LOCAL INVARIANCE AND BLOWING UP

by

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We fix a base field k. In this text all the schemes will be separated and of finite type over k and all the morphisms will be k-morphisms. The paper [V] is

Varshavsky, Yakov; Lefschetz-Verdier trace formula and a generalization of a theorem of Fujiwara. Geom. Funct. Anal. 17 (2007), no. 1, 271–319.

Let X be a scheme. Let $C \to X \times_k X$ be a morphism. Let $c_1, c_2 : C \to X$ be the two projections. Let $Z \hookrightarrow X$ be a closed subscheme. Let $U := X \backslash Z$.

We say that a closed subscheme $Z_0 \hookrightarrow X$ is locally C-invariant if for all $z_0 \in Z_0$, there is a (Zariski) neighborhood V of z_0 in X, such that we have a set-theoretic inclusion $c_1(c_2^{-1}(Z_0 \cap V)) \subseteq Z_0 \cup (X \backslash V)$. See [V], Def. 1.5.1. Notice that this is completely set-theoretic and does not depend on the scheme structure of Z_0 or C.

According to [V], 1.5.3 (d), Z_0 is locally C-invariant if and only if, for every irreducible component S of $c_2^{-1}(Z)\backslash c_1^{-1}(Z)$, we have $\overline{c_1(S)} \cap \overline{c_2(S)} = \emptyset$. Here $\bar{\cdot}$ refers to Zariski closure.

Accordingly, we shall say that a point P in $c_2^{-1}(Z_0)\backslash c_1^{-1}(Z_0)$ is *critical* (relatively to Z_0 and C), if $\overline{c_1(P)} \cap \overline{c_2(P)} \neq \emptyset$.

If $f : X_1 \to X$ is a morphism, we define a pull-back correspondence $f^*(C) \to X_1 \times_k X_1$ by base-change. More precisely, $f^*(C) \to X_1 \times_k X_1$ is uniquely determined by the requirement that the square

$$\begin{array}{ccc} f^*(C) & \longrightarrow & X_1 \times_k X_1 \\ \downarrow & & \downarrow f \times_k f \\ C & \longrightarrow & X \times_k X \end{array}$$

is cartesian.

Lemma 0.1. — *Suppose that c_2 is quasi-finite. Then there exists a proper morphism $\pi : \widetilde{X} \to X$, such that the induced morphism $\pi^{-1}(U) \to U$ is an isomorphism and such that $\widetilde{Z} := \pi^*(Z)$ is locally $\widetilde{C} := \pi^*(C)$-invariant.*

Proof. We define inductively a sequence of schemes X_i $(i \geqslant 0)$, together with subschemes $Z_i \hookrightarrow X_i$, and morphisms $C_i \to X_i \times_k X_i$.

Let $X_0 := X$, $Z_0 := Z$, $C_0 := C$.

If X_i, Z_i and C_i are given, we define $c_{1,i}, c_{2,i} : C_i \to X_i$ to be the first and second projections. We also define

$$W_i := \Big[\coprod_{\eta_T \in c_{2,i}^{-1}(Z_i)\backslash c_{1,i}^{-1}(Z_i) \text{ critical generic point}} \overline{c_{1,i}(\eta_T)}_{\text{red}} \Big] \cap \Big[\coprod_{\eta_T \in c_{2,i}^{-1}(Z_i)\backslash c_{1,i}^{-1}(Z_i) \text{ critical generic point}} \overline{c_{2,i}(\eta_T)}_{\text{red}} \Big]$$

Beware that the $\cap$ refers to the scheme-theoretic intersection. The coproduct refers to the union of closed subschemes of X_i (ie intersection of the corresponding ideal sheaves). Notice that set-theoretically

$$\coprod_{\eta_T \in c_{2,i}^{-1}(Z_i)\backslash c_{1,i}^{-1}(Z_i) \text{ critical generic point}} \overline{c_{2,i}(\eta_T)}_{\text{red}} \subseteq Z_i$$

So that W_i is naturally a closed subset of Z_i.

