Existence and stability of time-periodic solutions in a model for spherical flames with time-periodic heat losses

Jean-Michel ROQUEJOFFRE[a], Hélène ROUZAUD[b]

[a] *UMR CNRS 5640 (MIP) and Institut Universitaire de France, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex, France*

[b] *UMR CNRS 5640 (MIP) Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex, France*

Dedicated to Professor R. Wong, on his 60th birthday

Abstract

The problem under investigation is the study of the effect of periodicity when time-periodic heat losses are added in a model for spherical flames introduced by G. Joulin. In the present context, the flame either quenches in finite or infinite time, or its radius converges to an upper stable time-periodic solution, or to a lower unstable time-periodic solution.

Key words: stability, half derivatives, time-periodicity, spherical flames, heat losses

1 Introduction

The problem under study is the large-time behaviour of the solutions to an integro-differential equation describing the evolution of spherical flames with time-periodic heat losses. It has been acknowledged for some time (19) that spherical flames (or flame balls) are an important prototype of flames, and that a detailed study of their structure and propagation can lead to new insights in complex combustion processes. Hence the necessity of deriving simple models. G. Joulin (15), (16), (7), (5)... was a pionneer in this direction: starting from the classical thermo-diffusive model for flame propagation in 3 space dimensions, he derived, by means of formal matched asymptotic expansions, a series of models that are numerically tractable, but whose mathematical study needed to be done. The first attempt in this direction was (2), where the simplest case - evolution of a flame ball with no heat losses - was understood. The effect of constant heat losses, as well as a new class of time-asymptotic preserving numerical schemes, were studied in (21) and (3). See also (20) for the proof of a universal ignition threshold.

We are going to investigate here the large time behaviour of a spherical flame model with time-dependent heat losses, that comes directly from (5). The underlying question, which is of a certain practical importance when security issues are at stake, is the persistence of a flame, despite of the presence of heat losses. The study that follows is a first step in this direction; the problem under scrutiny here concerns the solutions to the following equation, with unknown $R(t) \in C(\mathbb{R}_+)$:

$$\partial_{1/2} R = \text{Log } R - \lambda(t) R^2 + \frac{Eq(t)}{R}, \quad R(0) = 0. \tag{1.1}$$

The notations are the following

- the operator $\partial_{1/2}$ is the classical Abel half derivative

$$\partial_{1/2} R(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{\dot{R}(\tau)}{\sqrt{t-\tau}}\, d\tau, \tag{1.2}$$

  whenever this expression makes sense;
- the function $\lambda(t)$ models the heat losses; it is smooth, positive and 1-periodic in t;
- the function $q(t)$ is smooth, connectedly supported and goes to 0 as $t \to +\infty$; moreover we have $\displaystyle\int_0^{+\infty} q(\tau)\, d\tau = 1$; it represents the heat source brought to the flame;
- the parameter $E > 0$ represents the strength of the heat source.

It is known (21) that, when $\lambda(t)$ is a constant function that we still call λ, that the following occurs.

- If $\lambda > \lambda_{cr} = e^{-1}$, then all solutions of (1.2) go to 0 - we also may say that they quench - in finite or infinite time.
- If $\lambda < \lambda_{cr}$, then the equation Log $R - \lambda R^2$ has two positive solutions $R_1 < R_2$. Given a smooth function $q(t)$ satisfying the above assumptions, there is $E_{cr}(q) > 0$ such that
  - · if $E < E_{cr}(q)$, then the solution $R(t)$ of (1.2) quenches in finite or infinite time;
  - · if $E > E_{cr}(q)$, then we have $\lim_{t\to+\infty} R(t) = R_2$;
  - · if $E = E_{cr}(q)$, then we have $\lim_{t\to+\infty} R(t) = R_1$.

Our goal is to prove that the above results persist if the function $\lambda(t)$ is 1-periodic in t. The difference is of course that the asymptotic states will be 1-periodic functions instead of constants. This is supported by the simulation displayed of Fig. 1. The result that we wish to prove is the following.

Theorem 1.1 *Consider $\lambda(t)$ 1-periodic, smooth, connectedly supported, satisfying the above-listed assumptions, and such that there are two constans $\underline{\lambda}$ and $\overline{\lambda}$ such that:*

$$0 < \underline{\lambda} \le \lambda(t) \le \overline{\lambda} < \lambda_{cr}.$$

Consider $\underline{R}_1$, $\overline{R}_1$, $\underline{R}_2$ and $\overline{R}_2$ the critical radii associated to $\underline{\lambda}$ and $\overline{\lambda}$ respectively.

Fig. 1. Extinction or stabilization according to te energy of the heat source

There are two functions $R_1(t) < R_2(t)$, smooth and 1-periodic, independent of q, such that

- *we have $\underline{R}_1 < R_1(t) < \overline{R}_1$, $\overline{R}_2 < R_2(t) < \underline{R}_2$;*
- *if $E < E_{cr}(q)$, then the solution $R(t)$ of (1.2) quenches in finite or infinite time;*
- *if $E > E_{cr}(q)$, then we have $\lim_{t\to+\infty}(R(t) - R_2(t)) = 0$;*
- *if $E = E_{cr}(q)$, then we have $\lim_{t\to+\infty}(R(t) - R_1(t)) = 0$.*

The plan of this paper comprises three further sections: in Section 2, we give some precise notions of stability of periodic solutions for (1.1). In Section 3 we prove Theorem 1.1; finally, Section 4 is devoted to the case of slightly super-critical heat losses, i.e. when the function $\lambda(t)$ is sometimes over λ_{cr}; this last point is important for modelling issues.

