Rational points of varieties with ample cotangent bundle over function fields

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Abstract

Let K be the function field of a smooth curve over an algebraically closed field k. Let X be a scheme, which is smooth and projective over K. Suppose that the cotangent bundle $\Omega_{X/K}$ is ample. Let $R := \mathrm{Zar}(X(K) \cap X)$ be the Zariski closure of the set of all K-rational points of X, endowed with its reduced induced structure. We prove that for each irreducible component R_0 of R, there is a projective variety X_0 over k and a finite and surjective K^{sep}-morphism $X_{0,K^{\mathrm{sep}}} \to R_{0,K^{\mathrm{sep}}}$, which is birational when $\mathrm{char}(K) = 0$.

Using our result, one can give the first examples of varieties, which are not embeddable in abelian varieties and satisfy a positive characteristic analog of the Bombieri-Lang conjecture.

1 Introduction

Recall that the Bombieri-Lang conjecture (see [17, middle of p. 108]) asserts that the set of rational points of a variety of general type over a number field is not dense. This conjecture can be proven in the situation where the variety is

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embeddable in an abelian variety (this is deep result of Faltings, see [5]) but it is not known to be true in any other situation, as far as the authors know.

Over function fields, it seems reasonable to make the following conjecture, which must have been part of the folklore for some time.

Conjecture 1.1 (”Bombieri-Lang” conjecture over function fields)**.** *Let K_0 be the function field of a smooth variety over an algebraically closed field k_0. Let Z be a variety of general type over K_0. Suppose that $\mathrm{Zar}(Z(K_0)) = Z$. Then there exists a variety Z_0 over k_0 and a rational, dominant, generically finite K_0^{sep}-morphism $g : Z_{0,K_0^{\mathrm{sep}}} \to Z_{K_0^{\mathrm{sep}}}$.*

One might speculate that the rational map g appearing in Conjecture 1.1 can also be taken to be generically purely inseparable.

In his article [10, p. 781], Lang gave a loose formulation of Conjecture 1.1 for $\mathrm{char}(k_0) = 0$. If Z is embeddable in an abelian variety and $\mathrm{char}(k_0) = 0$, Conjecture 1.1 can be proven (see [3] and [6]). It can also be proven in the situation where $\mathrm{char}(k_0) = 0$, the variety is smooth and its cotangent bundle is ample. This is a result of Noguchi, which was also proved independently by Martin-Deschamps (see [17] and [14]). When Z is embeddable in an abelian variety and $\mathrm{char}(k_0) > 0$, Conjecture 1.1 is a theorem of Hrushovski (see [9]). See also [18], [22] and [1] for different proofs of Hrushovski's theorem. When Z is of dimension 1 and $\mathrm{char}(k_0) = 0$, Conjecture 1.1 was first proven by Manin and Grauert and several other proofs were given in the course of the 1970s (eg by Parshin and Arakelov). See the articles [7], [13], [12] and [4, chap. I]. When Z is of dimension 1 and $\mathrm{char}(k_0) > 0$, Conjecture 1.1 was first proven by Samuel and other proofs were given later by Szpiro and Voloch. See the articles [19], [20] and [21].

In the following paper, we shall prove Conjecture 1.1 in the situation where K_0 has transcendence degree 1 over its prime field and Z is a subvariety of a larger variety Z', where Z' is smooth and has ample cotangent bundle over over K_0. See Theorem 1.2 below. Theorem 1.2 can be used to give the first examples (to the authors knowledge) of varieties of general type in positive characteristic that satisfy Conjecture 1.1 and are not embeddable in abelian varieties. Theorem 1.2 also provides a strengthening of Noguchi's result in the situation where K_0 has transcendence degree 1 over $\mathbb{Q}$ (because unlike Noguchi we do not assume that $Z = Z'$).

Here is a precise formulation of our result.

Let K be the function field of a smooth curve U over an algebraically closed field k. Let X be a scheme, which is smooth and projective over K.

We prove:

Theorem 1.2. *Suppose that the cotangent bundle* $\Omega_X := \Omega_{X/K}$ *is ample. Let* $R := \mathrm{Zar}(X(K) \cap X)$ *be the Zariski closure of the set of all* K*-rational points of* X*, endowed with its reduced induced structure. For each irreducible component* R_0 *of* R*, there is a projective variety* X_0 *over* k *and a finite and surjective* K^{sep}*-morphism* $h : X_{0,K^{\mathrm{sep}}} \to R_{0,K^{\mathrm{sep}}}$*. If* $\mathrm{char}(k) = 0$*, there exists a morphism* h *as above, which is birational.*

Remark. In Theorem 1.2, we assume that K is the function field of a curve (see also the discussion above). This restriction, which can probably be removed at the price of added technicality, comes from the fact that we need to consider a smooth compactification of U in the proof and also from the fact that we need to consider a Néron desingularisation of a scheme over U.

We now describe the strategy of the proof of Theorem 1.2, which can be viewed as a refinement of the method of Grauert (see [11, chap. VI] for a nice overview of this method), in which all the higher jet schemes are brought into the picture (unlike Grauert, who considers only the first jet scheme).

Here is a how Grauert and his followers proceed in characteristic 0, in the situation where the rational points are dense in the whole variety. One first shows that the rational points of X lift to rational points of the first jet scheme. Next, one shows that the "non-constant" rational points concentrate on a closed subscheme Σ (say), which is finite and generically inseparable over X. To establish this last fact, one needs to consider a non-singular compactification of X/K over a compactification of U and use the height machine over function fields. The proof can now be completed quickly because generically inseparable morphisms are birational in characteristic 0 and thus the projection of Σ onto X is an isomorphism. In view of the definition of the first jet scheme, this means that the Kodaira-Spencer class of X vanishes. Using the exponential map, one can conclude from this that X descends to k (up to a separable extension of K).

If one tries to carry through the above proof in positive characteristic, the first problem that one faces is that one cannot easily construct a non-singular compactification of X, unless one assumes the existence of resolutions of singularities in positive characteristic.

Next, even if one supposes that this first problem can be solved, one is faced with the basic problem that the projection $\Sigma \to X$ might not be an isomorphism.

Finally, even if the projection $\Sigma \to X$ can be shown to be an isomorphism, ie even if the Kodaira-Spencer class of X vanishes, one cannot conclude that X descends to k. For example, any smooth proper curve over K, which descends to K^p, has a vanishing Kodaira-Spencer class.

Here is how we deal with these issues. For the first problem, we replace the non-singular compactification by a Néron desingularisation, which is not compact, but suffices for our purposes. The second and third issues are dealt with simultaneously. We show that after a finite purely inseparable base-change the entire tower of jet schemes becomes trivial, in the following sense: the base-change of the first jet scheme has a section, the base-change of the second jet scheme has a section over the image of the first section, the base-change of the third jet scheme has a section over the image of the second section and so on. We show that the trivialisation of the tower of jet schemes is a consequence of a generalisation of a cohomological result of Szpiro and Lewin-Ménégaux (see before Proposition 2.3 below). Our proof of this generalisation is not based on the same principle as the result of Szpiro and Lewin-Ménégaux. Specializing all this to a closed point u_0 of U, we obtain a morphism of formal schemes between a constant formal scheme and the completion at u_0 of (a suitable model of) X. Applying Grothendieck's formal GAGA theorem and using the fact that the completion of U at u_0 is an excellent discrete valuation ring, we can construct the required morphism $X_{0,K^{\rm sep}} \to X_{K^{\rm sep}}$.