Now we define:

- X_{i+1} is the blow up of X_i along W_i, provided $W_i \neq \emptyset$; *otherwise the sequence stops at the index i*; denote the corresponding exceptional divisor by $E_{i+1} \hookrightarrow X_{i+1}$; denote by $\pi_{i+1,i} : X_{i+1} \to X_i$ the natural morphism;
- $C_{i+1} \to X_{i+1}$ is the pull-back of C_i by $\pi_{i+1,i}$;
- $Z_{i+1} := \pi_{i+1,i}^*(Z_i)$.

Now view $c_{2,i+1}^{-1}(Z_{i+1})\backslash c_{1,i+1}^{-1}(Z_{i+1})$ as a reduced locally closed subscheme of C_{i+1} and let

$$\lambda_{i+1,i} : c_{2,i+1}^{-1}(Z_{i+1})\backslash c_{1,i+1}^{-1}(Z_{i+1}) \to c_{2,i}^{-1}(Z_i)\backslash c_{1,i}^{-1}(Z_i)$$

be the natural morphism; notice that $c_{1,i} \circ \lambda_{i+1,i} = \pi_{i+1,i} \circ c_{1,i+1}$ and $c_{2,i} \circ \lambda_{i+1,i} = \pi_{i+1,i} \circ c_{2,i+1}$ and that $\lambda_{i+1,i}$ sends critical points into critical points.

Define for all $i \geqslant 0$

$$\pi_i := \pi_{1,0} \circ \pi_{2,1} \circ \cdots \circ \pi_{i,i-1} : X_i \to X_0 = X$$

and

$$\lambda_i := \lambda_{1,0} \circ \lambda_{2,1} \circ \cdots \circ \lambda_{i,i-1} : c_{2,i}^{-1}(Z_i)\backslash c_{1,i}^{-1}(Z_i) \to c_2^{-1}(Z)\backslash c_1^{-1}(Z).$$

Claim 1. We have

$$W_0 \supseteq \pi_1(W_1) \supseteq \pi_2(W_2) \supseteq \cdots$$

To prove the claim, let $\eta_S \in c_{2,i+1}^{-1}(Z_{i+1})\backslash c_{1,i+1}^{-1}(Z_{i+1})$ be a critical generic point.

By construction, we have $c_{1,i+1}(\eta_S) \notin Z_{i+1}$ and thus $\pi_{i+1}(c_{1,i+1}(\eta_S)) \notin Z$ and $c_{1,i+1}(\eta_S) \notin E_{i+1}$.

Now suppose first that $c_{2,i+1}(\eta_S) \notin E_{i+1}$. In that case, by construction, we have

$$\pi_{i+1,i}(c_{2,i+1}(\eta_S)) = c_{2,i}(\lambda_{i+1,i}(\eta_S)) \notin W_i.$$

Furthermore, by the above we also have

$$\pi_{i+1,i}(c_{1,i+1}(\eta_S)) = c_{1,i}(\lambda_{i+1,i}(\eta_S)) \notin W_i.$$

Also, $\lambda_{i+1,i}(\eta_S)$ is a critical point and hence a specialisation of a critical generic point of $c_{2,i}^{-1}(Z_i)\backslash c_{1,i}^{-1}(Z_i)$, which is none other than the generic point of the irreducible component in which $\lambda_{i+1,i}(\eta_S)$ lies. Hence

$$\pi_{i+1,i}(c_{1,i+1}(\eta_S)) \in \coprod_{\eta_T \in c_{2,i}^{-1}(Z_i)\backslash c_{1,i}^{-1}(Z_i) \text{ critical generic point}} \overline{c_{1,i}(\eta_T)}_{\text{red}}$$

and

$$\pi_{i+1,i}(c_{2,i+1}(\eta_S)) \in \coprod_{\eta_T \in c_{2,i}^{-1}(Z_i)\backslash c_{1,i}^{-1}(Z_i) \text{ critical generic point}} \overline{c_{2,i}(\eta_T)}_{\mathrm{red}}$$

Now, since blowing up separates closed subschemes (for details on this, see B. Conrad, "Notes on Nagata compactifications", lemma 1.4), this implies that $\overline{c_{1,i+1}(\eta_S)} \cap \overline{c_{2,i+1}(\eta_S)} = \emptyset$, which contradicts the fact that η_S is critical. Hence we must have $c_{2,i+1}(\eta_S) \in E_{i+1}$.

