline{T(g\circ p)})\phi^k(T(h^{\kappa_\cdot}))$$

where we have used 5.15 to compute $T(h^{\kappa_\cdot})$ from $T(h^{\mathcal{K}}_\cdot)$. **Q.E.D.**

Corollary 6.12 *Let ξ_i^∞ be endowed with any metric satisfying Bismut's assumption (A) with respect to $\overline{\eta}$ and $\overline{N}_\infty$. The equality*

$$g_*(\theta^k(\overline{N}_\infty^\vee)\psi^k(\overline{\eta})\alpha) = \sum_{i=0}^m (-1)^i (g\circ p)_*(\psi^k(\overline{\xi}_i^\infty)\alpha)+$$

$$\int_{Y/B} ch(\alpha)Td(Tg)ch(\psi^k(\eta)\theta^k(N^\vee))R(N)+\int_{P/B} kch(\alpha)Td(\overline{T(g\circ p)})\phi^k(T(h^{\xi_\cdot^\infty}))-$$

$$\int_{Y/B} k^{rg(N)}ch(\alpha)ch(\psi^k(\overline{\eta}))\phi^k(Td^{-1}(\overline{N}_\infty))\widetilde{Td}(g/g\circ p)$$

holds in $\widehat{K}_0(B)$ for all $k \geq 1$.

Proof: Let us put a split orthogonal hermitian metric on $\mathcal{S}$, in 6.5. Let us then denote the sequence of the i-th row in 6.5 by $\overline{\mathcal{E}}_i$. By the formula 5.12, we know that

$$\sum_{i=0}^m (-1)^i \overline{\xi}_i^\infty - T(h^{\kappa_\cdot}) + T(h^{\xi_\cdot}) =$$

$$\sum_{i=0}^m (-1)^i (\overline{\xi}_i^\infty - \widetilde{ch}(\overline{\mathcal{E}}_i)) =$$

$$\sum_{i=0}^{m}(-1)^i(\overline{\kappa}_i+\overline{\mathcal{S}}.)=\sum_{i=0}^{m}(-1)^i\overline{\kappa}_i$$

in $\widehat{K}_0(P)$. Therefore we can compute

$$\sum_{i=0}^{m}(-1)^i(g\circ p)_*(\psi^k(\overline{\kappa}_i)\alpha)+\int_{P/B}ch(\alpha)kTd(\overline{T(g\circ p)})\phi^k(T(h^{\kappa.}))=$$

$$(\sum_{i=0}^{m}(-1)^i(g\circ p)_*(\psi^k(\overline{\xi}_i^{\infty})-T(h^{\kappa.})+T(h^{\xi.}))\alpha)+\int_{P/B}ch(\alpha)kTd(\overline{T(g\circ p)})\phi^k(T(h^{\kappa.}))=$$

$$\sum_{i=0}^{m}(-1)^i(g\circ p)_*(\psi^k(\overline{\xi}_i^{\infty})\alpha)+\int_{P/B}ch(\alpha)kTd(\overline{T(g\circ p)})\phi^k(T(h^{\xi.^{\infty}}))$$

where the definition of the push-forward of forms was used from the second to the third line. If we reinsert this expression in the formula of 6.8, we get the result. **Q.E.D**.

6.2.3 The deformation theorem

Let $i:Y\to X$ be a regular closed immersion of arithmetic varieties and let $g:Y\to B$, $f:X\to B$ be p.f.s.r. maps to an arithmetic variety B such that $g=f\circ i$. Let the terminology of the geometric preliminaries 6.1 hold. From now on, we shall assume that the $\tilde{\xi}.$ are endowed with metrics such that Bismut's assumption (A) is satisfied on W and such that the sequence $0\to\xi_m\to\xi_{m-1}\to\dots\xi_0\to 0$ is orthogonally split on $\tilde{X}$. This is possible, since $\tilde{X}$ is disjoint with $j_*(Y\times\mathbf{P}^1)$.

Definition 6.13 *A metric h on W is said to be normal to the deformation if*

- **(a)** *It is Kähler;*
- **(b)** *the restriction $h|j_*(Y\times\mathbf{P}^1)$ is a product $h'\times h''$, where h' is a metric on Y and h'' a metric on $\mathbf{P}^1$;*
- **(c)** *the intersection of $i_{X*}X$ with $j_*(Y\times\mathbf{P}^1)$ and of $i_{P*}P$ with $j_*(Y\times\mathbf{P}^1)$ are orthogonal.*

Lemma 6.14 *There exists a metric on W, which is normal to the deformation.*

Proof: We use the terminology of 6.3. Since the last definition concerns the integral part of $W_{\mathbf{C}}$ only, we might suppose without loss of generality that $D=\mathbf{C}$ and that X and Y are integral. Choose an embedding $k:\mathbf{P}(E\oplus 1)\to X\times\mathbf{P}^r$

into a relative projective space over X. Composing maps, we get an embedding $W \to X \times \mathbf{P}^r \times \mathbf{P}^1$. If we choose a Kähler metric on X and Kähler metrics on $\mathbf{P}^r$ and $\mathbf{P}^1$, we can endow $X \times \mathbf{P}^r \times \mathbf{P}^1$ with the product metric, which induces a Kähler metric on W by restriction. This metric has the required properties. **Q.E.D.**

Theorem 6.15 (Deformation theorem) *Let W be endowed with a metric normal to the deformation. Let α be in the subgroup of $\widehat{K}_0(W)$ generated by hermitian bundles. Then the formula*

$$-g_*(\theta^k(\overline{N}^\vee)\psi^k(\overline{\eta})\alpha) + \sum_{i=0}^{m}(-1)^i f_*(\psi^k(\overline{\xi}_i)\alpha) + \int_{X/B} k.Td(\overline{Tf})\phi^k(T(h^{\xi}))ch(\alpha) -$$

$$\int_{Y/B} k.\phi^k(ch(\overline{\eta}))ch(\alpha)\phi^k(Td^{-1}(\overline{N}_0))\widetilde{Td}(g/f) =$$

$$-g_*(\theta^k(\overline{N}_\infty^\vee)\psi^k(\overline{\eta})\alpha) + \sum_{i=0}^{m}(-1)^i (g\circ p)_*(\psi^k(\overline{\xi}_i^\infty)\alpha) + \int_{P/B} k.Td(\overline{T(g\circ p)})\phi^k(T(h^{\xi^\infty}))ch(\alpha) -$$

$$\int_{P/B} k.\phi^k(ch(\overline{\eta}))ch(\alpha)\phi^k(Td^{-1}(\overline{N}_\infty))\widetilde{Td}(g/g\circ p)$$

holds in $\widehat{K}_0(B)$.

Proof: We choose once and for all sections of $\mathcal{O}(X)$, $\mathcal{O}(P)$, $\mathcal{O}(\tilde{X})$ whose zero-schemes are X, P and $\tilde{X}$. If D is a Cartier divisor and the bundle $\mathcal{O}(D)$ carries a hermitian metric, we shall often write $Td(\overline{D})$ for $Td(\overline{\mathcal{O}}(D))$. We shall also write $\psi^k(\overline{\xi}_.)$ for $\sum_{i=0}^m(-1)^i\psi^k(\overline{\xi}_i)$.

Lemma 6.16 *There are hermitian metrics on $\mathcal{O}(X)$, $\mathcal{O}(P)$ and $\mathcal{O}(\tilde{X})$ such that the isometry $\overline{\mathcal{O}}(X) \simeq \overline{\mathcal{O}}(P)\overline{\mathcal{O}}(\tilde{X})$ holds and such that the restriction of $\mathcal{O}(X)$ to X yields the metric of the normal bundle $N_{X/W}$, the restriction of $\mathcal{O}(\tilde{X})$ to $\tilde{X}$ yields the metric of the normal bundle $N_{\tilde{X}/W}$ and the restriction of $\mathcal{O}(P)$ to P yields the metric of the normal bundle $N_{P/W}$.*

Proof of 6.16: choose metrics on $\mathcal{O}(P)$ in a small neighborhood of P such that the restriction of $\mathcal{O}(P)$ to P yields the metric of the normal bundle. Do the same for $\mathcal{O}(\tilde{X})$. Since X is closed and disjoint from $\tilde{X}$ and P, we can extend these metrics via a partition of unity to metrics defined on W, so that the the restriction of the metric that $\mathcal{O}(X)$ inherits from the isomorphism $\mathcal{O}(X) \simeq \mathcal{O}(P)\mathcal{O}(\tilde{X})$ yields the metric of the normal bundle $N_{X/W}$. This completes the proof. **Q.E.D.**

From now on, we shall suppose that $\mathcal{O}(X)$, $\mathcal{O}(\tilde{X})$ and $\mathcal{O}(P)$ are endowed with hermitian metrics satisfying the hypotheses of the the last lemma. We shall compare push-forwards of Adams operations on X and P, by applying 6.7 to the resolutions

$$0 \to \mathcal{O}(-X) \to \mathcal{O}_W \to i_{X*}\mathcal{O}_X \to 0, \tag{8}$$

$$0 \to \mathcal{O}(-P) \to \mathcal{O}_W \to i_{P*}\mathcal{O}_P \to 0, \tag{9}$$

$$0 \to \mathcal{O}(-\tilde{X}) \to \mathcal{O}_W \to i_{\tilde{X}*}\mathcal{O}_{\tilde{X}} \to 0 \tag{10}$$

and to the resolution which is the tensor product of (9) and (10):

$$0 \to \mathcal{O}(-X) \otimes \mathcal{O}(-P) \to \mathcal{O}(-X) \oplus \mathcal{O}(-P) \to \mathcal{O}_W \to i_{P\cap\tilde{X}}\mathcal{O}_{P\cap\tilde{X}} \to 0 \tag{11}$$

They satisfy Bismut's assumption (A) (by 6.16 for (8), (9), (10), by 5.14 for (11)). The resolutions (8), (9), (10) are Koszul resolutions and we shall denote the associated Green currents by g_X, g_P and $g_{\tilde{X}}$, respectively. First note that the equality

$$\alpha\psi^k(\overline{\tilde{\xi}}.)((1-\overline{\mathcal{O}}(-X))-(1-\overline{\mathcal{O}}(-P))-(1-\overline{\mathcal{O}}(-\tilde{X}))+(1-\overline{\mathcal{O}}(-P))(1-\overline{\mathcal{O}}(-\tilde{X}))=0$$

holds in $\widehat{K}_0(W)$. We shall apply the push-forward map to both sides of this equality, and show that the resulting equality is equivalent to the statement of the theorem. Using 6.7, 5.14 and 5.13, we compute that this equality implies

$$f_*(\psi^k(\overline{\xi}.)\alpha) - \int_{X/B} ch(\alpha)ch(\psi^k(\xi.))R(N_{X/W})Td(Tf) -$$

$$\int_{W/B} ch(\alpha)Td(\overline{T(f\circ\phi)})ch(\psi^k(\overline{\tilde{\xi}}.))Td^{-1}(\overline{X})g_X) +$$

$$\int_{X/B} ch(\alpha)ch(\psi^k(\overline{\xi}.))Td^{-1}(\overline{N}_{X/W})\widetilde{Td}(f/f\circ\phi) -$$

$$(g\circ p)_*(\psi^k(\overline{\xi}.^{\infty})\alpha) - \int_{P/B} ch(\alpha)ch(\psi^k(\xi.^{\infty}))R(N_{P/W})Td(T(g\circ p)) -$$

$$\int_{W/B} ch(\alpha)Td(\overline{T(f\circ\phi)})ch(\psi^k(\overline{\tilde{\xi}}.))Td^{-1}(\overline{P})g_P +$$

$$\int_{P/B} ch(\alpha)ch(\psi^k(\overline{\xi}.^{\infty}))\widetilde{Td}(g\circ p/f\circ\phi)Td^{-1}(\overline{N}_{P/W})) -$$

$$\int_{W/B} ch(\alpha)Td(\overline{T(f\circ\phi)})ch(\psi^k(\overline{\tilde{\xi}}.))Td^{-1}(\overline{\tilde{X}})g_{\tilde{X}}$$

$$+\int_{W/B} ch(\alpha)Td(\overline{T(f\circ\phi)})ch(\psi^k(\overline{\tilde{\xi}}.))Td^{-1}(\overline{P}+\overline{\tilde{X}})(c_1(\overline{\mathcal{O}}(P))g_{\tilde{X}} + \delta_{\tilde{X}}.g_P) = 0$$

where we have dropped all the terms where an integral was taken over $\tilde{X}$, since $ch(\psi^k(\bar{\tilde{\xi}}.))$ vanishes on $\tilde{X}$. For the same reason the term

$$\int_{W/B} ch(\alpha)Td(\overline{T(f\circ\phi)})ch(\psi^k(\bar{\tilde{\xi}}.))Td^{-1}(\overline{P}+\overline{\tilde{X}})(\delta_{\tilde{X}}.g_P)$$

vanishes. For the next step, we shall need the cohomological Riemann-Roch theorem. Let $j: M' \to M$ be a projective smooth subvariety of a complex smooth projective variety M. Let $H(M)$ be the even real de Rahm cohomology of M. It can be viewed as the kernel of the operator dd^c acting on $\tilde{A}(M)$. In the next theorem, $j_*: H(M') \to H(M)$ will stand for the push-forward map in cohomology associated to j.

Theorem 6.17 *Let N be the normal bundle of the immersion j. The equality*

$$j_*(Td(N)^{-1}ch(x)) = ch(j_*(x))$$

holds in $H(M)$, for any virtual vector bundle x on M'.

