Some Asymptotic models in flame propagation

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1 Introduction

The goal of this mini-course is to present some singular limits arising in the modelling of flame propagation. Under rather reasonable physical assumptions, such as the low Mach Number and constant density assumptions, the propagation of a premixed flame, with simple chemistry $A \to B$, can be described by the following reaction-diffusion system.

$$(1.1) \qquad \begin{cases} T_t - \Delta T = & Y f_\varepsilon(T) \\ Y_t - \dfrac{\Delta Y}{Le} = & -Y f_\varepsilon(T) \end{cases} \qquad (\mathbb{R}^3)$$

The notations are classical: $T(t,x)$ is the temperature, $Y(t,x)$ is the mass fraction of the reactant. The number $Le > 0$ is the Lewis number, i.e. the ratio between thermal and molecular diffusion. The function $f_\varepsilon(T)$ is the Arrhenius term, where the exponential has been linearized around the burnt gases temperature that we call here T_b:

$$(1.2) \qquad f_\varepsilon(T) = \frac{1}{\varepsilon^2} \exp(\frac{T - T_b}{\varepsilon})$$

for T close to Le^{-1}. To avoid the cold boundary difficulty, the function f_ε is assumed to vanish for $T \le \theta$, with $0 < \theta < T_b$. We will very quickly see how to compute T_b. This singular limit was first introduced by Zeldovich in 1937; an account of the underlying physics can be found in [16] and [17]. The idea was to extract explicit expressions from (1.1), in order to make some prediction about, for instance, the velocity of a combustion wave. This asymptotics was subsequently used in an enormous amount of papers, in order to account for dynamical properties of (1.1). It is impossible to cite them all here; a pionneer in the domain is Sivashinsky [15].

Although it is extremely simple-looking, system (1.1) has in store dificult mathematical problems: free boundary problems, complicated dynamics... We wish to focus in this mini-course on some salient aspects of the structure that is revealed by the high activation energy assumption. We will adopt the following plan: in the first part, we will focus on the simplest propagation mode that model (1.1) has in store, namely: travelling waves. We will see how the singular limit allows, in the travelling wave regime, to compute explicitely the velocity of a wave. We will end this part by a more eleborate two-phase model. In the second part, we will undertake the study of a more difficult model, namely a class of equations describing the propagation of spherical flames, which was introduced by G. Joulin in [9]: a priviledged time-scale is identified, under which a nonlinear integro-diferential equation for the flame radius is derived. We will show how the formal derivation works, and we will give a brief account of the main steps for a mathematically rigorous proof. Finally, the last part will be devoted to the large-time behaviour of the latter model, and we will introduce the main ideas of its mathematical treatment. We will also study richer models, derived in the same fashion, including in particular heat loss, and collective effects.

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2 Singular limits for travelling wave models

When located in a long thin tube - thus considered to be infinite, a gaseous reacting mixture sees the propagation of a flame as a travelling wave phenomenon. In this context, the only relevantvariable is the longitudinal coordinate x, and a travelling wave solution to (1.1) is a function of the form $(T(x+ct), Y(x+ct))$; the couple (c, T, Y) - we insist here that the speed c is an unknown of the problem - satisfies the differential system

$$\left\{ \begin{array}{rl} -T'' + cT' = & Yf(T) \\ -\dfrac{Y''}{Le} + cY' = & -Yf(T) \end{array} \right. \qquad (x \in \mathbb{R}) \tag{2.1}$$

Moreover, a travelling wave connects the burnt state to the unburnt one; if we assume that the fresh state is on the left we add the condition

$$T(-\infty) = 0, \quad Y(-\infty) = 1. \tag{2.2}$$

We wish to study System (2.1)-(2.2) - in particular existence, uniqueness, estimates, and to apply to it the high activation energy assumption.

2.1 The burnt gas temperature and the term f_ε

Assuming - which is reasonable both from a physical point of view and also from the point of view of the dynamics of (2.1) - that everything is burnt at $+\infty$, namely $Y(+\infty) = 0$, we may add the two equations of (2.1) and integrate between $-\infty$ and $+\infty$ - the justification of this procedure is a minor exercise. This yields $c(T_b + 0 - 1 - 0) = 0$, hence $T_b = 1$. We therefore complement (2.1)-(2.2) by

$$T(+\infty) = 1, \quad Y(+\infty) = 0. \tag{2.3}$$

It remains to make the term f_ε precise; indeed the physical basis of expression (1.2) is the Arrhenius term, which vaguely - but not completely - looks like that. Here is the principle of the derivation of (1.2): having in mind that a normalisation has already taken place, the term $f(T)$ in (1.1) may be written as

$$f(T) = \frac{1}{\tau_{chem}} e^{-\beta/T}$$

where τ_{chem} is a characteristic time of the chemcal reaction - that is supposed to be short - and β the - normalised - activation energy. We make the point here that there is **no** definite modelling of τ_{chem}, based for instance a rigorous kinetic gas theory. This quantity is therefore adjusted to experiments or numerical computations, and this also gives us the freedom to decide its relative size with respect to the reduced activation energy. Bearing in mind that the maximum temperature is 1, and also that we have neglected the ignition temperature correction when T is far from 1, write

$$\frac{1}{T} = \frac{1}{1+T-1} \sim 1-(T-1) \text{ at least in a vicinity of 1.}$$