2 Diffusive formulation and stability definitions

It is known (2) that (1.2) can be lifted into the following parabolic formulation. Set

$$f_\lambda(t, u) = \text{Log } u - \lambda(t)u^2. \tag{2.1}$$

The function $R(t)$ satisfies (1.2) if and only if we have: $R(t) = u(t, 0)$ where

$$\begin{cases} u_t - u_{xx} = 0, \quad x \in]0, +\infty[, \\ u_x(t, 0) = -f_\lambda(t, u(t, 0)) - \dfrac{Eq(t)}{u(t, 0)} \\ u(0, x) = 0 \end{cases} \tag{2.2}$$

Antother equivalent formulation is to extend the solution $u(t, x)$ on the whole real line, and the parabolic formulation is then

$$\begin{cases} u_t - u_{xx} = 2\left(f_\lambda(t, u) + \dfrac{Eq(t)}{u} \right) \delta_{x=0}, \quad x \in \mathbb{R}, \\ u(0, x) = 0 \end{cases} \tag{2.3}$$

Both formulations (2.2) or (2.3), although they involve an additional variable - the space variable - are more tractable than (2.2): indeed, they are Cauchy problems whereas (1.2) is not. Further, the maximum principle is available in both formulations. In order to study the asymptotic states of (1.2), we have to study the global - i.e. defined on the whole time line - of the parabolic equation

$$u_t - u_{xx} = 2 f_\lambda(t, u) \delta_{x=0}, \quad x \in \mathbb{R} \tag{2.4}$$

or, equivalently:

$$\begin{cases} u_t - u_{xx} = 0, \quad x \in]0, +\infty[, \\ u_x(t, 0) = -f_\lambda(u(t, 0)) \end{cases} \tag{2.5}$$

Equation (2.5) generates a nonlinear discrete semigroup, denoted by S.

Let us now define a stability notion for (2.4). For any $a > 0$, let $u^a(t, x)$ 1-periodic in time solution of

$$\begin{cases} u^a_t - u^a_{xx} = 0, \quad x \in]0, a[, \\ u(t, a) = 0 \\ u^a_x(t, 0) = -f_\lambda(u^a(t, 0)), \end{cases} \tag{2.6}$$

Note that Equation (2.6) generates a nonlinear discrete semigroup, denoted by S_a. Consider the linearized problem around u^a:

$$\begin{cases} v_t - v_{xx} = 0, \quad x \in]0, a[, \\ v_x(t, 0) = -f'_\lambda(u^a) v, \\ v(t, a) = 0. \end{cases} \tag{2.7}$$

The fonction u^a will be called *stable* if the first Floquet exponent of (2.7) is nonnegative. In the opposite case, the function u^a will be said to be *unstable*. See (13) for general parabolic equations.

Let us extend this definition to unbounded domains: let a 1-periodic in time solution $u(t, x)$ of (2.5), and assume that it has been obtained as a limit of 1-periodic solutions u^a as $a \to +\infty$. For every $a' > 0$ and $a > a'$, let $\lambda^a_{a'} \geq 0$ the first Floquet exponent of

(2.7), posed on the interval $]0,a'[$ instead of $]0,a[$. Soit $\lambda_{a'} = \liminf_{a\to+\infty} \lambda^a_{a'} \geq 0$; in fact $\lambda^a_{a'}$ is a limit and not a liminf; it is - by Krein-Rutman Theorem - the first eigenvalue of the linearized problem:

$$\begin{cases} v_t - v_{xx} = 2f'_\lambda(u(t,0))v\delta_{x=0}, \\ v(t,\pm a') = 0. \end{cases} \tag{2.8}$$

The maximum princilpe is valid for equation (2.8) - and proved in a standard way by multiplications and integrations by parts. It implies that the sequence $(\lambda_{a'})_{a'}$ is nonincreasing; hence $\lambda_{a'} \to \lambda_\infty \geq 0$ as $a' \to +\infty$. Consider $\overline{w}_{a'}$ the eigenfunction associated to $\lambda_{a'}$. Once again by Krein-Rutman, it is positive except at $\pm a'$. Moreover, normalizing it to 1 at, say, $t = 0$ and $x = 1$ we may bound it uniformly from above and below; the right tool is in this case the Harnack inequalities up to the boundary; see (4). These bounds are actually independent of a', and are uniform on every compact $[0,b]$. This triggers the local uniform convergence of a subsequence $(w_{a'_n})_n$, from which one retrieves an eigenfunction of (2.8), posed this time on $\mathbb{R}$, with Floquet exponent λ_∞.

Definition 2.1 *The solution $u(t,x)$ is said to be stable if and only if $\lambda_\infty \geq 0$.*

The problem

$$u_t - \Delta u = f(t,u), \quad t > 0, \quad x \in [0,a],$$

f 1-periodic in its first variable, has been studied a lot in the literature. See for instance an existence proof of stable periodic solutions in (1) *via* a sub/super-solution method - Neumann boundary conditions. Dirichlet and Robin conditions are treated in (14). Unstable periodic solutions are studied in (18), a study extended in (9). The long term dynamics when $a = +\infty$ is treated by Feireisl-Polacik in (10), with the key assumption that $f'(0) < 0$. The case $a < +\infty$ was already treated in (12) : the global attractor for a Fisher-type equation is studied.

In our case, problem (2.2) has a very particular structure that does not make it reductible to any of the former ones: it may be viewed as a semilinear parabolic problem on $\mathbb{R}$ or $\mathbb{R}_+$, but the nonlinearity is degenerate. In particular, no extension of Krein-Rutman's theorem is available here.