For the proof, see [19, VI, 8.]. Recall that i is the immersion $Y \to X$ and i_∞ the immersion $Y \to \mathbf{P}(N\oplus 1)$ of the canonical model. Notice that the group endomorphism $\phi^k: H(X) \to H(X)$ (defined before 3.3) has an inverse in this case, which we shall denote by $\phi^{\frac{1}{k}}$; the map $\phi^{\frac{1}{k}}$ is $\mathbf{R}$-linear and defined by the formula $\phi^{\frac{1}{k}}(\sum_{i\geq 0} x_i) = \sum_{i\geq 0}\frac{1}{k^i}x_i$, for every element $x \in H(X)$. Using the projection formula in cohomology, we compute

$$i_{X*}(ch(\psi^k(\xi.))ch(\alpha)R(N_{X/W})Td(Tf)) = \tag{12}$$

$$i_{X*}\{\phi^k\{i_*\{Td^{-1}(N_{Y/X})ch(\eta)i^*\{\phi^{\frac{1}{k}}(R(N_{X/W})Td(Tf))\}\}\}\}ch(\alpha).$$

Similarly, we have at infinity

$$i_{P*}(ch(\psi^k(\xi.))ch(\alpha)R(N_{P/W})Td(T(g\circ p))) = \tag{13}$$

$$i_{P*}\{\phi^k\{i_{\infty*}\{Td^{-1}(N_{Y/P})ch(\eta)i_\infty^*\{\phi^{\frac{1}{k}}(R(N_{P/W})Td(T(g\circ p))\}\}\}\}ch(\alpha).$$

Now notice that the restriction of $N_{P/W}$ to Y_∞ is trivial and that the restriction of $N_{X/W}$ to Y_0 is trivial. To see this, notice that by construction $N_{Y_\infty/Y\times\mathbf{P}^1}$ and $N_{Y_0/Y\times\mathbf{P}^1}$ are trivial and by transversality $j_\infty^*N_{P/W} \simeq N_{Y_\infty/Y\times\mathbf{P}^1}$ and $j_0^*N_{X/W} \simeq N_{Y_0/Y\times\mathbf{P}^1}$. Thus $i_\infty^*(\phi^{\frac{1}{k}}(R(N_{P/W}))) = 0$ and $i^*(\phi^{\frac{1}{k}}(R(N_{X/W}))) = 0$ and the expressions in (12) and (13) vanish. Thus we are left with the equality

$$f_*(\psi^k(\overline{\xi}.)\alpha) - (g\circ p)_*(\psi^k(\overline{\xi}.^\infty)\alpha) =$$

$$\int_{W/B} ch(\alpha)Td(\overline{T(f\circ\phi)})ch(\psi^k(\bar{\tilde{\xi}}.))(Td^{-1}(\overline{X})g_X - Td^{-1}(\overline{P})g_P - Td^{-1}(\overline{\tilde{X}})g_{\tilde{X}}+$$

$$Td^{-1}(\overline{P}+\overline{\tilde{X}})c_1(\overline{\mathcal{O}}(P))g_{\tilde{X}})-$$

$$\int_{X/B} ch(\alpha)ch(\psi^k(\overline{\xi}.))Td^{-1}(\overline{N}_{X/W})\widetilde{Td}(f/f\circ\phi)+\int_{P/B} ch(\alpha)ch(\psi^k(\overline{\xi}.^{\infty}))\widetilde{Td}(g\circ p/f\circ\phi)Td^{-1}(\overline{N}_{P/W}).$$

Using the properties of the singular Bott-Chern current, we compute the equality of currents

$$ch(\psi^k(\overline{\tilde{\xi}}.))(Td^{-1}(\overline{X})g_X-Td^{-1}(\overline{P})g_P-Td^{-1}(\overline{\tilde{X}})g_{\tilde{X}}+Td^{-1}(\overline{P}+\overline{\tilde{X}})c_1(\overline{\mathcal{O}}(P))g_{\tilde{X}}) =$$

$$-\phi^k(dd^cT(h^{\tilde{\xi}})-ch(p_Y^*\overline{\eta})Td^{-1}(N_{Y\times\mathbf{P}^1/W})\delta_{Y\times\mathbf{P}^1})(Td^{-1}(\overline{X})g_X-Td^{-1}(\overline{P})g_P-$$

$$Td^{-1}(\overline{\tilde{X}})g_{\tilde{X}}+Td^{-1}(\overline{P}+\overline{\tilde{X}})c_1(\overline{\mathcal{O}}(P))g_{\tilde{X}}) =$$

$$-\{\,Td^{-1}(\overline{X})\phi^k(dd^cT(h^{\tilde{\xi}})g_X)-Td^{-1}(\overline{P})\phi^k(dd^cT(h^{\tilde{\xi}})g_P)-Td^{-1}(\overline{\tilde{X}})\phi^k(dd^cT(h^{\tilde{\xi}})g_{\tilde{X}})+$$

$$Td^{-1}(\overline{P}+\overline{\tilde{X}})\frac{1}{k}\phi^k(dd^cT(h^{\tilde{\xi}})c_1(\overline{\mathcal{O}}(P))g_{\tilde{X}})-$$

$$\{\phi^k(ch(p_Y^*\overline{\eta})Td^{-1}(N_{Y\times\mathbf{P}^1/W})\delta_{Y\times\mathbf{P}^1})\}\{Td^{-1}(\overline{X})g_X-$$

$$Td^{-1}(\overline{P})g_P-Td^{-1}(\overline{\tilde{X}})g_{\tilde{X}}+Td^{-1}(\overline{P}+\overline{\tilde{X}})c_1(\overline{\mathcal{O}}(P))g_{\tilde{X}}\}\,\}$$

The next lemma will evaluate the first part of the last expression.

Lemma 6.18 *The equality*

$$Td^{-1}(\overline{X})\phi^k(dd^cT(h^{\tilde{\xi}})g_X)-Td^{-1}(\overline{P})\phi^k(dd^cT(h^{\tilde{\xi}})g_P)-Td^{-1}(\overline{\tilde{X}})\phi^k(dd^cT(h^{\tilde{\xi}})g_{\tilde{X}})+$$

$$Td^{-1}(\overline{P}+\overline{\tilde{X}})\frac{1}{k}\phi^k(dd^cT(h^{\tilde{\xi}})c_1(\overline{\mathcal{O}}(P))g_{\tilde{X}}) =$$

$$k\phi^k(T(h^{\tilde{\xi}.}))(Td^{-1}(\overline{X})\delta_X-Td^{-1}(\overline{P})\delta_P-Td^{-1}(\overline{\tilde{X}})\delta_{\tilde{X}}+$$

$$Td^{-1}(\overline{P}+\overline{\tilde{X}})c_1(\overline{\mathcal{O}}(P))\delta_{\tilde{X}}))$$

holds in $P^W/P^{W,0}$.

For the proof we shall need the

Lemma 6.19 *Let* $\overline{E}$ *be a hermitian bundle of rank* r. *The identity of forms*

$$Td(\overline{E})ch(\lambda_{-1}(\overline{E}^{\vee})) = c_r(\overline{E})$$

holds.

Proof of 6.19: let Ω be the (local) curvature matrix associated to the hermitian holomorphic connection of $\overline{E}$. By construction the identity to be proved is equivalent to the identity

$$Det(\frac{\Omega.e^{\Omega}}{e^{\Omega}-I})Tr(\Lambda_{-1}(e^{-\Omega}))=Det(\Omega)$$

where $Tr(\Lambda_{-1}(e^{-\Omega}))=Tr(I)-Tr(\Lambda^1(e^{-\Omega}))+Tr(\Lambda^2(e^{-\Omega}))-\dots$. Here Λ^k refers to the k-th exterior power of the standard representation of $GL_n(\mathbf{C})$. Both sides are power series in the entries of Ω and invariant under matrix conjugation. Thus by the same density argument as in the proof of 6.11, we are reduced to prove that both sides coincide when evaluated on a diagonal matrix $M=(m_1,\dots m_r)$ with complex entries. By construction $Tr(\Lambda_{-1}(e^{-M}))=\prod_{i=1}^r(1-e^{-m_i})$. We can now compute

$$Det(\frac{M.e^M}{e^M-I})Tr(\Lambda_{-1}(e^{-M}))=\prod_{i=1}^r\frac{m_ie^{m_i}}{e^{m_i}-1}(1-e^{-m_i})=\prod_{i=1}^r m_i=Det(M)$$

and thus we are done. **Q.E.D**.

Proof of 6.18: using (7), we compute that the left hand of the equality gives

$$Td^{-1}(\overline{X})\phi^k(T(h^{\tilde{\xi}})(-c_1(\overline{\mathcal{O}}(X))+\delta_X))-Td^{-1}(\overline{P})\phi^k(T(h^{\tilde{\xi}})(-c_1(\overline{\mathcal{O}}(P))+\delta_P))-$$
$$Td^{-1}(\overline{\tilde{X}})\phi^k(T(h^{\tilde{\xi}})(-c_1(\overline{\mathcal{O}}(\tilde{X}))+\delta_{\tilde{X}}))+$$
$$Td^{-1}(\overline{P}+\overline{\tilde{X}})\frac{1}{k}\phi^k(T(h^{\tilde{\xi}})c_1(\overline{\mathcal{O}}(P))(-c_1(\overline{\mathcal{O}}(\tilde{X}))+\delta_{\tilde{X}}))=$$
$$-k\phi^k(T(h^{\tilde{\xi}}))(Td^{-1}(\overline{X})c_1(\overline{\mathcal{O}}(X))-Td^{-1}(\overline{P})c_1(\overline{\mathcal{O}}(P))-Td^{-1}(\overline{\tilde{X}})c_1(\overline{\mathcal{O}}(\tilde{X}))+$$
$$Td^{-1}(\overline{P}+\overline{\tilde{X}})c_1(\overline{\mathcal{O}}(\tilde{X}))c_1(\overline{\mathcal{O}}(P)))+$$
$$k\phi^k(T(h^{\tilde{\xi}}))(Td^{-1}(\overline{X})\delta_X-Td^{-1}(\overline{P})\delta_P-Td^{-1}(\overline{\tilde{X}})\delta_{\tilde{X}}+$$
$$Td^{-1}(\overline{P}+\overline{\tilde{X}})c_1(\overline{\mathcal{O}}(P))\delta_{\tilde{X}}))$$

Using the identity in 6.19, we compute that

$$Td^{-1}(\overline{X})c_1(\overline{\mathcal{O}}(X))-Td^{-1}(\overline{P})c_1(\overline{\mathcal{O}}(P))-Td^{-1}(\overline{\tilde{X}})c_1(\overline{\mathcal{O}}(\tilde{X}))+$$
$$Td^{-1}(\overline{P}+\overline{\tilde{X}})c_1(\overline{\mathcal{O}}(\tilde{X}))c_1(\overline{\mathcal{O}}(P))=0 \qquad (14)$$

This completes the proof. **Q.E.D**.

Lemma 6.20 *The equality*

$$\int_{W/B} ch(\alpha)Td(\overline{T(f\circ\phi)})\phi^k(ch(p_Y^*(\overline{\eta}))Td^{-1}(\overline{N}_{Y\times\mathbf{P}^1_D/W})\delta_{Y\times\mathbf{P}^1_D})(Td^{-1}(\overline{X})g_X - Td^{-1}(\overline{P})g_P -$$

$$Td^{-1}(\overline{\tilde{X}})g_{\tilde{X}} + Td^{-1}(\overline{P}+\overline{\tilde{X}})c_1(\overline{\mathcal{O}}(P))g_{\tilde{X}}) =$$

$$\int_{Y/B} ch(\alpha)k^{cod(Y)}\phi^k(ch(\overline{\eta})Td^{-1}(\overline{N}_0))\widetilde{Td}(g/f) -$$

$$\int_{Y/B} ch(\alpha)k^{cod(Y)}\phi^k(ch(\overline{\eta})Td^{-1}(\overline{N}_\infty))\widetilde{Td}(g/g\circ p) +$$

$$g_*(\theta^k(\overline{N}_0^\vee)\psi^k(\overline{\eta})\alpha) - g_*(\theta^k(\overline{N}_\infty^\vee)\psi^k(\overline{\eta})\alpha)$$

holds in $\widehat{K}_0(B)$.

Proof of 6.20: using the definition of $\widetilde{Td}$ and (7), we can rewrite the left side of the equality as

$$\int_{W/B} ch(\alpha)(dd^c\widetilde{Td}(g\circ p_Y/f\circ\phi) + Td(\overline{N}_{Y\times\mathbf{P}^1_D/W})Td(\overline{T(g\circ p_Y)})).$$

$$(Td^{-1}(\overline{X})g_X - Td^{-1}(\overline{P})g_P - Td^{-1}(\overline{\tilde{X}})g_{\tilde{X}} + Td^{-1}(\overline{P}+\overline{\tilde{X}})c_1(\overline{\mathcal{O}}(P))g_{\tilde{X}})$$

$$\delta_{Y\times\mathbf{P}^1_D}k^{cod(Y)}\phi^k(ch(p_Y^*(\overline{\eta}))Td^{-1}(\overline{N}_{Y\times\mathbf{P}^1_D/W})) =$$

$$\int_{W/B} ch(\alpha)\widetilde{Td}(g\circ p_Y/f\circ\phi)\delta_{Y\times\mathbf{P}^1_D}k^{cod(Y)}\phi^k(ch(p_Y^*(\overline{\eta}))Td^{-1}(\overline{N}_{Y\times\mathbf{P}^1_D/W}))$$

$$\{Td^{-1}(\overline{X})(\delta_X - c_1(\overline{X})) - Td^{-1}(\overline{P})(\delta_P - c_1(\overline{P})) -$$

$$Td^{-1}(\overline{\tilde{X}})(\delta_{\tilde{X}} - c_1(\overline{\tilde{X}})) + Td^{-1}(\overline{P}+\overline{\tilde{X}})c_1(\overline{\mathcal{O}}(P))(\delta_{\tilde{X}} - c_1(\overline{\tilde{X}}))\} +$$

$$\int_{W/B} ch(\alpha)Td(\overline{N}_{Y\times\mathbf{P}^1_D})\phi^k(Td^{-1}(\overline{N}_{Y\times\mathbf{P}^1_D/W}))Td(\overline{T(g\circ p_Y)})\phi^k(ch(p_Y^*(\overline{\eta})))k^{cod(Y)}$$

$$\delta_{Y\times\mathbf{P}^1_D}(Td^{-1}(\overline{X})g_X - Td^{-1}(\overline{P})g_P - Td^{-1}(\overline{\tilde{X}})g_{\tilde{X}} + Td^{-1}(\overline{P}+\overline{\tilde{X}})c_1(\overline{\mathcal{O}}(P))g_{\tilde{X}})$$

By 6.13, we have $Td^{-1}(\overline{N}_{Y\times\mathbf{P}^1_D/W})|_{Y_\infty} = Td^{-1}(\overline{N}_\infty)$, $Td(\overline{P})|_{Y_\infty} = 1$ and $Td^{-1}(\overline{N}_{Y\times\mathbf{P}^1_D/W})|_{Y_0} = Td^{-1}(\overline{N}_0)$, $Td(\overline{X})|_{Y_0} = 1$. Furthermore, remember that $\delta_{Y\times\mathbf{P}^1_D}\wedge\delta_{\tilde{X}} = 0$, $\delta_{Y\times\mathbf{P}^1_D}\wedge\delta_P = \delta_{Y_\infty}$, $\delta_{Y\times\mathbf{P}^1_D}\wedge\delta_X = \delta_{Y_0}$. With these equalities in hand and (14), we can evaluate the expression after the last equality as

$$\int_{Y/B} ch(\alpha)k^{cod(Y)}\phi^k(ch(\overline{\eta})Td^{-1}(\overline{N}_0))\widetilde{Td}(g\circ p_Y/f\circ\phi) -$$

$$\int_{Y/B} ch(\alpha)k^{cod(Y)}\phi^k(ch(\overline{\eta})Td^{-1}(\overline{N}_\infty))\widetilde{Td}(g\circ p_Y/f\circ\phi)+$$

$$\int_{Y\times\mathbf{P}^1_D/B} ch(\alpha)ch(\theta^k(\overline{N}^\vee_{Y\times\mathbf{P}^1_D/W}))Td(\overline{T(g\circ p_Y)})ch(\psi^k(p_Y^*(\overline{\eta})))(Td^{-1}(\overline{X})g_X - Td^{-1}(\overline{P})g_P -$$

$$Td^{-1}(\overline{\tilde{X}})g_{\tilde{X}} + Td^{-1}(\overline{P}+\overline{\tilde{X}})c_1(\overline{\mathcal{O}}(P))g_{\tilde{X}})$$