Now, plugging this formula in the exponential yields

$$f(T) \sim \frac{e^{-\beta}}{\tau_{chem}} e^{\beta(T-1)}. \tag{2.4}$$

The reduced activation energy is large, for obvious physical reasons. Once this is said, notice that the exponential the above expression is accurate for *all* values of T, for it is in both cases extremely small when T is not close to 1. It remains to decide the scale of τ_{chem} that we are going to choose; we *decide*:

$$\tau_{chem} = \beta^{-2} e^{-\beta} \tag{2.5}$$

which is indeed extremely small. This is exactly (1.2). We urge the reader to consult [16] for a more physically orthodox derivation of the singular reaction term.

2.2 Existence results and singular limit for (2.1)-(2.3)

A general existence result is available in [3]. Here is the statement:

Theorem 2.1 *(Berestycki, Nicolaenko, Scheurer [3]) If f is smooth, and as stated in the introduction, then the travelling wave problem has at least one solution (c, T, Y). The functions T and Y are respectively increasing and decreasing, and c in bounded in terms of the mass of f between 0 and 1 and the ignition temperature θ.*

A very simple case is when $Le = 1$; in this case we have $T + Y \equiv 1$, and the temperature satisfies

$$-T'' + cT' = (1-T)f(T) = g(T). \tag{2.6}$$

The existence result is here much older and goes back to Kanel (1960). Let us see what happens when the high activation assumption is turned on, and let us call $g_\varepsilon(T) := (1-T)f(T)$. Notice that g_ε converges, in the distribution sense, to the Dirac mass at 1. We claim that the velocity can be computed in a rigorous fashion: indeed, multiplying (2.6) by T' and integrating by parts yields

$$c\int_{\mathbb{R}} T'^2 = \int_0^1 g_\varepsilon(T)dT \sim 1. \tag{2.7}$$

The LHS is clearly above the integral of T'^2 taken over the set where T is less than θ; due to the monotonicity of T this is not less than $\dfrac{c^2\theta^2}{2}$: hence a uniform bound from above for c. Now, assume the normalisation

$$T(0) = 1 + 100\varepsilon \text{Log}\varepsilon \tag{2.8}$$

Introducing the classical scaling

$$\xi = \frac{x}{\varepsilon}, \quad p(\xi) = \frac{T(\varepsilon\xi) - 1}{\varepsilon} \tag{2.9}$$

we have

$$-p'' + \varepsilon c p' = -pe^p, \quad p(0) = -100\mathrm{Log}\varepsilon. \tag{2.10}$$

Due to the upper estimate on c, it is once again a simple exercise to prove that

$$p'(0) = T'(0^+) \sim \sqrt{2}. \tag{2.11}$$

For $x < 0$ we have $T^{-1}f(T) = O(\varepsilon^{98})$. A WKB argument - we have at this stage no lower bound for c, but it can be carried through with a bit of extra care - valid because c is bounded from above, yields

$$T(x) \sim e^{cx}, \text{ hence } T'(0^-) \sim c. \tag{2.12}$$

This yields the asymptotic value o c: $c \sim \sqrt{2}$. As a matter of fact, the above argument can be pushed further in order to give a *constructive* argument.

A general convergence theorem can be obtained for (2.1)-(2.3). Here is the result:

Theorem 2.2 *(Berestycki, Nicolaenko, Scheurer [3]) Let c_ε be the velocity of a wave solution for (2.1)-(2.3) with $f = f_\varepsilon$ - given by (2.2), and for a general positive Lewis number Le. There holds*

$$\lim_{\varepsilon \to 0} c_\varepsilon = \sqrt{2Le}. \tag{2.13}$$

2.3 A two-phase flame model

We wish to apply the high activation energy limit to a more sophisticated model, namely a flame in a reacting two-phase mixture. The physics is the following: liquid droplets of fuel can evaporate and yield some gaseous fuel A, which itself undegoes the chemical reaction $A \to B$. The liquid droplets are assumed - this simplifies the analysis but is by no means crucial - to have, at a given place, the same radius $R(x)$; in the experiment under consideration they are stored at $-\infty$. The travelling wave system is then written under the form

$$\left\{ \begin{array}{rl} -T'' + cT' = & Yf(T) \\ -\dfrac{Y''}{Le} = cY' = & -Yf(T) - \dfrac{4\pi}{3}(R^3)' \\ cR' = & -\varphi(T)H(R) \end{array} \right. \tag{2.14}$$

where the data are

- the term $f(T)$ is already described,
- the term $\varphi(T)$, which represents the intensity at which the droplets evaporate, is positive over a threshold temperature $\theta_v < \theta$ - the vaporization, or boiling, temperature - and vanishes below; moreover we have $\dfrac{d\varphi}{dT} \geq 0$.

- Moreover, $H(R)$ is the Heaviside function; it is placed here to prevent the (unphysical) appearance of negative radii.

