~~is positive.~~

III) The geometric Harnack inequality.

Harnack inequalities play a crucial role in the theory of elliptic PDE's of the second order. Roughly speaking, they improve the maximum principle and, thus, lead to regularity estimates.

~~Ex~~ For the simple example of the laplacian: one can prove very easily that a solution of

$$-\Delta u = 0 \quad (B_1)$$
$$u \geq 0 \quad (B_1)$$
$$u \not\equiv 0 \quad (B_1)$$

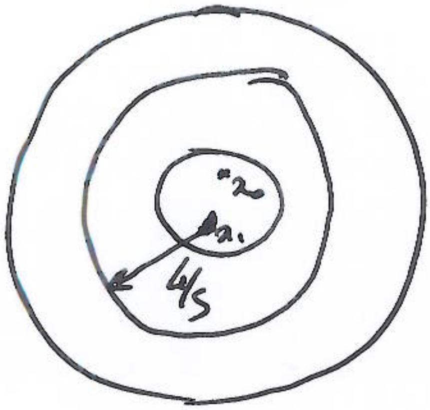
satisfies in fact:

$$\sup_{B_{1/3}} u \leq C \inf_{B_{1/3}} u$$

with C universal.

A way to see it is through the mean value formula for harmonic functions: if $B(x_0) \subset\subset B_1$, we have

$$u(x_0) = \fint_{B(x_0)} u(y)\, dy.$$



To see it:

$$u(x_0) = \max_{B_{1/3}} u$$

$$u(x_1) = \min_{B_{1/3}} u .$$

Then: $$u(x_1) = \fint_{B_{4/5}(x_1)} u(y)\,dy .$$

$$\geq \frac{1}{|B_{4/5}|} \int_{B_{1/10}(x_0)} u(y)\,dy .$$

$$= \frac{|B_{1/10}|}{|B_{4/5}|} \fint_{B_{1/10}(x_0)} u(y)\,dy = \frac{|B_{1/10}|}{|B_{4/5}|} u(x_0).$$

__Exercise__. One can replace $\frac{1}{3}$ by any radius $0<r<1$.

__Exercise__. Prove the existence of $\lambda \in (0,1)$ such that, if $\operatorname{osc}_B u$ (= the oscillation of u in B)

$:= \max_B u - \min_B u$, then:

$\operatorname{osc}_{B_{1/4}} u \leq \lambda \operatorname{osc}_{B_{1/2}} u$. Conclude that u is Hölder in u, with constant $\frac{\log \lambda}{\log 2}$.

Both exercises are solved in [GT] and many other places ...

Solution of the exercise.

Set $M_r = \sup_{B_r} u = \max_{\partial B_r} u$,

$$m_r = \inf_{B_r} u = \min_{\partial B_r} u,$$

by the scaling invariance we have:

$$\forall x \in B_{\frac{r}{2}} : M_r - u(x) \leq C \inf_{B_{\frac{r}{2}}} (M_r - u)$$

$$= C\left(M_r - M_{\frac{r}{2}}\right).$$

$$u(x) - m_r \leq C \inf_{B_{r/2}} (u - m_r)$$

$$= C\left(m_{r/2} - m_r\right).$$

Therefore $M_r - m_r \leq C\left(M_r - M_{\frac{r}{2}} + m_{\frac{r}{2}} - m_r\right)$.

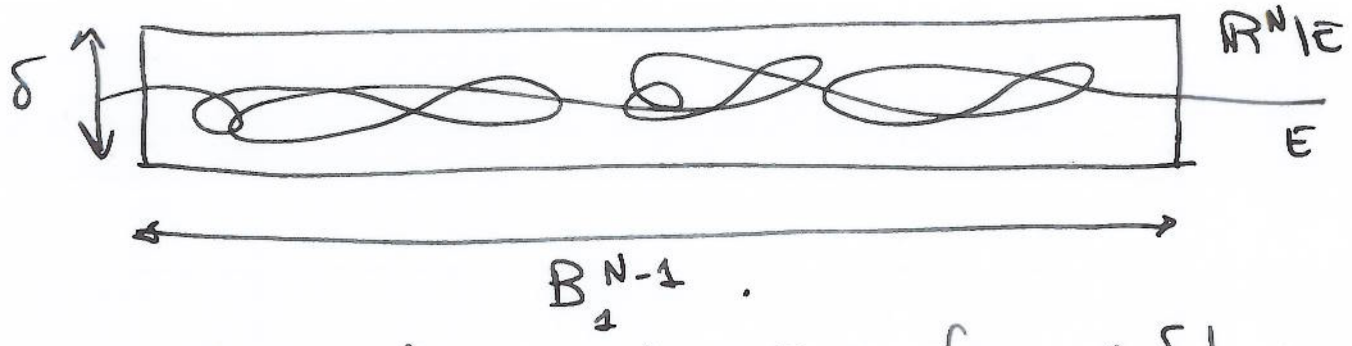
$$\boxed{M_{\frac{r}{2}} - m_{\frac{r}{2}} \leq \frac{C-1}{C}\left(M_r - m_r\right)}$$

Consequence: $M_{\frac{1}{2^n}} - m_{\frac{1}{2^n}} \leq \lambda^n (M_1 - m_1)$, $\lambda = \frac{C-1}{C}$

If $\frac{1}{2^{n+1}} \leq |x| \leq \frac{1}{2^n}$ we have:

$$|u(x) - u(0)| \leq \lambda^n (M_1 - m_1) \leq |x|^{+\frac{|\log \lambda|}{\log 2}} (M_1 - m_1)$$

The problem is now the following. Let us give ourselves a minimal set



.

such that $\{x_N \le -\delta\} \subset E \subset \{x_N \le \delta\}$.