Now, we can also compute that $\widetilde{Td}(g\circ p_Y/f\circ\phi)|_{Y\times\{\infty\}} = \widetilde{Td}(g/g\circ p)$ and $\widetilde{Td}(g\circ p_Y/f\circ\phi)|_{Y\times\{0\}} = \widetilde{Td}(g/f)$; indeed, by 6.13, the restriction of the normal sequence of $Y\times\mathbf{P}^1_D$ in W to $Y\times\{0\}$ (resp. $Y\times\{\infty\}$) is the orthogonal direct sum of the normal sequence of $Y\times\{0\}$ in X (resp. $Y\times\{\infty\}$ in P) with a sequence $0\to T\to T\to 0$, where T is a trivial line bundle endowed with a constant metric. From this, by the symmetry formula 5.12, the equalities follow. Furthermore, we can compute

$$\int_{Y\times\mathbf{P}^1_D/B} ch(\alpha)ch(\theta^k(\overline{N}^\vee_{Y\times\mathbf{P}^1_D/W}))Td(\overline{T(g\circ p_Y)})ch(\psi^k(p_Y^*(\overline{\eta})))(Td^{-1}(\overline{X})g_X - Td^{-1}(\overline{P})g_P -$$

$$Td^{-1}(\overline{\tilde{X}})g_{\tilde{X}} + Td^{-1}(\overline{P}+\overline{\tilde{X}})c_1(\overline{\mathcal{O}}(P))g_{\tilde{X}}) =$$

$$g_*(\theta^k(\overline{N}^\vee_0)\psi^k(\overline{\eta})\alpha) - g_*(\theta^k(\overline{N}^\vee_\infty)\psi^k(\overline{\eta})\alpha) \tag{15}$$

To see this, notice that that there are natural isomorphisms $j_0^*\mathcal{O}(-X)\simeq\mathcal{O}(-Y_0)$ and $j_\infty^*\mathcal{O}(-P)\simeq\mathcal{O}(-Y_\infty)$. Thus we have resolutions

$$0\to j^*\mathcal{O}(-X)\to\mathcal{O}_{Y\times\mathbf{P}^1}\to i_{Y0}\mathcal{O}_Y\to 0$$

and

$$0\to j^*\mathcal{O}(-P)\to\mathcal{O}_{Y\times\mathbf{P}^1}\to i_{Y\infty}\mathcal{O}_Y\to 0$$

where i_{Y0} is the embedding $Y\to Y\times\mathbf{P}^1$ at 0 and $i_{Y\infty}$ is the embedding $Y\to Y\times\mathbf{P}^1$ at ∞. The normal sequences of i_{Y0} and $i_{Y\infty}$ are clearly split orthogonal, the normal bundles of i_{Y0} and $i_{Y\infty}$ are trivial and the bundle $j^*\mathcal{O}(-\tilde{X})$ is trivial. Thus, if apply 6.7 to the equality

$$j^*(\alpha)\psi^k(\overline{\eta})\theta^k(\overline{N}^\vee_{Y\times\mathbf{P}^1/W})((1-j^*(\overline{\mathcal{O}}(-X)))-(1-j^*(\overline{\mathcal{O}}(-P)))-(1-j^*(\overline{\mathcal{O}}(-\tilde{X})))+$$

$$(1-j^*(\overline{\mathcal{O}}(-P)))(1-j^*(\overline{\mathcal{O}}(-\tilde{X})))) = 0$$

as at the beginning of the proof of the deformation theorem, we obtain (15). **Q.E.D.**

Combining the results of 6.18 and 6.20 yields the equality

$$f_*(\psi^k(\overline{\xi}_\cdot)\alpha) - (g\circ p)_*(\psi^k(\overline{\xi}^\infty_\cdot)\alpha) =$$

$$-\int_{W/B} ch(\alpha)Td(\overline{T(f\circ\phi)})k\phi^k(T(h^{\tilde{\xi}_\cdot}))(Td^{-1}(\overline{X})\delta_X - Td^{-1}(\overline{P})\delta_P -$$

$$Td^{-1}(\overline{\tilde{X}})\delta_{\tilde{X}}+Td^{-1}(\overline{P}+\overline{\tilde{X}})c_1(\overline{P})\delta_{\tilde{X}})-$$

$$\int_{Y/B}ch(\alpha)k^{cod(Y)}\phi^k(ch(\overline{\eta})Td^{-1}(\overline{N}_\infty))\widetilde{Td}(g/g\circ p)+$$

$$\int_{Y/B}ch(\alpha)k^{cod(Y)}\phi^k(ch(\overline{\eta})Td^{-1}(\overline{N}_0))\widetilde{Td}(g/f)+$$

$$g_*(\theta^k(\overline{N}_0^\vee)\psi^k(\overline{\eta})\alpha)-g_*(\theta^k(\overline{N}_\infty^\vee)\psi^k(\overline{\eta})\alpha)-$$

$$\int_{X/B}ch(\alpha)ch(\psi^k(\overline{\xi}.))Td^{-1}(\overline{N}_{X/W})\widetilde{Td}(f/f\circ\phi)+$$

$$\int_{P/B}ch(\alpha)ch(\psi^k(\overline{\xi}.^\infty))Td^{-1}(\overline{N}_{P/W})\widetilde{Td}(g\circ p/f\circ\phi).$$

The deformation theorem will follow from the next lemma, which evaluates the integrals appearing on the two last lines.

Lemma 6.21 *The equalities*

$$\int_{X/B}ch(\alpha)ch(\psi^k(\overline{\xi}.))Td^{-1}(\overline{N}_{X/W})\widetilde{Td}(f/f\circ\phi)=$$

$$\int_{X/B}ch(\alpha)k\phi^k(T(h^\xi))Td(\overline{Tf})-\int_{X/B}ch(\alpha)k\phi^k(T(h^\xi))Td^{-1}(\overline{N}_{X/W})Td(\overline{T(f\circ\phi)})$$

and

$$\int_{P/B}ch(\alpha)ch(\psi^k(\overline{\xi}.^\infty))Td^{-1}(\overline{N}_{P/W})\widetilde{Td}(g\circ p/f\circ\phi)=$$

$$\int_{P/B}ch(\alpha)k\phi^k(T(h^{\xi^\infty}))Td(\overline{T(g\circ p)})-\int_{P/B}ch(\alpha)k\phi^k(T(h^{\xi^\infty}))Td^{-1}(\overline{N}_{P/W})Td(\overline{T(f\circ\phi)})$$

hold in $\tilde{A}(B)$.

Proof of 6.21: we shall only prove the second one, the proof of the first one being similar. Using the definition of the singular Bott-Chern current, we compute

$$\int_{P/B}ch(\alpha)ch(\psi^k(\overline{\xi}.^\infty))Td^{-1}(\overline{N}_{P/W})\widetilde{Td}(g\circ p/f\circ\phi)=$$

$$-\int_{P/B}ch(\alpha)\phi^k(dd^cT(h^{\xi^\infty})-Td^{-1}(\overline{N}_\infty)ch(\overline{\eta})\delta_Y)Td^{-1}(\overline{N}_{P/W})\widetilde{Td}(g\circ p/f\circ\phi)=$$

$$-\int_{P/B}ch(\alpha)k(dd^c\phi^k(T(h^{\xi^\infty})))Td^{-1}(\overline{N}_{P/W})\widetilde{Td}(g\circ p/f\circ\phi)+$$

$$\int_{Y/B} ch(\alpha)k\phi^k(Td^{-1}(\overline{N}_\infty)ch(\overline{\eta}))Td^{-1}(\overline{N}_{P/W})\tilde{Td}(g\circ p/f\circ\phi).$$

The last integral vanishes, since the normal sequence of P in W is split orthogonal on $Y\times\{\infty\}$. Applying (7), we get

$$\int_{P/B} ch(\alpha)k\phi^k(T(h^{\xi^\infty}))Td^{-1}(\overline{N}_{P/W})(Td(\overline{N}_{P/W})Td(\overline{T(g\circ p)})-Td(\overline{T(f\circ\phi)})$$

which is the result. **Q.E.D.**

Q.E.D.

6.2.4 The general case

In this subsection, we state and prove the general form of the Adams-Riemann-Roch theorem for closed immersions, which provides a formula for direct images - composed with push-forward to a common base - of Adams operators acting on hermitian bundles. The proof is an immediate consequence of the deformation theorem and our computation for the canonical model.
If a regular immersion $Y\to X$ as above is given, we shall call *deformable* a Kähler metric on X which is extendable to a metric that is normal to the deformation.

Theorem 6.22 *Let $i: Y\to X$ be a regular closed immersion of arithmetic varieties and $g: Y\to B$, $f: X\to B$ be p.f.s.r. morphisms to an arithmetic variety B such that $g=f\circ i$. Let*

$$\Xi:\ 0\to\xi_m\to\xi_{m-1}\to\ldots\to\xi_0\to i_*\eta\to 0$$

be a resolution by vector bundles on X of a vector bundle η on Y. Suppose that X is endowed with a deformable Kähler metric, that Y carries the induced metric and that the normal bundle N of i carries the quotient metric. Suppose that η and the ξ_i are endowed with hermitian metrics satisfying Bismut's assumption (A) with respect to the metric of N. Let x lie in the subgroup of $\widehat{K}_0(X)$ generated by all the hermitian bundles. The equality

$$g_*(\theta^k(\overline{N}^\vee)\psi^k(\overline{\eta})i^*(x))=f_*(\psi^k(\overline{\xi}_\cdot)x)+$$

$$\int_{Y/B} Td(Tg)ch(i^*(x))ch(\psi^k(\eta)\theta^k(N^\vee))R(N)+\int_{X/B} kTd(\overline{Tf})\phi^k(T(h^{\xi_\cdot}))ch(x)-$$

$$\int_{Y/B} k^{rg(N)}ch(i^*(x))ch(\psi^k(\overline{\eta}))\phi^k(Td^{-1}(\overline{N}))\widetilde{Td}(g/f)$$

holds in $\widehat{K}_0(B)$ for all $k\geq 1$.

Proof: If we let $\alpha = \phi^*(x)$, the deformation theorem tells us that this formula holds, if it holds for the closed immersion i_∞ and the resolution $\overline{\xi}_.^{\infty}$. This is proved in 6.12, so we are done. **Q.E.D**.

Remark. Using 5.8 and going through a computation of the same type as the one appearing in the proof of 6.7, one can show that 6.22 holds for any Kähler metric on X. However, we do not prove this, since the proof doesn't use any new techniques and since we shall not need this fact in the proof of 3.6. If one is ready to give up the part $\frac{1}{k}$ over the torsion, then 6.22 can also be deduced from 3.6, by applying the operation ψ^k to both sides of the equality of 6.7.

7 The arithmetic Adams-Riemann-Roch theorem for local complete intersection p.f.s.r. morphisms

Let us recall the statement of Theorem 3.6:

Theorem 7.1 *Let $g : Y \to B$ be a p.f.s.r. local complete intersection morphism of arithmetic varieties. Suppose that Y is endowed with some Kähler metric. For each $k \geq 0$, let $\theta^k_A(\overline{Tg}^\vee)^{-1} = \theta^k(\overline{Tg}^\vee)^{-1}.(1 + R(Tg_{\mathbf{C}}) - k.\phi^k(R(Tg_{\mathbf{C}})))$. Then for the map $g_* : \widehat{K}_0(Y) \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{k}] \to \widehat{K}_0(B) \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{k}]$, the equality*

$$\psi^k(g_*(y)) = g_*(\theta^k_A(\overline{Tg}^\vee)^{-1}.\psi^k(y))$$

holds in $\widehat{K}_0(B)$ for all $k \geq 1$ and $y \in \widehat{K}_0(Y) \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{k}]$.

The entire section will be devoted to the proof of this theorem. The strategy goes as follows; we first define an error term which measures the difference between both sides of the equality in 7.1; we show that the error term vanishes for differential forms (7.5), that it is invariant under change of the Kähler metric (7.6), that it is additive up to an explicit cohomological term (7.9) and that it is base-change invariant (7.18). Next we prove that these properties suffice to prove the theorem for projective spaces, using a diagonal embedding argument. Using additivity again, we can then establish the full result.

Before starting with the core of the proof itself, we give a proof of 3.5, whithout which the statement of the theorem wouldn't be meaningful. For this, we first establish two propositions. Recall that the group homomorphism $ch : \widehat{K}_0(Y) \to Z(Y)$ is the map given by the formula $ch(\overline{E} + \eta) = ch(\overline{E}) + dd^c\eta$, where $ch(\overline{E})$ refers to the representative of the Chern character of $E_{\mathbf{C}}$ arising from the canonical hermitian holomorphic connection.

Proposition 7.2 *For any short exact sequence of hermitian bundles*

$$\mathcal{E} : 0 \to E' \to E \to E'' \to 0$$

on an arithmetic variety Y, *the equality*

$$\tilde{\theta}^k(\overline{\mathcal{E}}) = \theta^k((\overline{E}' \oplus \overline{E}'')^\vee)^{-1} - \theta^k(\overline{E}^\vee)^{-1}$$

holds in $\widehat{K}_0(Y) \otimes \mathbf{Z}[\frac{1}{k}]$.