The existence of travelling waves is proved in [4]. The interesting features of (2.14) appear once again when the activation energy assumption is made. The conditions at $-\infty$ are the following:

$$T(-\infty) = 0, \quad Y(-\infty) = Y_u \geq 0, \quad R(-\infty) = R_u \geq 0. \tag{2.15}$$

However, we first have to decide what the maximal temperature is; to see this we once again integrate the equation for $T + Y$ between $-\infty$ and $+\infty$; this yields, assuming once again that $Y(+\infty) = R(+\infty) = 0$:

$$T(+\infty) = Y_u + \frac{4\pi R_u^3}{3} := T_b(R_u, Y_u). \tag{2.16}$$

The theorem is the following.

Theorem 2.3 *(Berthonnaud, Domelevo [4]). Fix $T_b^0 > \theta$ and consider the curve $\mathcal{C}$ of states (Y_u, R_u) such that $T_b(Y_u, R_u) = T_b^0$ and $Y_u < \theta$. There exists $R_u^{cr}(T_b^0)$ such that, for every $(Y_u, R_u) \in \mathcal{C}$ we have*

- *If $R_u < R_u^{cr}$, then $\lim_{\varepsilon \to 0} c_\varepsilon = \frac{\sqrt{2Le}}{T_b^0}$.*
- *If $R_u \geq R_u^{cr}$, then $(c_\varepsilon)_\varepsilon$ converges to the unique solution of the equation*

$$cR_u = \int_{c^{-1}\mathrm{Log}(\theta_v \theta^{-1})}^{0} \varphi(\theta e^{cx}) \, dx. \tag{2.17}$$

The interest of the above result is that it displays the following alternative: when the droplets are small, the flame roughly behaves as a premixed flame, and the phenomenon is driven by the thermal and molecular diffusions. When the droplets are large, the phenomenon is slowed down by the evaporation. These results give a theoretical justification to some numerical simulations by P. Haldenwang.

3 An asymptotic model for spherical flames: formal and rigorous derivation

3.1 Motivation

In an important paper, G. Joulin [9] derives a nonlinear integro-differential equation for the slow motion of the radius of three-dimensional spherical flames. This equation is deduced from system (1.1), with the following conditions at infinity:

$$T(t, r = +\infty) = 0, \qquad Y(t, r = +\infty) = 1. \tag{3.1}$$

The goal of this second part is to explain Joulin's derivation [9] of the asymptotic equation

$$(1 - \sqrt{Le})\, \partial_{1/2}\, \rho = 2Le\mathrm{Log}(\sqrt{Le^3}\, \rho) + \phi(\tau) \tag{3.2}$$

for $R_\varepsilon(\tau/\varepsilon^2)$ where $R_\varepsilon(t)$ is the radius of the flame ball. Here $\phi(\tau)$ is a forcing term depending on the initial datum. The operator $\partial_{1/2}$ is the classical fractional derivative of order 1/2. For a C^1 function ρ it is defined by

$$\partial_{1/2}\,\rho = \frac{1}{\sqrt{\pi}}\int_0^\tau \frac{\dot\rho(s)}{\sqrt{\tau - s}}\,ds.$$

The method that is devised in [9] to derive (3.2) is to notice that, when $Le < 1$, one can identify a slow time scale of the order ε^{-2}. This allows an explicit computation of the temperature and mass fraction at finite distance; the slow motion of the flame is then controlled by the temperature and mass fraction fields at $\xi = +\infty$: if the flame were really steady, then the temperature - resp. the mass fraction - would not, in general, be 0 - resp. 1 - at $\xi = +\infty$, thus violating the conditions (3.2). This imposes the introduction of a large spatial scale, and the computation of the variations of the temperature and mass fraction on this large scale yields, to the first order in ε, an evolution equation for the radius of the flame. These considerations are typical of $Le < 1$; they indeed break down for $Le > 1$: it is conjectured in [9], and proved in the present paper in the context of the evolution of (1.2) around a steady solution, that the characteristic time scale is ε^2.

One interesting aspect of Joulin's method is that it has a wide applicability: it is possible to add complex chemistry, heat loss effects in the burnt or fresh gases, heat losses induced by a wall, heat losses due to turbulence... of course, the more numerous an effect one adds, the richer the dynamics of the asymptotic equation is. A whole class of models is thus available: [10] - preferential diffusion in the presence of more than one chemical reaction; [5] - heat losses, collective effets [1], heat losses trough convection and nonlinear periodic effects [11]... The only condition for the retrieval of a nontrivial relevant asymptotic equation is the existence of this characteristic ε^{-2} time scale.

Joulin's method is based on formal asymptotic expansions, i.e. explicit expressions for T and Y are sought, and the higher order terms in ε are dropped. The question that one therefore may ask is the rigorous derivation of all these models, namely: do we have the right to drop the higher order terms in ε? A possible answer is that, if the approximations were not justified, then the limiting equation would certainly not be well-posed. In our case, the asymptotic equation is well-posed if $Le < 1$, and ill-posed if $Le > 1$. Another - and perhaps more interesting - way to pose the problem is the following: can we understand mathematically what makes the difference between the well behaved case $Le < 1$ with the case $Le > 1$, where the high activation asymptotics simply seems to be irrelevant from the dynamical point of view? In other words, what is the mechanism responsible for the well-posedness of the limiting problem when $Le < 1$, and for its ill-posedness when $Le > 1$?