The method that we just outlined also allows us to treat the situation where the rational points are not Zariski dense. This was (apparently) not accessible before even in characteristic 0.

Here is the structure of the text.

In subsection 2.1, we recall various facts about the geometry of torsors under vector bundles (in particular, ample vector bundles). In subsection 2.2, we prove an injectivity criterion for purely inseparable pull-back maps between first cohomology groups of vector bundles (Corollary 2.2) and we prove a basic vanishing result (Proposition 2.3) for the group of global sections of a coherent sheaf, which is twisted by a sufficiently high power of Frobenius pull-backs of an ample bundle. In section 3, we prove Theorem 1.2. As explained above, our proof does not use the exponential map but uses formal schemes directly and thus differs in nature from the proofs of Noguchi and Martin-Deschamps (see op. cit.) even when $\mathrm{char}(k) = 0$.

The reader is advised to first read the proof with the supplementary assumption that U is proper over k and that X extends to a smooth and projective scheme over U. Many technicalities of the proof disappear when that (unrealistic...) supplementary assumption is made.

Notations. If S is a scheme of positive characteristic, we write F_S for the absolute Frobenius endomorphism of S. The acronym wrog stands for "without restriction of generality". If Y is an integral scheme, we write $\kappa(Y)$ for the function field of Y.

2 Preliminaries

2.1 The geometry of the compactifications of torsors under vector bundles

In this subsection, we recall various results proven in [14].

Let S be a scheme, which is of finite type over a field k_0.

If V a locally free sheaf over S, we shall write $\mathbb{P}(V)$ for the S-scheme representing the functor on S-schemes

$T \mapsto \{$iso. classes of surjective morphisms of $\mathcal{O}_T$-modules $V_T \to Q$, where Q is locally free of rank 1$\}$.

By construction, $\mathbb{P}(V)$ comes with a universal line bundle $\mathcal{O}_P(1)$. Let now

$$\mathcal{E} : 0 \to \mathcal{O}_S \to E \to F \to 0 \tag{1}$$

be an exact sequence of locally free sheaves over S. Consider the S-group scheme $\underline{\mathrm{F}} := \mathrm{Spec}(\mathrm{Sym}(F))$ representing the group functor on S-schemes sending T to $F^\vee_T(T)$. Let $R_\mathcal{E}$ be the functor from S-schemes to sets given by

$$T \mapsto \{\text{morphisms of } \mathcal{O}_T\text{-modules } E_T \mapsto \mathcal{O}_T \text{ splitting } \mathcal{E}_T\}. \tag{2}$$

There is an obvious (group functor-)action of $\underline{\mathrm{F}}$ on $R_\mathcal{E}$.

(1) The natural morphism $\mathbb{P}(F) \to \mathbb{P}(E)$ is a closed immersion and there is an isomorphism of line bundles $\mathcal{O}(\mathbb{P}(F)) \simeq \mathcal{O}_P(1)$.

(2) The complement $\mathbb{P}(E)\backslash\mathbb{P}(F)$ represents the functor $R_{\mathcal{E}}$. The isomorphism of functors on S-schemes $R_{\mathcal{E}} \to \mathbb{P}(E)\backslash\mathbb{P}(F)$ can be described as follows. There is a natural transformation of functors $R_{\mathcal{E}} \to \mathbb{P}(E)$ sending a morphism of $\mathcal{O}_T$-modules $E_T \mapsto \mathcal{O}_T$ splitting $\mathcal{E}_T$ to the same morphism $E_T \to \mathcal{O}_T$, viewed as a morphism from E_T onto a locally free sheaf of rank 1 (the latter being the trivial sheaf). This gives a morphism of schemes $R_{\mathcal{E}} \to \mathbb{P}(E)$, which is an open immersion onto $\mathbb{P}(E)\backslash\mathbb{P}(F)$.

Thus

(3) the scheme $R_{\mathcal{E}}$ with its $\underline{\mathrm{F}}$-action is an S-torsor under $\underline{\mathrm{F}}$.

Further, by (1):

(4) if E is ample then the scheme $\mathbb{P}(E)\backslash\mathbb{P}(F)$ is affine

(point (4) will actually not be used in the text).

Let us now suppose until the end of this section that F is ample.

(5) if $Z \hookrightarrow R_{\mathcal{E}}$ is a subscheme, which is closed in $\mathbb{P}(E)$, then the induced map $Z \to S$ is finite and has only a finite number of fibres that contain more than one point; in particular, if S is irreducible, then Z is irreducible and the morphism $Z \to S$ is generically radicial ;

(6) for all sufficiently large $n \in \mathbb{N}$, the line bundle $\mathcal{O}_P(n)$ is generated by its global sections and in this case the induced k_0-morphism

$$\phi_n : \mathbb{P}(E) \to \mathbb{P}(\Gamma(\mathcal{O}_P(n)))$$

is generically finite;

(7) for $n \in \mathbb{N}$ as in (6), the positive-dimensional fibres of the morphism ϕ_n are disjoint from $\mathbb{P}(F)$.

From the fact that fibre dimension is upper semi-continuous (see [8, IV, 13.1.5]) and (7) we deduce that

(8) the union I_{ϕ_n} of the positive dimensional fibres of ϕ_n is closed in $\mathbb{P}(E)$ and is contained in $R_{\mathcal{E}}$.

We endow I_{ϕ_n} with its reduced-induced structure. From (5) we deduce that

(9) the morphism $I_{\phi_n} \to S$ is finite and has only a finite number of fibres that contain more than one point; in particular, if S is irreducible, then I_{ϕ_n} is irreducible and the morphism $I_{\phi_n} \to S$ is generically radicial.

We shall also need the

(10) Every torsor under $\underline{\mathrm{F}}$ is isomorphic to a torsor $R_{\mathcal{E}}$ for some exact sequence $\mathcal{E}$ as in (1). The class in $H^1(S, F^\vee) \simeq \mathrm{Ext}^1(\mathcal{O}_S, F^\vee)$ corresponding to $R_{\mathcal{E}}$ is the image of $1 \in H^0(S, \mathcal{O}_S)$ in $H^1(S, F^\vee)$ under the connecting map in the long exact sequence

$$0 \to H^0(S, F^\vee) \to H^0(S, E^\vee) \to H^0(S, \mathcal{O}_S) \to H^1(S, F^\vee) \to \dots$$

associated with the dual exact sequence $\mathcal{E}^\vee$.