Proof: By 6.11, $dd^c\tilde{\theta}^k(\overline{\mathcal{E}}) = ch(\theta^k((\overline{E}' \oplus \overline{E}'')^\vee)^{-1}) - ch(\theta^k(\overline{E}^\vee)^{-1})$. Now consider the exterior product bundle $E'(1) := E'\Box\mathcal{O}(1)$ on $Y \times \mathbf{P}^1_{\mathbf{Z}}$. Let σ be the canonical section of $\mathcal{O}(1)$, which vanishes only at ∞. It defines a map of vector bundles $E' \to E'(1)$. Define the bundle $\tilde{E}$ as $(E \oplus E'(1))/E'$. We have an exact sequence on $Y \times \mathbf{P}^1_{\mathbf{Z}}$

$$\tilde{\mathcal{E}} : 0 \to E'(1) \to \tilde{E} \to E'' \to 0$$

(compare with [9, I, Par. f)]) and isomorphisms $j_0^*\tilde{E} \simeq E$, $j_\infty^*\tilde{E} \simeq E' \oplus E''$. Endow $\tilde{E}$ with a metric making these isomorphisms isometric. Endow $\mathcal{O}(1)$ with the Fubini-Study metric, $E'(1)$ with the product metric. Denote by p the projection $Y \times \mathbf{P}^1_{\mathbf{Z}} \to Y$. Using [20, Theorem, 4.4.6, p. 161], we can now compute

$$\theta^k((\overline{E}' \oplus \overline{E}'')^\vee)^{-1} - \theta^k(\overline{E}^\vee)^{-1} =$$
$$j_\infty^*\theta^k(\overline{\tilde{E}}^\vee)^{-1} - j_0^*\theta^k(\overline{\tilde{E}}^\vee)^{-1} =$$
$$-\int_{\mathbf{P}^1} ch(\theta^k(\overline{\tilde{E}}^\vee)^{-1})log|z|^2 =$$
$$\int_{\mathbf{P}^1} (\ ch(\theta^k((\overline{E}'(1) \oplus \overline{E}'')^\vee)^{-1}) - ch(\theta^k(\overline{\tilde{E}}^\vee)^{-1})\ )log|z|^2$$

The last equality is justified by the fact that

$$\int_{\mathbf{P}^1} ch(\theta^k((\overline{E}'(1) \oplus \overline{E}'')^\vee)^{-1})log|z|^2 = 0.$$

Indeed $ch(\theta^k((\overline{E}'(1) \oplus \overline{E}'')^\vee)^{-1})$ is by construction invariant under the change of variable $z \to 1/z$ and $log|1/z|^2 = -log|z|^2$. Therefore the integral changes sign under that change of variable. Resuming our computations, we get

$$\int_{\mathbf{P}^1} (\ ch(\theta^k((\overline{E}'(1) \oplus \overline{E}'')^\vee)^{-1}) - ch(\theta^k(\overline{\tilde{E}}^\vee)^{-1})\ )log|z|^2 =$$
$$\int_{\mathbf{P}^1} d_z d_z^c\tilde{\theta}^k(\overline{\tilde{\mathcal{E}}})log|z|^2 = \int_{\mathbf{P}^1} \tilde{\theta}^k(\overline{\tilde{\mathcal{E}}})d_z d_z^c log|z|^2 =$$
$$j_0^*\tilde{\theta}^k(\overline{\tilde{\mathcal{E}}}) - j_\infty^*\tilde{\theta}^k(\overline{\tilde{\mathcal{E}}}) = \tilde{\theta}^k(\overline{\mathcal{E}})$$

which ends the proof. **Q.E.D.**

Proposition 7.3 *For any short exact sequence of hermitian bundles*

$$\overline{\mathcal{E}} : 0 \to E' \to E \to E'' \to 0$$

on a complex manifold, $\tilde{\theta}^k(\overline{\mathcal{E}})$ *is equal to the expression*

$$\{\ k^{-rg(E)} Td^{-1}(\overline{E})\phi^k(Td(\overline{E}))k^{rg(E'')}\widetilde{Td}(\overline{\mathcal{E}}) - k^{1-rg(E')}\phi^k(\widetilde{Td}(\overline{\mathcal{E}}))\ \}.$$

$$Td^{-1}(\overline{E}')k^{-rg(E'')}Td^{-1}(\overline{E}'') \tag{16}$$

in $\tilde{A}(M)$.

Proof: A straightforward computation using the identity $\phi^k \circ dd^c = k.dd^c \circ \phi^k$ shows that we obtain

$$k^{-rg(E'+E'')}Td^{-1}(\overline{E}' \oplus \overline{E}'')\phi^k(Td(\overline{E}' \oplus \overline{E}'')) - k^{-rg(E)}Td^{-1}(\overline{E})\phi^k(Td(\overline{E}))$$

if we apply dd^c to the expression in 7.3. Furthermore the expression in 7.3 clearly vanishes when the sequence $\overline{\mathcal{E}}$ splits orthogonally. Thus, by the axiomatic caracterisation of secondary classes (see [9, I, Par. f)]), our claim is proved. **Q.E.D.**

Proof of 3.5.
Let $i_1 : Y \to X_1$ be a second factorisation like the one before 3.5. Let X' be the fiber product $X \times_B X_1$. Let $j : Y \to X'$ denote the diagonal embedding. If we endow X' with a Kähler metric and the normal bundle $N_{Y(\mathbf{C})/X'(\mathbf{C})}$ with any hermitian metric, then j gives a third factorisation. If we denote the natural projection morphism $X' \to X$ by p and by h the map $f \circ p$, then we have a commutative diagram of bundles on $Y(\mathbf{C})$.

$$\begin{array}{ccccccccc}
 & & & \mathcal{N}' & & \mathcal{N} & & & \\
 & & & 0 & & 0 & & & \\
 & & & \downarrow & & \downarrow & & & \\
 & & & Tg_{\mathbf{C}} & \overset{Id}{\simeq} & Tg_{\mathbf{C}} & & & \\
 & & & \downarrow & & \downarrow & & & \\
0 & \to & j^*Tp_{\mathbf{C}} & \to & j^*Th_{\mathbf{C}} & \to & i^*Tf_{\mathbf{C}} & \to & 0 \quad \mathcal{R}_1 \\
 & & \downarrow Id & & \downarrow & & \downarrow & & \\
0 & \to & j^*Tp_{\mathbf{C}} & \to & N_{Y(\mathbf{C})/X'(\mathbf{C})} & \to & N_{Y(\mathbf{C})/X(\mathbf{C})} & \to & 0 \quad \mathcal{R}_2 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

By a result of Gillet-Soulé in [23, Lemma 14, p. 501], we have

$$\widetilde{Td}(\mathcal{N}')Td(\overline{N}_{Y/X'})^{-1} - \widetilde{Td}(\mathcal{N})Td(\overline{N}_{Y/X})^{-1} +$$

$$\widetilde{Td}(\mathcal{R}_2)Td(j^*\overline{T}h)Td(\overline{N}_{Y/X'})^{-1}Td(\overline{N}_{Y/X})^{-1}Td(j^*\overline{T}p)^{-1} -$$

$$\widetilde{Td}(\mathcal{R}_1)Td(\overline{N}_{Y/X})^{-1}Td(j^*\overline{T}p)^{-1}=0.$$

Applying the formula 7.3 and carrying through a tedious but elementary calculation, we conclude that similarly

$$\tilde{\theta}^k(\mathcal{N}')\theta^k(\overline{N}_{Y/X'}{}^\vee)-\tilde{\theta}^k(\mathcal{N})\theta^k(\overline{N}_{Y/X}{}^\vee)+$$

$$\tilde{\theta}^k(\mathcal{R}_2)\theta^k(j^*\overline{T}h^\vee)^{-1}\theta^k(\overline{N}_{Y/X'}{}^\vee)\theta^k(\overline{N}_{Y/X}{}^\vee)\theta^k(j^*\overline{T}p^\vee)-$$

$$\tilde{\theta}^k(\mathcal{R}_1)\theta^k(\overline{N}_{Y/X}{}^\vee)\theta^k(j^*\overline{T}p^\vee)=0 \tag{17}$$

Another way to prove this identity is to consider that the proof of [23, Lemma 14, p. 501] can be carried through without change for $\tilde{\theta}^k$ in place of $\widetilde{Td}$. We can now compute

$$\theta^k(\overline{N}^\vee_{Y/X})\tilde{\theta}^k(\mathcal{N})+\theta^k(\overline{N}^\vee_{Y/X})\theta^k(i^*\overline{Tf}^\vee)^{-1}-$$

$$(\ \theta^k(\overline{N}^\vee_{Y/X'})\tilde{\theta}^k(\mathcal{N}')+\theta^k(\overline{N}^\vee_{Y/X'})\theta^k(i^*\overline{T}h^\vee)^{-1}\ )=$$

$$[\theta^k(j^*\overline{Tp}^\vee)^{-1}\theta^k(i^*\overline{Tf}^\vee)^{-1}-\theta^k(j^*\overline{Th}^\vee)^{-1}]\theta^k(\overline{N}^\vee_{Y/X}).\theta^k(j^*\overline{Tp}^\vee)+$$

$$[\theta^k(\overline{N}^\vee_{Y/X'})^{-1}-\theta^k((j^*\overline{Tp})^\vee)^{-1}\theta^k(\overline{N}^\vee_{Y/X})^{-1}]\theta^k(j^*\overline{Th}^\vee)^{-1}\theta^k(\overline{N}^\vee_{Y/X'})\theta^k(\overline{N}^\vee_{Y/X})\theta^k(j^*\overline{Tp}^\vee)+$$

$$\tilde{\theta}^k(\mathcal{N})\theta^k(\overline{N}^\vee_{Y/X})-\tilde{\theta}^k(\mathcal{N}')\theta^k(\overline{N}^\vee_{Y/X'}).$$

The expression after the last equality vanishes in view of 7.2 and (17), so we have proved that the arithmetic Bott class determined by i and the hermitian metrics on Tf and $N_{Y/X}$ coincides with the arithmetic Bott class determined by j and the hermitian metrics on Th and $N_{X'/Y}$. To complete the proof, note that by symmetry, the arithmetic Bott class determined by j also coincides with the arithmetic Bott class determined by i_1. **Q.E.D.**

If $A(x)=a_0+a_1x+a_2x^2+\ldots$ is a power series with real coefficients, we define A to be the unique additive characteristic class such that $A(L)=a_0+a_1c_1(L)+a_2c_1(L)^2+\ldots$ for every line bundle L.

Definition 7.4 *Let $A(x)$ be a power series with real coefficients. Let $g:Y\to B$ be a p.f.s.r. local complete intersection morphism of arithmetic varieties. Let h_Y be a Kähler metric on Y. Let $y_0\in\widehat{K}_0(Y)$. Define $\theta^k_A(\overline{Tg}^\vee)^{-1}:=\theta^k(\overline{Tg}^\vee)^{-1}(1-A(Tg_{\mathbf{C}}))$. The error term $\delta(A,g,h_Y,y_0)$ relative to A, g, h_Y and y_0 is the difference*

$$\psi^k(g_*(y_0))-g_*(\theta^k_A(\overline{Tg}^\vee)^{-1}\psi^k(y_0)).$$

Proposition 7.5 *For any morphism g, any power series A and any metric h_Y, the error term $\delta(A, g, h_Y, y_0)$ vanishes when $y_0 = \omega$, where $\omega \in \tilde{A}(Y)$.*

Proof: We compute

$$\psi^k(g_*(\omega)) = k.\phi^k \int_{Y/B} Td(\overline{Tg})\omega =$$

$$\int_{Y/B} k.k^{dimB-dimY}\phi^k(Td(\overline{Tg}))\phi^k(\omega) =$$

$$\int_{Y/B} k.ch(\theta^k(\overline{Tg}^\vee)^{-1})Td(\overline{Tg})\phi^k(\omega) =$$

$$g_*(\theta^k(\overline{Tg}^\vee)^{-1}\phi^k(\omega)) = g_*(\theta^k_A(\overline{Tg}^\vee)^{-1}\phi^k(\omega))$$

where we used the identity of 6.10 in the second line and and the definition of the product in $\tilde{A}(Y)$ in the third line. **Q.E.D.**

Proposition 7.6 *Fix a power series A, a morphism g and an element $y_0 \in \widehat{K}_0(Y)$. If h, h' are two Kähler metrics on Y, then $\delta(A, g, h, y_0) = \delta(A, g, h', y_0)$.*

Proof: In order to emphasize the dependence on the metric, we shall in this proof write g^h_* for the pushforward map $\widehat{K}_0(Y) \to \widehat{K}_0(B)$ associated to g and a Kähler metric h on Y. We write $\theta^k((T^h g)^\vee)^{-1}$ for the arithmetic Bott class associated to g and h. Let us write $\overline{\mathcal{MC}}$ for the sequence

$$0 \to Tg_{\mathbf{C}} \stackrel{Id}{\to} Tg_{\mathbf{C}} \to 0 \to 0$$

where the second term carries the metric induced by h and the third one the metric induced by h'.

Lemma 7.7 *The equality $\theta^k((T^h g)^\vee)^{-1} - \theta^k((T^{h'} g)^\vee)^{-1} = \tilde{\theta}^k(\overline{\mathcal{MC}})$ holds in $\widehat{K}_0(Y)$.*

Proof of 7.7: consider a factorisation $f = g \circ i$ as in 3.5, where i is a regular closed immersion into an arithmetic variety and f is a smooth map. Consider the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Tg_{\mathbf{C}} & \to & i^*Tf_{\mathbf{C}} & \to & N_{X(\mathbf{C})/Y(\mathbf{C})} & \to & 0 \\
 & & \downarrow Id & & \downarrow Id & & \downarrow Id & & \\
0 & \to & Tg_{\mathbf{C}} & \to & i^*Tf_{\mathbf{C}} & \to & N_{X(\mathbf{C})/Y(\mathbf{C})} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $Tg_{\mathbf{C}}$ is endowed with metric induced by h on the second row and the metric induced by h' on the third row. If we apply the general symmetry formula of [21, Prop. 1.3.4, p. 173] to this diagram and use the multiplicativity of $k^{-rg(.)}Td^{-1}(.)\phi^k(Td(.))$ we see that the difference between the secondary Bott-Chern form $\tilde{\theta}^k$ of the second row and the secondary form of the third row is equal to $\tilde{\theta}^k(\overline{\mathcal{MC}})ch(\theta^k(\overline{N}^\vee)^{-1})$ .The claim thus follows from the definition of the arithmetic Bott class. **Q.E.D.**

Lemma 7.8 *For any $y \in \widehat{K}_0(Y)$, the formula $g_*^{h'}(y) - g_*^h(y) = \int_{Y/B} ch(y)\widetilde{Td}(\overline{\mathcal{MC}})$ holds.*

Proof of 7.8: since the Grothendieck group of vector bundles $K_0(Y)$ is generated by g-acyclic vector bundles and both sides of the equality to be proved are additive, we can assume that $y = \overline{E}$, where $\overline{E}$ is a g-acyclic hermitian vector bundle or that $y = \kappa \in \tilde{A}(Y)$. For $y = \kappa$, we compute

$$g_*^{h'}(\kappa) - g_*^h(\kappa) = \int_{Y/B} (Td(T^{h'}g_{\mathbf{C}}) - Td(T^{h'}g_{\mathbf{C}}))\kappa =$$

$$\int_{Y/B} dd^c\widetilde{Td}(\overline{\mathcal{MC}})\kappa = \int_{Y/B} \widetilde{Td}(\overline{\mathcal{MC}})dd^c\kappa = \int_{Y/B} \widetilde{Td}(\overline{\mathcal{MC}})ch(\kappa).$$

For $y = \overline{E} = (E, h^E)$, we compute using 5.8

$$g_*^{h'}(\overline{E}) - g_*^h(\overline{E}) = (g_*E, g_*^{h'}h^E) - T(h', (E, h^E)) - (g_*E, g_*^h h^E) + T(h, (E, h^E)) =$$

$$-T(h', (E, h^E)) + T(h, (E, h^E)) + \widetilde{ch}(g_*^h h^E, g_*^{h'} h^E) = \int_{Y/B} \widetilde{Td}(\overline{\mathcal{MC}})ch(\overline{E}).$$

Combining our computations, we get the result. **Q.E.D.**

We resume the proof of 7.6. Let $\delta = dim(Y) - dim(B)$. Using the last Lemma, we compute that on the left side of the error term

$$\psi^k(g_*^{h'}(y_0)) - \psi^k(g_*^h(y_0)) = k.\phi^k(\int_{Y/B} ch(y_0)\widetilde{Td}(\overline{\mathcal{MC}})) =$$

$$\int_{Y/B} k^{1-\delta}\phi^k(ch(y_0))\phi^k(\widetilde{Td}(\overline{\mathcal{MC}}))$$

On the right side, we compute

$$g_*^{h'}(\theta_A^k(T^{h'}g^\vee)^{-1}\psi^k(y_0)) - g_*^h(\theta_A^k(T^h g^\vee)^{-1}\psi^k(y_0)) =$$

$$g_*^{h'}(\theta^k(T^{h'}g^\vee)^{-1}\psi^k(y_0)) - g_*^h(\theta^k(T^h g^\vee)^{-1}\psi^k(y_0)) =$$

$$(\ g_*^{h'}(\theta^k(T^{h'}g^\vee)^{-1}\psi^k(y_0)) - g_*^h(\theta^k(T^{h'}g^\vee)^{-1}\psi^k(y_0))\ )-$$

$$(g_*^h(\theta^k(T^h g^\vee)^{-1}\psi^k(y_0)) - g_*^h(\theta^k(T^{h'} g^\vee)^{-1}\psi^k(y_0)))$$

Using 7.7 and 7.3, we can see that the expression after the last equality equals

$$\int_{Y/B}\{ ch(\psi^k(y_0))ch(\theta^k((T^{h'}g)^\vee)^{-1})\widetilde{Td}(\overline{\mathcal{MC}}) - \tilde{\theta}^k(\overline{\mathcal{MC}})ch(\psi^k(y_0))Td(T^h g) \} \tag{18}$$

On the other hand, using 7.3 we compute that

$$\tilde{\theta}^k(\overline{\mathcal{MC}}) = Td^{-1}(T^h g)(ch(\theta^k((T^{h'}g)^\vee)^{-1})\widetilde{Td}(\overline{\mathcal{MC}}) - k^{1-rg(Tg)}\phi^k(\widetilde{Td}(\overline{\mathcal{MC}}))).$$

If we reinsert this in (18), we see that both sides coincide and we can conclude. **Q**.**E**.**D**.