This is a mathematical question that the formal asymptotics alone cannot explain. Therefore, given the breadth and variety of the different asymptotic models and nonlinear effects that can be retrieved from (1.1)-(3.1) - or slight modifications of it - it is worth spending some effort in trying to understand what properties of the original equations trigger the result.

Needless to say, a lot of the ideas present or underlying in [9] are of constant help: in particular, the Ariadne tread of our proof is the fact that, in our problem, the time-derivatives - which are usually a handicap in parabolic free boundary problems, because they are very difficult to control - help us: they are in general be one order of smallness - in ε - more than the unknowns themselves.

3.2 Setting of te problem and main results

Let us, for this whole session, assume that the functions are radially symmetric and let us give the following definition.

Definition 3.1 *Let $A > 0$ be large - for instance $A = 10^{10}$. For any solution (T, Y) of (1.1), the radius $R(t)$ of the solution - in other words, of the flame - is the largest $r > 0$ such that*

$$T(t, r) = T_b + A\varepsilon \mathrm{Log}\varepsilon. \tag{3.3}$$

Note that the maximum temperature is still to be determined. However, the fact that we are looking at a problem in $\mathbb{R}^3$ implies the presence of steady solutions to (1.1); this is a fact dating back to Zeldovich [17]. Consider the family of problems, parametrized by the the real number w, satisfying

$$\left\{ \begin{array}{l} -\Delta T = Y f_\varepsilon(T) \\ -\dfrac{\Delta Y}{Le} = -Y f_\varepsilon(T) \end{array} \right. \qquad (\mathbb{R}^3) \tag{3.4}$$

$$T(r = +\infty) = \varepsilon w, \qquad Y(r = +\infty) = 1. \tag{3.5}$$

There is a C^1 family $(T_w^\varepsilon, Y_w^\varepsilon)$ of solutions to the above problem - relatively simple exercise once the first part of this minicourse is understood. This also yields the maximal temperature that we are looking for: the function $W_w^\varepsilon = T_w^\varepsilon + \dfrac{Y_w^\varepsilon}{Le}$ being harmonic in $\mathbb{R}^3$, it is a constant, hence equal to 1 by the conditions at infinity. Inside the flame, the mass fraction of fresh gases can be assumed to be negligible; hence we have:

$$T_b = \frac{1}{Le}. \tag{3.6}$$

This closes our model. This being under control, the main result of this session, due to C. Lederman, N. Wolanski and the author, is the following.

Theorem 3.2 *([8]). Set $Le = 1 - \delta$ with $0 < \delta < 1$. Let $(T^\varepsilon(t, r), Y^\varepsilon(t, r))$ be a solution of (1.1), with initial datum $(T_0^\varepsilon, Y_0^\varepsilon)$ such that*

- *there exists $w_0 \in \mathbb{R}$, $\nu \in]0, 1[$ and a constant $C > 0$, such that*

$$|(T_0^\varepsilon, Y_0^\varepsilon)(x) - (T_{w_0}^\varepsilon, Y_{w_0}^\varepsilon)(x)|_\infty \leq C\varepsilon^3 \quad \text{for } \xi \leq \varepsilon^{-\nu} \tag{3.7}$$

- *$(T_0^\varepsilon, Y_0^\varepsilon)$ converges to $(0, 1)$ at a certain rate at $r = +\infty$, as $\varepsilon \to 0$.*

Set $\tau = \varepsilon^2 t$. Then there exist $\delta_0 > 0$, $\varepsilon_0 > 0$ and a smooth function $\phi(\tau)$, determined by the limiting behavior of the rescaled initial datum at infinity, such that, if $\delta < \delta_0$ and $\varepsilon < \varepsilon_0$, there exists $0 < \tau_{max} \leq +\infty$ such that, if $0 < \tau_0 < \tau_{max}$, the radius $R^\varepsilon(t)$ of the solution $(T^\varepsilon, Y^\varepsilon)$ satisfies that

$$R^\varepsilon\Big(\frac{\tau}{\varepsilon^2}\Big) \to \bar{R}(\tau), \qquad \text{uniformly on } [0, \tau_0],$$

and $\bar{R}(\tau)$ is the solution of

$$(1 - \sqrt{Le})\, \partial_{1/2}\bar{R} = 2Le\mathrm{Log}(\sqrt{Le^3}\, \bar{R}) + \phi(\tau), \tag{3.8}$$

with

$$R(0) = \frac{1}{\sqrt{Le^3}} \exp\Big(-\frac{w_0}{2}\Big). \tag{3.9}$$

Remark 3.3 *1. The actual hypotheses on $(T_0^\varepsilon, Y_0^\varepsilon)$ is that it be at distance at most $O(\varepsilon^3)$ in $\mathbb{R}^3$ of a pair (T_0, Y_0) constructed in Theorem 4.1. In particular, a pair that coincides with the Zeldovich flame $T_{w_0}^\varepsilon$ in $B(0, \varepsilon^{-\nu})$.*
2. τ_{max} is the lifetime of the maximal solution to (3.8)-(3.9).