3 Existence of time-periodic solutions

3.1 Existence of a stable periodic solution

This part, where the main tool is the use of sub and super solutions, is rather standard.

Theorem 3.1 *Problem (2.5) has a 1-periodic, stable solution.*

First, we study the truncated problem.

Lemma 3.2 *There is a stable, 1-periodic solution, to (2.6).*

Proof : by sub/super-solutions. Consider $\underline{u_{\overline{\lambda}}}$ a compactly supported sub-solution of

$$u_t - u_{xx} = 2\delta_{x=0}(\text{Log } u - \overline{\lambda}u^2);$$

see (21) for such a construction. For $a > 0$ large enough, $\underline{u_{\overline{\lambda}}}$ will also be a compactly supported sub-solution to (2.6). Hence, if u :

$$u(t+n) = S_a^n \underline{u_{\overline{\lambda}}}(t) = S_a(\underline{u_{\overline{\lambda}}})(t+n),\ t \in]0,1[,\ n \in \mathbb{N},$$

the sequence $(S^n u(t))_n$ is nondecreasing for all $t \in [0,1[$. On the other hand, $\underline{R}_2$ is a super-solution to (2.6), hence an upper bound for $S_a^n u(t)$. Thus $(u(t+n))_n$ converges as $n \to +\infty$. This yields a periodic solution to (2.6), such that $u_a(t,0)$ lies beteween $\overline{R}_2$ and $\underline{R}_2$. •

Proof of Theorem 3.1. The preceding lemma generates a sequence (u_λ^a) of 1-periodic, $\mathcal{C}^\infty$ solutions to (2.6), with:

$$\overline{R}_2 \le u_\lambda^a \le \underline{R}_2.$$

Ascoli's theorem applies, and yields a function u_λ 1-périodique, solution of (2.5). Because u_λ^a is generated by a sub/super-solution process, it is stable. Hence u_λ is salso stable. •

3.2 Existence of an unstable periodic solution

This part is of course more involved, as is usaul when we look for unstable solutions. The existence proof relies on a topological degree argument and approximation on a finite domain. Uniqueness is by no means a general feature, and holds thanks to the very special structure of the system under study.

Theorem 3.3 *Problem* (2.5) *has an unstable, 1-periodic solution.*

Proof : in three steps.

1. Approximation on a finite domain. The nonlinearity being singular at $u = 0$ we regularize the problem and consider the equation

$$\begin{cases} u_t^a - u_{xx}^a = 0, \quad x \in]0,a[, \\ u(t,a) = 0 \\ u_x^a(t,0) = -f_\lambda(u^a(t,0)) + \dfrac{\varepsilon}{u}, \end{cases} \tag{3.1}$$

where $\varepsilon > 0$. The same procedure as above yields a stable solution $u_{2,\varepsilon} > \overline{R}_2$, but also a stable, 1-periodic solution $u_{0,\varepsilon}$ which is close to 0. Let us denote by $\underline{R}_{0,\varepsilon}$, $\underline{R}_{1,\varepsilon}$

and $\underline{R}_{2,\varepsilon}$ (resp. $\overline{R}_{0,\varepsilon}$, $\overline{R}_{1,\varepsilon}$ and $\overline{R}_{2,\varepsilon}$) the zeroes of the function $\underline{f} : u \mapsto \text{Log } u - \underline{\lambda} u^2 + \frac{\varepsilon}{u}$ (resp. $\overline{f} : u \mapsto \text{Log } u - \overline{\lambda} u^2 + \frac{\varepsilon}{u}$).

Fig. 2. zeroes of $\overline{f}$ and $\underline{f}$

Choose $\eta \in]0, \overline{R}_{0,\varepsilon}[$. We seek 1-periodic solutions of (3.1) such that $u^a(t,0) \in [\overline{R}_{0,\varepsilon} - \eta, \underline{R}_{2,\varepsilon} + \eta]$. This amounts to looking for a solution $u(t,x)$ of (3.1) such that :

$$u(0,.) = u(1,.).$$

Define $\lambda_\tau = \tau\lambda + (1-\tau)\underline{\lambda}$, where $\tau \in [0,1]$, and consider the function space :

$$\mathcal{C} = \{u \in \mathcal{C}(\mathbb{R}, [\overline{R}_{0,\varepsilon} - \eta, \underline{R}_{2,\varepsilon} + \eta]),\ u(a) = 0\}.$$

The maximum principle allows us to define a mapping $\mathcal{F}_\tau$ by :

$$\begin{aligned} \mathcal{F}_\tau : \mathcal{C} &\to \mathcal{C}, \\ u_0 &\mapsto u(1), \end{aligned}$$

where $u(t,x)$ is the solution of (3.1) with initial datum u_0. We look for the zeroes of $Id_{\mathcal{C}} - \mathcal{F}_\tau$.