In view of the last proposition, we shall from now on drop the Kähler metric entry in the error term δ. Let $i : Y \to X$ be a regular closed immersion into an arithmetic variety X and $f : X \to B$ a p.f.s.r. map such that $g = f \circ i$. Let η be a locally free sheaf on Y and

$$0 \to \xi_m \to \xi_{m-1} \to \dots \xi_0 \to i_*\eta \to 0$$

be a locally free resolution on X of $i_*\eta$. We endow X with a deformable Kähler metric, Y with the induced metric and the normal bundle N of i the quotient metric. We endow the bundles η, ξ_i with hermitian metrics satisfying Bismut's assumption (A). The next Proposition studies the behaviour under immersions of the error term $\delta((.),(.),(.))$:

Proposition 7.9 *Let $A(x)$ be a formal power series with real coefficients. The formula*

$$\delta(A, g, \overline{\eta}) - \sum_{i=0}^{m}\delta(A, f, \overline{\xi_i}) =$$

$$\int_{Y/B} ch(\psi^k(\overline{\eta}))\{k.k^{dimB-dimY}\phi^k(R(N)Td(Tg))-$$

$$Td(Tf)A(Tf)ch(\theta^k(Tf^\vee)^{-1}).Td(N)^{-1})ch(\theta^k(N^\vee))-$$

$$Td(Tg)R(N)ch(\theta^k(N^\vee))ch(\theta^k(Tf^\vee)^{-1})+$$

$$Td(Tg)ch(\theta^k(Tg^\vee)^{-1})A(Tg)\}$$

holds in $\widehat{K}_0(B)$.

Proof: Let $\mathcal{N}$ be the normal sequence of the immersion, with the given metrics. Using 6.7, we compute

$$\psi^k(g_*(\overline{\eta})) =$$

$$\psi^k(f_*(\overline{\xi}.)) + k.\phi^k(\int_{X/B} T(h^{\xi.})Td(\overline{Tf}))$$

$$-k.\phi^k(\int_{Y/B} ch(\overline{\eta})Td^{-1}(\overline{N})\widetilde{Td}(g/f)) + k.\phi^k(\int_{Y/B} ch(\eta)R(N)Td(Tg))$$

and secondly, using the definition of the arithmetic Bott element,

$$g_*(\theta^k(\overline{Tg}^\vee)^{-1}(1 - A(Tg))\psi^k(\overline{\eta})) =$$

$$g_*(\theta^k(\overline{N}^\vee)\theta^k(\overline{T}^\vee f)^{-1}.\psi^k(\overline{\eta})) + \int_{Y/B} Td(\overline{Tg})ch(\theta^k(\overline{N}^\vee).\psi^k(\overline{\eta}))\tilde{\theta}^k(\overline{\mathcal{N}}) -$$

$$\int_{Y/B} Td(Tg)ch(\theta^k(Tg^\vee)^{-1})ch(\psi^k(\eta))A(Tg).$$

With these expressions in hand, we can compute:

$$\psi^k(g_*(\overline{\eta})) - g_*(\theta^k(\overline{Tg}^\vee)^{-1}(1 - A(Tg))\psi^k(\overline{\eta})) =$$

$$\psi^k(f_*(\overline{\xi}.)) + k.\phi^k(\int_{X/B} T(h^{\xi.})Td(\overline{Tf}))$$

$$-k.\phi^k(\int_{Y/B} ch(\overline{\eta})Td^{-1}(\overline{N})\widetilde{Td}(g/f)) + k.\phi^k(\int_{Y/B} ch(\eta)R(N)Td(Tg)) -$$

$$(\ g_*(\theta^k(\overline{N}^\vee)\theta^k(\overline{T}^\vee f)^{-1}.\psi^k(\overline{\eta})) +$$

$$\int_{Y/B}(Td(\overline{Tg})ch(\theta^k(\overline{N}^\vee).\psi^k(\overline{\eta}))\tilde{\theta}^k(\overline{\mathcal{N}})) -$$

$$\int_{Y/B} Td(Tg)ch(\theta^k(Tg^\vee)^{-1})ch(\psi^k(\eta))A(Tg)\ ).$$

Using 6.22, we compute

$$g_*(\theta^k(\overline{N}^\vee)\theta^k(\overline{T}^\vee f)^{-1}.\psi^k(\overline{\eta})) =$$

$$f_*(\theta^k(\overline{T}^\vee f)^{-1}\psi^k(\overline{\xi}.)) + \int_{Y/B} Td(Tg)R(N)ch(\theta^k(N^\vee))ch(\theta^k(Tf^\vee)^{-1})ch(\psi^k(\eta)) +$$

$$\int_{X/B} Td(\overline{Tf})k.\phi^k(T(h^{\xi.}))ch(\theta^k(\overline{T}^\vee f)^{-1}) -$$

$$\int_{Y/B} k^{rg(N)}ch(\theta^k(\overline{T}^\vee f)^{-1})ch(\psi^k(\overline{\eta}))\phi^k(Td^{-1}(\overline{N}))\widetilde{Td}(g/f).$$

Now notice that we can write

$$f_*(\theta^k(\overline{T}^\vee f)^{-1}\psi^k(\overline{\xi}.)) = f_*(\theta^k(\overline{T}^\vee f)^{-1}(1 - A(Tf))\psi^k(\overline{\xi}.)) +$$

$$\int_{X/B} Td(Tf)A(Tf)ch(\psi^k(\xi.))ch(\theta^k(T^\vee_{X/B})^{-1}).$$

Finally, returning to our expression for the substraction above, we get

$$\psi^k(g_*(\overline{\eta})) - g_*(\theta^k(\overline{Tg}^\vee)^{-1}(1 - A(Tg))\psi^k(\overline{\eta})) =$$

$$\psi^k(f_*(\overline{\xi}.)) + k.\phi^k(\int_{X/B} T(h^{\xi.})Td(\overline{Tf}))$$

$$-k.\phi^k(\int_{Y/B} ch(\overline{\eta})Td^{-1}(\overline{N})\widetilde{Td}(g/f)) + k.\phi^k(\int_{Y/B} ch(\eta)R(N)Td(Tg))-$$

$$(\ f_*(\theta^k(\overline{T}^\vee f)^{-1}(1-A(Tf))\psi^k(\overline{\xi}.))+\int_{X/B} Td(Tf)A(Tf)ch(\psi^k(\xi.))ch(\theta^k(T^\vee f)^{-1})+$$

$$\int_{Y/B} Td(Tg)R(N)ch(\theta^k(N^\vee))ch(\theta^k(Tf^\vee)^{-1})ch(\psi^k(\eta))+$$

$$\int_{X/B} Td(\overline{Tf})k.\phi^k(T(h^{\xi.}))ch(\theta^k(\overline{T}^\vee f)^{-1})-$$

$$\int_{Y/B} k^{rg(N)}ch(\theta^k(\overline{T}^\vee f)^{-1})ch(\psi^k(\overline{\eta}))\phi^k(Td^{-1}(\overline{N}))\widetilde{Td}(g/f)$$

$$+\int_{Y/B}(Td(\overline{Tg})ch(\theta^k(\overline{N}^\vee).\psi^k(\overline{\eta}))\tilde{\theta}^k(\overline{\mathcal{N}}))-$$

$$\int_{Y/B} Td(Tg)ch(\theta^k(Tg^\vee)^{-1})ch(\psi^k(\eta))A(Tg)\ )$$

We first reorder the expression, in order to gather integrals on closed forms, metrical terms (containing $\tilde{(.)}$) and singular current terms in separate groups. We obtain

$$\psi^k(g_*(\overline{\eta}))-g_*(\theta^k(\overline{Tg}^\vee)^{-1}(1-A(Tg))\psi^k(\overline{\eta}))-\psi^k(f_*(\overline{\xi}.))+f_*(\theta^k(\overline{T}^\vee f)^{-1}(1-A(Tf))\psi^k(\overline{\xi}.)) =$$

$$\{k.\phi^k(\int_{X/B} T(h^{\xi.})Td(\overline{Tf}))-$$

$$\int_{X/B} Td(\overline{Tf})k.\phi^k(T(h^{\xi.}))ch(\theta^k(\overline{T}^\vee f)^{-1})\}+$$

$$\{-k.\phi^k(\int_{Y/B} ch(\overline{\eta})Td^{-1}(\overline{N})\widetilde{Td}(g/f))+$$

$$\int_{Y/B} k^{rg(N)}ch(\theta^k(\overline{T}^\vee f)^{-1})ch(\psi^k(\overline{\eta}))\phi^k(Td^{-1}(\overline{N}))\widetilde{Td}(g/f)$$

$$-\int_{Y/B}(Td(\overline{Tg})ch(\theta^k(\overline{N}^\vee).\psi^k(\overline{\eta}))\tilde{\theta}^k(\overline{\mathcal{N}}))\}+$$

$$\{k.\phi^k(\int_{Y/B} ch(\eta)R(N)Td(Tg))-$$

$$\int_{X/B} Td(Tf)A(Tf)ch(\psi^k(\xi.))ch(\theta^k(T^\vee f)^{-1})-$$

$$\int_{Y/B} Td(Tg)R(N)ch(\theta^k(N^\vee))ch(\theta^k(Tf^\vee)^{-1})ch(\psi^k(\eta))+$$

$$\int_{Y/B} Td(Tg)ch(\theta^k(Tg^\vee)^{-1})ch(\psi^k(\eta))A(Tg)\}$$

The expression before the last equal sign is by definition equal to $\delta(A,g,\overline{\eta}) - \sum_{i=0}^m \delta(A,f,\overline{\xi_i})$. The proof of the Proposition will now follow from the next three lemmas, which evaluate the expressions in the brackets $\{\cdot\}$ separately.

Lemma 7.10 *The equality of differential forms*

$$k.\phi^k(\int_{X/B} T(h^{\xi.})Td(\overline{Tf})) - \int_{X/B} Td(\overline{Tf})k.\phi^k(T(h^{\xi.}))ch(\theta^k(\overline{T}^\vee f)^{-1}) = 0$$

holds.

Proof of 7.10: we compute

$$\int_{X/B} Td(\overline{Tf})k.\phi^k(T(h^{\xi.}))ch(\theta^k(\overline{T}^\vee f)^{-1}) =$$

$$\int_{X/B} Td(\overline{Tf})k.\phi^k(T(h^{\xi.}))k^{-rg(Tf)}Td(\overline{Tf})^{-1}\phi^k(Td(\overline{Tf})) =$$

$$\int_{X/B} k.\phi^k(T(h^{\xi.}))k^{-rg(Tf)}\phi^k(Td(\overline{Tf})) =$$

$$\int_{X/B} k.k^{-rg(Tf)}\phi^k(T(h^{\xi.})Td(\overline{Tf})) =$$

$$k.\phi^k(\int_{X/B} T(h^{\xi.})Td(\overline{Tf})).$$

Q.E.D.

Lemma 7.11 *The equality*

$$-k.\phi^k(\int_{Y/B} ch(\overline{\eta})Td^{-1}(\overline{N})\widetilde{Td}(g/f))+$$

$$\int_{Y/B} k^{rg(N)}ch(\theta^k(\overline{T}^\vee f)^{-1})ch(\psi^k(\overline{\eta}))\phi^k(Td^{-1}(\overline{N}))\widetilde{Td}(g/f)$$

$$-\int_{Y/B}(Td(\overline{Tg})ch(\theta^k(\overline{N}^\vee).\psi^k(\overline{\eta}))\tilde{\theta}^k(\overline{\mathcal{N}}))=0$$

holds in $\tilde{A}(B)$.

Proof of 7.11: apply 7.3 to the sequence $\overline{\mathcal{N}}$. **Q.E.D.**

For the next and last lemma, we shall need the Adams-Riemann-Roch theorem for the Grothendieck group of vector bundles. In the next theorem, let i_* denote the push-forward map $K_0(Y)\to K_0(X)$ associated to the immersion i (see [24, 2.12, p. 289]).

Theorem 7.12 *Let the definitions of the last theorem hold. The equality*

$$i_*(\theta^k(N^\vee)\psi^k(x))=\psi^k(i_*(x))$$

holds in $K_0(X)$.

For the proof, see [19, VI, 8.] or apply the forgetful map $\widehat{K}_0(B)\to K_0(B)$ to both sides of 6.22.

Lemma 7.13 *The equality*

$$\{k.\phi^k(\int_{Y/B}ch(\eta)R(N)Td(Tg))-$$

$$\int_{X/B}Td(Tf)A(Tf)ch(\psi^k(\xi.))ch(\theta^k(T^\vee f)^{-1})-$$

$$\int_{Y/B}Td(Tg)R(N)ch(\theta^k(N^\vee))ch(\theta^k(Tf^\vee)^{-1})ch(\psi^k(\eta))+$$

$$\int_{Y/B}Td(Tg)ch(\theta^k(Tg^\vee)^{-1})ch(\psi^k(\eta))A(Tg)\}=$$

$$\int_{Y/B}ch(\psi^k(\eta))\{k.k^{dimB-dimY}\phi^k(R(N)Td(Tg))-$$

$$Td(Tf)A(Tf)ch(\theta^k(Tf^\vee)^{-1}).Td(N)^{-1})ch(\theta^k(N^\vee))-$$

$$Td(Tg)R(N)ch(\theta^k(N^\vee))ch(\theta^k(Tf^\vee)^{-1})+$$

$$Td(Tg)ch(\theta^k(Tg^\vee)^{-1})A(Tg)\}$$

holds in $H(B)\subseteq\tilde{A}(B)$.

