Programme of the 2014 Alpbach summer school on the article of P. Colmez *Périodes des variétés abéliennes à multiplication complexe*

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1 Introduction

Let A be an abelian variety of dimension d defined over $\overline{\mathbf{Q}}$. Let $K \subseteq \mathbf{C}$ be a number field such that A is defined over K and such that the Néron model of A over $\mathcal{O}_K$ has semi-stable reduction at all the places of K. Let Ω be the $\mathcal{O}_K$-module of global sections of the sheaf of differentials of $\mathcal{A}$ over $\mathcal{O}_K$ and let α be a section of Ω^d. We write $A(\mathbf{C})_\sigma$ for the manifold of complex points of the variety $A \times_{\sigma(K)} \mathbf{C}$, where $\sigma \in \mathrm{Hom}(K, \mathbf{C})$ is an embedding of K in $\mathbf{C}$. The modular (or Faltings) height of A is the quantity

$$h_{\mathrm{Fal}}(A) := \frac{1}{[K:\mathbf{Q}]}\log(\#\Omega^d/\alpha.\Omega^d) - \frac{1}{2[K:\mathbf{Q}]}\sum_{\sigma:K\to\mathbf{C}}\log|\int_{\mathcal{A}(\mathbf{C})_\sigma}\alpha\wedge\overline{\alpha}|.$$

NB In the classical definition of the Faltings height, there is a multiplicative factor $\frac{1}{(2\pi)^d}$ in front of the $\int$ sign. We have chosen to follow Colmez's definition, because it makes more sense in the context of his conjecture (see below).

It does not depend on the choice of K or α. The modular height defines a height on some moduli spaces of abelian varieties and plays a key role in Falting's proof of the Mordell conjecture.

Suppose now that there exists a CM field E, of degree $2d$ over $\mathbf{Q}$ and an embedding of rings $\mathcal{O}_E \to \mathrm{End}(A)$ into the endomorphism ring of A. By definition, this means that A

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has complex multiplication by $\mathcal{O}_E$. We can suppose without loss of generality that the action of $\mathcal{O}_E$ is defined over K and that K contains all the conjugates of E in $\mathbf{C}$. Let Φ be the subset of $\tau \in \mathrm{Hom}(E, \mathbf{C})$ such that the subspace

$$\{t \in H^0(A, \Omega_{A/K}) : a^*(t) = \tau(a).t,\ \forall a \in \mathcal{O}_E\}$$

contains a non-zero element (this is the *type* of the CM abelian variety A). Using a refinement the theory of p-adic periods, a refinement of Shimura's period relations and explicit computations on Jacobians of Fermat curves, P. Colmez gave an explicit formula for $h_{\mathrm{Fal}}(A)$. To describe it, suppose furthermore that E is Galois over $\mathbf{Q}$ and let $G := \mathrm{Gal}(E|\mathbf{Q})$. Identify Φ with its characteristic function $G \to \{0, 1\}$ and define $\Phi^\vee$ by the formula $\Phi^\vee(\tau) := \Phi(\tau^{-1})$.

Theorem 1.1 (Colmez, with a final input by Obus)**.** *If G is abelian then*

$$\frac{1}{d} h_{\mathrm{Fal}}(A) = -\sum_{\chi,\ L(\chi,0)\neq 0} < \Phi * \Phi^\vee, \chi > [2\frac{L'(\chi,0)}{L(\chi,0)} + \log(f_\chi)]$$

holds.

Here χ runs through the Artin characters of E (which in this case are Dirichlet characters) and $L(\chi, s)$ refers to the associated L-function. The notation $< \cdot, \cdot >$ refers to the scalar product of complex valued functions on G and $*$ to the convolution product. The notation f_χ refers to the conductor of χ, which is an integer. This formula can be viewed as a generalisation of the formula of Chowla and Selberg (see [SC67]) to which it reduces when applied to a CM elliptic curve. Furthermore, Colmez *conjectures* that the theorem 1.1 holds even without the condition that G is abelian (see [Col98, Conjecture 3]).

The aim of this workshop is to got through the steps of Colmez's proof of a generalisation of Theorem 1.1, following his article [Col93]. The formula given in Theorem 1.1 (which is a direct consequence of the main result of [Col93]) can be found in Conj. 3 of his later article [Col98].