One would like to prove, as a 1^{st} move, that it is a graph if δ is small enough (~~or at~~ in other words if the cylinder in which ∂E is sandwiched is flat enough).

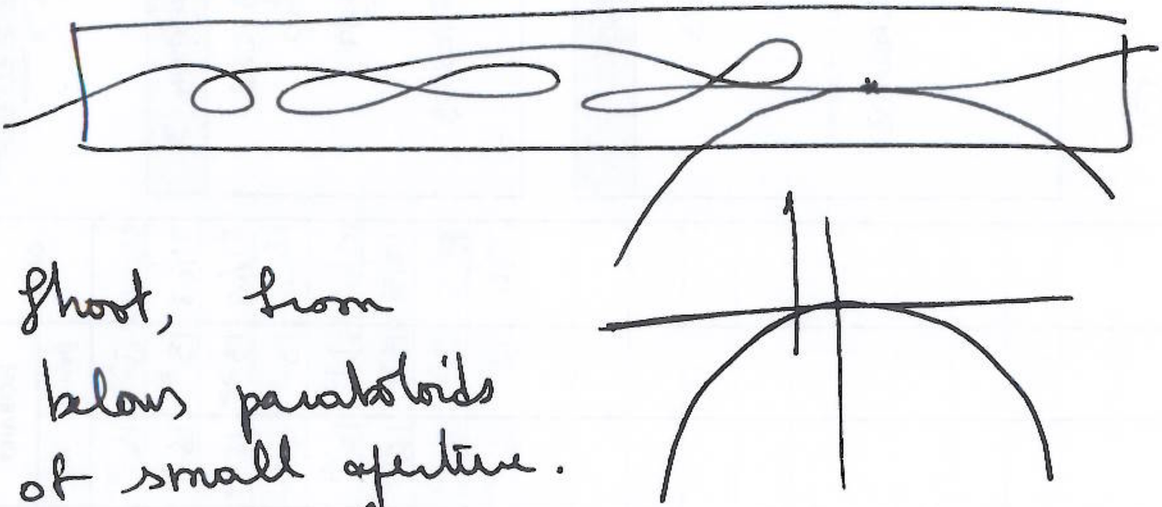
This is exactly the goal of the following Harnack inequality, that we are going to spend some time to prove.

<u>Th.</u> (Geometric Harnack inequality).

There is $\delta_0 > 0$ such that: there is $\vartheta \in (0,1)$ such that: for all $\delta \le \delta_0$, if $\{x_N \le -\delta\} \subset E \cap B_1 \subset \{x_N \le \delta\}$, and $0 \in \partial E$, then

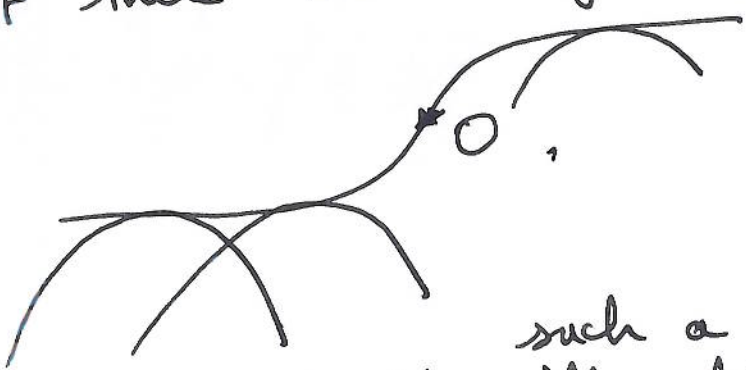
$$\{x_N \le -(1-\vartheta)\delta\} \subset E \cap B_{1/2} \subset \{x_N \le (1-\vartheta)\delta\}.$$

The strategy of the proof.



Shoot, from below paraboloids of small aperture. Stop until they touch the graph of u^-. At a contact point, u^- ~~is~~ is (philosophically) smooth from below. The viscosity relation implies that in fact u^- is smooth from above.

Of course one cannot conclude anything from that, it would be too easy! Nevertheless, even if this notion of "smoothness" is ~~very~~ essentially pointwise, one can infer something: if there are many contact points ~~even~~ and if



the surface is always far from O, this will be such a constraint on its area that it will stop being minimal.

And so, if ∂E is well localised in B_1, it is even better localised in $B_{1/2}$.

<u>Notation</u>. Before we forget: let us change $\mathbb{R}^N$ into $\mathbb{R}^{N+1}$. Let us consider E a minimal set of $\mathbb{R}^{N+1}$ such that

$$\{x_{N+1} \leq -\delta,\ x \in B_1\} \subset E \subset \{x_{N+1} \leq \delta,\ x \in B_1\}.$$

~~at~~ and $\mathbb{R}^{N+1} = \{(x, x_{N+1}) \in \mathbb{R}^N \times \mathbb{R}\}$.

Then we introduce $u_h^-(x)$, $u_h^+(x)$. u_h^- (resp. u_h^+) is a viscosity super (sub) solution of

$$-\operatorname{div} \frac{Du}{\sqrt{1+|Du|^2}} = 0.$$

(We drop the subscript h).

The plan of the proof is as follows.

1. A general estimate on the contact set (Alexandrov-Bakelman-Pucci).
2. Extension of the contact set. (how to go from 1 contact point to a set of nonzero measure).
3. Covering almost all $B_{1/2}$ w. contact points.

1°). Alexandrov - Bakelmann - Pucci -

- The paraboloids. For $a > 0$ and $y \in \mathbb{R}^N$ let $P_{a,c,y}$ be the paraboloid with equation $x_N = -\frac{a}{2}|x-y|^2 + c$.

