

An afterthought on the generalized Mordell-Lang conjecture

Damian Rössler *

December 2, 2005

Abstract

The generalized Mordell-Lang conjecture (GML) is the statement that the irreducible components of the Zariski closure of a subset of a group of finite rank inside a semi-abelian variety are translates of closed algebraic subgroups. In [6], M. McQuillan gave a proof of this statement. We revisit his proof, indicating some simplifications. This text contains a complete elementary proof of the fact that (GML) for groups of torsion points (= generalized Manin-Mumford conjecture), together with (GML) for finitely generated groups imply the full generalized Mordell-Lang conjecture.

Mathematics Subject Classification: 14G05, 14K15

1 Introduction

Let A be a semi-abelian variety over $\overline{\mathbb{Q}}$ (cf. beginning of Section 2 for the definition of a semi-abelian variety). We shall call a closed reduced subscheme of A *linear* if its irreducible components are translates of closed subgroup schemes of A by points of $A(\overline{\mathbb{Q}})$. Let Γ be a finitely generated subgroup of $A(\overline{\mathbb{Q}})$ and define $\text{Div}(\Gamma) := \{a \in A(\overline{\mathbb{Q}}) \mid \exists n \in \mathbb{Z}_{\geq 1} : n \cdot a \in \Gamma\}$. Let X be a closed reduced subscheme of A . Consider the following statement :

the variety $\text{Zar}(X \cap \text{Div}(\Gamma))$ is linear ()*.

The generalized Mordell-Lang conjecture (GML) is the statement that (*) holds for any data A, Γ, X as above. The statement (GML) with the supplementary requirement that $\Gamma = 0$ shall be referred to as (MM). The statement (GML) with $\text{Div}(\Gamma)$ replaced by Γ in (*) shall be referred to as (ML).

The statement (ML) was first proven by Vojta (who built on Faltings work) in [11]. The statement (MM) was first proven by Hindry (who built on work

*CNRS, Centre de Mathématiques de Jussieu, Université Paris 7 Denis Diderot, Case Postale 7012, 2, place Jussieu, F-75251 Paris Cedex 05, France, E-mail : dcr@math.jussieu.fr

of Serre and Ribet) in [3]. Finally, McQuillan (who built on the work of the previous) proved (GML) in [6].

The structure of McQuillan's proof of (GML) has three key inputs: (1) the statement (ML), (2) an extension of (MM) to families of varieties and (3) the Kummer theory of abelian varieties.

In this text, we shall indicate some simplifications of this proof. More precisely, we show the following. First, that once (MM) is granted, a variation of (2) sufficient for the purposes of the proof is contained in an automatic uniformity principle proved by Hrushovski. See Lemma 2.3 for the statement of this automatic uniformity principle and a reference for the proof, which uses nothing more than the compactness theorem of first order logic. Secondly, we show that one can replace the Kummer theory of abelian varieties (3) by an elementary geometrical argument. The core of the simplified proof is thus an elementary proof of the following statement:

*if (ML) and (MM) hold then (GML) holds (**)*

and the proof of (GML) is then obtained by combining (**) with the statements (MM) and (ML), which are known to be true by the work of Hindry and Vojta.

We stress that our proof of (**) is independent of the truth or techniques of proof of either (ML) or (MM).

Notice that yet another proof of (GML) was given by Hrushovski in [4, Par. 6.5]. His proof builds on a generalisation of his model-theoretic proof of (MM) (which is based on the dichotomy theorem of the theory of difference fields) and on (ML). It also avoids the Kummer theory of abelian varieties but it apparently doesn't lead to a proof of (**). Finally, we want to remark that a deep Galois-theoretic result of Serre (which makes an earlier statement of Bogomolov uniform), which is used in McQuillan's proof of (GML) as well as in Hindry's proof of (MM) (see [3, Lemme 12]), was never published. Now different proofs of (MM), which do not rely on Serre's result, were given by Hrushovski in [4] and by Pink-Rössler in [8]. Our proof of (**) thus leads to another proof of (GML) which is independent of Serre's result.

The structure of the article is as follows. Section 3 contains the proof of (**) and section 2 recalls the various facts from the theory of semi-abelian varieties that we shall need in Section 3. The reader is encouraged to start with Section 3 and refer to Section 2 as necessary.