The weaker form of the result of Colmez, which follows when one views both sides of the equality only up to an element of $\log(|\bar{\mathbb{Q}}^*|)$ (so that in particular the conductors become irrelevant) is much easier to prove and follows from the work of Anderson (see [And82]). See also the article by B.H. Gross [Gro78].

In the article by K. Köhler and D. Rössler [KR03], the theorem of Colmez is proven up to $\bar{\mathbb{Q}}^* \log(f_\chi)$ factors using methods from higher dimensional Arakelov theory. This is not as strong as Colmez's results but is stronger than Anderson's. Another proof (as a consequence of a more general ”motivic” result) is given in the paper [MR04], which I suggest could be discussed in Talk 12.

In the (not yet published) PhD thesis of J. Frésan, a proof of a motivic generalisation of Anderson's result is given, which is based on the work of Saito-Terasoma. See http://arxiv.org/pdf/1403.4105.pdf for this. The idea behind this proof comes from a unpublished letter from S. Bloch to H. Esnault.

In [MR02], a partial generalisation of Colmez's conjecture is proposed, which provides an interpretation of the values $L'(\chi, 1-n)/L(\chi, 1-n)$ for all $n \geqslant 0$ such that $L(\chi, 1-n) \neq 0$ (see Talk 13). A full generalisation, which would include a conjectural generalisation of the $\log(f_\chi)$ factors yet has to be found.

There is some speculation in the introduction of [Col93] on possible purely p-adic analogs of Theorem 1.1 (see the bottom of p. 633 in [Col93]). Something in this direction was done in the unpublished thesis of F. Urfels. See http://www.theses.fr/1998STR13210.

The only known confirmation of the conjecture of Colmez in a non abelian situation (ie when the CM field is a non abelian extension of $\mathbb{Q}$) is a result of T. Yang (see [Yan10a]). He proves a weak version of Colmez's conjecture (formulated in (3) on p. 634 of [Col93]) for certain CM abelian surfaces.

There seems to be no other introduction to Colmez's article than the original article. The introduction to the article of Obus (see [Obu13]) provides a useful survey of the results and methods of the article of Colmez. Otherwise, one could have a look at par. 3 of [Yan10b].[1]

2 Programme

Here is a user's guide for the programme proposed below.

- The talks carry stars (one or two ”*”). The number of stars indicates the level of difficulty of the talk.

- Talk 12 and 13 are concerned with the Arakelov-theoretic approach to the result of Colmez. Talk 12, which concerns motivic generalisations of the result of Colmez could be skipped if some of the speakers of Talks 0 to 11 needed more time (for instance, if the speaker for Talk 1 on absolute Hodge cycles [which is important] needed more time). Talk 13 is more speculative in nature but is important because it shows that Colmez's

[1] my thanks to J. Frésan for bringing this text to my attention

work is part of a larger framework.

- It is important that every speaker stick as much as possible with Colmez's notations. This rule might have to be broken in Talks 0, 2 and 3, if the speakers wanted to use Anderson's notations rather than Colmez's.

- The speakers do not have to follow the directions given below precisely if they do not wish to do so. Nevertheless, all the results mentioned in the descriptions of the talks should at least be formulated by the speakers, because the talks build on each other.

- I have compiled a folder containing electronic copies of all the bibliographical references mentioned in this report, with very few exceptions. The exceptions are: [KRY06] and [Fon77]. It can be downloaded from the web address

http://www.math.univ-toulouse.fr/ rossler/mypage/pdf-files/Colmez-Source-Files.dmg.

The folder Colmez-Source-Files.dmg is encrypted. The password to decrypt it is *Shimura*.

2.1 Sunday. Talk 0^{**}. Introduction and overview

This talk will first introduce the subject, following the introduction given above. Note that the main result of Colmez's article is Th. III.2.9 (ii) in [Col93]. The $\log(2)$ indeterminacy appearing there can be shown to vanish, thanks to the work of Obus (see Talk 11). This result is more precise than Theorem 1.1 above but I suggest not to formulate Th. III.2.9 (ii) of [Col93] in this introduction because the definition of the map ht appearing there is somewhat technical. Note also that Colmez gives an interpretation of Th. III.2.9 (ii) in the introduction to his article as a product formula for the valuations of all the periods (p-adic and classical) of CM abelian varieties (see Conjecture 0.1 in [Col93]) but I suggest to concentrate on the formulation given in Th. III.2.9 (ii), because the formulation in terms of a product over the valuations of the periods involves renormalisation issues that are unpleasant to keep track of.