Proof of 7.13: we can compute, using the Adams-Riemann-Roch and cohomological Riemann-Roch theorems for coherent sheaves, that

$$ch(\psi^k(\xi.)) = ch(i_*(\theta^k(N^\vee)\psi^k(\eta)) = i_*(Td(N)^{-1}ch(\theta^k(N^\vee))ch(\psi^k(\eta))).$$

Thus, using the projection formula for the push-forward in cohomology, we can compute

$$\int_{X/B} Td(Tf)A(Tf)ch(\psi^k(\xi.))ch(\theta^k(T^\vee f)^{-1}) =$$

$$\int_{Y/B} Td(Tf)A(Tf)Td(N)^{-1}ch(\theta^k(N^\vee))ch(\psi^k(\eta))ch(\theta^k(T^\vee f)^{-1}). \qquad (19)$$

Reinserting (19) in the expression on the left hand of the equality of 7.13, we obtain the right hand. **Q.E.D**.

Q.E.D.

In the next corollary, $(.)|_Y$ means restriction to Y.

Corollary 7.14 *Let the terminology of 7.9 hold. If* $A(Tf)|_Y = k.\phi^k(R(Tf))|_Y - R(Tf)|_Y$ *and* $A(Tg) = k.\phi^k(R(Tg)) - R(Tg)$ *then*

$$\delta(A, g, \overline{\eta}) - \sum_{i=0}^{m} \delta(A, f, \overline{\xi_i}) = 0$$

Proof: If we compute the right side of the equality of 7.9, we get

$$\int_{Y/B} ch(\psi^k(\eta))\{k.k^{dimB-dimY}\phi^k(R(N)Td(Tg))+$$
$$Td(Tf)(R(Tf) - k.\phi^k(R(Tf)))ch(\theta^k(Tf^\vee)^{-1}).Td(N)^{-1})ch(\theta^k(N^\vee))-$$
$$Td(Tg)R(N)ch(\theta^k(N^\vee))ch(\theta^k(Tf^\vee)^{-1})-$$
$$Td(Tg)ch(\theta^k(Tg^\vee)^{-1})(R(Tg) - k\phi^k(R(Tg)))\}.$$

Using 6.11, this expression can be rewritten as

$$\int_{Y/B} ch(\psi^k(\eta))\{k.k^{dimB-dimY}\phi^k(R(N))\phi^k(Td(Tg)) + Td(Tf)(R(Tf)-$$
$$k.\phi^k(R(Tf)))k^{dimB-dimX}Td(Tf)^{-1}\phi^k(Td(Tf))Td(N)^{-1}k^{rg(N)}Td(N)\phi^k(Td(N)^{-1})-$$
$$Td(Tg)R(N)k^{rg(N)}Td(N)\phi^k(Td(N)^{-1})k^{dimB-dimX}Td(Tf)^{-1}\phi^k(Td(Tf))-$$
$$Td(Tg)k^{dimB-dimY}Td(Tg)^{-1}\phi^k(Td(Tg))(R(Tg) - k.\phi^k(R(Tg)))\}.$$

Using the multiplicativity of the Todd class and the additivity of $R(.)$, the last expression can be evaluated to be

$$\int_{Y/B} ch(\psi^k(\eta))\{k.k^{dimB-dimY}\phi^k(R(N))\phi^k(Td(Tg))+$$

$$(R(Tf)-k.\phi^k(R(Tf)))k^{dimB-dimX}\phi^k(Td(Tf))k^{rg(N)}\phi^k(Td(N)^{-1})-$$

$$R(N)k^{rg(N)}\phi^k(Td(N)^{-1})k^{dimB-dimX}\phi^k(Td(Tf))-$$

$$k^{dimB-dimY}\phi^k(Td(Tg))(R(Tg)-k.\phi^k(R(Tg)))\}=$$

$$\int_{Y/B} ch(\psi^k(\eta))\{k.k^{dimB-dimY}\phi^k(R(N))\phi^k(Td(Tg))+$$

$$R(Tf)k^{dimB-dimX}\phi^k(Td(Tf)).k^{rg(N)}\phi^k(Td(N)^{-1})-$$

$$k.\phi^k(R(Tf))k^{dimB-dimX}\phi^k(Td(Tf))k^{rg(N)}\phi^k(Td(N)^{-1})-$$

$$R(N)k^{rg(N)}\phi^k(Td(N)^{-1})k^{dimX-dimB}\phi^k(Td(Tf))-k^{dimB-dimY}\phi^k(Td(Tg))R(Tg)+$$

$$k^{dimB-dimY}\phi^k(Td(Tg))k\phi^k(R(Tg))\}=$$

$$\int_{Y/B} ch(\psi^k(\eta))k^{dimB-dimY}\{k.\phi^k(Td(Tg))\phi^k(R(N)-R(Tf)+R(Tg))+$$

$$\phi^k(Td(Tg))(R(Tf)-R(N)-R(Tg))\}=0$$

which ends the proof. **Q.E.D.**

The next lemma is needed to prove the projection formula stated after it.

Lemma 7.15 *Let M be a complex Kähler manifold. Let $n>0$ and let $p_M : M\times\mathbf{P}^n(\mathbf{C})\to M$ and $p_P : M\times\mathbf{P}^n(\mathbf{C})\to\mathbf{P}^n(\mathbf{C})$ be the projection maps. Fix a Kähler metric on M and endow $M\times\mathbf{P}^n(\mathbf{C})$ with the product of the Kähler metric on M and the Fubini-Study metric on $\mathbf{P}^n(\mathbf{C})$. Let $\overline{\eta}$ be a hermitian bundle on M. Endow the tautological bundle $\mathcal{O}(1)$ on $\mathbf{P}^n(\mathbf{C})$ with the Fubini-Study metric. The formula*

$$T(h_{M\times\mathbf{P}^n(\mathbf{C})},p_M^*(\overline{\eta})\otimes p_P^*(\overline{\mathcal{O}}(k)))=ch(\overline{\eta})\tau(\overline{\mathcal{O}}(k))$$

holds for all $k>>0$, where $\tau(\overline{\mathcal{O}}(k))$ is the Ray-Singer analytic torsion of $\overline{\mathcal{O}}(k)$.

Proof: For $u>0$, recall that the Bismut superconnection of the fibration defined by $h_{M\times\mathbf{P}^n(\mathbf{C})}$ and p_M is the differential operator

$$B_u=\nabla^E+\sqrt{u}(\overline{\partial}^Z+\overline{\partial}^{Z*})-\frac{1}{2\sqrt{2u}}c(T)$$

where E is the infinite dimensional bundle on M whose fibers are the C^∞ sections of the bundle $\Lambda(T^{*(0,1)}p_M) \otimes p_P^*(\mathcal{O}(n)) \otimes p_M^*(\eta) \simeq p_P^*(\Lambda(T^{*(0,1)}\mathbf{P}^n) \otimes \mathcal{O}(k)) \otimes p_M^*(\eta)$, T is the torsion of a certain connection and $\overline{\partial}^Z$ is the Dolbeaut operator along the fibers. Recall that for $Re(s) > 1$

$$\zeta^1(s) = -\frac{1}{\Gamma(s)} \int_0^1 u^{s-1}\{\phi(Tr_s(N_u.exp(-B_u^2)) - Tr_s(N_V.exp(-(\nabla^{p_{M*}(p_P^*(\mathcal{O}(k))\otimes p_M^*(\eta))})^2))\}du$$

and similarly for $Re(s) < 1/2$

$$\zeta^2(s) = -\frac{1}{\Gamma(s)} \int_1^\infty u^{s-1}\{\phi(Tr_s(N_u.exp(-B_u^2)) - Tr_s(N_V.exp(-(\nabla^{p_{M*}(p_P^*(\mathcal{O}(k))\otimes p_M^*(\eta))})^2))\}du.$$

where N_u is the operator defined in 5.4. The functions ζ^1 and ζ^2 have meromorphic continuations to the whole complex plane, which are holomorphic at the origin. By definition, the higher analytic torsion $T(h_{M\times\mathbf{P}^n(\mathbf{C})}, h^{p_P^*(\mathcal{O}(k))\otimes p_M^*(\eta)})$ equals $\frac{\partial}{\partial s}(\zeta_1 + \zeta_2)(0)$. First notice that the term $\frac{1}{2\sqrt{2u}}c(T)$ vanishes, since the horizontal bundle $T^H p_M$ is integrable. Let ω be the Kähler form of the metric $h_{M\times\mathbf{P}^n(\mathbf{C})}$ and ω' the Kähler form of the metric h_M. The forms ω and $\omega - p_M^*\omega'$ induce the same metrics on the bundle Tp_M and thus by 5.8, we have

$$T(h_{M\times\mathbf{P}^n(\mathbf{C})}, h^{p_P^*(\mathcal{O}(k))\otimes p_M^*(\eta)}) = T(\omega, h^{p_P^*(\mathcal{O}(k))\otimes p_M^*(\eta)}) = T(\omega - p_M^*\omega', h^{p_P^*(\mathcal{O}(k))\otimes p_M^*(\eta)}).$$

This shows that we can assume that $N_u = N_V$. Now let s be a section of $\Lambda(T^{*(0,1)}\mathbf{P}^n) \otimes \mathcal{O}(k)$ over $\mathbf{P}^n(\mathbf{C})$ and t a section of $\eta \otimes \Lambda(T^*_{\mathbf{C}}M)$ over M. Since the bundle $p_P^*(\Lambda(T^{*(0,1)}\mathbf{P}^n) \otimes \mathcal{O}(k))$ is trivial in horizontal directions, we have

$$B_u(p_P^*(s) \otimes p_M^*(t)) = p_P^*(\sqrt{u}(\overline{\partial} + \overline{\partial}^*)(s)) \otimes p_M^*(t) + (-1)^{|s|} p_P^*(s) \otimes p_M^*(\nabla^\eta(t))$$

where $|s|$ is the graded degree of s. Squaring both sides of this formula, we get

$$B_u^2(p_P^*(s) \otimes p_M^*(t)) = p_P^*(u.\Delta(s)) \otimes p_M^*(t) + p_P^*(s) \otimes p_M^*((\nabla^\eta)^2(t))$$

where Δ is the Laplacian $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$. Thus, using the fact that B_u is a vertical differential operator, we get

$$N_V exp(-B_u^2)(p_P(s) \otimes p_M^*(t)) = p_P^*(N_V exp(-u\Delta)(s)) \otimes p_M^*(t) + p_P^*(s) \otimes p_M^*(exp(-\nabla^\eta)^2(t))$$

which proves that $Tr_s(N_V exp(-B_u^2))$ is the mean of $p_P^*(Tr_s(N_V exp(-u\Delta))).p_M^*(\phi^{-1}(ch(\overline{\eta})))$ over the fibers of $p_M^*(\Lambda T_{\mathbf{C}} M)$ (ϕ^{-1} is the inverse of the map ϕ). Thus $Tr_s(N_V exp(-B_u^2)) = \phi^{-1}(ch(\overline{\eta}))Tr_s(N_V exp(-u\Delta))$. On the other hand, we clearly have

$$Tr_s(N_V.exp(-(\nabla^{p_{M*}(p_P^*(\mathcal{O}(k))\otimes p_M^*(\eta))})^2)) = \phi^{-1}(ch(\overline{\eta}))Tr_s(N_V.exp(-(\nabla^{p_{M*}(p_P^*(\mathcal{O}(k))})^2)) =$$

$$\phi^{-1}(ch(\overline{\eta}))s_{n,k}$$

where $s_{n,k}$ is the dimension of the space of global holomorphic sections of $\mathcal{O}(k)$ over $\mathbf{P}^n(\mathbf{C})$. So we finally get

$$\zeta_1(s) = ch(\overline{\eta}).(-\frac{1}{\Gamma(s)}\int_0^1 \{u^{s-1}\phi(Tr_s(N_V exp(-u\Delta)) - s_{n,k})\}du)$$

and we have a similar equation for ζ_2. From this and 5.5 the claim of the lemma follows. **Q.E.D.**

We shall need the following special case of the projection formula:

Proposition 7.16 *Let B be any arithmetic variety. Let $p_B : \mathbf{P}^n_B \to B$ be the projection from some relative projective. Endow $\mathbf{P}^n_B$ with the product Kähler metric. The formula*

$$p_{B*}(p_B^*(b)a) = b.p_{B*}(a)$$

holds for all $b \in \widehat{K}_0(B)$ and all $a \in \widehat{K}_0(\mathbf{P}^n_B)$.

Proof: If b is represented by an element of $\tilde{A}(B)$ or a is represented by an element of $\tilde{A}(\mathbf{P}^n_B)$ then the claim of the Proposition follows from the projection formula for fiber integrals (see [5, p. 31]). If we remember that $K_0(\mathbf{P}^n_B)$ is generated by elements of the type $p^*_{\mathbf{P}^n_\mathbf{Z}}(\mathcal{O}(k)) \otimes p^*_B(b)$, we are thus reduced to the case where $a = p^*_{\mathbf{P}^n_\mathbf{Z}}(\overline{\mathcal{O}}(k))$ and b is represented by an acyclic hermitian bundle $\overline{V}$. Let E be the trivial bundle of rank $n+1$ over M, with trivial metric. Using 7.15, we compute $p_{B*}(p_B^*(b)a) = Sym^k(\overline{E}) \otimes \overline{V} - T(h_{\mathbf{P}^n_B}, h^{p_B^*(\overline{V}) \otimes p_P^*(\overline{\mathcal{O}}(k))}) = Sym^k(\overline{E}) \otimes \overline{V} - ch(\overline{V})\tau(\mathcal{O}(k)) = b.p_{B*}(a)$, which ends the proof. **Q.E.D.**

We shall also need the following special case of a "base change" formula for the push-forward map:

Proposition 7.17 *Let the terminology of 7.16 hold. Let $f_n : \mathbf{P}^n_\mathbf{Z} \to Spec\mathbf{Z}$ and $p_{\mathbf{P}^n_\mathbf{Z}} : \mathbf{P}^n_B \to P$ be the natural projections. Endow $\mathbf{P}^n_\mathbf{Z}$ with the Fubini-Study metric and $\mathbf{P}^n_B$ with the product metric. The formula*

$$f_n^* f_{n,*}(p) = p_{B*}p^*_{\mathbf{P}^n_\mathbf{Z}}(p)$$

holds in $\widehat{K}_0(\mathbf{P}^n_\mathbf{Z})$ for all $p \in \widehat{K}_0(\mathbf{P}^n_\mathbf{Z})$.