Basic notational conventions. A (closed) subvariety of a scheme S is a (closed) reduced subscheme of S . If X is a closed subvariety of a $\overline{\mathbb{Q}}$ -group scheme A , we shall write $\text{Stab}(X)$ for the stabilizer of X in A , which is a closed group subscheme of A such that $\text{Stab}(X)(\overline{\mathbb{Q}}) = \{a \in A(\overline{\mathbb{Q}}) | X + a = X\}$. If H is a commutative group, we write $\text{Tor}(H) \subseteq H$ for the subgroup consisting of the elements of finite order in H . If T is a noetherian topological space, denote by $\text{Irr}(T)$ the set of its irreducible components.

Acknowledgments. Many thanks to P. Vojta for his careful reading of a first

version of this text and his detailed comments. My thanks also go to R. Pink, for his comments and suggestions of improvement and for many interesting conversations on matters related to this article.

2 Preliminaries

A *semi-abelian variety* A over an algebraically closed field k is by definition a commutative group scheme over k with the following properties: it has a closed subgroup scheme G which is isomorphic to a product of finitely many multiplicative groups over k and there exists an abelian variety B over k and a surjective morphism $\pi : A \rightarrow B$, which is a morphism of group schemes over k and whose kernel is G .

In the next lemma, let A be a semi-abelian variety as in the previous definition.

Lemma 2.1. *Let $n \in \mathbb{Z}_{\geq 1}$. The multiplication by n morphism $[n]_A : A \rightarrow A$ is quasi-finite.*

Proof: we must prove that the fibers of $[n]_A$ have finitely many points or equivalently that they are of dimension 0. Moreover, since the function $\dim([n]_{A,a})$ (= dimension of the fiber of $[n]_A$ over a) is a constructible function of $a \in A$ (see [2, Ex. 3.22, chap. II]), it is sufficient to prove that the fibers of $[n]_A$ over closed points are finite. A fiber of $[n]_A$ over a closed point can be identified with the fiber $\ker [n]_A$ of $[n]_A$ over $0 \in A$. The scheme $\ker [n]_A$ is naturally fibered over $\ker [n]_B$. The scheme $\ker [n]_B$ consists of finitely many closed points because multiplication by n in B is a finite morphism, as B is an abelian variety (see [7, Prop. 8.1 (d)]). It will thus be sufficient to prove that the fibers of the morphism $\ker [n]_A \rightarrow \ker [n]_B$ are finite and furthermore each of these fibers can be identified with the fiber of $\ker [n]_A \rightarrow \ker [n]_B$ over 0. By construction this fiber is the closed subscheme $\ker [n]_A \times_A G = \ker [n]_G$ of A . To prove that $\ker [n]_G$ has finitely many closed points, choose an identification $G \simeq G_m^\rho$ of G with a product of ρ multiplicative groups over k . The closed points of $\ker [n]_G$ then correspond to ρ -tuples of n -th roots of unity in k . This set is finite and this concludes the proof of the lemma. **Q.E.D.**

Let now A be a semi-abelian variety over $\overline{\mathbb{Q}}$.

Theorem 2.2 (Kawamata-Abramovich). *Let X be a closed irreducible subvariety of A . The union $Z(X)$ of the irreducible linear subvarieties of positive dimension of X is Zariski closed. The stabilizer $\text{Stab}(X)$ of X is finite if and only if the complement of $Z(X)$ in X is not empty.*

For the proof see [1, Th. 1 & 2].

Let Y be a variety over $\overline{\mathbb{Q}}$ and let $W \hookrightarrow A \times_{\overline{\mathbb{Q}}} Y$ be a closed subvariety.

Lemma 2.3 (Hrushovski). *If (MM) holds then the quantity*

$$\text{Sup}\{\#\text{Irr}(\text{Zar}(W_y \cap \text{Tor}(A(\overline{\mathbb{Q}}))))\}_{y \in Y(\overline{\mathbb{Q}})}$$

is finite.

Notice that (MM) predicts that the irreducible components of $\text{Zar}(W_y \cap \text{Tor}(A(\overline{\mathbb{Q}})))$ are linear for each W_y , $y \in Y(\overline{\mathbb{Q}})$. In words, the content of the lemma is that if this is the case, then the number of these irreducible components can be bounded independently of $y \in Y(\overline{\mathbb{Q}})$. A self-contained proof of Lemma 2.3 can be found in [4, Postscript, Lemma 1.3.2, p. 52-53]. For an extension of Lemma 2.3, see [10].

Suppose now that A has a model A_0 over a number field K .