Ascoli's theorem implies that $\mathcal{F}_\tau$ is a compact operator. On the other hand, $\lambda_\tau \in [\underline{\lambda}, \overline{\lambda}]$; λ_τ is a 1-periodic function, and we know that the zeroes of $Id_{\mathcal{C}} - \mathcal{F}_\tau$ are such that their trace at $x = 0$ is in $[\overline{R}_{0,\varepsilon}, \underline{R}_{2,\varepsilon}] \subset]\overline{R}_{0,\varepsilon} - \eta, \underline{R}_{2,\varepsilon} + \eta[= V$. Consequently, by homotopy,

$$\deg(Id_{\mathcal{C}} - \mathcal{F}_1, 0, V) = \deg(Id_{\mathcal{C}} - \mathcal{F}_0, 0, V).$$

The mapping $Id_{\mathcal{C}} - \mathcal{F}_0$ has 3 zeroes: $\underline{R}_{0,\varepsilon}$, $\underline{R}_{1,\varepsilon}$, $\underline{R}_{2,\varepsilon}$ - hereafter we shall make the abuse of notations consisting in identifying a zero of $Id_{\mathcal{C}} - \mathcal{F}_1, 0, V$ to its trace at $x = 0$. The constant functions $\underline{R}_{0,\varepsilon}$ et $\underline{R}_{2,\varepsilon}$ are not only stable solutions, but their first Floquet exponents are positive (21); hence the degree of $Id_{\mathcal{C}} - \mathcal{F}_0$ relative to each of these functions is 1. In order to determine the degree of $Id_{\mathcal{C}} - \mathcal{F}_0$ relative to

$\underline{R}_{1,\varepsilon}$,we investigate the number of nonpositive eigenvalues to:

$$\begin{cases} -v'' = \nu v, \quad x \in]0, a[, \\ v'(0) = -\left(\dfrac{1}{\underline{R}_{1,\varepsilon}} - 2\lambda \underline{R}_{1,\varepsilon} + \dfrac{\varepsilon}{\underline{R}_{1,\varepsilon}}\right) v(0), \;\; v(a) = 0. \end{cases} \tag{3.2}$$

The quantity $\nu = -\theta^2$ is an eigenvalue if and only if:

$$th\, \mu = \frac{1}{L\left(\dfrac{1}{\underline{R}_{1,\varepsilon}} - 2\lambda \underline{R}_{1,\varepsilon} + \dfrac{\varepsilon}{\underline{R}_{1,\varepsilon}}\right)}\mu,$$

with the notation $\mu = a\theta$. There is only one possible nonpositive eignevalue, which is in fact negative. Hence the degree of $Id_{\mathcal{C}} - \mathcal{F}_0$ relative to $\underline{R}_{1,\varepsilon}$ is (-1).

2. Approximation on a bounded domain: computation of the degrees and existence of the unstable solution. We already know that (2.6), with a general λ, has two stable solutions; let us prove that their first Floquet exponents are positive.

- The upper solution. Let $u_u^{a,\varepsilon,+}$ be the sequence of stable maximal upper solutions; by the strong maximum principle the sequence $(u^{a,\varepsilon,+})_a$ is increasing; by the concavity of $u \mapsto f_\lambda(t, u)$, the first Floquet exponent of the upper solution with general λ is a decreasing sequence.
- *The lower solution* The lower solution, denited by $u_l^{a,\varepsilon,+}$is of order $\varepsilon\text{Log } \varepsilon^{-1}$; hence $f_u(t, .)$ is always < 0 at that value. The strong maximum principle implies the positivity of the first Floquet exponent.

Conseqently, these two solutions have unit topological degrees.

We should still have to prove that the upper and lower solutions are the only stable ones. This is deferred to the next subsection, and we take it for granted. As a result, should there be a $\tau \in (0, 1]$ at which $Id_{\mathcal{C}} - \mathcal{F}_\tau$ has no unstable solution in V, the degree of this mapping relative to V should be 2. On the other hand, we have

$$\deg(Id_{\mathcal{C}} - \mathcal{F}_0, 0, V) = 1.$$

This contradicts the degree invariance by homotopy; hence the existence of an intermedate, unstable solution.

3. The limits $a \to +\infty$, $\varepsilon \to 0$. Let us call $u^{a,\varepsilon,-}$ the so constructed unstable solution. We wish to prove that we may pass to the limit and still keep an unstable solution. From the maximum principle and elementary sub/super-solution theory, we know that (21) $u_l^{a,\varepsilon,+}$ (resp. $u^{a,\varepsilon,+}$) attracts all solutions of (3.1) whose trace at $x = 0$ is permanently under $\underline{R}_{1,\varepsilon}$ (resp. over $\overline{R}_{1,\varepsilon}$); hence there is $t_\varepsilon^a \in (0, 1)$ such that $u^{a,\varepsilon,-}(t_\varepsilon^a, 0) \in [\underline{R}_{1,\varepsilon}, \overline{R}_{1,\varepsilon}]$. On the other hand, we have $\lim\limits_{\varepsilon\to 0} \underline{R}_{1,\varepsilon} = \underline{R}_1$ et $\lim\limits_{\varepsilon\to 0} \overline{R}_{1,\varepsilon} = \overline{R}_1$. This is sufficient - with the help of the boundary Harnack inequalities and (21) - to conclude that

- the sequence $(u^{a,\varepsilon,-}(t,0))$ is uniformly bounded with respect to a;
- the sequence $(u^{a,\varepsilon,-}(t,0))$ is uniformly bounded away from 0 with respect to a.

Hence the sequence $(u^{a,\varepsilon,-})_{a,\varepsilon}$ is relatively compact in $C_{loc}([0,1]\times\mathbb{R})$; passing to the limit up to a subsequence yields a solution that has to be, at $x=0$, within $[\underline{R}_1, \overline{R}_1]$ at least for some times. The uniqueness part shows that there can only be one stable solution to (2.5). Hence the limit cannot be stable. •

4 Uniqueness of the stable and unstable periodic solutions and large time behaviour

We will in this section present two uniqueness proofs. The first one is valid only when λ has small variations, but does not use the special form of f_λ; hence it is more general. The second one uses the concavity of $u \mapsto \mathrm{Log}\, u - \lambda u^2$, but is valid for any size of λ, provided it stays below λ_{cr}.

