3. For $R>0$ large, compute a subsolution $\underline{u}_R$ to $(-\Delta)^\alpha u = 0$, $u|_{\partial B_R} = 0$, $u>0$ in ~~B_R~~ in the vicinity of ∂B_R.

4. Cut this subsolution in a neighbourhood of ∂B_R.

Exercise. Let u be a solution of $(-\Delta)^\alpha u = 0$ in Ω, $u = \varphi$ outside $\partial\Omega$.
Assume that $\varphi \geq 0$, and $\exists x_0 \in \partial\Omega : \varphi(x_0)=0$.
Then there is $q>0$ such that

$$u(x) \sim q|(x-x_0)\cdot \nu(x_0)|^\alpha \quad \text{as } x \to x_0.$$

All this to say that it is perfectly legitimate to consider problems of the form $(S)_\alpha$. They may not have any solutions, but they are not entirely stupid.

2°). The Caffarelli - Silvestre extension.

Recall that, if we solve

$$-\Delta v = 0 \qquad \mathbb{R}^2_+ = (x,y) \in \mathbb{R}\times\mathbb{R}_+.$$

$$v(x,0) = u(x)$$

we have $v(x,y) = \displaystyle\int \frac{y}{y^2+(x-x')^2}\, u(x')\,dx'.$

How can we see it?

The solution of

$$-\Delta w = 0 \qquad \frac{\partial w}{\partial y} = -u$$

is $\displaystyle\int \frac{u(x')}{(|x-x'|^2+y^2)^{\frac{N-2}{2}}}\,dx'$.

Consider then $v = -\partial_y w$:

(i). we have $\Delta v = \Delta w = 0$ in $\mathbb{R}^2_+$.

(ii). $v(x,0) = u(y)$.

And thus, up once again to constant coefficients (that we can adjust so as to have a unit integral):

$$v(x,y) = \int \frac{y}{(|x-x'|^2+y^2)^{\frac{N}{2}}}\,dy .$$

(convolution with the Poisson kernel). ~~and~~ Moreover ~~that~~ $\frac{\partial v}{\partial y}(x,0) = -(-\Delta)^{1/2} u(x)$. Indeed:

$$\frac{v(x,y)-v(x,0)}{y} = \int \frac{u(x')-u(x)}{y^2+(x-x')^2}\,dx'.$$

$$= \text{P.V.} \int \frac{u(x')-u(x)}{(x'-x)^2}\,dx'.$$

Here we have neglected the constants.

Now, ~~set~~ $\beta = 1-2\alpha$.

$\beta = 0$ if $\alpha = \frac{1}{2}$.

< 0 if $\alpha > \frac{1}{2}$.

> 0 if $\alpha < \frac{1}{2}$.

Solve: $(\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times \mathbb{R}_+) = \{(x,y)\}$.

$$\frac{-\operatorname{div}(y^\beta \nabla v) = 0}{v(0,x) = u(x)}.$$

Is there an explicit expression? Expand,

$$\frac{-\Delta_x v - \frac{\beta}{y} v_y - v_{yy} = 0}{v(0,x) = u(x)}.$$

Suppose $\beta \in \mathbb{N}$ (which is of course false).

Then y can be thought of a radial variable in $\beta+1$ dimensions.

Hence the fundamental solution:

$$G_\beta(x,y) = \frac{1}{\left(|x|^2+y^2\right)^{\frac{N+\beta-1}{2}}}$$

$$= \frac{1}{\left(|x|^2+y^2\right)^{\frac{N-2\alpha}{2}}}.$$

How do we access the Poisson kernel?

<u>Exercise.</u> If v solves $-\mathrm{div}(y^a \nabla v) = 0$ in $\mathbb{R}^N_+$, then $w = y^a \partial_y v$ solves $-\mathrm{div}(y^{-a} \nabla w) = 0$ in $\mathbb{R}^N_+$.

<u>Consequence.</u> Consider $w(x,y) = G_{-\beta}(x,y) *_x u$;

$$v(x,y) = y^{-\beta} \partial_y \left[G_{-\beta}(x,y) * u \right].$$

(i). Then $-\mathrm{div}(y^\beta \nabla v) = 0$ in $\mathbb{R}^2_+$.