The formulation of this result is a good example of what can be expected for such a problem: an $O(\varepsilon)$ perturbation of a steady configuration at $O(\varepsilon^{-1})$ distance implies nontrivial $O(1)$ changes on the ε^{-2} time scale.

The merit of this theorem is to give a justification of equation (3.8), but it is only a first step to a complete understanding of the dynamics of system (3.2), and some questions linger around: on the one hand, it would be desirable to remove the assumption "Lewis number close to 1". This is far from obvious, for it involves a stability analysis in the flame sheet that is completely open. Also, it would be interesting to investigate the large time behaviour of the solutions of (1.1); namely for times much larger than ε^{-2}. A possible scenario is that, when the flame radius is large enough, then a travelling wave behaviour is picked up. It would be very nice to prove - or disprove - such a result.

It is to be noted here that the case under study differs radically from the case $Le = 1$. When $Le = 1$, the system becomes, setting $W = T + Y - 1$ and $g_\varepsilon(T) = (1 - T) f_\varepsilon(T)$:

$$\left\{ \begin{array}{rl} T_t - \Delta T = & g_\varepsilon(T) + W f_\varepsilon(T) \\ W_t - \Delta W = & 0 \end{array} \right. \qquad (\mathbb{R}^3) \tag{3.10}$$

If $W(0, x) = 0$, then $W(t, x) = 0$ and (3.10) reduces to the scalar equation

$$T_t - \Delta T = g_\varepsilon(T). \tag{3.11}$$

The limit $\varepsilon \to 0$ in (3.11) is studied in [6], [7] for very general initial data. The outcome is that the solutions T_ε of (3.11) converge, in a suitable sense, to viscosity solutions of the free boundary problem

$$\left\{ \begin{array}{rll} T_t - \Delta T = & 0 & (t > 0, x \in \Omega_t := \{x : \ T(t,x) < 1\}) \\ T = 1, \ \ \partial_\nu T = & C & (t > 0, x \in \partial\Omega_t) \end{array} \right. \tag{3.12}$$

where ν is the outer normal vector to Ω_t and C is a constant determined by the function g_ε. In any case, the correct time scale for (3.10) and (3.11) is 1, contrary to what happens in the case that we study. Coming back to the full problem with $Le \neq 1$, we note that the complete characterisation of the behaviour of the solutions to (1.1) for $\varepsilon \to 0$ is still a very much open problem, and that we are exploring only a small subset of what may happen. In particular, it would be very interesting to know whether a reasonable singular limit can be derived; from our conclusions this is far from obvious.

3.3 Strategy of the proof of Theorem 3.2

The first thing to do is to produce steady solutions; this is already under control. The first crucial part of the analysis is the stability of the Zeldovich solution. Due to the highly singular behaviour of the nonlinearity in terms of ε, unstable eigenvalues of the order ε^{-2} are *a priori* possible. Let $\mathcal{L}^\varepsilon$ be the linearized operator about the Zeldovich solution:

$$\mathcal{L}^\varepsilon(u, v) = \begin{pmatrix} -\Delta u - f'_\varepsilon(T_0) Y_0 u - f_\varepsilon(T_0) v \\ \dfrac{\Delta v}{Le} + f'_\varepsilon(T_0) Y_0 u + f_\varepsilon(T_0) v \end{pmatrix} \tag{3.13}$$

We are interested in the eigenvalue problem

$$\mathcal{L}^\varepsilon(u,v) = \lambda(u,v), \qquad (u,v) \in L^2(\mathbb{R}^3). \tag{3.14}$$

Theorem 3.4 *Assume $Le < 1$: $Le = 1-\delta$, $\delta > 0$. There exists $\delta_0 > 0$ such that, for all $\delta \in]0,\delta_0[$, the following is true: there exist three constants $m > 0$, $M > 0$ and $\theta \in [0, \frac{\pi}{2}[$, possibly depending on δ, such that the only eigenvalue of (3.14) outside the set*

$$\{\lambda \in \mathcal{C} : \ \arg(\lambda + M) \le \theta, \quad \mathrm{Re}\lambda \ge m\}$$

is a complex number $\varepsilon^2\lambda_\varepsilon$ such that $\lim\limits_{\varepsilon\to 0}(\varepsilon^{-2}\lambda_\varepsilon)$ is a finite negative number.

As opposed to this situation, $\mathcal{L}^\varepsilon$ has an unstable eigenvalue of size ε^{-2} when $Le > 1$. This explains why the case $Le > 1$ cannot produce quasi-steady solutions.