4.1 Uniqueness for small variations of λ

Let $S_+(\lambda)$ the set defined by:

$$S_+(\lambda) = \{1\text{-periodic solutions of (2.6) that are } \geq \underline{u}_{\overline{\lambda}}\}.$$

From the preceding section, $S_+(\lambda) \neq \emptyset$, $S_+(\lambda)$ is bounded from above and below; hence it admits a minimal element, called u^a_λ. For all $\tau \in]0,1]$, call $u^a_{\tau\lambda}$ the minimal element of $S_+(\tau\lambda)$. In what follows, we take $a > 0$ large enough, and the above proofs can be carried on to $a = +\infty$.

Lemma 4.1 *We have* $\lim\limits_{\tau\to 0} u^a_{\tau\lambda} = +\infty$ *uniformly on* $[0, \dfrac{a}{2}]$.

Proof : Because both $u_{\tau\lambda}$ and $\underline{u}_{\tau\overline{\lambda}}$ can be attained from below starting from a compactly supported sub-solution, we have $u_{\tau\lambda} \geq \underline{u}_{\tau\overline{\lambda}}$. •

Lemma 4.2 *Define* τ_0 *by:*

$$\tau_0 = \sup\{0 < \tau < 1, \forall \tau' < \tau,\ u_{\tau'\lambda} \geq u_\lambda\}.$$

Then $\tau_0 = 1$.

Proof :Due to the preceding lemma we have $\tau_0 > 0$. Assume therefore that $\tau_0 < 1$. Then there is (t_0, x_0) such that:

$$u_{\tau_0\lambda}(t_0,x_0) = u_\lambda(t_0,x_0).$$

On the other hand,

$$\partial_x(u_{\tau_0\lambda} - u_\lambda)(t_0, x_0) = (\tau_0 - 1)\lambda u_\lambda(t_0, x_0)^2 < 0,$$

which is impossible because $u_{\tau_0\lambda} \geq u_\lambda$. Once again from the Hopf Lemma, $\partial_x(u_{\tau_0\lambda} - u_\lambda)(t_0, x_0) > 0$. Finally, the assumption $x_0 \in]0, a[$ is against the strong maximum principle. •

Lemma 4.3 *Consider* $0 \leq u_1 \leq u_2$ *two* 1*-periodic functions, solving - recall that we may take* $a = +\infty$ *:*

$$\begin{cases} u^i_t - u^i_{xx} = 2\delta_{x=0} f_i(u^i), & x \in [-a, a], \\ u^i(0, x) = u^i_0(x), \end{cases}$$

where

$$\forall \varepsilon > 0,\ \exists \eta > 0,\ \forall x, y \in \mathbb{R},\ y - x \geq \eta,\ f_2(y) - f_1(x) \geq \varepsilon. \tag{4.1}$$

Also assume that :

$$\forall \alpha \in \mathbb{R}_+,\ u_1 + \alpha \not\leq u_2.$$

Then, $\exists t_0 > 0$ *such that* $u_1(t_0, 0) = u_2(t_0, 0)$.

Proof : Assume that, for all $t \in \mathbb{R}$, $u_1(t, 0) < u_2(t, 0)$. Then, there is $\eta > 0$ such that:

$$u_1(t, 0) \leq u_2(t, 0) - \eta.$$

Set $v = u_2 - u_1$. Let $G_a(s, t, x, y)$ be the Green function of the heat operator on $(-a, a)$ with Dirichlet condition; let us compute v :

$$v(t, x) = \int_{-a}^{a} G_a(0, t, 0, y)(u^0_2 - u^0_1)(x) + \int_0^t G_a(s, t, 0, 0)(f_2(u_2(s, 0)) - f_1(u_1(s, 0)))ds.$$

The first term goes to 0 as $t \to +\infty$; moreover, by assumption we have

$$f_2(u_2(s, 0)) - f_1(u_1(s, 0)) \geq \varepsilon > 0.$$

Hence,

$$\liminf_{t \to +\infty} v(t, x) = 0 \text{ uniformly on every compact of } (-a, a),$$

wich contradicts the assumptions of the lemma. •

These three lemmas imply that there is only one solution of (2.6) which is $\underline{u}_{\overline{\lambda}}$. Indeed, choose

$$f_1(x) = \text{Log } x - \lambda x^2,\ f_2(x) = \text{Log } x - \tau\lambda x^2,$$

assumption (4.1) is indeed satisfied, for $u_\lambda(t, 0) \geq R_2(\overline{\lambda})$. Hence, by Lemma 4.3, there is $t_0 > 0$ such that $u_{\tau\lambda}(t_0, 0) = u_\lambda(t_0, 0)$; otherwise we would contradict the maximality of τ_0. But then we contradict the Hopf Lemma: therefore we have $\tau_0 = 1$. Now, call $u = \lim_{\tau \to 1} u_{\tau\lambda}$. Should there be a contact point between u and u_λ we may prove, by the same argument as in the case $\tau_0 < 1$, that $u = u_\lambda$. If there is no contact point between u and u_λ, then $\lim\limits_{\tau \to 1} u_{\tau\lambda} - u_\lambda > 0$, and u is also a periodic solution of (2.6). On the other hand,

$$\partial_x (u - u_\lambda)(t, 0) = -(\text{Log } u - \lambda u^2) + (\text{Log } u_\lambda - \lambda u_\lambda^2) > 0 \tag{4.2}$$

as soon as λ does not vary too much. Indeed, $u \mapsto \text{Log } u - \lambda u^2$ is decreasing from $u = \dfrac{1}{\sqrt{2\lambda}}$. For the above inequality to be valid, we must therefore have $\overline{R_2} > \dfrac{1}{\sqrt{2\underline{\lambda}}}$, as is illustrated on Figure 3.