We summarize:

if $\eta_S \in c_{2,i+1}^{-1}(Z_{i+1})\backslash c_{1,i+1}^{-1}(Z_{i+1})$ is a critical generic point then $c_{2,i+1}(\eta_S) \in E_{i+1}$ and in particular $\pi_{i+1}(c_{2,i+1}(\eta_S)) \in \pi_i(W_i)$.

This is an important fact that we will refer to as (*).

A consequence of (*) is the weaker fact that $W_{i+1} \subseteq E_{i+1}$ and hence that $\pi_{i+1}(W_{i+1}) \subseteq \pi_i(W_i)$, which proves the claim.

If the sequence stops for some index i then by the discussion before the lemma, we may take $\widetilde{X} := X_i$ and $\pi = \pi_i$.

So we assume to obtain a contradiction that the sequence does not stop.

Claim 2. There exists an i_0 such that for $i \geqslant i_0$, we have $\dim(\pi_{i+1}(W_{i+1})) < \dim(\pi_i(W_i))$.

We prove Claim 2. By noetherianity and Claim 1, there exists an i_0 such that $\pi_{i+1}(W_{i+1}) = \pi_i(W_i)$ for all $i \geqslant i_0$. We shall show that this i_0 works.

So let $i \geqslant i_0$. We write out the fact that $\pi_{i+1}(W_{i+1}) = \pi_i(W_i)$. This is

$$\pi_{i+1}\Big(\Big[\coprod_{\eta_T \in c_{2,i+1}^{-1}(Z_{i+1})\backslash c_{1,i+1}^{-1}(Z_{i+1}) \text{ critical generic point}} \overline{c_{1,i+1}(\eta_T)}_{\mathrm{red}}\Big] \cap \Big[\coprod_{\eta_T \in c_{2,i+1}^{-1}(Z_i)\backslash c_{1,i+1}^{-1}(Z_{i+1}) \text{ critical generic point}} \overline{c_{2,i+1}(\eta_T)}_{\mathrm{red}}\Big]\Big)$$

$$= \pi_i(W_i)$$

In particular, in view of fact (*), we have set-theoretically

$$\begin{aligned} & \pi_{i+1}\Big(\coprod_{\eta_T \in c_{2,i+1}^{-1}(Z_{i+1})\backslash c_{1,i+1}^{-1}(Z_{i+1}) \text{ critical generic point}} \overline{c_{2,i+1}(\eta_T)}_{\mathrm{red}}\Big) \\ = & \bigcup_{\eta_T \in c_{2,i+1}^{-1}(Z_{i+1})\backslash c_{1,i+1}^{-1}(Z_{i+1}) \text{ critical generic point}} \overline{\pi_{i+1}(c_{2,i+1}(\eta_T))} \\ = & \bigcup_{\eta_T \in c_{2,i+1}^{-1}(Z_{i+1})\backslash c_{1,i+1}^{-1}(Z_{i+1}) \text{ critical generic point}} \overline{c_2(\lambda_{i+1}(\eta_T))} \\ = & \pi_i(W_i). \end{aligned}$$

Thus there exists a critical generic point η_S of $c_{2,i+1}^{-1}(Z_{i+1})\backslash c_{1,i+1}^{-1}(Z_{i+1})$ such that $c_2(\lambda_{i+1}(\eta_S))$ is the generic point of an irreducible component of maximal dimension of $\pi_i(W_i)$, ie such that $\dim(\overline{c_2(\lambda_{i+1}(\eta_S))}) = \dim(\pi_i(W_i))$.

Let now $\eta_{S'}$ be any other critical generic point of $c_{2,i+1}^{-1}(Z_{i+1})\backslash c_{1,i+1}^{-1}(Z_{i+1})$.