The graph of u^- satisfies:

$$\forall x \in B_1, \quad -2\delta \leq u^-(x) \leq 2\delta.$$

We slide from below paraboloids $P_{a,c,y}$ until they touch the graph of u^-. This determines a unique c, called c_y. This determines a contact point. ~~and~~ Let $(x, u^-(x))$ be such a contact point.

The result is the

Prop. Let $B \subset B_{3/4}$ a set of nonzero measure. Let A be the set of contact points with ∂E obtained by taking $y \in B$, and $a \in (0,1]$. There exists $q > 0$, universal, such that

$$|A| \geq q\,|B|.$$

provided that $A \subset B_{3/4}$.

Before the proof, some remarks.
let $u^\varepsilon(x)$ be the unique viscosity solution of

$$\begin{cases} \partial_t u + |Du|^2 = 0. & x \in B_{3/4} \\ u(0,x) = u^-(x) \end{cases}$$

taken at $t = \varepsilon$. If $\varepsilon > 0$ is small enough there is (finite speed of propagation) indeed a unique solution in $B_{3/4}$.

We have

$$u^\varepsilon(x) = \inf_{y \in \mathbb{R}^N} \left(u^-(y) + \frac{1}{\varepsilon}|x-y|^2 \right).$$

And the lemma is:

<u>lemma</u>. For all small $\varepsilon > 0$, u^ε is a viscosity super-solution to $-\operatorname{div} \frac{Du}{\sqrt{1+|Du|^2}} = 0$.

<u>Proof</u>. let $y_\varepsilon(x)$ realise the minimum (exercise: prove its existence). We have

$$u^\varepsilon(x) = u^-(y_\varepsilon(x)) + \frac{1}{\varepsilon}|x - y_\varepsilon(x)|^2.$$

Let now be $\varphi \in C^2(B_1)$ and x_0 be a minimum for $u^\varepsilon - \varphi$. Then we have

$$u^\varepsilon(x_0) - \varphi(x_0) \leq u^\varepsilon(x) - \varphi(x).$$

In particular:

$$u^-(z_\varepsilon(x_0)) + \frac{1}{\varepsilon}|x_0 - z_\varepsilon(x_0)|^2 - \varphi(x_0)$$
$$\leq u^-(z_\varepsilon(x)) + \frac{1}{\varepsilon}|z_\varepsilon(x) - x|^2 - \varphi(x) \qquad \forall x.$$

Hence $u^-(z_\varepsilon(x_0)) + \frac{1}{\varepsilon}|x_0 - z_\varepsilon(x_0)|^2 - \varphi(x_0)$
$$\leq u^-(z) + \frac{1}{\varepsilon}|z - x|^2 - \varphi(x), \quad \forall (x,z).$$

~~If $x = x_0$ we have:~~

Choose: $x = z - z_\varepsilon(x_0) + x_0$. We have:

$$u^-(z_\varepsilon(x_0)) + \frac{1}{\varepsilon}|x_0 - z_\varepsilon(x_0)|^2 - \varphi(x_0)$$
$$\leq u^-(z) + \frac{1}{\varepsilon}|x_0 - z_\varepsilon(x_0)|^2 - \varphi(z - z_\varepsilon(x_0) + x_0).$$

Apply the viscosity relation to the test function

$$\psi(z) = \varphi(z - z_\varepsilon(x_0) + x_0). \text{ We have:}$$

$$-\operatorname{div} \frac{D\psi}{\sqrt{1+|D\psi|^2}}\Bigg|_{z = z_\varepsilon(x_0)} \geq 0.$$

Hence $$-\operatorname{div} \frac{D\varphi}{\sqrt{1+|D\varphi|^2}}\Bigg|_{x = x_0} \geq 0. \qquad \blacksquare$$

The function u^ε is semi-concave. Hence (Alexandrov's Theorem) there is a set Z

of zero measure such that, for all $x_0 \in B_{3/4} \setminus Z$, there is a matrix (that we denote by $D^2 u^\varepsilon(x_0)$) such that, for all x in the vicinity of x_0:

$$u^\varepsilon(x) - u^\varepsilon(x_0) = (x - x_0) . Du^\varepsilon(x_0) + \frac{1}{2} D^2 u^\varepsilon(x_0) . (x - x_0)^{(2)} + o(|x - x_0|)^2 .$$

In particular, u^ε is a.e. differentiable.

Proof of Alexandrov - Bakelman - Pucci.

We first try to touch the graph of u^ε — that we call u by commodity.

Let $(x, u(x))$ be a contact point with the paraboloid $\{ y = -\frac{a}{2} |x - y|^2 + c_y \}$.

Assume moreover that $x_0 \in B_{3/4} \setminus Z$. Then:

[i]. We have $Du(x_0) = D\left[-\frac{a}{2} |x - y|^2 \right] \Big|_{x = x_0}$

$$= -a(x_0 - y).$$

In particular: $|Du(x_0)| = O(a)$.

[ii]. We moreover know that

$$D^2 u(x_0) \geq -aI$$

by definition of the contact point (to be comfortable let us take

$$D^2 u(x_0) \geq -\frac{a}{2} I \;).$$

Apply the viscosity relation at x_0 : we have

$$-\frac{\Delta u(x_0)}{\sqrt{1+|Du(x_0)|^2}} + \frac{D^2u(x_0).Du(x_0).Du(x_0)}{(1+|Du(x_0)|^2)^{3/2}} \geq 0.$$

We claim that, if $\lambda(x_0)$ is an eigenvalue of $D^2u(x_0)$ we have :

$$\lambda(x_0) \leq Ca, \quad C \text{ universal}.$$

Indeed, if $e = Du(x_0)$ we have :

$$-\frac{\Delta u(x_0)}{\sqrt{1+|e|^2}} + \frac{D^2u(x_0).e.e}{(1+|e|^2)^{3/2}}$$

$$= \frac{\sum \lambda_i}{\sqrt{1+|e|^2}} - \frac{\sum \lambda_i e_i^2}{(1+|e|^2)^{3/2}}$$

$$\geq \frac{1}{C}\left(\sum \lambda_i\right)\left(1 - \frac{C^2a^2}{(1+C^2a^2)}\right) \geq \frac{1}{2C}\sum \lambda_i$$

(assume $a \leq 1$)

if $a > 0$ is small enough.