Then we rederive Joulin's equation. In fact, using Joulin's ideas, we construct in a rigorous fashion a class of functions $(T_J(t,x), Y_J(t,x))$ satisfying the system

$$\left\{ \begin{array}{rl} T_t - \Delta T = & Y f_\varepsilon(T) + O(\varepsilon^C)(\mathbf{1}_{|x|\le\varepsilon^{-\nu}} + \delta_{|x|=\varepsilon^{-\nu}}) \\ Y_t - \dfrac{\Delta Y}{Le} = & -Y f_\varepsilon(T) + O(\varepsilon^C)(\mathbf{1}_{|x|\le\varepsilon^{-\nu}} + \delta_{|x|=\varepsilon^{-\nu}}) \end{array} \right. \qquad (\mathbb{R}^3) \tag{3.15}$$

$$T(t, r = +\infty) = 0, \qquad Y(t, r=+\infty) = 1. \tag{3.16}$$

for a large $C > 0$, independent of ε, to be adapted to our needs. In the end, the solution that is constructed is Lipschitz in space and time and, if $R_J(t)$ is its radius, then $R_J(\tau/\varepsilon^2)$ satisfies the Joulin equation (3.2) up to $O(\varepsilon^{C/2})$ errors.

The next step is then to show that, on time intervals of order ε^{-2}, the solutions of (1.1) remain ε^2-close to the ones of (3.15), provided the initial data are ε^2-close. This is a nonlinear stability result, with the additional inconvenient that the essential spectrum of the linearized operator around a Zeldovich steady solution - or a frozen Joulin solution - contains a segment $[0, M]$. Rather precise decay estimates on the linear semigroup associated to the linearized operator around a frozen Joulin solution, together with the fact that the linear operator evolves on the ε^{-2} time scale, allow us to conclude.

4 Large time dynamics of spherical flame models

In this third part, we analyze the propagation of a spherical flame according to the model (3.2), that we rewrite as

$$R\,\partial_{1/2}R = R\,\mathrm{Log}R + E\,q(t), \quad R(0) = 0. \tag{4.1}$$

The forcing term consists of a function $q(t) \ge 0$ which represents the source of energy and a factor $E > 0$ which stands for the intensity of the heat source. In this last part, we will always assume that q is smooth, and normalized to mass 1, i.e., $\displaystyle\int_0^\infty q(t)\,dt = 1$. We also wish to include additional effects, such as heat losses.

4.1 Propagation versus extinction in model 4.1

There are at least two questions that we may ask.

- Determine whether there is a critical value $E_{cr}(q)$ such that the flame does not propagate if $E < E_{cr}(q)$.
- Is there a uniform lower bound for $E_{cr}(q)$ independent of q? If this is so, we would like to know whether there exists minimizer for the critical energy and what it looks like. Is it an L^1 function or a measure, for instance a Dirac mass?

These questions have a practical importance. First, they play a role in assessing the qualitative relevance of the model: experimental evidence shows that a flame needs a certain amount of energy to propagate, except in extremely specific materials. Second, they play are important role in safety considerations, if one belives that the model is accurate. A preliminary numerical investigation on the existence of a uniform bound was performed in [9], where the critical energy is computed for a 2-parameter family of exponentially decreasing driving forces, normalized to 1, and a global minimum for the critical energy in this family of exponentials is found.

The first question has a positive answer, as is provided by the following

Theorem 4.1 *([2]). Given $q(t)$ as above, there exists a critical value $E_{cr} = E_{cr}(q) > 0$ such that, as time increases, the flame radius*

- *goes to infinity for $E > E_{cr}$,*
- *goes to 1 for $E = E_{cr}$,*
- *quenches in finite or infinite time if $E < E_{cr}$.*

The definition of *quenching at time* $t_0 \in (0, +\infty]$ is the following: there is a sequence $(t_n)_n$ going to t_0 such that $\lim_{n\to+\infty} R(t_n) = 0$. A by-product of the analysis of [2] is that we in fact have $\lim_{t\to t_0} R(t) = 0$.

The main tool in the analysis is the lifting of the unknown function $R(t)$ to (4.1) into the solution of a parabolic equation: $R(t)$ satisfies (4.1) if and only if it is the value at $x = 0$ of the solution of the following PDE problem

$$\left\{\begin{array}{ll} u_t - u_{xx} = 0 & x > 0,\ t > 0 \\ u_x = -2(\operatorname{Log} u + \dfrac{E\,q(t)}{u}) & x = 0,\ t > 0 \\ u(0, x) = 0 & x > 0. \end{array}\right. \tag{4.2}$$

The second question also has a positive answer. Here is the precise result.

Theorem 4.2 *([12]) Consider equation* (4.1). *There is $E_0 > 0$ such that, for all $q \in C(\mathbb{R}_+)$ with $q \geq 0$ and $\int_0^{+\infty} q(s)\ ds = 1$, we have*

$$E_{cr}(q) \geq E_0.$$

We wish to emphasize that the problem is not totally trivial, if we only have the L^1 bound for q in hand: there is indeed no theory for (4.1) with q only in L^1, since we do not know what sense to give to $\frac{q}{R}$ if $R(0) = 0$. On the other hand, we have to be able to treat these kinds of functions q, since we wish to be able to model highly transient phenomena like sparks. In fact, a by-product of our study is that there is no reasonable solution to (4.1) when q is a Dirac mass at $t = 0$.