Fig. 3. Cases where inequality (4.2) is satisfied

Remark : this proof is also valid for the uniqueness of the unstable solution; indeed the function $u \mapsto \text{Log } u - \lambda u^2$ is increasing on $[\underline{R}_1, \overline{R}_1]$. Take

$$S_-(\lambda) = \{\text{periodic solutions of (2.6) with trace} \in [\underline{R}_1, \overline{R}_1]\},$$

$v \in S_\lambda$ and $(u_\tau)_{\tau \in (0,1)}$ a family of unstable solutions which is above v for small τ. Then, as long as $u_\tau > v$, we have

$$\text{Log } u_\tau - \tau\lambda u_\tau + \text{Log } v - \lambda v > 0,$$

and we proceed as above.

4.2 Uniqueness under the concavity assumption.

4.2.1 Uniqueness of the stable solution

As is now usual, we start by working on a bounded interval :

$$\begin{cases} u_t - u_{xx} = 0, \quad x > 0, \\ u_x(t,0) = -f_\lambda(u(t,0)), \\ u(t,a) = 0, \end{cases}$$

where $a > 0$, $f_\lambda(x) = \mathrm{Log}\ x - \lambda x^2$, and u is 1-periodic in time. We may assume u to be the minimal stable solution, and let v be another stable solution. Set $w = v - u \geq 0$. Then

$$\begin{cases} w_t - w_{xx} \leq 2\delta_{x=0} f'_\lambda(u) w, \quad x \in [-a,a], \\ w(t, \pm a) = 0, \end{cases}$$

thanks to the concavity of f_λ. By assumption, the first Floquet exponent of the linearized operator, λ_a, is nonnegative.

- $\lambda_a > 0$.
  Call $\overline{w}$ the solution of :

$$\begin{cases} \overline{w}_t - \overline{w}_{xx} = 2\delta_{x=0} f'_\lambda(u)\overline{w}, \\ \overline{w}(t, \pm a) = 0, \\ \overline{w}(0,x) = w(0,x). \end{cases} \tag{4.3}$$

  From the maximum principle, $\overline{w} > 0$ on $]-a,a[$, and, on the other hand, $\overline{w}(t,x)$ decays exponentially to 0 as $t \to +\infty$. Once again by the maximum principle, $w(t) \leq \overline{w}(t)$. The function w is 1-periodic in time and $w(t) \to 0$ as $t \to +\infty$. Consequently, $w \leq 0$ and $u = v$.

- $\lambda_a = 0$.
  The function $\overline{w}$, solution of (4.3), is the eigenfunction associated to 0. By Krein-Rutman, it is > 0; moreover it is 1-periodic in t. Assume that we may choose $C > 0$ such that $w(0,.)$ and $C\overline{w}(0,.)$ have a unique contact point $x_0 \in]-a,a[$. Set $z = w - C\overline{w}$.

$$\begin{cases} z_t - z_{xx} = 0, \quad x \in]0,a[, \\ z(0,x_0) = 0, \\ z(t,a) = 0. \end{cases}$$

  Because of the periodicity of z, we have $z(0,x_0) = z(1,x_0) = 0$, which is against the strong maximum principle. If the contact point is $\pm a$, we may use the Hopf Lemma. Consequently, for all $C > 0$, we have $w - C\overline{w} \leq 0$ and $w \leq 0$.

Let us turn to the unbounded interval case. The case $\lambda_\infty > 0$ being solved exactly as in the compact interval case, let us investigate the case $\lambda_\infty = 0$. We still may consider the real number C, smallest $C' > 0$ such that $w(0,.) \leq C'\overline{w}(0,.)$. Either there is a contact point, and we come to a contradiction at $t = 1$. Or there is no contact point; consider then $z = w - (C - \delta)\overline{w}$. For small δ we have $z(t, 0) < 0$. This function z, besides being 1-periodic in time, satisfies:

$$\begin{cases} z_t - z_{xx} = 0, \quad x > 0, \\ z(t,0) \leq -\alpha < 0. \end{cases}$$

This is impossible: when $t \to +\infty$, we have $z(t, 0) \to 0$. Hence, $w \leq 0$ and $v \leq u$. •

Remark : When λ is constant, there is a $\dfrac{1}{\sqrt{t}}$ convergence to the upper radius R_2: we conjecture therefore that $\lambda_\infty = 0$ in most cases.

4.2.2 Uniqueness of the unstable solution

Here, we only prove the interesting case $a = +\infty$, he case of finite a being even easier. Consider u_1 and u_2 two unstable solutions. Two cases may occur.

- Either $u_1 < u_2$. It is a standard - at least in the compact interval case - fact that two unstable solutions cannot be ordered; let us say why in this particular case of inifinite domain. First, because $u_2 - u_1$ is bounded away from 0 at $x = 0$ at all times we have, letting $t \to +\infty$, the existence of a $\delta > 0$ such that