Thus, if $\lambda_{max} = \max_{i \in [1,N]} \lambda_i$:

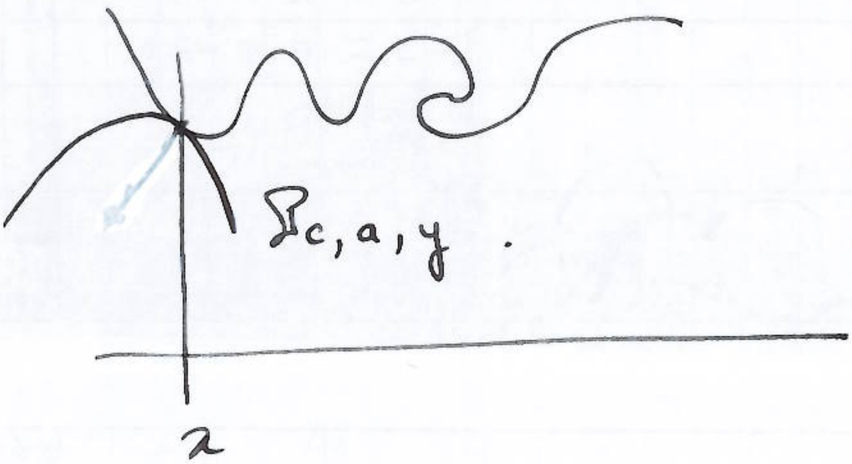
$$\lambda_{max} + \frac{1}{2C} \sum_{\lambda \neq \lambda_{max}} \lambda \leq 0 .$$

We know that $\lambda \geq -Ca$, hence

$$\lambda_{max} \leq + Ca .$$

<u>Conclusion</u> : $\| D^2 u(x_0) \| \leq Ca$.

Let us now write the contact condition, i.e the normal vectors coincide; in other words:



$$-\frac{Du(x_0)}{\sqrt{1+|Du(x_0)|^2}} = -\frac{D\left(-\frac{a}{2}|x_0-y|^2\right)}{\sqrt{1+D\frac{a}{2}|x_0-y|^2}} .$$

$$= a \frac{x_0 - y}{\sqrt{1+a^2|x_0-y|^2}} .$$

Thus: $\dfrac{|x_0-y|}{\sqrt{1+a^2|x_0-y|^2}} = \dfrac{1}{a} \dfrac{|Du(x_0)|}{\sqrt{1+|Du(x_0)|^2}} \Rightarrow |x_0-y| = |Du(x_0)|$

and

$$x_0 - y = -\frac{1}{a} Du(x_0) .$$

It especially implies

$$y = x_0 + \frac{1}{\alpha} Du(x_0);$$

Consider now the map

$$x_0 \in A \xmapsto{\ \varphi\ } x_0 + \frac{1}{\alpha} Du(x_0) \in B.$$

This is a surjection from A to B. Moreover this is a $C^{1,1}$ map: we have

$$\|D\varphi\|_{L^\infty(A)} \leq \frac{1}{\alpha} \|D^2 u\|_{L^\infty(A)} \leq C.$$

The co-area formula for lipschitz maps implies:

$$|B| = \int_{\varphi(A)} dx \leq \int_A |\det D\varphi|\, dx \leq C|A|.$$

This is the estimate for u_ε.

We now replace u by u^-. By definition of u_ε, for every x which is a contact point with

$\mathcal{P}_{c,a,y}$, there is a sequence $(x_\varepsilon)_\varepsilon$ of contact points of the family $\mathcal{P}_{c,a,y}$ with the graph of u_ε. More precisely: if we touch the graph of u_ε with a paraboloid, we will touch the graph of u^- at a very close point. Let A_h be the set of contact points and A_ε the set of contact points with the graph of u_ε. We have:

$$A \supset \bigcap_{n=1}^{+\infty} \bigcup_{p=n}^{+\infty} E_{1/p}$$

Each set $\displaystyle\bigcup_{p=n}^{+\infty} E_{1/p}$ has at least the measure $q|B|$ and the sequence is decreasing.

Hence $|A_h| \geq q|B|$. Same argument for A.

Remarks.

1. The regularisation tricks works for any elliptic equation of the form

$$-F(D^2u, Du, u, x) = 0$$

where F is weakly elliptic, i.e.

$$F(M+N, p, u, x) \geq F(M, p, u, x)$$

provided that $N \geq 0$ (in the matrix sense).

2. The main body of the argument (i.e. the measure of the contact set) is ubiquitous in the elliptic theory. The classical ABP. estimates states the following: let

$$L = -a_{ij}(x)\partial_{ij}$$

and u solve $Lu = f$ in Ω. Then

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u^+ + C\|f\|_{L^N(\Omega)}.$$

3. We cheated: we need the theorem on the contact set of semiconcave functions, explained in Section IV - 1°) -

2°). <u>Increase of the contact set</u>. Here: u^- is a super-solution of the curvature equation.

As said before, the idea is the following.