In [9, Par. 1.2 and Prop. 2] it is shown that there exists a variety \overline{A}_0 projective over K and an open immersion $A_0 \hookrightarrow \overline{A}_0$ such that for all $n \in \mathbb{Z}_{\geq 1}$ the multiplication by n morphism $[n]_{A_0} : A_0 \rightarrow A_0$ extends to a K -morphism $[n]_{\overline{A}_0} : \overline{A}_0 \rightarrow \overline{A}_0$. Furthermore, it is shown in [9, Prop. 3] that the corresponding diagram

$$\begin{array}{ccc} A_0 & \hookrightarrow & \overline{A}_0 \\ \downarrow [n]_{A_0} & & \downarrow [n]_{\overline{A}_0} \\ A_0 & \hookrightarrow & \overline{A}_0 \end{array}$$

is then cartesian.

Let Γ be a finitely generated subgroup of $A_0(K)$.

Lemma 2.4 (McQuillan). *The group generated by the set $\text{Div}(\Gamma) \cap A_0(K)$ is finitely generated.*

Lemma 2.4 is McQuillan's Lemma 3.1.3 in [6]. As the proof given there is somewhat sketchy, we shall provide a proof of Lemma 2.4.

Proof: let B be an abelian variety over $\overline{\mathbb{Q}}$ and $\pi : A \rightarrow B$ be a $\overline{\mathbb{Q}}$ -morphism whose kernel G is isomorphic to a product of tori over $\overline{\mathbb{Q}}$. These data exist since A is a semi-abelian variety. Notice that for the purposes of the proof we may enlarge the field of definition K of A if necessary, since that operation will also enlarge the set $\text{Div}(\Gamma) \cap A_0(K)$. Hence we may assume that B (resp. G) has a model B_0 (resp. G_0) over K and that π has a model π_0 over K . Furthermore, we may assume that the isomorphism of G with a product of tori descends to a K -isomorphism of G_0 with a product of split tori over K . Now fix a compactification \overline{A}_0 of A_0 over K as above. We consider the following situation. The symbol V refers to an open subscheme of the spectrum $\text{Spec } \mathcal{O}_K$ of the ring of integers of K and \mathcal{A}_0 is a semi-abelian scheme over V , which is a model of A_0 . The symbol $\overline{\mathcal{A}}_0$ refers to a projective model over V of \overline{A}_0 and we suppose given an open immersion $\mathcal{A}_0 \hookrightarrow \overline{\mathcal{A}}_0$, which is a model of the open immersion $A_0 \hookrightarrow \overline{A}_0$. We also suppose that the multiplication by n morphism $[n]_{\mathcal{A}_0}$ on \mathcal{A}_0 extends to a V -morphism $[n]_{\overline{\mathcal{A}}_0} : \overline{\mathcal{A}}_0 \rightarrow \overline{\mathcal{A}}_0$ and we suppose that the corresponding diagram

$$\begin{array}{ccc} \mathcal{A}_0 & \hookrightarrow & \overline{\mathcal{A}}_0 \\ \downarrow [n]_{\mathcal{A}_0} & & \downarrow [n]_{\overline{\mathcal{A}}_0} \\ \mathcal{A}_0 & \hookrightarrow & \overline{\mathcal{A}}_0 \end{array}$$

is cartesian. Let \mathcal{B}_0 (resp. \mathcal{G}_0) be a model of B_0 (resp. G_0) over V and let $\tilde{\pi}_0$ be a model of π_0 over V . Furthermore, we assume that the K -isomorphism of G_0 with a product of split tori extends to a V -isomorphism of \mathcal{G}_0 with a product of split tori over V . We leave it to the reader to show that there are objects V , \mathcal{A}_0 etc. satisfying the described conditions.