After that, an overview of the proof of Anderson's result should be given. More precisely

1. Introduce Anderson's period map (p. 317 in [And82]).
2. Formulate Anderson's result (Th. 2.1 in [And82]) directly for a an abelian variety with CM by an abelian extension of $\mathbb{Q}$ (without mentioning the distribution).
3. Formulate Shimura's result on monomial relations (to be discussed in Talk 2).

4. Explain briefly why in view of Shimura's relations, it is sufficient to compute the periods of Fermat curves (this is Th. 3.1 and Th. 4.7 and the computation at the bottom of p. 325 in [And82]; this will be discussed in Talk 3) to prove Th. 2.1.

Finally, explain the following:

- The proof of Colmez's theorem (ie Th. III.2.9 (ii) in [Col93]) has a similar structure but he considers a refined period distribution, where p-adic and complex periods are bundled in one package (via the map ht, see Talk 10); in particular, he needs p-adic analogs of Shimura's monomial relations (see Talk 9).

- To define this refined period distribution, he needs a general result expressing the p-adic valuations of p-adic periods of CM formal groups (and thus of CM abelian varieties) in terms of the integral structure of de Rham cohomology (see Talk 7). It is this new expression that is ultimately computed on some quotients of the Fermat curves, using results of Coleman and Obus (see Talk 11).

NB There is a sign mistake in the diagram of equalities at the top of p. 633 of [Col93]. *In the first line of the diagram, the term* $+\frac{1}{2}\log\ D$ *should be replaced by* $-\frac{1}{2}\log\ D$.

2.2 Monday. The periods mod $\bar{\mathbb{Q}}^*$.

2.2.1 Talk 1^{**}. Deligne's theorem on absolute Hodge cycles on abelian varieties.

This talk is a survey of the results of [DMOS82, Chap. I]. Another reference for this material is [Pan94]. See also [Del80].

First introduce the various comparison isomorphisms (as on p. 19 of [DMOS82, Chap. I]). Briefly recall the Hodge conjecture (see [Gro69]) and its striking numerical consequences on periods (ie Prop. 1.5, p. 23 in [DMOS82, Chap. I]).

Introduce the notion of absolute Hodge cycle, following par. 2 (p. 28) in [DMOS82, Chap. I]. If the speaker wishes, he might take l-adic cohomology out of the definition of an absolute Hodge cycle, because l-adic cohomology plays no role in Colmez's article.

Formulate the conjecture 'tout cycle de Hodge l'est absolument' (Open Question 2.4 in [DMOS82, Chap. I]). Explain that the main result of the article (Deligne's theorem) is that this conjecture is a theorem for abelian varieties.

Introduce the category of homological pure motives for absolute Hodge cycles. This category is defined exactly as the category of classical homological pure motives (see [Sch94]), with homological correspondences replaced by absolute Hodge cycles. You should not spend much time on this and assume that the audience has already heard of motives. Follow for instance the presentation given in [Del80] or [Del79, 0. Motifs]. Explain that motives for absolute Hodge cycles have periods and that two isomorphic motives for absolute Hodge cycles have the same periods.

Explain that Deligne's theorem has the same numerical consequence as the Hodge conjecture for periods (Prop. 7.1 in [DMOS82, Chap. I]).

Formulate principles A (=Th. 3.8 in [DMOS82]) and B (=Th. 2.12 in [DMOS82]) for absolute Hodge cycles. You may give some indications about their proofs, which are not very difficult.

Explain that the main theorem is proven in four steps:

- first one proves Lemma 4.5 (formulate and prove it);

- then one uses a deformation argument together with principle B to extend Lemma 4.5 to CM abelian varieties; this is Th. 4.8, which should be formulated (including the definition of polarizations) but not proven;

- now principle A is used to prove that all Hodge cycles on CM abelian varieties are absolute; this is the core of the argument and a fair amount of detail should be given. Here are the main steps: 1) Introduce the groups G^{H} and G^{AH} 2) Show that $G^{\mathrm{H}} \subseteq G^{AH}$ and that G^{H} and G^{AH} are both tori 3) Use Th. 4.8 together with the existence of some algebraic cycles (ie (a), (b) on p. 67) to describe several vanishing relations (see p. 70 in [DMOS82]) in the character group of G^{AH} viewed as a quotient of $X(\prod E^{\times} \times \mathbb{G}_m)$ 4) Show that these relations together with the known explicit description of G^{H} suffice to show that the character groups of G^{H} and G^{AH} coincide, which implies that $G^{\mathrm{H}} = G^{\mathrm{AH}}$.