Assume that we can touch the graph of u^- by a paraboloid of aperture $\frac{1}{a}$ ($a \ll 1$) inside $B_{3/4}$, at a point where

$u^-(x) \le -(1-a)\delta$.

Then, possibly by increasing a bit the aperture ($a \longrightarrow Ca$) and the height at which we are allowed to touch ($-(1-a)\delta \to -(1-Ca)\delta$) we have a contact set of nonzero measure in B_1.

For $a \in (0,1)$ small let us set:

$$A_a = \{x \in B_{3/4} : u^-(x) \le -(1-a)\delta\ ;\ (x, u^-(x)) \text{ is a contact point with some } P_{x,a,y}\}.$$

prop. Let $x_0 \in B_{1/2}$ such that
$$A_a \cap \overline{B}_r(x_0) \neq \phi \quad (r>0 \text{ small}).$$
Then there exists $q>0$ universal such that
$$\frac{|A_a \cap B_{r/8}(x_0)|}{|B_r|} \geq q .$$

Proof.

Let x_1 be a contact point in $\overline{B}_r(x_0)$, we always may assume that $x_1 \in B_r(x_0)$ (otherwise we argue w. $B_{r+\varepsilon}(x_0)$ and let $\varepsilon \to 0$). The plan is the following:

[i]. Find a point x_2 such that

- x_2 is close to x_0.
- $u(x_2) - P(x_2, y) \leq$ a small constant

($P(x_2, y)$: touching paraboloid).

[ii]. Vary the base point of the paraboloid in a small ball and realise that all touching points occur in B_1.

[iii]. Apply the preceding proposition.

Let us therefore denote by
$$P(x, y_1) = -\frac{a}{2}|x - y_1|^2 + c_{y_1}$$

the touching paraboloid at $(x_1, u^-(x_1))$.
(and $x_1 \in B_r(x_0)$).

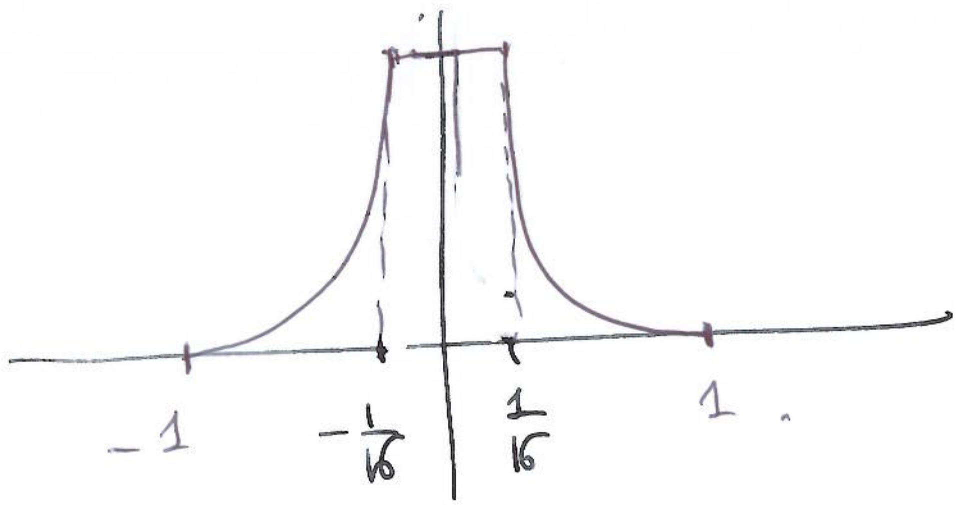
[i]. We claim the existence of $x_2 \in B_{r/16}(x_0)$
such that

$$u^-(x_2) - P(x_2, y_1) \leq C a r^2,$$

C: large and universal.
For this we construct a sub-solution. Let

$$\varphi(r) = \begin{cases} \frac{1}{\alpha}\left(r^{-\alpha} - 1\right) & \frac{1}{16} \leq |r| \leq 1. \\ \frac{1}{\alpha}\left(16^{\alpha} - 1\right) & r \leq \frac{1}{16}. \end{cases}$$

We choose $\alpha \gg 1$. The graph of φ is



(φ is very steep around $\frac{1}{16}$ and very flat around 1.)

And we set $\psi(x) = P(x, y_1) + \underbrace{a r^2 \varphi\left(\frac{x - x_0}{r}\right)}_{\tilde{\varphi}(x)}$.

For $x \in B_{\frac{r}{16}}(x_0)$:

- $\left(DP(x, y_1), D^2P(x, y_1)\right) \sim \left(O(ar), O(a)\right)$.

- $D\varphi(x) = O(ar)$.

- For $\frac{r}{16} < |x - x_0| \leq r$ we have

$$-D^2\tilde{\varphi}(x) = \begin{pmatrix} a\,\varphi''\left(\frac{|x-x_0|}{r}\right) & 0 \\ 0 & \frac{ar}{|x-x_0|}\varphi'\left(\frac{|x-x_0|}{r}\right) I_{N-1} \end{pmatrix}$$

in the (e_r, e_θ) basis (e_θ is a basis of S^{N-1})

Hence
$$D^2\tilde{\varphi}(x) = a\left(\frac{|x-x_0|}{r}\right)^{\alpha-2} \begin{pmatrix} \alpha - 1 & 0 \\ 0 & ar^2 I_{N-1} \end{pmatrix}.$$

and we have:
$$-\operatorname{div}\frac{D\psi}{\sqrt{1+|D\psi|^2}} = \left(-\Delta\psi - \frac{D^2\psi\, D\psi\cdot D\psi}{1+|D\psi|^2}\right)\sqrt{\ }$$