(ii). The trace of w on $\mathbb{R}_+$ is obtained as follows: we have

$$\lim_{y\to 0} y^{-\beta} \frac{\partial w}{\partial y}(x,y) = \lim_{y \to 0} \left(y^{-\beta} \partial_y G_{-\beta}(x,y) *_x u \right)$$

$$= v(x,0) = u(x).$$

Hence we have the Poisson kernel,

$$w(x,y) = \int \frac{u(x')}{(|x-x'|^2+y^2)^{\frac{N-\beta-1}{2}}}\,dx'$$

$$v(x,y) = y^{-\beta}\int \frac{u(x')\,y}{(|x-x'|^2+y^2)^{\frac{N-\beta+1}{2}}}\,dx'.$$

~~Once again, we have~~ $\cancel{\int}$

$$= \int \frac{y^{2\alpha}u(x')}{(|x-x'|^2+y^2)^{\frac{N+2\alpha}{2}}}\,dx'.$$

Once again,

$$\int \frac{y^{2\alpha}}{(|x-x'|^2+y^2)^{\frac{N+2\alpha}{2}}}\,dx' = O(1).$$ Hence the constant can be adjusted to have $\int = 1$.

<u>Exercise</u>. Prove that

$$(-\Delta)^\alpha u(x) = -\lim_{y\to 0^+} y^\beta v_y(x,y),$$

$$v(x,y) = c_\alpha \int \frac{y^{2\alpha}\,u(x')}{((x-x')^2+y^2)^{\frac{N+2\alpha}{2}}}\,dx'.$$

3°). <u>The weak form of $(S)_\alpha$</u>.

Unfortunately, there is no real hope to have strong solutions to $(S)_\alpha$. Worse, it is known that, for the usual laplacian, there can be cone-like singularities in dimensions ≥ 8.

Hence we need a weak form for $(S)_2$.

Let us take help a last time from $(S)_1$. Consider the energy:

$$J_1(u, B) = \int_B |Du|^2 + \mathcal{L}_N(\{u>0\}).$$

B: any sufficiently nice set of $\mathbb{R}^N$.

Idea: look at local minimisers of J_1, i.e. a function u satisfying:

$$\forall B \subset \mathbb{R}^N, \quad \forall v \in H^1(B),$$

$$v = u \text{ on } \partial B \Rightarrow J_1(u, B) \leq J_1(v, B).$$

Assume: u ~~defin~~ $\in H^1(B_1) \cap C(B_1)$.

Then: (i) $-\Delta u = 0$ inside $\{u>0\}$.

(ii). If u is smooth up to $\partial\{u>0\}$ ~~and have~~ $\partial\{u>0\}$ is smooth, we have $\frac{\partial u}{\partial \nu} = -1$.

Actually, this is not too surprising. Perturbations of u consist, in particular, in moving the free boundary. Therefore it is not unexpected to have such a F.B.-

These pbs were studied by Alt-Caffarelli.

Generalisation to $(S)_\alpha$.

We are going to consider, in the light of the extension, energies of the form

$$J_\alpha(u, B) = \int_B |y|^\beta |\nabla u|^2 + \int_{B \cap \mathbb{R}^N} \mathbb{1}_{\{u>0\}} \, dx,$$

where: $B \subset \mathbb{R}^{N+1} = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}\}$; $\beta = 1-2\alpha$

Here, we have just switched from $\mathbb{R}^{N+1}_+$ to $\mathbb{R}^{N+1}$ by an even extension of the functions under consideration. We will always do this ~~sur~~ without further mention.

A few words on the functional framework.