NB There is a typo on l. 13, p. 71 of [DMOS82]. *One should have* $Y(G^{\mathrm{AH}}) \subset Y(G^{\mathrm{H}})$ *and not* $Y(G^{\mathrm{H}}) \subset Y(G^{\mathrm{AH}})$.

- finally a deformation argument together with principle B is used to reduce the proof of the main theorem to CM abelian varieties. Give the desired amount of detail but do not describe in detail the construction of the Shimura variety needed in the deformation argument.

2.2.2 Talk 2^*. Anderson's period distribution. Shimura's monomial relations. The Gamma distribution.

The aim of this talk is to present Anderson's period distribution and in particular to formulate Shimura's theorem giving relations between periods of CM abelian varieties.

Recall the definition of Anderson's periods (p. 317 in [And82]).

Formulate Shimura's theorem on monomial relations between periods (Th. 1.3 in [And82]).

Show that Shimura's theorem follows from the Hodge conjecture. Show that Deligne's result on absolute Hodge cycles has the same consequence. See [Del80] for this.

Introduce the period distribution (Prop. 1.5 in [And82]) and formulate Th. 2.1 (which is the main result of [And82]).

Give the explicit description of M^{ab} appearing in Th. 3.1. Prove Th. 3.1.

With a view to proving Th. 4.7 in [And82], prove the reduction lemma on p. 324.

2.2.3 Talk 3^*. Fermat curves. Proof of Anderson's theorem.

The aim of this talk is to prove Th. 4.7 in [And82].

Introduce the curve with function field $\bar{\mathbb{Q}}(x, x^a(1-x)^b)$ and show that its Jacobian has complex multiplication. Compute its CM type as at the bottom of p. 325. Deduce that by the reduction lemma, it is sufficient to compute the periods of this Jacobian to prove Th. 4.7.

Give the formula before eq. (4.10) and explain its proof (with the amount of detail that the speaker desires).

Finally, prove the reformulation of Th. 4.7 in terms of derivatives of Dirichlet L-functions given in paragraph 5 of [And82]. This uses (3.3) and (3.4) in [And82], that must have appeared in the previous talk (in the explicit description of M^{ab}).

2.3 Tuesday. Formal groups.

2.3.1 Talk 4^{**}. Commutative formal groups. The Dieudonné and Tate modules of a commutative formal group. Tate's results.

In this talk, there will be many definitions but few proofs. The aim is to present several basic facts and results in the theory of p-divisible groups.

Define finite group schemes and p-divisible groups as in Tate's article [Tat67, (2.1)]. Introduce Cartier duality and the fundamental exact sequence as in [Tat67, (1.4)] and explain that these generalise to p-divisible groups ((2.3) and ?). Formulate but don't prove the equivalence between commutative formal groups and connected p-divisible groups (Prop. 1). Prove the all important Prop. 3 and introduce the two fundamental Galois modules $\Phi(G)$ and $T(G)$ as on p. 169. Formulate but don't prove Prop. 8. This proposition concerns the Galois cohomology of $\mathbb{C}_p$ and is at the root of all the further investigations in p-adic Hodge theory.

Now formulate Prop. 11 and then Th. 3. Prove that Cor. 2 follows from them (this requires an orthogonality result proved in the proof of Th. 3, at the bottom of p. 179, which should be made explicit). Explain that Cor. 2 is the p-adic Hodge decomposition for commutative formal groups over the ring of integers of local fields.

Now formulate Prop. I.1.1 (a theorem of Fontaine - see [Fon77]) and Prop. I.1.4 in [Col93]. Give the basic properties of the ring $B_p = B^+_{\mathrm{DR,p}}$ but don't spend much time on it (the precise structure of the ring B_p does not play an important role in [Col93]; it will reappear in Talk 7; see [Ber04, II.2] for more details about this ring). Explain that Prop. I.1.4 is the analog *for de Rham cohomology* of Tate's p-adic Hodge decomposition, which holds for *Hodge cohomology* (where Hodge cohomology is the direct sum of the quotients of the Hodge filtration on de Rham cohomology).

2.3.2 Talk 5^*. Lubin-Tate formal groups.

The aim of this talk is to define Lubin-Tate formal groups, which are certain commutative formal groups with complex multiplication and to give some of their basic properties.