Setting $D\psi = (e_1, -, e_N)$ in ~~the~~ an eigenbasis of $D^2\psi$, λ_i the eigenvalues of $D^2\psi$:

$$+\operatorname{div}\frac{D\psi}{\sqrt{1+|D\psi|^2}}\sqrt{1+|D\psi|^2}$$

$$= \sum\lambda_i - \frac{\sum\lambda_i e_i^2}{1+|e|^2}$$

$$= \sum\lambda_i \frac{1+|e|^2 - e_i^2}{1+|e|^2}.$$ Now: $e = O(ar)$, thus

$$= \sum\lambda_i \frac{1+O(a^2r^2)}{1+O(a^2r^2)} \geq \frac{1}{2}\sum\lambda_i = \frac{1}{2}\Delta\psi.$$

And we have:

$$\Delta\varphi = a\left(\frac{|x-x_0|}{r}\right)^{\alpha-2}(\alpha-1+Nar^2)+O(a).$$

Hence $\Delta\varphi > 0$ for α large enough.

Conclusion: φ is a strict sub-solution in

$$B_r(x_0)\setminus B_{\frac{r}{16}}(x_0).$$

Let us now try to see where

$$\min_{\overline{B}_R(x_0)} (u^- - \varphi)$$

is attained.

* On $\partial B_r(x_0)$ we have

$u^- - \varphi = u^- - P(\cdot, y_1)$.

We always may assume that the minimum of $u^- - P$ is strict, therefore the min is not attained on $\partial B_r(x_0)$.

* In $B_r(x_0)\setminus\overline{B_{\frac{r}{16}}}(x_0)$: because u^- is a super-solution, we have

$$-\operatorname{div}\frac{D\varphi}{\sqrt{1+|D\varphi|^2}}\Big|_{x=x_0} \geq 0.$$

We just saw that it was < 0; impossible.

Thus the min is attained some where inside $B_{\frac{1}{16}}(x_0)$.

The minimum has to be <0, since we have:

$$u^-(x_1) - \psi(x_1) = \underbrace{u^-(x_1) - P(x_1, y)}_{0} - ar^2\varphi\left(\frac{x_1 - x_0}{r}\right)$$

Therefore, if x_2 is the sought for minimum we have:

$$u(x_2) \leq \psi(x_2) = P(x_2, y) + ar^2\varphi\left(\frac{x_0 - x_2}{r}\right)$$

$$\leq P(x_2, y) + Car^2.$$

[ii]. Let us consider the family of paraboloids

$$P(x, y_1) - \frac{C'a}{2}|x - y|^2 + c_y := Q(x, y)$$

$$|y - x_2| \leq \frac{r}{64}.$$

For each $y \in B_{\frac{r}{64}}(x_2)$, c_y is adjusted

so that

$$u^- - Q \geqq 0 \text{ in } \overline{B_1}.$$

$u^- - (Q + \varepsilon)$ takes negative values for all $\varepsilon > 0$.

We are going to prove that the contact points occur inside $B_{\frac{r}{16}}(x_0)$.

- We have $\cancel{u - Q}$ $u^-(x_2) - Q(x_2, y) \geq 0$.

Thus $u^-(x_2) - P(x_2, y_1) + \frac{C'a}{2}|x_2 - y|^2 \geq c_y$.

Hence: $c_y \leq Car^2 + \frac{C'a}{2}\left(\frac{r}{64}\right)^2$.

- If now $|x - x_0| \geq \frac{r}{16}$ we have:

$\cancel{P(}$ $u^-(x) - Q(x, y)$

$$= \underbrace{u^-(x) - P(x, y_1)}_{\geq 0} + \frac{C'a}{2}|x - y|^2 - c_y$$

$$\geq \frac{C'a}{2}\left(|x - x_2| - |x_2 - y|\right)^2 - c_y$$

$$\geq \frac{C'a}{2}\left(\frac{r}{32}\right)^2 - \frac{C'a}{2}\left(\frac{r}{64}\right)^2 - Car^2.$$

Thus, if C' is large enough, $u^- - Q > 0$ at a point $x \notin B_{\frac{r}{16}}(x_0)$.

- Finally it remains to see that $Q \in \Pi_{a,c,y}$ for some a, c, y. However:

$$\frac{a}{2}|x - y_1|^2 + \frac{C'a}{2}|x - y|^2$$

$$= (1 + C')\frac{a}{2}|x|^2 - a\, x \cdot (y_1 + C'y) + h(y, y_1)$$

$$= (1 + C')\frac{a}{2}\left(|x|^2 - 2x \cdot \left(\frac{y_1}{1 + C'} + \frac{C'}{1 + C'}y\right)\right) + h(y, y_1)$$

Thus $Q = P(x, \bar{y})$ with a replaced by $\frac{1+C}{2} a$

and $\bar{y} = \frac{y_1}{1+C'} + \frac{C' y}{1+C'}$.

- The set $\left\{ \frac{y_1}{1+C'} + \frac{C' y}{1+C'} y,\ y \in B_{\frac{r}{64}}(x_2) \right\} := B$

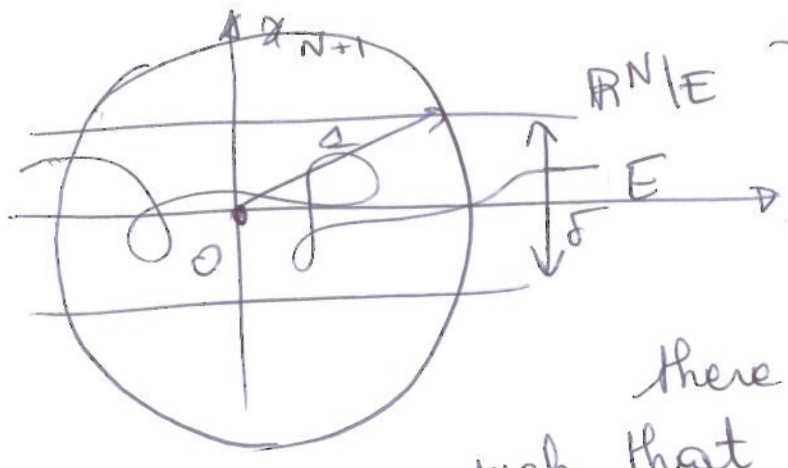
is
- within B_1.
- has a measure $\sim |B_r|$ if r is small.