Proof: Suppose first that p is represented by an element $\eta \in \tilde{A}(\mathbf{P}^n_\mathbf{Z})$; using Fubini's theorem for the integration on product spaces, we compute

$$p_{B*}p^*_{\mathbf{P}^n_\mathbf{Z}}(\eta) = \int_{B(\mathbf{C})\times\mathbf{P}^n(\mathbf{C})/B(\mathbf{C})} Td(\overline{Tp_B})p^*_{\mathbf{P}^n_\mathbf{Z}}(\eta) =$$

$$\int_{\mathbf{P}^n(\mathbf{C})/B(\mathbf{C})} p^*_{\mathbf{P}^n_\mathbf{Z}}(Td(\overline{T\mathbf{P}^n_\mathbf{Z}}).\eta) = \int_{\mathbf{P}^n(\mathbf{C})} Td(\overline{T\mathbf{P}^n_\mathbf{Z}}).\eta = f_n^* f_{n,*}(\eta)$$

which shows that the claim holds in this case. To prove the general case, we can again assume that p is represented by the bundle $\overline{\mathcal{O}}(k)$. The claim then follows immediately from the definition of the push-forward (after 3.1) and 7.15 (with $\overline{\eta}$ taken to be the trivial bundle). **Q.E.D.**

Remark. The two last propositions hold for more general fiber products than relative projective spaces; this can be proved either by observing that both equalities are compatible with 6.7 or by generalizing 7.15.

Proposition 7.18 *Let the terminology of 7.17 and 7.16 hold. Fix a power series A. Then if* $\delta(A, f_n, (.))$ *vanishes on all the elements of* $\widehat{K}_0(\mathbf{P}^n_{\mathbf{Z}})$ *then* $\delta(A, p_B, (.))$ *vanishes on all the elements of* $\widehat{K}_0(\mathbf{P}^n_B)$.

Proof: We again endow $\mathbf{P}^n_B$ with the product metric. Let $\mathcal{O}(1)$ be the tautological line bundle on $\mathbf{P}^n_{\mathbf{Z}}$, endowed with its Fubini-Study metric. Write $\theta_{BP} := \theta^k(\overline{Tp_B}^{\vee})^{-1}(1 - A(Tp_B))$ and $\theta_P := \theta^k(\overline{Tf_n}^{\vee})^{-1}(1 - A(Tf_n))$. Write $p_P = p_{\mathbf{P}^n_{\mathbf{Z}}}$. By construction, we have $\theta_{BP} := p_P^*(\theta_P)$. Using 7.17 and 7.16, we can compute for all $p \in \widehat{K}_0(\mathbf{P}^n_{\mathbf{Z}})$ and all $b \in \widehat{K}_0(B)$

$$p_{B*}(\theta_{BP}\psi^k(p_P^*(p)p_B^*(b))) = p_{B*}(\theta_{BP}\psi^k(p_P^{-1}(p)))\psi^k(b) =$$

$$p_{B*}(p_P^*(\theta_P\psi^k(p)))\psi^k(b) =$$

$$f^*(f_*(\theta_P\psi^k(p)))\psi^k(b) = f^*(\psi^k(f_*(p)))\psi^k(b) = \psi^k(p_{B*}(p_P^*(p)))\psi^k(b) =$$

$$\psi^k(p_{B*}(p_P^*(p)p_B^*(b)))$$

which shows that $\delta(A, p_B, p_P^*(p)p_B^*(b)) = 0$ holds for all the elements $p_P^*(p)p_B^*(b)$. Since $\widehat{K}_0(\mathbf{P}^n_B)$ is generated by such elements and elements of $\tilde{A}(\mathbf{P}^n_B)$, we are done. **Q.E.D.**

Proposition 7.19 *If* $A = k.\phi^k(R) - R$ *then* $\delta(A, f_n, y_0) = 0$ *for all* $y_0 \in \widehat{K}_0(\mathbf{P}^n_{\mathbf{Z}})$ *and for all* $n \geq 0$.

Proof: We first need two lemmas.

Lemma 7.20 *Let A be a power series with real coefficients. Suppose that* $\delta(A, f_n, y_0) = 0$ *for all hermitian bundles* $y_0 \in \widehat{K}_0(\mathbf{P}^n_{\mathbf{Z}})$. *Then* $A(T\mathbf{P}^n_{\mathbf{Z}}) = k.\phi^k(R(T\mathbf{P}^n_{\mathbf{Z}})) - R(T\mathbf{P}^n_{\mathbf{Z}})$.

Proof of 7.20: let $\Delta : \mathbf{P}^n_{\mathbf{Z}} \to \mathbf{P}^n_{\mathbf{Z}} \times \mathbf{P}^n_{\mathbf{Z}}$ be the diagonal embedding and $f : \mathbf{P}^n_{\mathbf{Z}} \times \mathbf{P}^n_{\mathbf{Z}} \to \mathbf{P}^n_{\mathbf{Z}}$ be the projection on the first factor and $g : \mathbf{P}^n_{\mathbf{Z}} \to \mathbf{P}^n_{\mathbf{Z}}$ be the identity. Endow $\mathbf{P}^n_{\mathbf{Z}} \times \mathbf{P}^n_{\mathbf{Z}}$ with a deformable Kähler metric, endow $\mathbf{P}^n_{\mathbf{Z}}$ with the metric induced by Δ and the normal bundle N of Δ with the quotient metric. Using the hypothesis and 7.18, we see that $\delta(A, f, y_0) = 0$ for all hermitian vector bundles $y_0 \in \widehat{K}_0(\mathbf{P}^n_{\mathbf{Z}} \times \mathbf{P}^n_{\mathbf{Z}})$. Also, $\delta(A, g, y_0) = 0$ for every hermitian vector bundle $y_0 \in \widehat{K}_0(\mathbf{P}^n_{\mathbf{Z}})$, since g is the identity. Applying 7.9 with $\overline{\eta}$ the trivial hermitian bundle, we get the equation

$$k.\phi^k(R(N)) - Td(Tf)A(Tf)ch(\theta^k(Tf^\vee)^{-1})Td(N)^{-1}ch(\theta^k(N^\vee)) -$$

$$R(N)ch(\theta^k(N^\vee))ch(\theta^k(Tf^\vee)^{-1}) = 0$$

where N is the normal bundle of the immersion Δ. It is shown in [18, Ex. 8.4.2, p. 146] that we have $N \simeq T\mathbf{P}^n_{\mathbf{Z}}$. Furthermore, we clearly have $Tf = p_2^* T\mathbf{P}^n_{\mathbf{Z}}$, where p_2 is the projection on the second factor of $\mathbf{P}^n_{\mathbf{Z}} \times \mathbf{P}^n_{\mathbf{Z}}$. Thus $\Delta^* Tf = \Delta^* p_2^* T\mathbf{P}^n_{\mathbf{Z}} = (p_2 \circ \Delta)^* T\mathbf{P}^n_{\mathbf{Z}} = T\mathbf{P}^n_{\mathbf{Z}}$. Therefore, we can compute

$$k.\phi^k(R(T(\mathbf{P}^n)) - Td(T\mathbf{P}^n)A(T\mathbf{P}^n)ch(\theta^k((T\mathbf{P}^n)^\vee)^{-1})Td(T\mathbf{P}^n)^{-1}ch(\theta^k((T\mathbf{P}^n)^\vee)) -$$

$$R(T\mathbf{P}^n)ch(\theta^k((T\mathbf{P}^n)^\vee))ch(\theta^k((T\mathbf{P}^n)^\vee)^{-1}) = k.\phi^k(R(T\mathbf{P}^n)) - A(T\mathbf{P}^n) - R(T\mathbf{P}^n) = 0$$

which proves our claim. **Q.E.D.**

The use of the diagonal immersion in the above proof was suggested to us by Nicusor Dan. The next lemma might be compared to the lemma [26, Lemma 1.7.1], which gives a determination of the Todd genus.

Lemma 7.21 *Let $A(x)$ be a power series with real coefficients. The $n+1$ first coefficients $a_0, \dots a_n$ of A are uniquely determined by the conditions $\delta(A, f_i, \overline{\mathcal{O}_{\mathbf{P}^i_{\mathbf{Z}}}}) = 0$ $(i = 0 \dots n)$, where the $\overline{\mathcal{O}_{\mathbf{P}^i_{\mathbf{Z}}}}$ are the trivial line bundles endowed with their trivial metrics.*

Proof of 7.21: let $k > 0$. Writing out the conditions stated in the lemma, we get the following system of equations for the coefficients of A:

$$f_{i*}(\theta^k(\overline{Tf_i}^\vee)^{-1}A(Tf_i)) = \psi^k(f_{i*}(1)) - f_{i*}(\theta^k(\overline{Tf_i}^\vee)^{-1}) \qquad (20)$$

where $0 \le i \le n$. Notice that we have an exact sequence

$$0 \to \mathbf{R} \to \widehat{K}_0(\mathbf{Z}) \to K_0(\mathbf{Z}) \to 0$$

over $\mathbf{Z}$ (see [21, Th. 6.2, (i), p. 213]). In view of the algebraic Adams-Riemann-Roch theorem for local complete intersection morphisms (see [19, Th. 7.6, p. 149]), the image in $\widehat{K}_0(\mathbf{Z}) \otimes \mathbf{Z}[\frac{1}{k}]$ of the right side of (20) lies in $\mathbf{R}$. Since the left side is by construction in $\mathbf{R}$ we can consider (20) as a system of linear equations over $\mathbf{R}$. Let y_i be the real number corresponding to $\psi^k(f_{i*}(1)) - f_{i*}(\theta^k(\overline{Tf_i}^\vee)^{-1})$. We have to solve the following system:

$$\int_{\mathbf{P}^i(\mathbf{C})} k^{-i}\phi^k(Td(Tf_i))A(Tf_i) = y_i, \ 0 \le i \le n$$

Recall that on $\mathbf{P}^i(\mathbf{C})$ there is an exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus i+1} \to T_{\mathbf{P}_i(\mathbf{C})} \to 0 \tag{21}$$

Let $x = c_1(\mathcal{O}(1))$. Using the additivity of R and the the multiplicativity of the Todd class, we are thus reduced to

$$\int_{\mathbf{P}^i(\mathbf{C})} k^{-i}\phi^k(Td(x)^{i+1})(i+1)A(x) = y_i.$$

For each $i \geq 0$, this is a system of equations in the variables $a_0, \dots a_i$ and the coefficient of a_i is the real number $k^{-i}(i+1)\int_{\mathbf{P}^i(\mathbf{C})} x^i = k^{-i}(i+1)$. Thus we are done. **Q.E.D.**

The preceding lemma provides us with a unique power series $A(x)$ such that the $\delta(A, f_i, \overline{\mathcal{O}_{\mathbf{P}^i_{\mathbf{Z}}}}) = 0$ for all $i \geq 0$. Until the end of the proof, let $A(x)$ denote that uniquely determined series. We make the following inductive hypothesis on n: the term $\delta(A, f_i, y_0)$ vanishes for any virtual hermitian bundle y_0 on $\mathbf{P}^i_{\mathbf{Z}}$, for all non-negative $i < n$ and the coefficients $a_0, a_1, \dots a_{n-1}$ coincide with the n first coefficients $q_0, q_1, \dots q_{n-1}$ of the series $k.\phi^k(R) - R$. This hypothesis is clearly true for $n = 0$.
We carry out the first part of the inductive step. Let $\mathcal{O}(1)$ be the tautological line bundle on $\mathbf{P}^n_{\mathbf{Z}}$. Let s be the canonical section of $\mathcal{O}(1)$ vanishing on the hyperplane at ∞. The section s determines a resolution

$$0 \to \mathcal{O}(-1) \to \mathcal{O}_{\mathbf{P}^n} \to i_*\mathcal{O}_{\mathbf{P}^{n-1}} \to 0$$

If we tensorize this sequence with $\mathcal{O}(l)$, wet get the sequence

$$0 \to \mathcal{O}(l-1) \to \mathcal{O}(l) \to i_*(\mathcal{O}(l)) \to 0.$$

Let f be the projection $\mathbf{P}^n_{\mathbf{Z}} \to Spec\ \mathbf{Z}$ and g the projection $\mathbf{P}^{n-1}_{\mathbf{Z}} \to Spec\ \mathbf{Z}$. By construction, $\delta(A, f, \overline{\mathcal{O}}_{\mathbf{P}^n_{\mathbf{Z}}})$ vanishes. Applying induction on l, we suppose that $\delta(A, f, \overline{\mathcal{O}}_{\mathbf{P}^n_{\mathbf{Z}}}(l-1)) = 0$. Let N be the normal bundle of the immersion i of the hyperplane. Using the induction hypothesis on the coefficients of A, the fact that N is a line bundle and the fact that $c_1(N)^i = 0$ for all $i > n-1$ (since the cohomology vanishes in degree greater than the dimension of $\mathbf{P}^{n-1}_{\mathbf{C}}$), we see that $A(N) = R(N) - k.\phi^k(R(N))$. Using the same argument and the exact sequence (21), we also see that $A(Tg) = (n+1)A(\mathcal{O}(1)) = (n+1)(R(\mathcal{O}(1)) - k.\psi^k(\mathcal{O}(1))) = R(Tg) - k.\phi^k(R(Tg))$. From this we deduce that $i^*(A(Tf)) = A(N) + A(Tg) = i^*(R(Tf) - k.\phi^k(R(Tf)))$. So by 7.14 $f_*(\theta^k(\overline{Tf}^\vee)^{-1}(1 - A(Tf))\psi^k(\overline{\mathcal{O}}(l))) - \psi^k(f_*(\overline{\mathcal{O}}(l))) = 0$, which means that $\delta(A, f, \overline{\mathcal{O}}(l))$ vanishes as well. By induction on l, it thus holds for all $\overline{\mathcal{O}}(l)$ ($l \geq 0$). Since these generate $K_0(\mathbf{P}^n_{\mathbf{Z}})$, any hermitian bundle can be represented in $\widehat{K}_0(\mathbf{P}^n_{\mathbf{Z}})$ as a linear combination of elements of $\tilde{A}(\mathbf{P}^n_{\mathbf{Z}})$ and bundles $\overline{\mathcal{O}}(l)$. Using 7.5 and additivity, we conclude that $\delta(A, f, y_0) = 0$ holds for all hermitian vector bundles y_0 on $\mathbf{P}^n_{\mathbf{Z}}$. This settles the first part of the inductive step on n.