B open subset of $\mathbb{R}^{N+1}$,

$$H^1(\beta, B) = \{ v \in L^2(B), \int_B |y|^\beta |\nabla v|^2 \, dx\, dy < +\infty \}.$$

$$J_\alpha(v, B) = \int_B |y|^\beta |\nabla v|^2 + \mathcal{L}_N(\mathbb{R}^N \cap \{u>0\}).$$

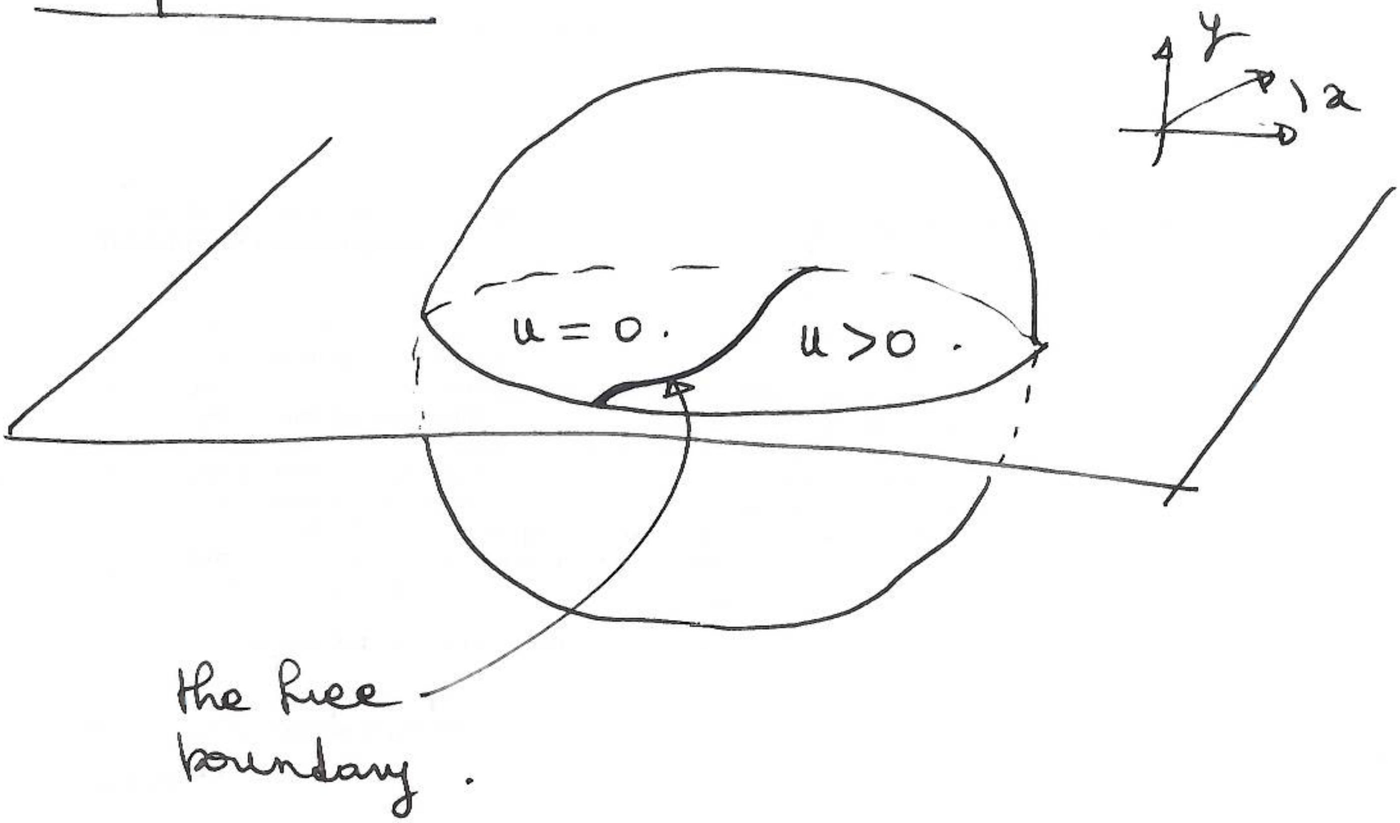
Minimisation problem.

We want to study the properties of $u \in H^1(\beta, B_1)$ (B_1 = unit ball of $\mathbb{R}^{N+1}$) such that: $\forall B \subset\subset B_1$

$$\boxed{\forall v \in H^1(\beta, B), \quad v = u \text{ on } \partial B \Rightarrow J_\alpha(u, B) \leq J_\alpha(v, B)}$$

We will ~~sw~~ from now on switch to this problem.