Therefore we may apply Alexandrov – Bakelman – Pucci

$$|A_{C'a} \cap B_{\frac{r}{8}}(x_0)| \geq q|B_r| .$$

3°) – Proof of the Harnack inequality –

Recall the situation



$\partial E \cap B_1$ is trapped in a cylinder $B_1^N \times \{-\delta, \delta\}$. We assume that there exist points of $\partial E \cap B_{1/2}$ such that $\{ (x, x_{N+1}) \in \partial E,\ \cancel{x_{N+1} \leq} -\delta \leq x_{N+1} \leq -(1-a)\delta \}$

with $a \ll 1$. For $\varepsilon \ll a$, there is a point of $\{ d(x, x_{N+1}, \partial E) \in [-\varepsilon, -\frac{\varepsilon}{2}] \}$ such that: $-(1+\varepsilon)\delta \leq x_{N+1} \leq -(1-a)\delta$.

Consider the paraboloid $x_{N+1} = -\delta + \frac{a}{2}|x|^2 =: P_a(x, y)$ start sliding it upwards. We have, if $\delta > 0$ is less than $\frac{1}{2}$: $P_a(x,0) + \frac{a}{2} = -\delta + \frac{a}{2}(1-|x|^2)$

$$\geq -(1-a\delta) \text{ if } x = 0.$$

$$= -\delta \text{ if } |x| = 1.$$

Thus it has contact points within B_1. If now y is close enough to 0 ($|y| \leq \frac{1}{4}$ for instance) $P_a(\cdot, y)$ has contact points also inside B_1.

Let now: $D_k = A_{C^k a}$. The constant C is that of the "increase of contact set" proposition. Hence we have $k \leq \frac{|\log a|}{\log C}$, and k can be increased very much if a becomes very small.

First, a covering lemma.

<u>lemma</u>. Suppose the existence of a finite sequence of sets $(D_k)_{0 \leq k \leq k_0}$ such that

(i). $D_0 \neq \emptyset$.

(ii). For all $x_0 \in B_{1/2}$, for all $r < \frac{1}{2}$,

$$\left(D_k \cap \overline{B_r(x_0)} \neq \emptyset\right) \Longrightarrow \left(|D_{k+1} \cap B_{r/8}(x_0)| \geq q|B_r|\right)$$

with q universal.

Then there exists $\mu > 0$ universal such that:

$\| \; |B_{1/2} \setminus D_k| \leq (1-\mu)^k |B_{1/2}|$.

<u>Proof</u>. Let $d_k(x) = d(x, D_k)$. Let I be a countable subset of $B_{1/2} \setminus D_k$ such that:

(i). $B_{1/2} \setminus D_k = \bigcup_{x \in I} B_{d_k(x)}(x)$.

(ii). $x \neq x' \Rightarrow B_{\frac{d_k(x)}{3}}(x) \cap B_{\frac{d_k(x')}{3}}(x') = \emptyset$.

Then we write:

$$
\begin{aligned}
|D_{k+1} \setminus D_k| &= \Big| \bigcup_{x \in I} \big(D_{k+1} \cap B_{d_k(x)}(x)\big) \Big| \\
&\geq \Big| \bigcup_{x \in I} \big(D_{k+1} \cap B_{\frac{d_k}{3}(x)}(x)\big) \Big| \\
&= \sum_{x \in I} |D_{k+1} \cap B_{\frac{d_k}{3}(x)}(x)| \\
&\geq q \sum_{x \in I} |B_{\frac{d_k}{3}(x)}(x)| \\
&\geq \frac{q}{3^N} \sum_{x \in I} |B_{d_k(x)}(x)| \\
&\geq \frac{q}{3^N} |B_{1/2} \setminus D_k|.
\end{aligned}
$$

Hence $|B_{1/2} \setminus D_{k+1}| = |B_{1/2} \setminus D_k| - |(D_{k+1} \setminus D_k) \cap (B_{1/2} \setminus D_k)|$

$$\leq \left(1 - \frac{q}{3^N}\right) |B_{1/2} \setminus D_k|. \quad \blacksquare$$

Remark. At this point, it is worth asking ourselves what we are doing. We have just proved that the projection of the contact set of u^- (or u^-_ε, whatever ...) almost covers $B_{1/2}$ in measure. How can it be?

Once again, go back to the classical theory of elliptic equations. Assume u to be smooth, then it is a classical solution to

$$-\operatorname{div} \frac{Du}{\sqrt{1+|Du|^2}} = 0,$$

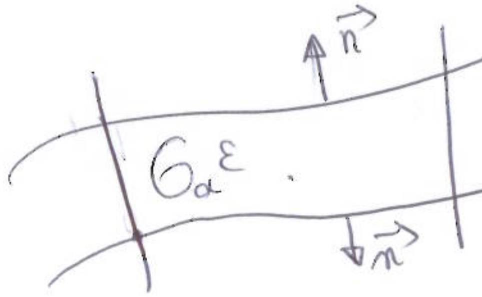
Moreover it is very close to have a minimum. Moreover, a classical super-solution having a minimum is constant. And we are very close to this situation!