The morphism $[n]_{\mathcal{A}_0}$ is then proper, because $[n]_{\overline{\mathcal{A}}_0}$ is proper (as $\overline{\mathcal{A}}_0$ is proper over V) and properness is invariant under base change. The morphism $[n]_{\mathcal{A}_0}$ is therefore finite, as it is quasi-finite by 2.1 (applied to each fiber of \mathcal{A}_0 over V). We may suppose without restriction of generality that Γ lies in the image of $\mathcal{A}_0(V)$ in $A_0(K)$; indeed this condition will always be fulfilled after possibly removing a finite number of closed points from V . Let $a \in \text{Div}(\Gamma) \cap A_0(K)$. Choose an $n \in \mathbb{Z}_{\geq 1}$ such that $n \cdot a \in \Gamma$. Let E be the image of the section $V \rightarrow \mathcal{A}_0$ corresponding to $n \cdot a$. Consider the reduced irreducible component C of $[n]_{\mathcal{A}_0}^* E$ containing the image of a . The image of a is the generic point of C and by assumption the natural morphism $C \rightarrow E$ identifies the function fields of C and E . Furthermore, the morphism $C \rightarrow E$ is finite. Now let R_0 be the ring underlying the affine scheme V . In view of the above, we can write $C = \text{Spec } R$, where R is a domain and the morphism $C \rightarrow E$ identifies R with an integral extension of R_0 inside the integral closure of R_0 in its own field of fractions. As R_0 is integrally closed (it is even a Dedekind ring) $C \rightarrow E$ is an isomorphism. Hence $a \in \mathcal{A}_0(V)$. Thus we only have to show that $\mathcal{A}_0(V)$ is finitely generated. This follows from the fact that $\mathcal{B}_0(V)$ is finitely generated by the Mordell-Weil theorem applied to B_0 and the fact that $\mathcal{G}_0(V)$ is finitely generated by the generalized Dirichlet unit theorem (see [5, chap. V, par. 1]).

Q.E.D.

Lemma 2.5. *Let $C > 0$. The set $\{a \in \text{Tor}(A(\overline{\mathbb{Q}})) \mid [K(a) : K] < C\}$ is finite.*

Proof: if A_0 is an abelian variety over K then this follows from the fact that the Néron-Tate height of torsion points vanishes and from Northcott's theorem. We leave the general case as an exercise for the reader. **Q.E.D.**

3 Proof of (**)

In this section we shall prove (GML) using the results listed in Section 2 as well as (ML) and (MM).

First notice that to prove (*), one may assume without loss of generality that X is irreducible and that $\text{Zar}(X \cap \text{Div}(\Gamma)) = X$. The statement (*) then becomes *the variety X is a translate of a connected closed group subscheme of A (*)*.

In the remaining of this section, we shall prove that (ML) and (MM) imply (*). The overall structure of our proof will be similar to McQuillan's proof of (GML), with some simplifications that we shall point out along the way.

First, we may suppose without restriction of generality that $\text{Stab}(X) = 0$.

To see the latter, consider the closed subvariety $X/\text{Stab}(X)$ of the quotient variety $A/\text{Stab}(X)$. The image of the group $\text{Div}(\Gamma)$ in $(A/\text{Stab}(X))(\overline{\mathbb{Q}})$ lies inside the group $\text{Div}(\Gamma_1)$, where Γ_1 is the image of Γ and the image of $\text{Div}(\Gamma)$ is dense in $X/\text{Stab}(X)$. So the assumptions of (*) hold for $A/\text{Stab}(X)$, Γ_1 and $X/\text{Stab}(X)$. Furthermore, by construction $\text{Stab}(X/\text{Stab}(X)) = 0$. Now if (*)' holds in this situation, $X/\text{Stab}(X)$ is the translate of a connected closed group subscheme of $A/\text{Stab}(X)$. Hence $X/\text{Stab}(X)$ is a closed point. This implies that X is a translate of $\text{Stab}(X)$, thus proving (*)' for A , Γ and X . It is thus sufficient to prove (*)' for $A/\text{Stab}(X)$, Γ_1 and $X/\text{Stab}(X)$ and we may thus replace A by $A/\text{Stab}(X)$, Γ by Γ_1 , X by $X/\text{Stab}(X)$. We then have $\text{Stab}(X) = 0$.

We may also assume without loss of generality that A (resp. X) has a model A_0 (resp. X_0) over a number field K such that $\Gamma \subseteq A_0(K)$ and such that the immersion $X \rightarrow A$ has a model over K as an immersion $X_0 \rightarrow A_0$.

Let U be the complement in X of the union of the irreducible linear subvarieties of positive dimension of X . By Theorem 2.2 the set U is non-empty and open in X and thus $U \cap \text{Div}(\Gamma)$ is dense in X .

Let $a \in U \cap \text{Div}(\Gamma)$ and let $\sigma \in \text{Gal}(\overline{\mathbb{Q}}|K)$. By construction $\sigma(U) \subseteq U$ and we thus have $\sigma(a) - a \in U - a$. Now by definition, there exists $n = n(a) \in \mathbb{Z}_{\geq 1}$ such that $n \cdot a \in \Gamma \subseteq A_0(K)$. We calculate

$$n \cdot (\sigma(a) - a) = \sigma(n \cdot a) - n \cdot a = n \cdot a - n \cdot a = 0.$$

Thus $\sigma(a) - a \in \text{Tor}(A(\overline{\mathbb{Q}})) \cap (U - a)$. The statement (MM) implies that $\text{Tor}(A(\overline{\mathbb{Q}})) \cap (U - a)$ is finite and using Theorem 2.3, we see that $\#(\text{Tor}(A(\overline{\mathbb{Q}})) \cap (U - a)) < C$ for some constant $C \in \mathbb{Z}_{\geq 1}$, which is independent of a . Now this implies that $\#\{\tau(a) | \tau \in \text{Gal}(\overline{\mathbb{Q}}|K)\} = \#\{\tau(a) - a | \tau \in \text{Gal}(\overline{\mathbb{Q}}|K)\} < C$.