The picture.



4°). General results. (Work w. L. Caffarelli and Y. Sire).

Let u be a minimiser in B_1.

Th. 1. (Optimal regularity).

$u \in C^\alpha(B_1)$.

We cannot do better: see $(x_+)^\alpha$.

Why do we insist in proving an optimal regularity? Because we wish to perform a study of $\Gamma = \partial\{u > 0\}$. And, to do it, we need rescalings that are ~~independent~~ keep the energy invariant (or leave it fixed by a constant).

Th. 2. (Nondegeneracy). There is $C>0$ such that, if $0 \in \Gamma$:

(i). If ~~u(x)~~ $u(x,0)>0$, then
$$u(x,0) \geq C\, d(x,\Gamma)^{\alpha}.$$

(ii). Not exactly equivalent.
$$u(x,0) \geq C |x|^{\alpha}.$$

Once again, satisfied by $(x_+)^{\alpha}$.

Suppose now that $0 \in \Gamma$, and set
$$u_r(x) = \frac{1}{r^{\alpha}} u(rx, ry). \text{ Then,}$$

- u_r is a minimiser in $B_{1/r}$.
- A subsequence of $(u_r)_r$ will converge to some function $w(x,y) \in C^{\alpha}(\mathbb{R}^{N+1})$ which
  - has optimal growth.
  - which is nontrivial.

Thus we may start studying the free boundary.

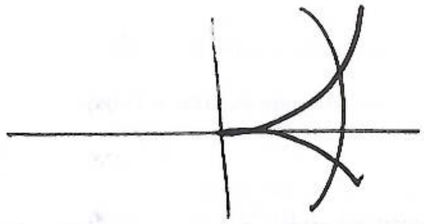
def. Such a function w is called a blow-up limit of u at $x=0$. Of course can be done at any other point.

Th. 3. (Regularity of the F.B. in the geome-

tric measure theory sense). 0 F. B. point.
There is $C>0$ such that: $\forall r<1$,
$\mathcal{L}_N(\{u>0\}) \cap B_r) \geq C r^N$.
· $\mathcal{L}_N(\{u=0\} \cap B_r) \geq C r^N$.

<u>Geometric interpretation.</u>

A cusp cannot occur. Of course the free boundary can have a much more ~~...~~ nasty imagination than 2 hypersurfaces meeting at a cusp.



<u>Refinement</u>:

<u>Th. 4.</u> (Clean ball estimate). For every $r>0$, there is a ball of size comparable to r inside $B_r \cap \{u>0\}$ and another one of the same size inside $\{u=0\}$.

In particular, this forbids a situation of the type (not forbidden by the positive density estimate.

<u>Consequence</u>. The free boundary has "a lot" of regular points.

The question is, however: is the regular set ~~...~~ of full Hausdorff measure? Does the F.B. have finite perimeter?

5°). Smoothness results.

First, we show that our variational problem yields a free boundary relation.

Th. 5. Set $A_\alpha = c_{1,\alpha} \int_{-1}^{0} \frac{(1+x)^\alpha}{(-x)^\alpha} dx \int_{1}^{+\infty} \frac{(1+x)^\alpha}{x^{1+2\alpha}} dx$

the constant defining $(-\Delta)^\alpha$ for $N=1$.

Let x_0 be a point at which the free boundary is differentiable. Then:

$$\lim_{x \to x_0} \frac{u(x) - u(x_0)}{|(x-x_0) \cdot \nu(x_0)|^\alpha} = A_\alpha .$$

By " differentiability of the free boundary at $x = x_0$, we mean that Γ is, in a neighbourhood of x_0, a graph that is differentiable at x_0.

Finally, here is a smoothness result.