2.2 Torsors under vector bundles and purely inseparable base-change

If W is a quasi-coherent $\mathcal{O}_Y$-module on a integral scheme Y, we shall write

$$\Gamma(Y, W)_g := \{e \in W_{\kappa(Y)} \,|\, \exists\, \sigma \in \Gamma(Y, W) : \sigma_{\kappa(Y)} = e\}.$$

Lemma 2.1. *Let Y be a normal and integral scheme. Let W be a vector bundle over Y. Let $T \to Y$ be a torsor under W and let $Z \hookrightarrow T$ be a closed immersion, where Z is an integral scheme. Suppose that the induced morphism $f : Z \to Y$ is quasi-finite, separated, radicial and dominant. Suppose that $\Gamma(Z, \Omega_f \otimes f^*W)_g = 0$. Then $f|_Z$ is an open immersion.*

Proof. Let $\pi : T \times_Y T \to Y$. We consider the scheme $T \times_Y (T \times_Y T)$. Via the projection on the second factor $T \times_Y T$, this scheme is naturally a torsor under the vector bundle $\pi^* W$. This torsor has two sections:

- the section σ_1 defined by the formula $t_1 \times t_2 \mapsto t_1 \times (t_1 \times t_2)$;

- the section σ_2 defined by the formula $t_1 \times t_2 \mapsto t_2 \times (t_1 \times t_2)$.

Since $T \times_Y (T \times_Y T)$ is a torsor under $\pi^* W$, there is a section $s \in \Gamma(T \times_Y T, \pi^* W)$ such that $\sigma_1 + s = \sigma_2$ and by construction $s(t_1 \times t_2) = 0$ iff $t_1 = t_2$. In other words, s vanishes precisely on the diagonal of $T \times_Y T$.

Now let $\Delta_Z : Z \hookrightarrow Z \times_Y Z$ be the diagonal immersion. Let $\Delta_Z^{(1)} : Z^{(1)} \hookrightarrow Z \times_Y Z$ be the first infinitesimal neighborhood of Δ_Z. By the definition of the differentials, we have an exact sequence of sheaves

$$0 \to \Omega_f \to \mathcal{O}_{Z^{(1)}} \to \mathcal{O}_Z \to 0$$

which gives rise to an exact sequence

$$0 \to \Gamma(Z, \pi^* W|_Z \otimes \Omega_f) \to \Gamma(Z^{(1)}, \pi^* W|_{Z^{(2)}}) \to \Gamma(Z, \pi^* W|_Z)$$

Now by construction, the image of the section $s \in \Gamma(T \times_Y T, \pi^* W)$ in $\Gamma(Z, \pi^* W|_Z)$ vanishes. Hence the image of the section $s \in \Gamma(T \times_Y T, \pi^* W)$ in $\Gamma(Z^{(1)}, \pi^* W|_{Z^{(2)}})$ is the image of a section $s_0 \in \Gamma(Z, \pi^* W|_Z \otimes \Omega_f)$. By assumption, we have $s_{0,\kappa(Z)} = 0$ and thus by the construction of s the immersion $Z \hookrightarrow Z^{(1)}$ is generically an isomorphism. Hence $\Omega_{f,\kappa(Y)} = 0$ and f is birational. Zariski's main theorem now implies that f is an open immersion. $\square$

Corollary 2.2. *Let Y be a normal and integral scheme. Let W be a vector bundle over Y. If $\Gamma(Y, \Omega_{F_Y} \otimes F_Y^*(W))_g = 0$ then the natural map of abelian groups $H^1(Y, W) \to H^1(S, F_Y^* W)$ is injective.*

Proof. Consider an element in the kernel of the map $H^1(Y, W) \to H^1(S, F_Y^* W)$. Let $T \to Y$ be a torsor under W corresponding to this element. By assumption, there is a Y-morphism $Y \to F_Y^* T$ and from this we may construct a closed integral subscheme $Z \hookrightarrow T$ together with a factorisation $Y \xrightarrow{\phi} Z \xrightarrow{f} Y$, where the arrow f is the natural projection and $f \circ \phi = F_Y$. Now using [15, Th. 26.5, p. 202], we see that we have an exact sequence

$$0 \to \phi^* \Omega_{f,\kappa(Y)} \to \Omega_{F_Y,\kappa(Y)} \to \Omega_{\phi,\kappa(Y)} \to 0.$$

Now from the existence of this sequence and from the fact that $\Gamma(Y, \Omega_{F_Y} \otimes F_Y^*(W))_g = 0$, we may conclude that $\Gamma(Z, \Omega_f \otimes f^* W)_g = 0$. Thus, by Lemma 2.1, the torsor $T \to Y$ must be trivial. $\square$

Note that if Y is projective over an algebraically closed field, then a weaker form of Corollary 2.2 (which is not sufficient for our purposes) is contained in [20, exp. 2, Prop. 1]. The proof given there depends on the existence of the Cartier isomorphism.

Let now S be an integral scheme, which is projective over a field k_0 of characteristic p.

Proposition 2.3. *Suppose that* $\dim(S) > 0$. *Let* V *be an ample vector bundle of rank* r *over* S. *Let* V_0 *be a coherent sheaf over* S. *Then we have* $H^0(S, F_S^{n,*}(V^\vee) \otimes V_0)_g = 0$ *for all sufficiently large* $n \geqslant 0$.

Proof. We may suppose without restriction of generality that k_0 is algebraically closed.

The proof is by induction on the dimension $d \geqslant 1$ of S.

Suppose that $d > 1$. Consider a pencil of hypersurfaces in S and let $b : \widetilde{S} \to S$ the total space of the pencil, so that we are given a birational morphism $m : \widetilde{S} \to \mathbb{P}^1_{k_0}$. Let $\eta \in \mathbb{P}^1_{k_0}$ be the generic point. Let n_0 be sufficiently large, so that

$$H^0(\widetilde{S}_\eta, F^{n,*}_{\widetilde{S}_\eta}((b^*V^\vee)_\eta) \otimes (b^*V_0)_\eta)_g = 0$$

for all $n \geqslant n_0$. This is possible by the induction hypothesis and because $(b^*V^\vee)_\eta$ is ample. The fact that $(b^*V^\vee)_\eta$ is ample is a consequence of the fact that the restriction of V to any closed fibre of m is ample and of the fact that ampleness on the fibre of m is a constructible property. Now if we had $H^0(S, F_S^{n,*}(V^\vee) \otimes V_0)_g \neq 0$ for some $n \geqslant n_0$ then the pull-back $b^*(F_S^{n,*}(V^\vee) \otimes V_0)$ would have a section that would not vanish at the generic point of $\widetilde{S}_\eta$, which is a contradiction.

Thus we are reduced to prove the statement for $d = 1$. We may replace wrog S by its normalisation. Since S is now a non-singular curve, we know that V is cohomologically p-ample (see [16, Rem. 6), p. 91]). Also, V_0 is now the direct sum of a torsion sheaf and of a locally free sheaf so we may assume wrog that V_0 is locally free. Now using Serre duality, we may compute

$$H^0(S, F_S^{n,*}(V^\vee) \otimes V_0) = H^1(S, F_S^{n,*}(V) \otimes V_0^\vee \otimes \Omega_{S/k_0})^\vee$$

and the vector space $H^1(S, F_S^{n,*}(V) \otimes V_0^\vee \otimes \Omega_{S/k_0})$ vanishes for $n >> 0$ because V is cohomologically p-ample. □

Corollary 2.4. *Suppose that V is an ample bundle on S and that S is normal. Let $n_0 \in \mathbb{N}$ be such that $H^0(S, F_S^{n,*}(V^\vee) \otimes \Omega_{F_S})_g = 0$ for all $n > n_0$. Let S' be an irreducible scheme and let $\phi : S' \to S$ be a finite surjective morphism, which is generically inseparable. Then the map*

$$H^1(S, F_S^{n_0,*}(V^\vee)) \to H^1(S', \phi^*(F_S^{n_0,*}(V^\vee)))$$

is injective.