Since $c_2(\lambda_{i+1}(\eta_{S'})) \in \pi_i(W_i)$ by fact (*), we have that $\overline{c_2(\lambda_{i+1}(\eta_{S'}))} \leqslant \dim(\pi_i(W_i))$ and thus

$$\dim(\overline{c_1(\lambda_{i+1}(\eta_{S'}))}) \leqslant \dim(\overline{\lambda_{i+1}(\eta_{S'})}) = \dim(\overline{c_2(\lambda_{i+1}(\eta_{S'}))}) \leqslant \dim(\pi_i(W_i)),$$

because c_2 is quasi-finite. Thus

$$\dim(\overline{c_2(\lambda_{i+1}(\eta_S))} \cap \overline{c_1(\lambda_{i+1}(\eta_{S'}))} < \dim(\overline{c_2(\lambda_{i+1}(\eta_S))}) = \dim(\pi_i(W_i)) \tag{1}$$

for otherwise $\overline{c_2(\lambda_{i+1}(\eta_S))} = \overline{c_1(\lambda_{i+1}(\eta_{S'}))}$, which would imply that $c_1(\lambda_{i+1}(\eta_{S'})) \in \pi_i(W_i) \subseteq Z$, a contradiction.

Now notice that we a set-theoretic identification

$$W_{i+1} = \bigcup_{\eta_{T_1},\eta_{T_2} \text{ critical generic points of } c_{2,i+1}^{-1}(Z_{i+1})\backslash c_{1,i+1}^{-1}(Z_{i+1})} \overline{c_{1,i+1}(T_1)} \cap \overline{c_{2,i+1}(T_2)}$$

and by (1), the closed set

$$\pi_{i+1}(\overline{c_{1,i+1}(\eta_{T_1})} \cap \overline{c_{2,i+1}(\eta_{T_2})}) \subseteq \overline{c_1(\lambda_{i+1}(\eta_{T_1}))} \cap \overline{c_2(\lambda_{i+1}(\eta_{T_2}))}$$

is of dimension $< \dim(\pi_i(W_i))$ if $\dim(\overline{c_2(\lambda_{i+1}(\eta_{T_2}))}) = \dim(\pi_i(W_i))$. On the other hand if

$$\dim(\overline{c_2(\lambda_{i+1}(\eta_{T_2}))}) < \dim(\pi_i(W_i))$$

then the closed set $\pi_{i+1}(\overline{c_{1,i+1}(\eta_{T_1})} \cap \overline{c_{2,i+1}(\eta_{T_2})})$ is also of dimension $< \dim(\pi_i(W_i))$. This proves that $\dim(\pi_{i+1}(W_{i+1})) < \dim(\pi_i(W_i))$ and proves Claim 2.

Proof of Lemma 0.1. Claim 2 contradicts Claim 1 so the sequence must stop. □

Remark. To see the construction of the lemma at work in a simple example, suppose that X is irreducible of dimension 2 and that Z is of dimension 1. In that case W_0 must be a finite set of closed points. Let η_T be a critical generic point of $c_{2,1}^{-1}(Z_1)\backslash c_{1,1}^{-1}(Z_1)$. Then by fact (*), $c_{2,1}(\eta_T) \in E_1$ and thus $c_2(\lambda_1(\eta_T)) \in W_0$ is a closed point and thus by quasi-finiteness, $\lambda_1(\eta_T)$ is a closed point. Thus $c_1(\lambda_1(\eta_T))$ is a closed point, which by construction does not lie on W_0. Thus η_T cannot be a critical point, a contradiction. So in this case, one blow up suffices.

Corollary 0.2. — *Same assumptions as in the lemma. Identify U and $\pi^{-1}(U)$. Suppose furthermore that $c_1^{-1}(U) \to U$ is proper, so that $\widetilde{c}_1^{-1}(U) \to U$ is also proper. Then there exists*
– an open k-immersion $\widetilde{X} \hookrightarrow \overline{\widetilde{X}}$, where $\overline{\widetilde{X}}$ is a scheme, which is proper over k;
– a morphism $\overline{\widetilde{C}} \to \overline{\widetilde{X}} \times_k \overline{\widetilde{X}}$ extending $\widetilde{C} \to \widetilde{X} \times_k \widetilde{X}$.
such that $\overline{\widetilde{X}}\backslash U$ is locally $\overline{\widetilde{C}}$-invariant.

Proof. This is a consequence of [V], Lemma 1.5.4. □

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