  $$u_2 - u_1 \geq \delta \text{ on } \mathbb{R}^2.$$

  This can be seen by a straightforward computation on the heat equation on $\mathbb{R}_+ \times \mathbb{R}_+$ with Dirichlet conditions at $x = 0$. This fact being at hand, the rest of the proof is standard - see for instance (12); let us only recall it: let $\phi(t, x)$ be an eigenfunction of the first - negative Floquet exponent of the linearized system; then, for small $\varepsilon > 0$ the function $u_1 + \varepsilon\phi_1$ is a subsolution to (2.5), which therefore grows up in time, while being bounded by u_2; hence it converges to some 1-periodic solution $u_3 \leq u_2$. By the uniqueness of stable solutions, u_3 is unstable, hence has a neagtive Floquet exponent λ_3. However, if $\mathbf{T}_3$ is the linearized Poincaré map around u_3, the mapping $\mathbf{T}_3 - \lambda_3 I$ is Fredholm; moreover from (11), Chap. 5, λ_3 is an isolated eigenfunction of $\mathbf{T}_3 - \lambda_3 I$. A spectral projector on the null space of $\mathbf{T}_3 - \lambda_3 I$ can therefore be defined, from which one may use the classical stability theory to build a solution of (2.5) that starts close to u_3 and that is $O(1)$ far from u_3 at a large time. From the positivity of the first eigenfunction, this latter solution might be taken to be below u_3, but above $u_1 + \varepsilon\phi_1$. Hence the two solutions that we have constructed collide at some time, which is a contradiction.
- For all $t \in [0, 1]$ the function $x \mapsto u_1(t, x) - u_2(t, x)$ has zeroes. The lap number decay - see (21) for its application - implies that the set of zeroes of $u_1 - u_2$ is a finite or infinite number of smooth curves $(x_i(t))_{1 \leq i \leq N}$, with $x_{i+1} - x_i$ bounded away from 0 and x_i 1-periodic in time. This is possible because the functions u_1 and u_2 are time-global solutions. Consider any curve $\{x = x_i(t)\}$; the function

$v = u_1 - u_2$ satisfies the heat equation on $\{x = x_i(t)\}$, with Dirichlet conditions at the boundary of this smooth open subset of $\mathbb{R}_+ \times \mathbb{R}$. Hence it goes to 0 as $t \to +\infty$; consequently - another very easy computation with the explicit solution of the heat equation - $u_1 \equiv u_2$ on that set. By Cauchy-Kovalewskaya Theorem, $u_1 \equiv u_2$ everywhere.

Now that we have periodic solutions, we may prove Theorem 1.1. However, once uniqueness is known, the proof resembles very much to (2) and (21); so we only indicate the main steps.

Proof of Theorem 1.1. The first thing to prove is the fact that, for large E, the solution of (1.1) tends to the upper solution as $t \to +\infty$. Let $u^{a,\varepsilon,+}$ be the stable solution of (2.8); extend it by 0 outside $(-a, a)$ into a compactly supported subsolution of (2.5). Our task can be reduced to proving that the solution $u(t, x)$ of (2.5) with Cauchy datum 0 exceeds $u^{a,\varepsilon,+}$ in finite time; however this is done exactly as in (21).

The next step consists in proving that, if a solution $R(t)$ of (1.1) becomes too small, it goes to 0 in finite or infinite time. This is detailed in (2).

We conclude by a shooting method. Let $u^{\pm}$ be the stable (resp. unstable) 1-periodic solutions of (2.5), with radii $R^{\pm}(t)$. If $R_E(t)$ is the solution of (1.1) and $u_E(t, x)$ is diffusive extension, we introduce the sets

$$X_+ = \{E > 0 : \ \lim_{t\to+\infty} |R_E(t) - R^+(t)| = 0\}$$
$$X_- = \{E > 0 : \ \exists t_0 \in (0, +\infty] \text{ such that } \lim_{t\to t_0} R_E(t) = 0\}.$$

From (2), these two sets are open in $\mathbb{R}_+$. Hence the set $X_0 = \mathbb{R}_+ \backslash (X_+ \cup X_-)$ is non void. The last step is therefore to prove that; if $E \in X_0$, then $R_E(t) - R^-(t)$ tends to 0; this is done by studying the zero set of $u_E - u^-$; once again it mimicks (2). Finally, the instability of u^- implies that X_0 is reduced to one point. •

5 Partially supercritical heat losses

In this section, we wish to show the possibility of non-extinction of spherical flames when the function λ is sometimes supercritical, in a periodic fashion. For a given $\lambda(t)$, let us come back to the solution $R(t)$ of

$$\begin{cases} \partial_{1/2} R = \text{Log } R - \lambda(t) R^2 + \dfrac{Eq(t)}{R}, & t > 0, \\ R(0) = 0, \end{cases}$$

and its diffusive extension, $u(t,x)$, solution of

$$\begin{cases} u_t - u_{xx} = 2\delta_{x=0}\left(\text{Log } u - \lambda(t)u^2 + \dfrac{Eq(t)}{u}\right), & x \in \mathbb{R}, \quad t > 0, \\ u(0) = 0, \end{cases}$$

where $\lambda_{min} \leq \lambda(t) \leq \lambda_{max}$, with $\lambda_{min} < \lambda_{cr} < \lambda_{max}$. Thus $u \geq v$; where v solves

$$\begin{cases} v_t - v_{xx} = 2\delta_{x=0}\left(\text{Log } v - \lambda_{max}v^2 + \dfrac{Eq(t)}{v}\right), & x \in \mathbb{R}, \quad t \in [0,1], \\ v(0,.) = 0. \end{cases} \tag{5.1}$$

Consider $\overline{\lambda} = \dfrac{\lambda_{cr} + \lambda_{min}}{2}$. We may bound λ by the 1-periodic function $\tilde{\lambda}$, represented on Fig. 4.