Proof. (of Corollary 2.4). Let H be the function field of S and let $H'|H$ be the (purely inseparable) function field extension given by ϕ. Let ℓ_0 be sufficiently large so that there is a factorisation $H^{p^{-\ell_0}}|H'|H$. We may suppose that S' is a normal scheme, since we may replace S' by its normalization without restriction of generality. On the other hand the morphism $F_S^{\ell_0} : S \to S$ gives a presentation of S as its own normalization in $H^{p^{-\ell_0}}$. Thus there is a natural factorization $S \to S' \overset{\phi}{\to} S$, where the morphism $S \to S$ is given by $F_S^{\ell_0}$. Using Corollary 2.2, we see that there is a natural injection $H^1(S, F_S^{n_0,*}(V^\vee)) \hookrightarrow H^1(S, F_S^{\ell_0,*}(F_S^{n_0,*}(V^\vee)))$ and thus an injection $H^1(S, F_S^{n_0,*}(V^\vee)) \to H^1(S', \phi^*(F_S^{n_0,*}(V^\vee)))$. $\square$

3 Proof of Theorem 1.2

In this section, we assume that the hypotheses and the notation of Theorem 1.2 are in force.

The jet scheme construction described in [18, sec. 2] provides a covariant functor

$$\mathcal{Y} \mapsto J^i(\mathcal{Y}/U)$$

from the category of quasi-projective schemes $\mathcal{Y}$ over U to the category of schemes over U.

The construction also provides an infinite tower of U-morphisms

$$\cdots \to J^2(\mathcal{Y}/U) \overset{\Lambda_{2,\mathcal{Y}}}{\to} J^1(\mathcal{Y}/U) \overset{\Lambda_{1,\mathcal{Y}}}{\to} \mathcal{X}.$$

If Y is smooth over U, then the scheme $J^i(\mathcal{Y}/U)$, viewed as a $J^{i-1}(\mathcal{Y}/U)$-scheme via $\Lambda_{i,\mathcal{Y}}$, is a torsor under the vector bundle $\mathrm{T}\mathcal{Y} \otimes \mathrm{Sym}^i(\Omega_{U/k})$, where $\mathrm{T}\mathcal{Y} := (\Omega^1_{\mathcal{Y}/U})^\vee$. Here the bundle $\mathrm{T}\mathcal{Y}$ (resp. $\mathrm{Sym}^i(\Omega_{U/k})$) is implicitly pulled back from $\mathcal{Y}$ (resp. U) to the scheme $J^{i-1}(\mathcal{Y}/U)$.

The functor $J^i(\cdot/U)$ preserves closed immersions and smooth morphisms. In particular, for any $i \in \mathbb{N}$, there is a natural map

$$\lambda_{i,\mathcal{Y}} : \mathcal{Y}(U) = \mathrm{Mor}_U(U, \mathcal{Y}) \to \mathrm{Mor}_U(J^i(U/U), J^i(\mathcal{Y}/U)) = \mathrm{Mor}_U(U, J^i(\mathcal{Y}/U)), \tag{3}$$

and these maps are compatible with the morphisms $\Lambda_{i,\mathcal{Y}}$.

We refer to [18, sec. 2] for more details on jet schemes.

We now suppose wrog that X/K extends to a (not necessarily proper) smooth scheme $\pi : \mathcal{X} \to U$. We shall write $J^i(X/K)$ for $J^i(\mathcal{X}/U)_K$.

We shall divide the proof into steps. The main part of the proof will take place over K and will not make use of the model $\mathcal{X}/U$ of X/K.

By a compactification $\bar{T}$ of an S-scheme T, we shall mean a proper scheme $\bar{T} \to S$, which comes with an open immersion $T \hookrightarrow \bar{T}$ with dense image.

Let $\bar{U} \to \operatorname{Spec} k$ be a smooth compactification of U.

Recall that we suppose that the cotangent bundle of X over K is ample.

Step I. Compactifications.

First choose any projective compactification $\bar{\mathcal{X}}_0$ of $\mathcal{X}$ viewed as a scheme over $\bar{U}$. By applying Néron desingularization to $\bar{\mathcal{X}}_0$ (see [2, chap. 3, th. 2])), we obtain another projective compactification $\bar{\mathcal{X}}_{00}$ of $\mathcal{X}$ over $\bar{U}$, with the property that the injection $\bar{\mathcal{X}}_{00}^{\mathrm{sm}}(\bar{U}) \hookrightarrow \bar{\mathcal{X}}_{00}(\bar{U}) = X(K)$ is a bijection. Here $\mathcal{X}_{00}^{\mathrm{sm}} \subset \mathcal{X}_{00}$ is the largest open subset $\mathcal{X}_{00}^{\mathrm{sm}}$ of $\mathcal{X}_{00}$, such that $\mathcal{X}_{00}^{\mathrm{sm}} \to \bar{U}$ is smooth. Since $X(K) \neq \emptyset$, this shows that there exists a model of X over $\bar{U}$ (ie the model $\bar{\mathcal{X}}_{00}^{\mathrm{sm}}$), which is smooth and surjective onto $\bar{U}$.

Thus, we may (and do) suppose that $U = \bar{U}$ and that $\mathcal{X}$ is surjective (and smooth) over U. We let $\bar{\mathcal{X}}$ be any compactification of $\mathcal{X}$ over U.

We now choose specific compactifications $\bar{J}^i(\mathcal{X}/U)$ over U for the jet schemes $J^i(\mathcal{X}/U)$.

For $i = 0$, we let $\bar{J}^0(\mathcal{X}/U) = \bar{\mathcal{X}}$ and we define them inductively for $i \geqslant 0$.

So suppose that the compactification $\bar{J}^i(\mathcal{X}/U)$ has already been constructed. As said above, the $J^i(\mathcal{X}/U)$-scheme $J^{i+1}(\mathcal{X}/U)$ is a torsor under $F_i^\vee$, where $F_i := (\mathrm{T}\mathcal{X} \otimes \mathrm{Sym}^{i+1}(\Omega_{U/k}))^\vee$ (viewed as a vector bundle over $J^i(\mathcal{X}/U)$). We shall denote this torsor by T_i. Let

$$0 \to \mathcal{O}_{J^i(\mathcal{X}/U)} \to E_i \to F_i \to 0 \tag{4}$$

be an extension (unique up to non-unique isomorphism) associated with the class of T_i in $H^1(J^i(\mathcal{X}/U), F_i^\vee)$. It was explained in (2) subsection 2.1 that the $J^i(\mathcal{X}/U)$-scheme $J^{i+1}(\mathcal{X}/U)$ can be realized as the complement $\mathbb{P}(E_i)\backslash\mathbb{P}(F_i)$. We now define the compactification $\bar{J}^{i+1}(\mathcal{X}/U)$ to be some $\bar{J}^i(\mathcal{X}/U)$-compactification of $\mathbb{P}(E_i)$, such that the diagram

$$\begin{array}{ccc} \mathbb{P}(E_i) & \hookrightarrow & \bar{J}^{i+1}(\mathcal{X}/U) \\ \downarrow & & \downarrow \\ J^i(\mathcal{X}/U) & \hookrightarrow & \bar{J}^i(\mathcal{X}/U) \end{array}$$

is cartesian. This is possible because we may extend E_i to a coherent sheaf $\bar{E}_i$ on $\bar{J}^i(\mathcal{X}/U)$ and define $\bar{J}^{i+1}(\mathcal{X}/U) := \mathbb{P}(\bar{E}_i)$.