To prove the second part, we apply 7.20 and conclude that $A(T\mathbf{P}^n) = k.\phi^k(R(T\mathbf{P}^n)) - R(T\mathbf{P}^n)$. Using the exact sequence (21), we compute

$$\int_{\mathbf{P}^n_{\mathbf{C}}} A(T\mathbf{P}^n) = (n+1)\int_{\mathbf{P}^n_{\mathbf{C}}} A(x) = (n+1)a_n \int_{\mathbf{P}^n_{\mathbf{C}}} x^n = (n+1)a_n.$$

Carrying out a similar computation for $R(T\mathbf{P}^n) - k.\phi^k(R(T\mathbf{P}^n))$ in place of A, we get

$$\int_{\mathbf{P}^n_{\mathbf{C}}} k.\phi^k(R(T\mathbf{P}^n)) - R(T\mathbf{P}^n) = (n+1)q_n.$$

Thus $a_n = q_n$ and we are through with the inductive step on n. **Q.E.D.**

Corollary 7.22 *The statement 7.1 holds.*

Proof: Apply the Propositions 7.14, 7.18 and 7.19. **Q.E.D.**

Let us notice that the "diagonal trick" we use to prove 7.1 for the projective spaces works in the algebraic case as well. In the arithmetic case, the advantage of this method over the original method of Gillet and Soulé (which gave birth to the R-genus) is that it avoids any explicit computation of the analytic torsion. In the algebraic case, it avoids the computation of the group $K_0(\mathbf{P}^n_{\mathbf{Z}})$. About this, see also [17]. J.-B. Bost told us that he knew a proof of the analog of 7.19 for arithmetic Chow groups, using explicit resolutions of the diagonal.

8 The arithmetic Grothendieck-Riemann-Roch theorem for local complete intersection p.f.s.r. morphisms

In this section, we shall define a graded ring which arises from the γ-filtration on arithmetic Grothendieck groups, define a Chern character with values in that ring, state and prove a relative Riemann-Roch theorem for that Chern character and finally discuss shortly the relationship between that ring and the arithmetic Chow ring.
Let R be a λ-ring endowed with an augmentation $rk : R \to \mathbf{Z}$. We also suppose that R is locally nilpotent. The following definition appears in [24, V, 1.11, p. 308 and 3.10, p. 331].

Definition 8.1 *The group GrR is the direct sum $\bigoplus_{i=0}^{\infty} F^i R/F^{i+1}R$.*

Since the γ-filtration is a ring filtration, the group GrR carries a natural ring structure, which is compatible with its natural grading.

Definition 8.2 *Let $y \in R$. The i-th Chern class $c_i^{gr}(y)$ of y is the element $\gamma^i(y - rk(y))$ mod $F^{i+1}R \in Gr^iR$.*

Let σ_n denote the n-th symmetric function in the variables $T_1, \dots T_n$. Let P be the unique power series with rational coefficients such that $P(\sigma_1, \dots \sigma_n) = \sum_{i=1}^n e^{T_i}$. The Chern character $ch^{gr}(y) \in GrR_{\mathbf{Q}}$ is the element $P(c_1^{gr}(y), \dots c_n^{gr}(y))$. Let also Q be the unique power series with rational coefficients such that $Q(\sigma_1, \dots \sigma_n) = \prod_{i=1}^n \frac{T_i}{1-e^{-T_i}}$. The Todd class $Td^{gr}(y) \in GrR_{\mathbf{Q}}$ is the element $Q(c_1^{gr}(y), \dots c_n^{gr}(y))$. For each $j \geq 0$, let us denote by R_k^j the eigenspace in $R_{\mathbf{Q}}$ associated to the eigenvalue k^j of the $\mathbf{Q}$-vector space endomorphism of $R_{\mathbf{Q}}$ given by the k-th Adams operation ψ^k. The proof of the following proposition can be found in [4, Th. 4.3, p. 119 and Th. 1, p. 97]:

Proposition 8.3 **(a)** *The space R_k^j is independent of k; it will thus henceforth be denoted by R^j;*

(b) *if $GrR_{\mathbf{Q}}$ is endowed with the λ-ring structure arising from its grading then the Chern character induces a λ-ring isomorphism $ch^{gr} : R_{\mathbf{Q}} \to GrR_{\mathbf{Q}}$; if $x \in R^j$, then $ch^{gr}(x) = x$ mod $F^{j+1}R_{\mathbf{Q}}$.*

Notice that in view of 4.1 and (b), the equality $F^jR_{\mathbf{Q}} = \oplus_{l\geq 0}R^{j+l}$ holds, where the direct sum is interior.
We now specialize to the case $R = \widehat{K}_0(Y)$. If $\omega \in \tilde{A}(Y)$, we shall abbreviate $ch^{gr}(\omega)$ by ω. The following lemma is well-known; because we can't give a reference for a proof, we shall include one.

Lemma 8.4 *Let $A = \oplus_{i=0}^{\infty} A_i$ be a graded commutative $\mathbf{Q}$-algebra. Let $C \in 1 + \oplus_{i=1}^{\infty} A_i$. For $k > 1$, the equation $a^{-1}.\phi^k(a) = C$ has a unique solution in $1 + \oplus_{i=1}^{\infty} A_i$.*

Proof: Let $C = C_0 + C_1 + C_2 + \dots$, $a = a_0 + a_1 + a_2 + \dots$ be the representations of C and a arising from the grading (the sums are finite). In terms of the a_i and the C_i, the equation reads

$$\sum_{i=0}^{\infty} k^i.a_i = (\sum_{i=0}^{\infty} a_i)(\sum_{i=0}^{\infty} C_i)$$

which is equivalent to the linear system of equations

$$a_0C_i + a_1C_{i-1} + \dots + a_{i-1}C_1 + (1 - k^i)a_i = 0.$$

Let us fix $a_0 = 1$. The fact that $1-k^i \neq 0$ for $i > 0$ then implies that the system has a unique solution, which can be determined recursively. This completes the proof. **Q.E.D.**

Let $Y \to B$ be a local complete intersection p.f.s.r. morphism. We suppose that Y is endowed with a Kähler metric.

Definition 8.5 *The arithmetic Todd genus $Td_A^{gr}(\overline{Tg})$ of g is the unique element of $Gr\widehat{K}_0(Y)_{\mathbf{Q}}$ determined via the last lemma by the equation*

$$ch^{gr}(\theta_A^k(\overline{Tg}^{\vee})^{-1}) = k^{dim(B)-dim(Y)} Td_A^{gr}(\overline{Tg})^{-1}\phi^k(Td_A^{gr}(\overline{Tg})).$$

To see that how the R-genus appears in the arithmetic Todd genus, let us define the element $Td^{gr}(\overline{Tg})$, which is uniquely determined via the last lemma by the equation

$$ch^{gr}(\theta^k(\overline{Tg}^{\vee})^{-1}) = k^{dim(B)-dim(Y)} Td^{gr}(\overline{Tg})^{-1}\phi^k(Td^{gr}(\overline{Tg}))$$

(if g is smooth, it can proved that $Td^{gr}(\overline{Tg})$ is the Todd class of the hermitian bundle $\overline{Tg}$). Let us now look for an additive real characteristic class A, such that the equation

$$ch^{gr}(\theta_A^k(\overline{Tg}^{\vee})^{-1}) = k^{dim(B)-dim(Y)}(Td^{gr}(\overline{Tg})(1-A(Tg)))^{-1}\phi^k(Td^{gr}(\overline{Tg})(1-A(Tg)))$$

is satisfied. Were are lead to the equation in cohomology

$$1 + R(Tg) - k.\phi^k(R(Tg)) = (1 - A(Tg))^{-1}.\psi^k(1 - A(Tg)).$$

Using the definition of the product in arithmetic K_0-theory, the expression after the last equality can be evaluated to be

$$(1 + A(Tg)).(1 - k.\phi^k(A(Tg))) =$$

$$1 - k.\phi^k(A(Tg)) + A(Tg) - dd^c A(Tg).k.\phi^k(A(Tg)) = 1 - k.\phi^k(A(Tg)) + A(Tg)$$

an thus using the last lemma, we can conclude that $A(Tg) = R(Tg)$ and thus that $Td_A^{gr}(\overline{Tg}) = Td^{gr}(\overline{Tg})(1 - R(Tg))$. We can now state the main result of this section.

Theorem 8.6 *Let $g : Y \to B$ be a local complete intersection p.f.s.r. morphism. Let $d = dim(Y) - dim(B)$.*

(a) *The inclusion $g_* F^i\widehat{K}_0(Y)_{\mathbf{Q}} \subseteq F^{i-d}\widehat{K}_0(B)_{\mathbf{Q}}$ holds for all $i \in \mathbf{Z}$. Thus the push-forward map induces a group map $g_* : Gr\widehat{K}_0(Y)_{\mathbf{Q}} \to Gr\widehat{K}_0(B)_{\mathbf{Q}}$.*

(b) *Let $y \in \widehat{K}_0(Y)$. The equality $ch^{gr}(g_*(y)) = g_*(Td_A^{gr}(\overline{Tg})ch^{gr}(y))$ holds in $Gr\widehat{K}_0(B)_{\mathbf{Q}}$.*

Proof: Before beginning the proof, notice that the element $\theta^k_A(\overline{Tg}^\vee)^{-1}$ is invertible in $\widehat{K}_0(Y)_{\mathbf{Q}}$. This follows from 4.2 and the fact that for any differential form $\omega \in \tilde{A}(Y)$, an inverse in $\widehat{K}_0(Y)$ of the element $1-\omega$ is given by the finite sum $1+\omega+dd^c\omega.\omega+dd^c\omega.dd^c\omega.\omega+\dots$.
(a) Let $e = ch^{gr,-1}(Td^{gr}_A(\overline{Tg})^{-1})$. Let $k>1$. From the definitions, we have $\theta^k_A(\overline{Tg}^\vee)^{-1}.\psi^k(e) = e.k^{-d}$. Let $y \in \widehat{K}_0(Y)^j$. We compute

$$\psi^k(g_*(e.y)) = g_*(\theta^k_A(\overline{Tg}^\vee)^{-1}\psi^k(e).\psi^k(y)) =$$

$$g_*(k^{-d}e.k^j y) = k^{j-d}g_*(e.y).$$

In view of 8.3, this implies that $g_*(e.y) \in F^{j-d}\widehat{K}_0(B)_{\mathbf{Q}}$. Since $e.y \in F^j\widehat{K}_0(Y)_{\mathbf{Q}}$, it is thus sufficient to show that every element of $F^j\widehat{K}_0(Y)_{\mathbf{Q}}$ is of the form $e.(y_1+\dots y_r)$, where for all $1\leq i\leq r$, $\psi^k(y_i)=k^{j_i}y_i$ for some $j_i\geq j$. This is a consequence of the fact that e is invertible in $\widehat{K}_0(Y)_{\mathbf{Q}}$ and of the remark after 8.3 (b), so we are done.
(b) Continuing with the same terminology, we compute

$$ch^{gr}(\psi^k(g_*(e.y))) = k^{j-d}g_*(e.y) \ mod \ F^{j-d+1}_{\mathbf{Q}} =$$

$$k^{j-d}g_*(y) \ mod \ F^{j-d+1}_{\mathbf{Q}} =$$

$$k^{j-d}g_*(ch^{gr}(y)) = \phi^k(g_*(ch^{gr}(y)))$$

where the first equality follows from (a) and the second one from the fact that by construction e is the sum of 1 and an element of $F^1\widehat{K}_0(Y)_{\mathbf{Q}}$. Notice now that by additivity, the resulting equality $ch^{gr}(\psi^k(g_*(e.y))) = \phi^k(g_*(ch^{gr}(y)))$ is valid for all $y\in\widehat{K}_0(Y)_{\mathbf{Q}}$. Thus we might choose $y=e^{-1}.y'$ and we obtain the equality $\phi^k(ch^{gr}(g_*(y'))) = \phi^k(g_*(Td^{gr}_A(\overline{Tg})ch^{gr}(y')))$ and thus the result of (b). **Q.E.D.**

The part (b) of the last theorem is formally identical to the arithmetic Riemann-Roch theorem in all degrees stated in [16, Th. 6.1, p. 77]. Analogously to the arithmetic Riemann-Roch theorem [23, Th. 8, p. 534], it can be used to estimate asymptotically the covolumes of twisted hermitian bundles. More precisely, let $B = Spec\mathbf{Z}$ and let $\overline{E}$ be a hermitian bundle on Y. Let L be an ample line bundle on Y, endowed with a positive hermitian metric. For any hermitian $\mathbf{Z}$-module $\overline{V}$, let $Vol(\overline{V})$ denote the volume of a fundamental domain of the lattice $V\subset V_{\mathbf{C}}$, for the unique Haar measure which gives volume 1 to the unit ball. It follows from the definitions that there is an isomorphism $Gr^1\widehat{K}_0(\mathbf{Z})\simeq\mathbf{R}$, which sends elements of $\tilde{A}(\mathbf{Z})$ on the corresponding real number and hermitian $\mathbf{Z}$-modules $\overline{V}$ on $-\log(Vol(\overline{V}))$. Let $\Gamma(.)$ take the global sections of a hermitian bundle, endowed with the metric integrated along the fibers. From (b), we obtain that

$$-\log(Vol(\Gamma(\overline{E}\otimes\overline{L}^{\otimes n}))) =$$

$$\tau(\overline{E_{\mathbf{C}}}\otimes\overline{L_{\mathbf{C}}}^{\otimes n})+g_*(Td_A^{gr}(\overline{Tg})ch^{gr}(\overline{E}\otimes\overline{L}^{\otimes n}))$$

when $n >> 0$. By a theorem of Bismut and Vasserot [14], the asymptotic estimate $\tau(\overline{E_{\mathbf{C}}}\otimes\overline{L_{\mathbf{C}}}^{\otimes n}) = O(n^{dim(Y(\mathbf{C}))}log(n))$ holds. For degree reasons, the term $g_*(Td_A^{gr}(\overline{Tg})ch^{gr}(\overline{E}\otimes\overline{L}^{\otimes n})) = g_*(Td_A^{gr}(\overline{Tg})ch^{gr}(\overline{E})exp(n.c_1^{gr}(\overline{L})))$ is a polynomial of degree $dim(Y)$, with leading coefficient $\frac{1}{dim(Y)!}rk(E)g_*(c_1^{gr}(\overline{L})^{dim(Y)})$. As a consequence, the equality of real numbers

$$-\lim_{n\to\infty}\frac{\log(Vol(\Gamma(\overline{E}\otimes\overline{L}^{\otimes n})))}{n^{dim(Y)}}=\frac{1}{dim(Y)!}rk(E)g_*(c_1^{gr}(\overline{L})^{dim(Y)})$$

holds, which is a variant of an arithmetic analog of the Hilbert-Samuel theorem. About this, see [23, Th. 9, p. 539] and [1, Intro.].
The group $Gr\widehat{K}_0(.)_{\mathbf{Q}}$ is naturally isomorphic to the arithmetic Chow theory defined in [20], as a covariant and contravariant functor. The contravariance statement follows immediately from the functoriality of the λ-operations and the fact that arithmetic Chow theory and arithmetic K_0-theory are isomorphic as λ-rings (modulo torsion) via the arithmetic Chern character (see [21, Th. 7.3.4, p. 235] for the proof). The covariance statement can be deduced from the unpublished arithmetic Riemann-Roch theorem in all degrees for arithmetic Chow groups mentioned in the introduction, the just mentioned isomorphism statement and the unicity of arithmetic Chern classes proved in [21, Th. 4.1, p. 187]. However, we shall not carry out the details of the proof of covariance, in view of the inofficial character of the just mentioned arithmetic Riemann-Roch theorem.

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