Therefore, no worry about the result.

Before proving the Harnack inequality, we need a final lemma. Let F_a^ε be the set of contact points of $\{d(X,\partial E) = -\varepsilon\}$ and Ω_a^ε be its sub-level set, i.e. $\{d(X, F_a^\varepsilon) < 0\}$. Then

<u>lemma</u>. We have $Per(\Omega_a^\varepsilon, B_{1/2}) \le Per(E, B_{1/2})$

This lemma is due to Caffarelli-Cordoba. Its proof is omitted (in order to avoid complete exhaustion of the audience). The geometric situation is the following:



$F_a^\varepsilon \subset \{d(X,\partial E) = -\varepsilon\}$.

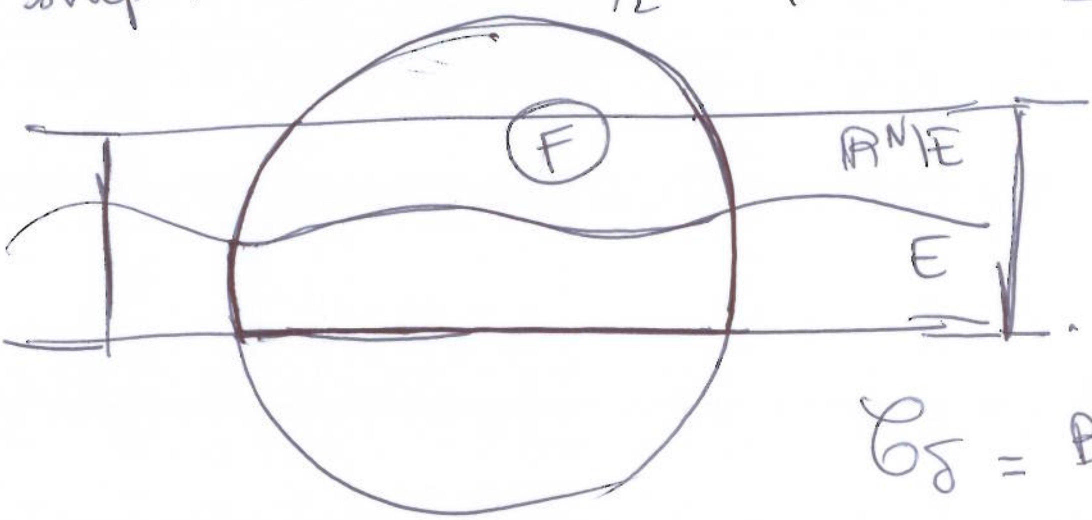
O_a^ε is bounded by ∂E, $\cancel{E}'$ $\{d(X,\partial E) = -\varepsilon\}$ and the set of rays from $\partial B_{1/2} \cap \{d(X,\partial E) = -\varepsilon\}$ to ∂E. Integration of $-\Delta d \le 0$ in O_a^ε would yield, if everything was smooth:

$$-\int_{\partial O_a^\varepsilon} Dd \cdot n \, d\sigma \le 0.$$

This time we have

$$-\int_{\partial E} Dd \cdot n \, d\sigma = -Per(E, B_{1/2}), \quad \int_{\{d(X,\partial E) = -\varepsilon\}} -Dd \cdot n \, d\sigma = +Per(\Omega_a^\varepsilon, B_{1/2})$$

Proof of the Harnack inequality - Time to use the fact that ∂E is sandwiched in a small strip. Let $F = B^{N+1}_{1/2} \cap \{x_{N+1} \geq -\delta\}$.



$$\mathcal{C}_\delta = B^N_1 \times [-\delta, \delta].$$

We have: $\operatorname{Per}(E, \mathcal{C}) \leq \operatorname{Per}(E \setminus F, \mathcal{C})$.

Hence: $\operatorname{Per}(E, B^N_{1/2}) \leq \operatorname{area}(\partial F \cap E)$

$$\leq |B_{1/2}| + C\delta.$$

Assume the existence of a point of $\partial E \cap B_{1/2}$ such that is height is less than $-(1-a)\delta$. Let k be such that $C^k a \leq \frac{1}{2}$. We have, for all $\varepsilon > 0$:

$$\operatorname{Per}\left(E \cap \{x_{N+1} \leq -\tfrac{1}{2}\}, B_{1/2}\right)$$

$$\geq \operatorname{Per}\left(\{d(X, \partial E) = -\varepsilon\} \cap \{x_{N+1} \leq -\tfrac{1}{2}\}, B_{1/2}\right)$$

$$\geq \operatorname{Per}\left(F^{\varepsilon}_{C^k a}, B_{1/2}\right)$$

$$\geq \left(1-(1-\mu)^k\right) |B_{1/2}|.$$

(F_{C^k} is a Lipschitz graph) ...

On the other hand, slide paraboloids of the form

$$\left\{ x_{N+1} = \frac{C}{2} |x-y|^2 + \alpha \right\}$$

from above, to touch $\partial E \cap \{x_{N+1} \geq -1/4\}$. With the Alexandrov-Bakelman-Pucci estimate, we find two constants: $q>0$ small, and $C>0$ large, universal, such that the ~~contact~~ set of points $x \in B_{1/2}$, such that $(x, u_\varepsilon^+(x))$ is a contact point with a paraboloid of the above class, has measure larger than q.

Going through the whole argument, we have:

$$\mathrm{Per}\left(E \cap \{x_{N+1} \geq -\tfrac{1}{4}\}, B_{1/2}\right) \geq q .$$

Conclusion:

$$\mathrm{Per}(E, B_{1/2}) \geq q + (1-(1-\mu)^k)|B_{1/2}| -$$

Contradiction.