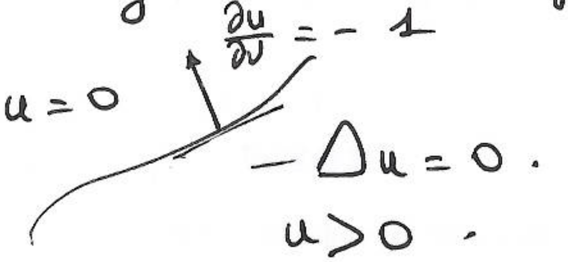
Th. 6. Let $N=2$ and x_0 a point at which there is a neighbourhood where Γ is a Lipschitz graph (for instance in $B_1(x_0)$).

Then, in $B_{1/2}(x_0)$, Γ is a C^1 graph.

What we cannot do:

- $N > 2$.
- Reach $C^{1,\gamma}$ regularity.

However, some progress is certainly possible. There is at the moment a nice result by D. de Silva on questions related to $(S)_1$. Let us briefly recall things. We are solving:



De Silva's result is the

<u>Theorem</u>. Assume Γ to be δ-flat in B_1, i.e.:

Then, if δ is small enough, Γ is $C^{1,\gamma}$ in $B_{1/2}$.

The ~~method so~~ idea is to adapt a very general argument of O. Savin (described in the last chapter of this course), itself inspired from the de Giorgi theorem on the regularity of minimal surfaces.

We are working on the problem and it seems that, at least for $\alpha = \frac{1}{2}$, some progress is possible.

5°) - <u>Some proofs</u> -

We are going to sketch the proof of optimal regularity, just to see how the minimisation process is made. We are going to prove that:

if u is a minimiser of J_α in B_1, then $u \in C^\alpha(\overline{B_{1/2}})$.

We use the Morrey characterisation of Hölder functions: if $u \in C(B_1)$ and if, for all $r \in (0, \frac{1}{2})$ and one $p \geq 1$ and all $(x_0, y_0) \in \mathbb{R}^{N+1}$:

$$\int_{B_r(x_0,y_0)} |\nabla u|^p \leq C r^{N+1-p+p\alpha}$$

then $u \in C^\alpha(\overline{B_{1/2}})$.

⚠ We are writing $N+1$ because we are placing in $N+1$ space dimensions.

So, let us chose $(x_0, y_0) \in B_1$ and let $\bar{u}_{x_0,y_0}$ the harmonic replacement of u in $B_r(x_0, y_0)$:

$$\begin{cases} -\operatorname{div} |y|^\beta \nabla \bar{u} = 0 & (B_r(x_0,y_0)). \\ \bar{u} = u & (\partial B_r(x_0,y_0)). \end{cases}$$

We study how the Dirichlet energy of u decreases in concentric balls.

$0 < r < \rho$, we ~~have~~ want to compare $\int_{B_r} |y|^\beta |\nabla u|^2$ et $\int_{B_\rho} |y| \, |\nabla u|^2$. We have:

$$\int_{B_r} |y|^\beta |\nabla u|^2 = \int_{B_r} |y|^\beta |\nabla u - \nabla \bar{u}_\rho + \nabla \bar{u}_\rho|^2$$

$$\leq 2 \int_{B_r} |y|^\beta |\nabla u - \nabla \bar{u}_\rho|^2 + \int_{B_r} |\nabla \bar{u}_\rho|^2 |y|^\beta$$

$$\leq 2 \int_{B_r} |y|^\beta |\nabla \bar{u}_\rho|^2 + 2 \int_{B_\rho} |y|^\beta |\nabla u - \nabla \bar{u}_\rho|^2 .$$

* $$\int_{B_\rho} |y|^\beta |\nabla u - \nabla \bar{u}_\rho|^2 = \int_{B_\rho} |y|^\beta |\nabla u|^2$$

$$- 2 \int |y|^\beta \nabla u \cdot \nabla \bar{u}_\rho$$

$$+ \int |y|^\beta |\nabla \bar{u}_\rho|^2 .$$

$$\leq 2 \int_{B_\rho} |y|^\beta |\nabla \bar{u}_\rho| + \underbrace{\mathcal{L}_N(\{\bar{u}_\rho > 0\})}_{\leq C\rho^N.}$$

$$- 2 \int |y|^\beta \nabla u \cdot \nabla \bar{u}_\rho$$

$$= C\rho^N + 2 \int_{B_\rho} |y|^\beta (\nabla u - \nabla \bar{u}_\rho) \cdot \nabla \bar{u}_\rho .$$

$= C\rho^N$ by def. of harm. repla-

cement.