Define Lubin-Tate formal groups in dimension 1. Here you may follow the original article by Lubin-Tate (see [LT65]), which is very well written or the more succinct exposition given in [Neu99, (4.5), V.4, P. 343], which is perhaps more suitable for this talk.

Prove Theorem V.4.6 (p. 344) in [Neu99], using Prop. 2.2 (p. 328), which should be proven first.

After that, explain quickly the content of Cor. 5.7 (p. 350), which justifies the introduction of Lubin-Tate formal groups.

Do Exercise 4 in V.4 (p. 345) in [Neu99] (this is important in the context of Colmez's article).

Finally, formulate the generalisation of Exercise 4 mentioned at the bottom of p. 641 and at the top of p. 642 in [Col93]. This concerns higher-dimensional generalisations of Lubin-Tate formal groups; these were introduced in [Car76].

2.3.3 Talk 6^*. CM formal groups. The main structure theorem.

The aim of this talk is to prove Theorem I.3.2 in [Col93]. This theorem is a structure theorem for CM formal groups, similar to Prop. I.1.1 in [Col93]. It consists in some clever computations with formal power series. The speaker should present the proof of Th. I.3.2 (ie Lemmes I.3.2 to I.3.14), which is self-contained, with the desired amount of detail.

2.4 Wednesday. Talk 7^{**}. The p-adic periods of CM formal groups.

The aim of this talk is to formulate Th. I.3.15, which computes the valuations of the periods of CM formal groups, and to prove it in the case of Lubin-Tate formal groups (I do not suggest that the speaker prove Th. I.3.15 in full).

Formulate Th. I.1.4 and Formulate Th. I.3.15 (in particular, introduce the symbols Z_p and $\mu_{\mathrm{Art},p}$).

Now formulate and prove Th. I.2.1, going through Lemme I.2.2 and I.2.3. Finally, prove Lemme I.2.4 and show (Prop. I.2.7) that it implies that Th. I.3.15 holds for Lubin-Tate formal groups.

2.5 Thursday. Colmez's period distribution. Formulation of the main theorem.

2.5.1 Talk 8^*. Crystalline cohomology. The action of the crystalline Weil group.

The aim of this talk is to define crystalline cohomology and formulate its main properties. Furthermore, the action of the crystalline Weil group of varieties with good reduction should be described. More precisely:

(1) Define crystalline cohomology. This is a cohomology theory arising as the sheaf cohomology of a natural sheaf on the so-called crystalline site. You may follow [Ill94] for this, looking up the earlier report [Ill76] for more details. The basic reference here is Berthelot's thesis [Ber74].

(2) Explain that in the presence of a smooth lifting over an absolutely unramified discrete valuation ring of mixed characteristic, the crystalline cohomology coincides with the de Rham cohomology of the lifting (see (1.3.8, p. 47 in [Ill94]).

(3) Now explain the more refined independence results described in par. 2.3, p. 51 of [Ill94]. These results are proved in [BO83].

(4) Finally define the crystalline Weil group as in par. 4 p. 187 of [BO83] and formulate Th. 4.2, p. 188 in [BO83], which describes the action of the crystalline Weil group on varieties with good reduction. Prove this result assuming the independence results mentioned above and the general theorem of Gillet-Messing 4.3.

2.5.2 Talk 9^{**}. p-adic absolute Hodge cycles.

The aim of this talk is to go through the proof of the main result of [Bla94]. A variant of Blasius's argument is presented in [Ogu90]. We shall follow Ogus's argument.

(1) First formulate Faltings p-adic comparison theorem (see 2.1.2 in [Ill90] or [Ber04, II]) and explain briefly its link with the results on formal groups described in Talk 4.

(2) Formulate Th. 4.2 in [Ogu90]. Explain why it implies that two isomorphic homological motives for absolute Hodge cycles have the same classical and p-adic periods.

(3) Explain and prove the analog of Principle A in this setting (see the beginning of the proof of Th. 4.2).

(4) Give the argument of the proof of the analog of principle B (Prop. 4.3). Here the existence of the element $\bar{\eta}$ requires a reference to Deligne's article on mixed Hodge structures, which is beyond the scope of these conference so the proof of the existence of $\bar{\eta}$ should be skipped or only outlined.