We call $\bar{\Lambda}_{i+1} : \bar{J}^{i+1}(\mathcal{X}/U) \to \bar{J}^i(\mathcal{X}/U)$ the corresponding morphism.

The following diagram summarizes the resulting geometric configuration:

$$\begin{array}{ccccc} J^{i+1}(\mathcal{X}/U) & \hookrightarrow & \mathbb{P}(E_i) & \hookrightarrow & \bar{J}^{i+1}(\mathcal{X}/U) \\ {\scriptstyle \Lambda_{i+1}}\downarrow & & \downarrow & & \downarrow{\scriptstyle \bar{\Lambda}_{i+1}} \\ J^i(\mathcal{X}/U) & = & J^i(\mathcal{X}/U) & \hookrightarrow & \bar{J}^i(\mathcal{X}/U) \end{array}$$

Here the hooked horizontal arrows are open immersions and the square on the right is cartesian.

Recall the following key properties. The scheme U is proper over k and the schemes $\bar{J}^i(\mathcal{X}/U)$ are proper over U. The morphisms $\bar{J}^{i+1}(\mathcal{X}/U) \to \bar{J}^i(\mathcal{X}/U)$ are proper. The schemes $J^i(\mathcal{X}/U)$ and $\mathbb{P}(E_i)$ are smooth and surjective onto U.

By the valuative criterion of properness, there is a natural map $\mathcal{X}(U) \to \bar{J}^i(\mathcal{X}/U)(U)$ extending the map $\lambda_i : \mathcal{X}(U) \to J^i(\mathcal{X}/U)(U)$. By unicity, this map is none other than the map λ_i composed with the open immersion $J^i(\mathcal{X}/U) \hookrightarrow \bar{J}^i(\mathcal{X}/U)$. Abusing notation, we shall therefore also denote it by λ_i.

Define $\bar{J}^i(X/K) := \bar{J}^i(\mathcal{X}/U)_K \simeq \mathbb{P}(E_i)_K$.

Step II. The schemes $Z_i \hookrightarrow J^i(X/K)$.

We shall inductively construct closed integral subschemes $Z_i \hookrightarrow J^i(X/K)$ with the following properties. They are sent onto each other by the morphisms Λ_i.

The morphisms $Z_{i+1} \to Z_i$ are finite, surjective and generically radicial and Z_i is proper over U (in particular, Z_i is closed in $\bar{J}^i(X/K)$). Furthermore the image of $\mathcal{X}(U) = X(K)$ by $\lambda_{i,K}$ in $J^i(X/K)$ meets Z_i in a dense set.

The schemes Z_i are defined via the following inductive procedure.

Define Z_0 as the reduced closed subscheme of X associated with an arbitrary irreducible component of the closed set $\mathrm{Zar}(X(K))$.

To define Z_{i+1} from Z_i notice that by the Step I, we have an identification $\bar{J}^{i+1}(\mathcal{X}/U)_{Z_i} = \mathbb{P}(E_{i,Z_i})$.

Notice also that F_{i,Z_i} is ample (over K) since $Z_i \to X$ is finite. Thus by (6) in subsection 2.1, we are given a K-morphism

$$\phi_{n_i} : \mathbb{P}(E_{i,Z_i}) \to \mathbb{P}^{n_i}_K,$$

for some $n_i \in \mathbb{N}$ and ϕ_{n_i}. Call $I_{\phi_{n_i}}$ the union of the positive dimensional fibres of ϕ_{n_i}.

Write $H_i \subseteq \mathbb{P}^{n_i}_K$ for a hyperplane such that $\phi_{n_i}^{-1}(H_i) = \mathbb{P}(F_{i,Z_i})$. This exists by (1) in subsection 2.1.

Let $\bar{\mathbb{P}}(E_{i,Z_i})$ be the Zariski closure of $\mathbb{P}(E_{i,Z_i})$ in $\bar{J}^{i+1}(\mathcal{X}/U)$ and let $\bar{\mathbb{P}}(F_{i,Z_i})$ be the Zariski closure of $\mathbb{P}(F_{i,Z_i})$ in $\bar{J}^{i+1}(\mathcal{X}/U)$.

Now call $\Sigma_i \subseteq \mathcal{X}(U) = X(K)$ the set of sections $\sigma \in \mathcal{X}(U)$ such that $\lambda_{i,K}(\sigma) \in Z_i(K)$.

Lemma 3.1. *Let $\sigma \in \Sigma_i$. We have*

$$\lambda_{i+1}(\sigma) \in \bar{\mathbb{P}}(E_{i,Z_i})(U)$$

and

$$\lambda_{i+1}(\sigma)(U) \cap \bar{\mathbb{P}}(F_{i,Z_i}) = \emptyset. \tag{5}$$

Proof. The first equation follows from the definitions. The second equation follows from the fact that $\lambda_{i+1}(\sigma)(U) \subseteq J^{i+1}(\mathcal{X}/U) = \mathbb{P}(E_i) \backslash \mathbb{P}(F_i)$ and from the fact that we have a set-theoretic identity $\bar{\mathbb{P}}(F_{i,Z_i}) \cap \mathbb{P}(E_i) \subseteq \mathbb{P}(F_i)$, since $\mathbb{P}(F_i)$ is a closed subset of $\mathbb{P}(E_i)$. $\square$

Now choose a proper birational U-morphism $b_i : \bar{\mathbb{P}}'(E_{i,Z_i}) \to \bar{\mathbb{P}}(E_{i,Z_i})$, which is an isomorphism over K and such that there exists a proper U-morphism

$$\widetilde{\phi}_i : \bar{\mathbb{P}}'(E_{i,Z_i}) \to \mathbb{P}^{n_i}_U,$$

with the property that $\phi_{n_i} \circ b_{i,K} = \widetilde{\phi}_{i,K}$. Furthermore, we suppose that $\bar{\mathbb{P}}'(E_{i,Z_i})$ is integral and normal.

Let $\bar{\mathbb{P}}'(F_{i,Z_i})$ be the Zariski closure of $b_{i,K}^{-1}(\mathbb{P}(F_{i,Z_i}))$ in $\bar{\mathbb{P}}'(E_{i,Z_i})$. Let $\bar{H}_i \subseteq \mathbb{P}_U^{n_i}$ be the Zariski closure of H_i.