Fig. 4. Représentation des fonctions λ et $\tilde{\lambda}$

Denote by $\overline{R_1} < \overline{R_2}$ the two critical radii attached to $\overline{\lambda}$. We wish to construct $\tilde{\lambda}$ on $[1,2]$ so that $R(t) \geq \dfrac{\overline{R_1} + \overline{R_2}}{2}$. From the maximum principle, $u(t) \geq w(t)$, with

$$\begin{cases} w_t - w_{xx} = 2\delta_{x=0}(\text{Log } w - \tilde{\lambda}w^2), & t \in [1,2], \quad x \in \mathbb{R}, \\ w(1,.) = v(1,.). \end{cases}$$

When E is large enough and $\varepsilon > 0$ small enough so that $w(1+\varepsilon,.)$ is above a compactly supported subsolution of

$$u_t - u_{xx} = 2\delta_{x=0}(\text{Log } u - \overline{\lambda}u^2),$$

the function w is above an increasing function on the time-interval $[1+\varepsilon, 2]$; up to shrinking our parameter ε we may iterate this process over the time-intervals $[n, n+1]$, $n \geq 2$. Hence we have constructed a class of time-periodic, partially supercritical heat losses, for which a spherical flame will survive eternally, provided that the initial energy input is large enough.
This phenomenon is represented on figures 5 and 6; they respectively represent

a function $\lambda(t)$ constructed as above, and the time-evolution of the radii for two different energy inputs : for $E = 10$, we get quenching; for $E = 20$, and for the same heat loss term, the flame stabilizes to an upper 1-periodic solution, despite the fact that the heat losses are supercritical on an infinite time-interval. The integro-differential equation is integrated with the numerical scheme devised in (3), which is known to preserve - at least in the constant coefficient case - the large-time dynamics.

Fig. 5. Heat loss coefficient

Fig. 6. Time-evolution of the flame

References

[1] A. Alikakaos, P. Hess, H. Matano, Discrete order preserving semigroups and stability for periodic differential equations, *J. Diff. Eqs.*, **82**:322–341, 1989.

[2] J. Audounet, V. Giovangigli, J.-M. Roquejoffre, A threshold phenomenon in the propagation of a point source initiated flame, *Physica D*, **121**:295–316, 1998.

[3] J. Audounet, J.-M. Roquejoffre, H. Rouzaud, Numerical simulation of point-source initiated flame ball with heat losses, *Mathematical Modelling and Numerical Analysis*, **36**, 2:273–291, 2002.

[4] H. Berestycki, L. A. Caffarelli, L. Nirenberg, Uniform estimates for regularization of free boundary problems, *Analysis and PDE,* 567–619, Marcel Dekker, New York, 1990.

[5] J.D. Buckmaster, G. Joulin, Flame balls stabilized by suspension in fluid with a steady linear ambient velocity distribution, *J. Fluid. Mechanics*, **227**:407–427, 1991.

[6] J.D. Buckmaster, G. Joulin, Influence of boundary-induced losses on the structure and dynamics of flame-balls, *Comb. Sci. Tech*, **89**:57–69, 1993.

[7] M. Champion, B. Deshaies, G. Joulin, Relative influences of convective and diffusive transports during spherical flame initiation, *Combustion and Flame*, **74**:161–170, 1988.

[8] X.-Y. Chen, H. Matano, Convergence, asymptotic periodicity, and finite-point blow-up, *J. Diff. Eqs.,* **78**, 1:160–190, 1989.

[9] E. N. Dancer, P. Hess, Stability of fixed points for order-preserving discrete-time dynamical systems, *J. Reine Angew. Math.,* **419**:125–139, 1991.

[10] E. Feireisl, P. Polacik, Structure of periodic solutions and asymptotic behavior for time-periodic reaction-diffusion equations on $\mathbb{R}$, *Adv. Diff. Eqs.,* **5**, 4-6:583–622, 2000.

[11] D. Henry, *Geometric theory of semilinear parabolic equations,* Lecture Notes in Mathematics. Springer, New-York.

[12] P. Hess, H. Weinberger, Convergence to spatial-temporal clines in the Fisher equation with time-periodic fitnesses, *J. Math. Biol.,* **28**, 1:83–98, 1990.

[13] P. Hess, *Periodic parabolic boundary value problems and positivity,* Longman Scientific and Technical, Harlow, Essex, 1991.

[14] P. Hess, Asymptotics in semilinear periodic diffusion equations with Dirichlet or Robin boundary conditions, *Arch. Rational Mech. Anal.,* **116**, 1:91–99, 1991.

[15] G. Joulin, Point source initiation of lean spherical flames of light reactants: an asymptotic theory, *Comb. Sci. Tech*, **43**:99–113, 1985.

[16] G. Joulin, Preferential diffusion and the initiation of lean flames of light fuels, *Siam. J. Appl. Math.*, **47**:998–1016, 1987.

[17] C. Lederman, J.-M. Roquejoffre, N. Wolanski, Mathematical justification of a nonlinear integro-differential equation for the propagation of spherical flames, Ann. Mat. Pura Appl, **183**:173–239, 2004.

[18] H. Matano, Existence of nontrivial unstable sets for equilibriums of strongly order-preserving systems, *J. Fac. Sci. Univ. Kyoto,* **30**: 89–110, 1984.

[19] P. D. Ronney, M. S. Wu, H. G. Pearlman, K. J. Weiland, Experimental study of flame balls in space: preliminary results from STS-83 space flight experiments, *AIAA J.,* **36**:1361–1368, 1998.

[20] J.-M. Roquejoffre, J.-L. Vazquez, Ignition and propagation in an integro-differential model for spherical flames, *Discrete Contin. Dyn. Syst. Ser. B,* **2**, 3:379–387, 2002.

[21] H. Rouzaud, Long-time dynamics of an integro-differential equation describing the evolution of a spherical flame, *Revista Matamatica Complutense,* **16**, 1:207–232, 2003.