4.2 Proof of Theorem 4.2

We give here a complete proof of Theorem 4.2, for it is very short. In the sequel, we will denote, as is usual, by $*$ the convolution product on $\mathbb{R}_+$; the following result is well-known:

Lemma 4.3 *Consider the solution $\phi(t)$ of the equation*

$$\partial_{1/2}\phi + \phi = q(t). \tag{4.3}$$

where $q \in L^1(\mathbb{R}_+)$. Then, for all $p \in [1, 2)$, we have $\phi \in L^p(\mathbb{R}_+)$; moreover the mapping $q \mapsto \phi$ is continuous from $L^1(\mathbb{R}_+)$ to $L^p(\mathbb{R}_+)$.

Indeed, we have $\phi = H_1 * q$, with $H_1(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-ts}\frac{\sqrt{s}}{1+s}\,ds$. This implies

$$H_1(t) \leq C\min\left(\frac{1}{\sqrt{t}}, \frac{1}{t^{3/2}}\right),$$

which concludes the lemma. Then, the cornerstone of the proof is

Lemma 4.4 *Given (i) two real positive numbers $\beta > 0$ and $\delta \in (0, 1/4]$, and (ii) a function $q \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ with unit total mass, consider the equation with unknown ϕ:*

$$\partial_{1/2}\phi + \phi = \beta\phi^{1+\delta} + Eq(t). \tag{4.4}$$

There exists $E_0 > 0$ such that, for $E \in [0, E_0]$, this problem has a unique solution $\phi_E(t)$ over $\mathbb{R}_+$ which satisfies

$$\|\phi_E\|_{L^{1+\delta}(\mathbb{R}_+)} \leq CE, \tag{4.5}$$

with a constant $C > 0$ independent of the function q.

Proof. We use a classical Banach fixed point argument. For $v \in L^{1+\delta}(\mathbb{R}_+)$ we define $\mathcal{T}v$ by

$$\mathcal{T}v = H_1 * (\beta v^{1+\delta} + Eq). \tag{4.6}$$

We define X_E as the intersection of the ball of centre 0 and radius $E^{1-\delta/2}$ in $L^{1+\delta}(\mathbb{R}_+)$ with the cone of nonnegative functions; let us prove that $\mathcal{T}$ is a contraction in X_E.

1. We have

$$\begin{aligned}\|\mathcal{T}v\|_{1+\delta} \leq\ & C_\delta(E\|q\|_1 + \beta\|v^{1+\delta}\|_1)\\ =\ & C_\delta(E\|q\|_1 + \beta\|v\|_{1+\delta}^{1+\delta})\\ \leq\ & C_\delta(E + \beta E^{(1+\delta)(1-\delta/2)}),\end{aligned}$$

which is less than $E^{1-\delta/2}$ as soon as $E > 0$ is small enough. Then we notice that we always have $\mathcal{T}v \geq 0$ if $v \geq 0$, which implies that $\mathcal{T}$ maps X_E into itself.

2. Using the following inequality

$$|v^{1+\delta} - w^{1+\delta}| \leq C_\delta(|v|^\delta + |w|^\delta)|v - w|$$

for a possibly different C_δ, we obtain

$$\begin{aligned} \|\mathcal{T}v - \mathcal{T}w\|_{1+\delta} \leq \ & C_\delta \|v^{1+\delta} - w^{1+\delta}\|_1 \\ \leq \ & C_\delta(\|v\|^\delta_{1+\delta} + \|w\|^\delta_{1+\delta})\|v - w\|_{1+\delta} \\ \leq \ & C_\delta E^{\delta(1-\delta/2)}\|v - w\|_{1+\delta} \end{aligned}$$

which proves that $\mathcal{T}$ is a contraction of X_E as soon as E is small enough. This ends the proof of the lemma. □

We may now conclude. Indeed, let $u(t, .)$ be the lifting into (4.2) of a solution of (4.1). Then set $v(t, x) = u^2(t, x)$; the function v satisfies

$$v_t - v_{xx} + 2(\partial_x \sqrt{v})^2 = 2(\sqrt{v}\,\mathrm{Log} v + 2Eq(t))\delta_{x=0}, \tag{4.7}$$

with $v(0, x) = 0$ for all $x > 0$. Choose $\delta \in (0, 1/4]$. There exists $\beta > 0$ such that we have:

$$\forall v > 0, \quad 2\sqrt{v} \log v \leq -v + \beta v^{1+\delta}.$$

It then follows from the Maximum Principle that $v(t, x) \leq \bar{v}(t, x)$, where $\bar{v}(t, x)$ is the solution of

$$\begin{aligned} \bar{v}_t - \bar{v}_{xx} \ &= 2(-\bar{v} + \beta\bar{v}^{1+\delta} + 2Eq(t))\delta_{x=0} \\ \bar{v}(0, x) \ &= 0. \end{aligned} \tag{4.8}$$

Using again the equivalence of formulations we see that the function $\phi(t) := \bar{v}(t, 0)$ solves equation (4.4). Hence, if $E > 0$ is small enough, then $\|\phi\|_{1+\delta} \leq CE$, independently of q. Therefore, $R(t) = u(t, 0)$ cannot go to $+\infty$ as $t \to +\infty$.