By the valuative criterion of properness, there is a natural map

$$\bar{\lambda}'_{i+1} : \mathcal{X}(U) \to \bar{\mathbb{P}}'(E_{i,Z_i})(U),$$

such that $b_i \circ \bar{\lambda}'_{i+1} = \lambda_{i+1}$.

Now let $\bar{\mathbb{P}}''(E_{i,Z_i})$ be the normalisation of $\widetilde{\phi}_i(\bar{\mathbb{P}}'(E_{i,Z_i}))$ in $\bar{\mathbb{P}}'(E_{i,Z_i})$. By construction, we have a factorisation

$$\bar{\mathbb{P}}'(E_{i,Z_i}) \xrightarrow{\rho_i} \bar{\mathbb{P}}''(E_{i,Z_i}) \xrightarrow{\bar{\nu}_i} \mathbb{P}_U^{n_i}$$

of $\widetilde{\phi}_i$ and the morphism ρ_i is birational.

Lemma 3.2. *There exists a constant $\beta_{i+1} \geqslant 0$, which is independent of σ, such that for all $\sigma \in \Sigma_i$, we have*

$$\text{length}(\rho_i(\bar{\lambda}'_{i+1}(\sigma))(U) \cap \bar{\nu}_i^*(H_i)) \leqslant \beta_{i+1}. \tag{6}$$

Here $\cap$ refers to the scheme-theoretic intersection.

Proof. Notice that $\bar{\nu}_i^*(\bar{H}_i)$ has a finite number of irreducible components. Among those, the only horizontal (over U) irreducible component is $\rho_i(\bar{\mathbb{P}}'(F_{i,Z_i}))$. Furthermore, we have $\bar{\lambda}'_{i+1}(\sigma) \cap \bar{\mathbb{P}}'(F_{i,Z_i}) = \emptyset$ (by equation (5)). Thus $\rho_i(\bar{\lambda}'_{i+1}(\sigma))$ meets only the horizontal irreducible components of $\bar{\nu}_i^*(\bar{H}_i)$. Finally, the intersection multiplicity of $\bar{\lambda}'_{i+1}(\sigma)$ with a fixed horizontal component of $\bar{\nu}_i^*(\bar{H}_i)$ can be bounded independently of σ. $\square$

Now we have

Lemma 3.3. *There exist a scheme M_i, which is quasi-projective over k and a U-morphism $\mu_i : M_i \times_k U \to \bar{\mathbb{P}}''(E_{i,Z_i})$, with the following properties:*

- *for all $P \in M_i(k)$, we have*

$$\deg(\mu_i(P \times \text{Id}_U)^*(\bar{\nu}_i^*(\mathcal{O}(H_i))) \leqslant \beta_{i+1};$$

- *for any U-morphism $\kappa : U \to \bar{\mathbb{P}}''(E_{i,Z_i})$ such that*

$$\deg(\mu_i(P \times \mathrm{Id}_U)^*(\bar{\nu}_i^*(\mathcal{O}(H_i))) \leqslant \beta_{i+1}$$

there is a $P \in M_i(k)$ such that $\kappa = \mu_i(P \times \mathrm{Id}_U)$.

Proof. Notice that the morphism $\bar{\nu}_i$ is finite and thus the divisor $\bar{\nu}_i^*(H_i)$ is ample. The existence of M_i is now a consequence of the theory of Hilbert schemes. □

Corollary 3.4. *For almost all the sections $\sigma \in \Sigma_i$, we have $\lambda_{i+1,K}(\sigma)(\mathrm{Spec}\, K) \in I_{\phi_{n_i}}$.*

Proof. Let M_i be as in Lemma 3.3. In view of (6) and the assumptions, we know that M_i has positive dimension. Let $C \hookrightarrow M_i$ be a smooth curve inside M_i such that $C(k)$ contains infinitely many k-points corresponding to elements of Σ_i. We have by construction a K-morphism $C_K \to \bar{\mathbb{P}}''(E_{i,Z_i})_K$. Let $U_i'' \subseteq \mathbb{P}''(E_{i,Z_i})_K$ be the largest open set such that $\rho_i|_{U_i''}$ is finite (and thus an isomorphism, by Zariski's main theorem). To obtain a contradiction to the conclusion of the Corollary, suppose that the image of C_K intersects U_i''. Restricting the size of C, we may then suppose that the image of $C_K \to \bar{\mathbb{P}}''(E_{i,Z_i})_K$ lies inside U_i'' and thus we obtain a morphism $l_C : C_K \to \mathbb{P}(E_{i,Z_i}) \subseteq J^{i+1}(X/K)$.

Now consider that we also have by functoriality a morphism

$$J^{i+1}(C_K/K) \to J^{i+1}(X/K)$$

arising from the morphism $C_K \to X$ coming from the composition of l_C with the projection $\mathbb{P}(E_{i,Z_i}) \to X$. The morphism $J^{i+1}(C_K/K) \to J^{i+1}(X/K)$ can be composed with the natural section $C_K \to J^{i+1}(C_K/K)$ to obtain a second K-morphism

$$l'_C : C_K \to J^{i+1}(X/K).$$

By construction, the morphisms l_C and l'_C coincide on a dense set of K-points and thus $l_C = l'_C$.

Now consider a smooth compactification $\bar{C}$ of C over k. By the valuative criterion of properness, there is a unique K-morphism $\bar{C}_K \to X$ extending the morphism $C_K \to X$ described in the last paragraph. Following the steps of the construction given above, we again obtain a K-morphism

$$l'_{\bar{C}} : \bar{C}_K \to J^{i+1}(X/K),$$

which extends l'_C. On the other hand, $l'_{\bar{C}}(\bar{C}_K) \subseteq \mathbb{P}(E_{i,Z_i})\backslash\mathbb{P}(F_{i,Z_i})$ by construction and the image $\phi_{n_i}(l'_{\bar{C}}(\bar{C}_K))$ is closed in $\mathbb{P}^{n_i}_K$. Furthermore, by the definition of H_i,

$$\phi_{n_i}(l'_{\bar{C}}(\bar{C}_K)) \cap H_i = \emptyset.$$

Thus $\phi_{n_i}(l'_{\bar{C}}(\bar{C}_K))$ is the underling set of an integral scheme, which is affine and proper over K and thus consists of a point. We conclude that $l'_C(C_K) \subseteq I_{\phi_{n_i}}$, which contradicts the assumption that $l'_C(C_K)$ meets the (open) locus of the finite fibres of ϕ_{n_i}. $\square$

Since $\lambda_{i,K}(\Sigma_i)$ is dense in Z_i by assumption, we deduce from (9) in subsection 2.1 and from Corollary 3.4 that the base-change $I_{\phi_{n_i},Z_i}$ has a single irreducible component of positive dimension, which is finite and generically radicial over Z_i. Furthermore, from (7) in subsection 2.1, we know that $I_{\phi_{n_i}}$ is proper over Spec K.

We have shown that $Z_{i+1} := I^{\mathrm{red}}_{\phi_{n_i},Z_i}$ has all the required properties.

Step III. Purely inseparable trivialization.