• The term $\int_{B_r} |y|^\beta |\nabla \bar{u}_\rho|^2$.

We use the monotonicity formula:

<u>Lemma</u>. $\int_{B_r} |y|^\beta |\nabla \bar{u}_\rho|^2 \leq \left(\frac{r}{\rho}\right)^{N+1+\beta} \int_{B_\rho} |y|^\beta |\nabla \bar{u}|^2$.

~~Once again it has a classical counterpart~~.

<u>Why should this be true?</u>

• $a = \frac{1}{2}$. Then we want

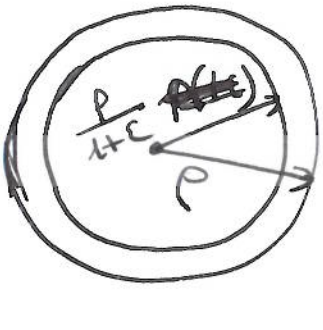
$$\int_{B_r} |\nabla \bar{u}_\rho|^2 \leq \cancel{\delta} \left(\frac{r}{\rho}\right)^{N+1} \int_{B_\rho} |\nabla \bar{u}_\rho|^2 .$$

<u>Exercise</u>. Use the fact that $|\nabla \bar{u}_\rho|^2$ ~~is a subsolution~~ satisfies $-\Delta |\nabla \bar{u}_\rho|^2 \leq 0$.

• <u>General case</u>.

<u>Exercise</u>. Set $v := \bar{u}_\rho$, we have

$$-\mathrm{div}\,(|y|^\beta \nabla v) = 0 \text{ in } B_\rho .$$



Set. $v_\varepsilon(x,y) = \dfrac{u((1+\varepsilon)(x,y))}{1+\varepsilon}$ if $|(x,y)| \leq \frac{1}{1+\varepsilon}$

• extend it in a linearly radial fashion to coincide with v on ∂B_ρ.

Write down $\lim_{\varepsilon \to 0} \dfrac{\int_{B_\rho} (|\nabla v_\varepsilon|^2 - |\nabla v|^2)}{\varepsilon} \geq 0$.

<u>Exercise.</u> We may (but it takes more time!) prove the formula with the aid of ~~div~~ $|y|^\beta|\nabla v|$

- for $\alpha \leq \frac{1}{2}$, show that
$$-\operatorname{div}\left[|y|^\beta \nabla\left(|y|^\beta|\nabla v|^2\right)\right] \leq 0.$$

- for $\alpha \geq \frac{1}{2}$, compute
$$-\operatorname{div}\left[|y|^\beta \nabla\left(|y|^\beta|\nabla v|^2\right)\right]$$
in terms of the derivatives of v. Integrate against the fundamental solution of $-\operatorname{div}(|y|^\beta \nabla .)$.

<u>Back to our proof.</u> Because $\bar{u}_\rho$ is the harmonic replact. of u in B_ρ:

$$\int_{B_r} |y|^\beta |\nabla \bar{u}_\rho|^2 \leq \left(\frac{r}{\rho}\right)^{N+\beta+1} \int_{B_\rho} |y|^\beta |\nabla \bar{u}_\rho|^2 \leq \left(\frac{r}{\rho}\right)^{N+1+\beta} \int_{B_\rho} |\nabla u|^2 |y|^\beta$$

<u>Thus</u>:
$$\int_{B_r} |\nabla u|^2 |y|^\beta \leq \left(\frac{r}{\rho}\right)^{N+1+\beta} \int_{B_\rho} |y|^\beta |\nabla u|^2 + C r^N.$$

This yields:
$$\int_{B_r} |\nabla u|^2 |y|^\beta \leq C r^N.$$

Argue then according to whether $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$.

One could stay much longer on this pb, but we just wanted to give a taste. This is, anyway, a good introduction to the last topic of this course.