(5) Explain that the fact that the analogs of principles A and B hold implies that it suffices to show that the Hodge cycles constructed in Lemma 4.5 in [DMOS82, Lemma] (this Lemma appears in Talk 1) have the property asserted in Th. 4.2, which is obvious.

2.5.3 Talk 10^{*}. The period distribution ht. The main result. Link with the Faltings height.

The aim of this talk is to introduce the map ht and to prove Lemme II.2.9 and Th. II.2.10 in [Col93]. Furthermore, the aim of this talk is to formulate the main result Th. III.2.9 (ii) of Colmez article (see below).

(1) Formulate Th. II.1.1 and explain that it follows from Th. I.3.15, with a bit more work (skip the proof).

(2) Introduce the map ht as defined in Lemme II.2.9. Prove Lemme II.2.9. It is a direct consequence of

- the product formula for the absolute values of rational numbers (ie the product of all the absolute values of a rational number equals 1)

- Lemme II.2.1, which should be proved (the proof is elementary)

- Th. II.1.1

(3) Prove Th. II.2.10 (i). This is a refinement of Shimura's period relations and is a very important step in the proof of the main result of Colmez's article. The crux of the proof is the fact that two specific motives with absolute Hodge cycles are equivariantly (under the action of a CM field) isomorphic (see l.-9, p. 666 in [Col93]). Since they are isomorphic, the main result of Talk 1 and the main result of Talk 9 shows that they have (loosely speaking) the same classical as well as p-adic periods. More precisely, the equality at the bottom of p. 666 in [Col93] holds, which quickly implies Th. II.2.10 (i).

(4) Give an outline (with the amount of detail desired) of the proof of Th. II.2.10 (ii). This is a computation, where the main point is Lemme II.2.13.

Finally explain that the main result Th. III.2.9 (ii) of the article [Col93] is that Conjecture II.2.11 (i) holds if a lies in $\mathcal{CM}^{\mathrm{ab}}$, up to a factor $\mathbb{Q}\log(2)$, which was proven to be 0 by the subsequent work of A. Obus (see [Obu13]), who computed the Frobenius matrices of Fermat curves in char. 2 (see Talk 11). As a consequence of this and Th. II.2.10 (ii), one gets a formula for the Faltings height of CM abelian varieties with complex multiplication by a CM field, which is an abelian extension of $\mathbb{Q}$ (see Conjecture II.2.11 (ii)). This formula is made more explicit in [Col98, Conjecture 3].

2.6 Friday. Fermat curves. Completion of the proof. Outlook.

2.6.1 Talk 11^{**}. Coleman's results on the action of the crystalline Weil group on the cohomology of Fermat curves. Completion of the proof.

The aim of this talk is to give a complete proof of Colmez's main result Th. III.2.9 (ii) in [Col93], on the basis of Th. II.2.10 (i).

Explain that if Th. II.2.10 (i) (ie the refined monomial relations) could be proven by general arguments without going through a direct calculation of the valuations of the p-adic periods of CM abelian varieties, then the proof of the main result could be considerably simplified. More precisely, the proof of the main result would then be reduced to the proof of Lemme III.2.5 (see below) and it should in principle be possible to prove this Lemme III.2.5 in a less round about way then in Colmez's paper (but the author of this report does not know how to provide such a proof).

(1) Formulate Prop. III.1.2 in [Col93]. The proof of this proposition is an application of the formulae of Hurwitz for the first terms of the Taylor development of Hurwitz zeta functions, together with the explicit description of $\mathcal{CM}^{\mathrm{ab}}$ given in Talk 1 (ie Th. 3.1 in [And82]). It is similar to formula (5.1) in [And82]. Give as many details as desired (it is not very important to give a detailed proof).

(2) Prove Lemme III.2.1. This requires the computation of some explicit classical periods on a certain quotient of the Fermat curve. This was already treated in Talk 1. Then Prove Cor. III.2.2, whose proof depends on Prop. III.1.2

(3) Prove Lemme III.2.3. This again relies on some explicit evaluations of periods of quotients of the Fermat curve. The proof may be skipped, if desired.

(4) Explain why Lemme III.2.5 suffices to complete the proof, provided one has Lemma III.2.4. The argument is similar to the "Reduction Lemma" on p. 326 of [And82]. Then prove Lemme III.2.4.