Let $\widetilde{Z}_0 \to Z_0$ be the normalisation of Z_0. For every $i \geqslant 0$, write $\widetilde{Z}_i \to \widetilde{Z}_0$ for the $\widetilde{Z}_0$-scheme obtained by base-change. Similarly, write $\widetilde{J}^i(X/K)$ for the pull-back of $J^i(X/K)$ to $\widetilde{Z}_0$. We denote by $\widetilde{F}_0$ the pull-back to $\widetilde{Z}_0$ of the vector bundle $F_0 \simeq \mathrm{T}X \otimes \Omega_{K/k}$.

Suppose that $\mathrm{char}(k) = 0$.

The scheme $\widetilde{J}^1(X/K)$ is by construction a torsor under $\widetilde{F}_0$. Furthermore, the morphism $\widetilde{Z}_1^{\mathrm{red}} \to \widetilde{Z}_0$ is an isomorphism by Zariski's main theorem and it trivializes the torsor $\widetilde{J}^1(X/K)$ by construction.

Repeating this reasoning for $\widetilde{Z}_2^{\mathrm{red}}$ over $\widetilde{Z}_1^{\mathrm{red}} \simeq \widetilde{Z}_0$, $\widetilde{Z}_3^{\mathrm{red}}$ over $\widetilde{Z}_2^{\mathrm{red}}$ and so forth, we see that the natural morphisms $\widetilde{Z}_{i+1}^{\mathrm{red}} \to \widetilde{Z}_i^{\mathrm{red}}$ are isomorphisms for all $i \geqslant 0$.

Now suppose until the end of Step III that $\mathrm{char}(k) > 0$.

Write $F_X^{m_0,*}\widetilde{Z}_i$ for the base-change of $\widetilde{Z}_i \to \widetilde{Z}_0$ by $F^{m_0}_{\widetilde{Z}_0}$. Let

$$\pi_{i,m_0} : (F^{m_0,*}_{\widetilde{Z}_0}\widetilde{Z}_i)^{\mathrm{red}} \to \widetilde{Z}_0$$

be the natural morphism.

Similarly, write $F^{m_0,*}_{\widetilde{Z}_0}\widetilde{J}^i(X/k)$ for the base-change of $\widetilde{J}^i(X/k)$ by $F^{m_0}_{\widetilde{Z}_0}$.

We now choose m_0 large enough so that

$$H^0(\widetilde{Z}_0, F^{m,*}_{\widetilde{Z}_0}(\widetilde{F}_0) \otimes \Omega_{F_{\widetilde{Z}_0}})_g = 0$$

for all $m \geqslant m_0$. The existence of m_0 is predicted by Proposition 2.3.

The scheme $F_{\widetilde{Z}_0}^{m_0,*}\widetilde{J}^1(X/k)$ is by construction a torsor under $F_{\widetilde{Z}_0}^{m_0,*}(\widetilde{F}_0)$. Furthermore, the morphism π_{1,m_0} trivializes this torsor by construction, since there is a $\widetilde{Z}_0$-morphism $(F_{\widetilde{Z}_0}^{m_0,*}\widetilde{Z}_1)^{\mathrm{red}} \to F_{\widetilde{Z}_0}^{m_0,*}\widetilde{J}^1(X/k)$. By the assumption on m_0 and because π_{1,m_0} is finite and generically radicial, the $F_{\widetilde{Z}_0}^{m_0,*}(\widetilde{F}_0)$-torsor $F_{\widetilde{Z}_0}^{m_0,*}\widetilde{J}^1(X/k)$ is actually trivial (use Corollary 2.4). Let

$$t : \widetilde{Z}_0 \to F_{\widetilde{Z}_0}^{m_0,*}\widetilde{J}^1(X/K)$$

be a section. The scheme $t_*(\widetilde{Z}_0)$ is proper over K and thus property (5) in subsection 2.1 implies that $t_*(\widetilde{Z}_0) = (F_{\widetilde{Z}_0}^{m_0,*}\widetilde{Z}_1)^{\mathrm{red}}$ and thus the morphism $(F_{\widetilde{Z}_0}^{m_0,*}\widetilde{Z}_1)^{\mathrm{red}} \to \widetilde{Z}_0$ is an isomorphism.

Repeating this reasoning for $(F_{\widetilde{Z}_0}^{m_0,*}\widetilde{Z}_2)^{\mathrm{red}}$ over $(F_{\widetilde{Z}_0}^{m_0,*}\widetilde{Z}_1)^{\mathrm{red}} \simeq \widetilde{Z}_0$, $(F_{\widetilde{Z}_0}^{m_0,*}\widetilde{Z}_3)^{\mathrm{red}}$ over $(F_{\widetilde{Z}_0}^{m_0,*}\widetilde{Z}_2)^{\mathrm{red}}$ and so forth, we see that the natural morphisms $(F_{\widetilde{Z}_0}^{m_0,*}\widetilde{Z}_{i+1})^{\mathrm{red}} \to (F_{\widetilde{Z}_0}^{m_0,*}\widetilde{Z}_i)^{\mathrm{red}}$ are isomorphisms from all $i \geqslant 0$.

Step IV. Formalization and utilization of Grothendieck's GAGA.

In this final step, the argument will be written up under the assumption that $\mathrm{char}(k) > 0$. The argument goes through verbatim when $\mathrm{char}(k) = 0$, if one sets $m_0 = 0$ in the text below.

We now choose an affine open subset $U_0 \subseteq U$ and a normal integral scheme $\widetilde{\mathcal{Z}}_0$, which is of finite type over U_0 and comes with a finite U_0-morphism $\widetilde{\mathcal{Z}}_0 \to \mathcal{X}_{U_0}$ extending the morphism $\widetilde{Z}_0 \to X$. We let $\mathcal{Z}_0$ be the (reduced) Zariski closure of Z_0. Furthermore, we assume wrog that the natural map

$$H^1(\widetilde{\mathcal{Z}}_0, F_{\widetilde{\mathcal{Z}}_0}^{m_0,*}(\mathrm{T}\mathcal{X}_{U_0}) \otimes \Omega_{U_0/k}) \to H^1(\widetilde{Z}_0, F_{\widetilde{Z}_0}^{m_0,*}(\mathrm{T}X) \otimes \Omega_{K/k})$$

is injective. Here as before, $\mathrm{T}\mathcal{X}_{U_0}$ and $\Omega_{U_0/k}$ are identified with their pull-backs to $\widetilde{\mathcal{Z}}_0$.

We write $\widetilde{J}^i(\mathcal{X}_{U_0}/U_0)$ for the pull-back of $J^i(\mathcal{X}_{U_0}/U_0)$ to $\widetilde{\mathcal{Z}}_0$ and $F_{\widetilde{\mathcal{Z}}_0}^{m_0,*}\widetilde{J}^i(\mathcal{X}_{U_0}/U_0)$ for the base-change of $\widetilde{J}^i(\mathcal{X}_{U_0}/U_0)$ by $F_{\widetilde{\mathcal{Z}}_0}^{m_0}$.