A consequence of this conclusion is reached by McQuillan at the beginning of Par. 3.3 (p. 157) of [6] using Theorem 3.2.2 of that article. This last theorem is replaced by Theorem 2.3 in our context.

By Galois theory, we thus have $[K(a) : K] < C$. We deduce from this last inequality that

$$[K(\sigma(a) - a) : K] \leq [K(a, \sigma(a)) : K] < C^2.$$

By Lemma 2.5, we see that this implies that $\sigma(a) - a \in T$, where $T \subseteq A(\overline{\mathbb{Q}})$ is a finite set, which is dependent on C but independent of either a or σ . For each $b \in A(\overline{\mathbb{Q}}) \setminus A_0(K)$, choose $\sigma_b \in \text{Gal}(\overline{\mathbb{Q}}|K)$ such that $\sigma_b(b) \neq b$. We know that either the set

$$\{b \in U \cap \text{Div}(\Gamma) | b \in A(\overline{\mathbb{Q}}) \setminus A_0(K)\}$$

or the set

$$\{b \in U \cap \text{Div}(\Gamma) | b \in A_0(K)\}$$

is dense in X . First suppose the former. In that case, there exists $t_0 \in T$, $t_0 \neq 0$ such that the set

$$\{b \in U \cap \text{Div}(\Gamma) | b \in A(\overline{\mathbb{Q}}) \setminus A_0(K), \sigma_b(b) - b = t_0\}$$

is dense in X . Since $\sigma_b(b) \in U$ for all $b \in U$ such that $b \in A(\overline{\mathbb{Q}}) \setminus A_0(K)$, we see that this implies that $t_0 \in \text{Stab}(X)(\overline{\mathbb{Q}})$. But $t_0 \neq 0$ so this contradicts our hypothesis that $\text{Stab}(X) = 0$. Thus we deduce that the set $\{b \in U \cap \text{Div}(\Gamma) \mid b \in A_0(K)\}$ is dense in X . By Lemma 2.4, the elements of this set are contained in a finitely generated group, so using (ML) we deduce (*).

The last part of our proof of (*)' is similar to McQuillan's final argument in Par. 3.3 of [6]. His argument involves the Kummer theories of abelian varieties and tori; this is replaced in our context by the above elementary geometrical construction, based on Lemma 2.5.

References

- [1] Abramovich, D.: *Subvarieties of semiabelian varieties*. Compositio Math. **90**, no. 1, 37–52 (1994).
- [2] Hartshorne, R.: *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag: New York-Heidelberg 1977.
- [3] Hindry, M.: *Autour d'une conjecture de Serge Lang*. Invent. Math. **94**, no. 3, 575–603 (1988).
- [4] Hrushovski, E.: *The Manin-Mumford conjecture and the model theory of difference fields*. Annals of Pure and Applied Logic **112**, 43–115 (2001).
- [5] Lang, S.: *Algebraic Number Theory*. Second Edition. Springer 1994
- [6] McQuillan, M.: *Division points on semi-abelian varieties*. Invent. Math. **120**, no. 1, 143–159 (1995).
- [7] Milne, J.: *Abelian Varieties*. In: *Arithmetic Geometry* (Storrs, Conn., 1984), 103–150. Springer: New York 1986.
- [8] Pink, R.; Rössler, D.: *On psi-invariant subvarieties of semiabelian varieties and the Manin-Mumford conjecture*. J. Algebraic. Geom. **13** (2004), no. 4, 771–798.
- [9] Serre, J.-P.: *Quelques propriétés des groupes algébriques commutatifs*. Appendix in *Astérisque* **69-70**, 191–202 (1979).
- [10] Scanlon, T.: *Automatic uniformity*. Int. Math. Res. Not. **62**, 3317–3326 (2004).
- [11] Vojta, P.: *Integral points on subvarieties of semiabelian varieties. I*. Invent. Math. **126**, no. 1, 133–181 (1996).