(5) Finally, prove Lemme III.2.5. The point of the Lemma is to compute $v_p(\omega_{gq})$. This computation relies on two facts: $v_p(g(\omega_q)) = v_p(\omega_q)$, which is consequence of Th. II.1.1 (iii) (see Talk 10), and the computation of the valuation of the quotient $g(\omega_q)/\omega_{gq}$, which was computed by Coleman in [Col90]. Notice that the restrictions on p in Lemma III.2.5 can be lifted by the work of Obus (see [Obu13]).

NB. The formula for the Faltings height of the Jacobian of the curve with equation $y^2 = x^5 + 1$ in the middle of p. 680 of [Col93] *is incorrect. The term $\Gamma(1/5)^{-1}$ in that formula should be replaced by $\Gamma(4/5)^{-1}$.*

2.6.2 Talk 12^*. Complements. The motivic setting.

The aim of this talk is to survey the results presented in Soulé's Bourbaki talk [Sou07]. These results partially generalise Colmez's theorem to a motivic setting.

(1) Give a very quick survey of Arakelov theory as in par. 1 of [Sou07] (for more information, see [Bos91] or [Sou92] and the references therein).

(2) Introduce all the elements (R-genus, char. classes, etc.) appearing in Th. 3.1 of [Sou07] and formulate Th. 3. Note that Th. 3.1 is the main result of [KR01].

(3) Formulate Prop. 4.1, insisting on Prop. 4.1 (i) (proven in [RS73, Th. 3.1]), which is the key vanishing result. Show that Th. 3.1 together with Prop. 4.1 implies equality (3) in [Sou07].

(4) Explain that some linear algebra (give the amount of detail you wish) implies that

equality (3) implies Prop. 4.4.

(5) Explain that Prop. 4.4 leads to the conjecture given in 7.1, which is a generalisation of the conjecture of Colmez to all pure motives with complex multiplication, provided one disregards $\bar{\mathbb{Q}}\log(p)$ factors, for p a prime dividing the discriminant of the CM field.

The main reference for Soulé's Bourbaki talk is [MR04].

2.6.3 Talk 13^{**}. Complements. The logarithmic derivatives of Dirichlet L-functions at negative integers.

The aim of this talk is to formulate the conjecture presented in [MR02]. This conjecture gives a formula relating the logarithmic derivatives of Artin L-functions at negative integers to arithmetic Chern classes of subbundles of the Hodge bundle of abelian schemes with complex multiplication. It specialises to a weak form of Colmez's conjecture (ie Conjecture II.2.11 (i) in [Col93]) when the Artin L-functions are evaluated at 0.

We recommend the following: formulate the conjecture only in the setting of abelian schemes, not semiabelian schemes. In this way, one does not have to introduce the (partially conjectural) generalised arithmetic Chow theory described in the first paragraph of [MR02, 1. Préliminaires].

(1) Describe the setting: polarised abelian schemes over an arithmetic variety, together with the ring of integers of a number field K. Explain that the polarisation induces a metric on the Hodge bundle. *Suppose from here onwards that the discriminant of K is invertible in the arithmetic ring underlying the base arithmetic variety*. This assumption is not made in the article [MR02] but this extra degree of generality now seems doubtful to the author of this program. A correct formulation of the conjecture with this assumption removed has yet to be found.

(2) Formulate Conjecture 2.1 in [MR02]. Explain that this conjecture should also be true for semiabelian schemes but that this generalisation requires the generalised Chow theory described in [MR02, 1. Préliminaires] (note that this generalisation is now almost in existence: see [BGKK07, Introduction] and [BGKK05, par. 5.4]). Explain that Conjecture 2.1 in the situation where K is an abelian extension of $\mathbb{Q}$ is a consequence of the general relative arithmetic Lefschetz formula proven in [Tan12], together with the vanishing of the equivariant analytic torsion form of the relative de Rham complex proven in general in [Bis04] (in the situation of abelian schemes, one can give a direct proof of the latter). The proof is outlined in [MR02, 4. Résultats]. This computation is analogous to the computation presented in Talk 12.

(3) Give several special cases of this formula (up to some $\log(p)$ factors, for p dividing the discriminant of the number field) that have been considered in the literature, for example [Küh01] and [KRY06, Th. 1.0.5]. Further references to known special cases are given after Conjecture 2.1. When the base arithmetic variety is the arithmetic ring itself and the generic fibre of the abelian scheme has (maximal) complex multiplication, the formula reduces to the main result of Colmez, up to some $\bar{\mathbb{Q}}\log(p)$ factors, for p dividing the discriminant of the CM field.

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