By construction the $F_{\widetilde{\mathcal{Z}}_0}^{m_0,*}(\mathrm{T}\mathcal{X}_{U_0})\otimes\Omega_{U_0/k}$-torsor $F_{\widetilde{\mathcal{Z}}_0}^{m_0,*}\widetilde{J}^1(\mathcal{X}_{U_0}/U_0)$ is trivial. If we choose a section $\widetilde{\mathcal{Z}}_0 \hookrightarrow F_{\widetilde{\mathcal{Z}}_0}^{m_0,*}\widetilde{J}^1(\mathcal{X}_{U_0}/U_0)$ then the pull-back by this trivialisation of the $F_{\widetilde{\mathcal{Z}}_0}^{m_0,*}(\mathrm{T}\mathcal{X}_{U_0})\otimes\Omega_{U_0/k}$-torsor $F_{\widetilde{\mathcal{Z}}_0}^{m_0,*}\widetilde{J}^2(\mathcal{X}_{U_0}/U_0)$ is also trivial. Continuing in this way, we get a sequence of compatible sections $\widetilde{\mathcal{Z}}_0 \hookrightarrow F_{\widetilde{\mathcal{Z}}_0}^{m_0,*}\widetilde{J}^i(\mathcal{X}_{U_0}/U_0)$, for all $i \geqslant 0$.

Now write $\widetilde{J}^i(\mathcal{Z}_0/U_0)$ for the pull-back of $J^i(\mathcal{Z}_0/U_0)$ to $\widetilde{\mathcal{Z}}_0$ and $F^{m_0,*}_{\widetilde{\mathcal{Z}}_0}\widetilde{J}^i(\mathcal{Z}_0/U_0)$ for the base-change of $\widetilde{J}^i(\mathcal{Z}_0/U_0)$ by $F^{m_0}_{\widetilde{\mathcal{Z}}_0}$. Notice that each of the sections $\widetilde{\mathcal{Z}}_0 \hookrightarrow F^{m_0,*}_{\widetilde{\mathcal{Z}}_0}\widetilde{J}^i(\mathcal{X}_{U_0}/U_0)$ factors through $F^{m_0,*}_{\widetilde{\mathcal{Z}}_0}\widetilde{J}^i(\mathcal{Z}_0/U_0)$. This follows simply from the fact that $\lambda_{i,K}(Z_0(K)) \cap Z_i$ is (by construction) dense in Z_i and from the fact that

$$\lambda_{i,K}(Z_0(K))) := \{J^i(z) \in J^i(Z_0/K)(K) \subseteq J^i(X/K)(K) \,|\, z \in Z_0(K)\}.$$

Now choose a closed point $u_0 \in U_0$. View u_0 as a closed subscheme of U_0. For any $i \geqslant 0$, let u_i be the i-th infinitesimal neighborhood of $u_0 \simeq \operatorname{Spec} k$ in U_0 (so that there is no ambiguity of notation for u_0). Notice that u_i has a natural structure of k-scheme. Suppose wrog that the u_0-scheme $\mathcal{X}_{0,u_0}$ is proper and smooth and that the morphism $\widetilde{\mathcal{Z}}_{0,u_0} \to \mathcal{Z}_{u_0}$ is birational. Recall that by the definition of jet schemes (see [18, sec. 2]), the scheme $J^i(\mathcal{Z}_0/U_0)_{u_0}$ represents the functor on k-schemes

$$T \mapsto \operatorname{Mor}_{u_i}(T \times_k u_i, \mathcal{Z}_{0,u_i}).$$

Thus the infinite chain of trivialisations $\widetilde{\mathcal{Z}}_0 \hookrightarrow F^{m_0,*}_{\widetilde{\mathcal{Z}}_0}\widetilde{J}^i(\mathcal{Z}_0/U_0)$ gives rise to u_m morphisms

$$\widetilde{\mathcal{Z}}^{(p^{-m_0})}_{0,u_0} \times_{k_0} u_i \to \mathcal{Z}_{0,u_i} \tag{7}$$

compatible with each other under base-change. Here $\widetilde{\mathcal{Z}}^{(p^{-m_0})}_{0,u_0}$ denotes the base-change of the k-scheme $\widetilde{\mathcal{Z}}_{0,u_0}$ by the m_0-th power of the inverse of the Frobenius automorphism of k.

Let $\widehat{U}_{u_0}$ be the completion of the local ring of U at u_0. View the $\widehat{U}_{u_0}$-schemes $\widetilde{\mathcal{Z}}^{(p^{-m_0})}_{0,u_0} \times_k \widehat{U}_{u_0}$ and $\mathcal{Z}_{0,\widehat{U}_{u_0}}$ as formal schemes over $\widehat{U}_{u_0}$ in the next sentence. The family of morphisms (7) provides us with a morphism of formal schemes

$$\widetilde{\mathcal{Z}}^{(p^{-m_0})}_{0,u_0} \times_k \widehat{U}_{u_0} \to \mathcal{Z}_{0,\widehat{U}_{u_0}}$$

and since both schemes are projective over $\widehat{U}_{u_0}$, Grothendieck's GAGA theorem shows that this morphism of formal schemes comes from a unique morphism of schemes

$$\iota : \widetilde{\mathcal{Z}}^{(p^{-m_0})}_{0,u_0} \times_k \widehat{U}_{u_0} \to \mathcal{Z}_{0,\widehat{U}_{u_0}}.$$

By construction, at the closed point u_0 of $\widehat{U}_{u_0}$, the morphism ι specializes to the morphism $F^{m_0}_{\widetilde{\mathcal{Z}}_{0,u_0}}$ composed with the finite morphism $\widetilde{\mathcal{Z}}_{0,u_0} \to \mathcal{Z}_{u_0}$. Since the set of points of $\mathcal{X}_{\widehat{U}_{u_0}}$, where the fibres of ι are non-empty and of dimension 0 is open, we see that the morphism ι is finite over the generic point of $\widehat{U}_{u_0}$.

Let $\widehat{K}$ be the function field of $\widehat{U}_{u_0}$. Since k is an excellent field, we know that the field extension $\widehat{K}|K$ is separable. On the other hand the just constructed finite morphism $\widetilde{\mathcal{Z}}_{0,u_0}^{(p^{-m_0})} \times_k \widehat{K} \to Z_{0,\widehat{K}}$ is defined over a finitely generated subfield K' (as a field over K) of $\widehat{K}$. The field extension $K'|K$ is then still separable, so that by the theorem on separating transcendence bases, there exists a variety U'/K, which is smooth over K and whose function field is K'. Furthermore, possibly replacing U' by one of its open subschemes, we may assume that the morphism $\widetilde{\mathcal{Z}}_{0,u_0}^{(p^{-m_0})} \times_k K' \to Z_{0,K'}$ extends to a finite morphism

$$\alpha : \widetilde{\mathcal{Z}}_{0,u_0}^{(p^{-m_0})} \times_k U' \to Z_{0,U'}.$$

Let $P \in U'(K^{\mathrm{sep}})$ be a K^{sep}-point over K (the set $U'(K^{\mathrm{sep}})$ is not empty because U' is smooth over K). The morphism $P^*\alpha$ gives a morphism

$$\widetilde{\mathcal{Z}}_{0,u_0}^{(p^{-m_0})} \times_k K^{\mathrm{sep}} \to Z_{0,K^{\mathrm{sep}}}.$$

Letting Z_0 run through all the irreducible components of $\mathrm{Zar}(X(K))$, we obtain the morphism h advertised in Theorem 1.2.

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