4.3 Further models

We start by including a heat loss term in model (1.1); then (4.1) becomes - see [5]:

$$\partial_{1/2}R = \mathrm{Log}R - \lambda R^2 + \frac{Eq(t)}{R}, \quad R(0) = 0. \tag{4.9}$$

The term accounting for the heat losses is of course $-\lambda R^2$. It is readily seen that, if $\lambda < \lambda_{cr} = e^{-1}$, then the equation

$$\mathrm{Log}R = \lambda R^2$$

has two solutions $0 < R_-(\lambda) < R_+(\lambda)$; moreover $R_-(0) = 1$. They collapse into a single R_{cr} for $\lambda = \lambda_{cr}$. The theorem that can be obtained id the following:

Theorem 4.5 *([13]) Assume $q(t)$ to be smooth and have unit mass. For $\lambda \geq \lambda_{cr}$, then $R(t)$ always quenches in finite or infinite time. If $\lambda < \lambda_{cr}$, then there is $E_{cr}(\lambda, q) > 0$ such that $R(t)$*

- *quenches if $E < E_{cr}(q, \lambda)$,*

- *goes to $R_-(\lambda)$ if $E = E_{cr}(q,\lambda)$,*
- *goes to $R_+(\lambda)$ if $E > E_{cr}(q,\lambda)$.*

Equation (4.9) may itself be made more sophisticated if we ask the heat losses to depend on time. This is once again an important question from the point of view of applications to security issues. The model problem that we investigate is the following - see once again [5]

$$\partial_{1/2} R = \mathrm{Log} R - \lambda(t) R^2 + \frac{Eq(t)}{R}, \quad R(0) = 0. \tag{4.10}$$

Theorem 4.6 *([14]) Assume $\lambda(t)$ to be 1-periodic in t, while remaining sub-critical, i.e. $\lambda(t) < \lambda_{cr}$ for all t. Then there are two 1-periodic functions $0 < R_-(\lambda,t) < R_+(\lambda,t)$ such thatis $E_{cr}(\lambda,q) > 0$ such that: if $q(t)$ is smooth and has unit mass, then*

- *$R(t)$ quenches if $E < E_{cr}(q,\lambda)$,*
- *we have $\lim_{t\to+\infty}(R(t) - R_-(\lambda,t)) = 0$ if $E = E_{cr}(q,\lambda)$,*
- *we have $\lim_{t\to+\infty}(R(t) - R_+(\lambda,t)) = 0$ if $E > E_{cr}(q,\lambda)$.*

Theorems 4.5 and 4.6 imply that the heat losses, if not strong enough, are not sufficient to quench a spherical flame, and that an eternal flame may survive despite the heat losses. The final model, however, deals with the collective behaviour of a population of spherical flames that exchange heat from one another, which results in a collective quenching. The context is described in d'Angelo-Joulin [1], and the model that we are studying comes from their paper. Consider a population of spherical flames, that is parametrized by the initial radius ρ. If the size of the flames is relatively small with respect to the typical spatial scale, the population can be described by the sole unknown $R(t,\rho)$ which designates the radius at time t of the flames that had radius ρ at time 0. The equation is

$$\partial_{1/2} R = \mathrm{Log} R - \lambda R^2 - \nu \mathcal{F}(t)[R], \quad R(0,\rho) = \rho \tag{4.11}$$

where the operator $\mathcal{F}(t)$ is given by

$$\mathcal{F}(t)[R] = \int_0^t \int_{\rho>0} N_0(\rho) R(s,\rho)\, d\rho ds. \tag{4.12}$$

Here, N_0 is a smooth probability density, and the operator $\mathcal{F}(t)$ accounts for the heat lost by a sample of the population of spherical flames to the rest of the poulation. We insist here that this strange-looking term can be derived in a physically rigorous fashion, from the first principles. The real number $\nu > 0$ is a coupling parameter, and we are interested in what happens for small ν. Recall that, if $\nu = 0$, then $R(t,\rho)$ goes to $R_+(\lambda)$ as soon as $\rho \geq R_+(\lambda)$.

Theorem 4.7 *([14]) Assume $\nu > 0$. Then there is $t_{quench}(\rho) \geq C\rho$ such that $R(t,\rho)$ quenches at time $t_{quench}(\rho)$. Moreover, for $\nu > 0$ small and for a given $\rho > R_2(\lambda)$, the following scenario is valid:*

- *there is a function $R_\nu(t)$, independent of ρ, such that $R(t,\rho)$ stays close to $R_\nu(t)$ during a duration of time $t_{max}(\nu) \sim \dfrac{\tau_0}{\nu}$, τ_0 explicit depending only on data;*

- *then $R(t)$ quenches; we have $t_{quench}(\rho) \geq t_{max}(\nu) + O(\frac{1}{\sqrt{\nu}})$.*

An intersting problem would be to derive a kinetic-like equations of the time 4.12, where the flames are ignited by a population of heat sources; the qualitative behaviour would, most probably, look equivalent to the one derived in Theorem 4.7.

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