V). Nonlocal minimal surfaces.

Given the remaining time, it will be more a seminar than a course. However we will motivate the subject and explain some general principles.

The question is here to study the boundary of sets ~~minim~~ whose indicator function minimises, locally, the H^α norm with $\alpha < \frac{1}{2}$.

This does not seem to mean anything, unless one knows about minimal surfaces. And, indeed, this is the theory that we wish to parallel.

Everything that will be exposed here is a joint work with L. Caffarelli and O. Savin.

We will first explain the background and motivations of this work, finishing the paragraph with the issue that we wish to treat, i.e. ~~your~~ the regularity of the sets under consideration. Then we will review the de Giorgi regularity theory for minimal surfaces, explaining the main steps of a recent approach due to Caffarelli and Savin. We will then apply this approach to our setting.

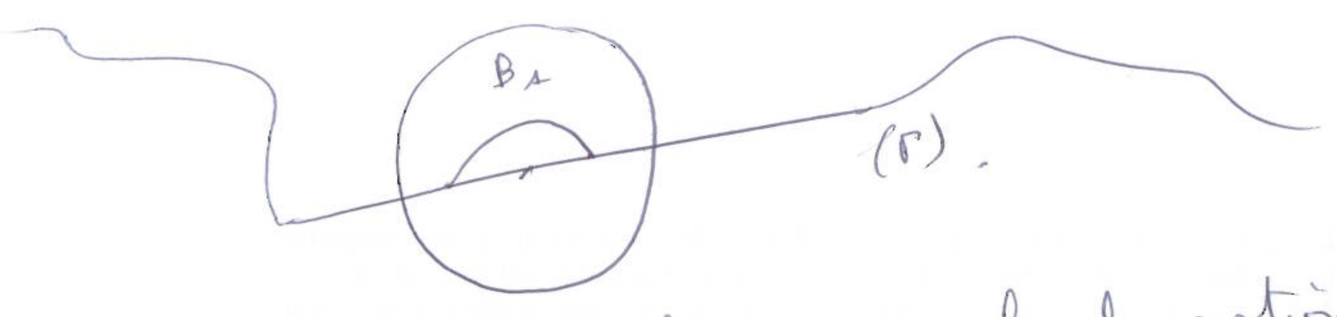
1º). Background and motivations.

As said before, the motivation of this chapter is to devise a theory that parallels the theory of minimal surfaces.

- Minimal surfaces: naive definition.

Γ: surface of $\mathbb{R}^N$, perhaps with corners (but we may compute its area).

We will say that Γ is ~~un~~ minimal in B_1 if any perturbation of it increases its area.



We notice that this is a local notion.
There are fascinating issues about this very simple definition, one of them being regularity. The intuition is that this minimality property will force regularity, but: will it allow singularities, where?

The 1st serious result in this direction is due to Bernstein, at the beginning of the XX th century. He proves that ~~the~~ a minimal graph in 3D, defined on the whole space, is necessarily a plane (this can be viewed as a regularity result). He uses complex analysis arguments.

Closer to us are the works of the geometric measure theory school: Almgren, Federer, Fleming... The main effort was to understand how to replace the complex analysis arguments in higher space dimensions.

For instance, Almgren proved regularity of a 3D minimal surface.

However, the definite blow came from de Giorgi. Since everything we are going to say is based on his theory, we are going to give a precise definition of his setting and result.

1. Admissible sets.

We work in the sets of finite perimeter. Let E be a set of $\mathbb{R}^N$, we will say that E has finite perimeter in B_1 if $D\mathbb{1}_E$, ~~has finite~~ is a Radon measure such that $\int_{B_1} |D\mathbb{1}_E| < +\infty$ (total mass of of the measure). We call $\int_{B_1} |D\mathbb{1}_E|$ the perimeter of E in B_1. (We say that $\mathbb{1}_E \in BV(B_1)$)

Remark / Exercise. If ∂E is a C^1 surface, then the perimeter of E in B_1 is just the area of ∂E in B_1.

2. Minimal sets -

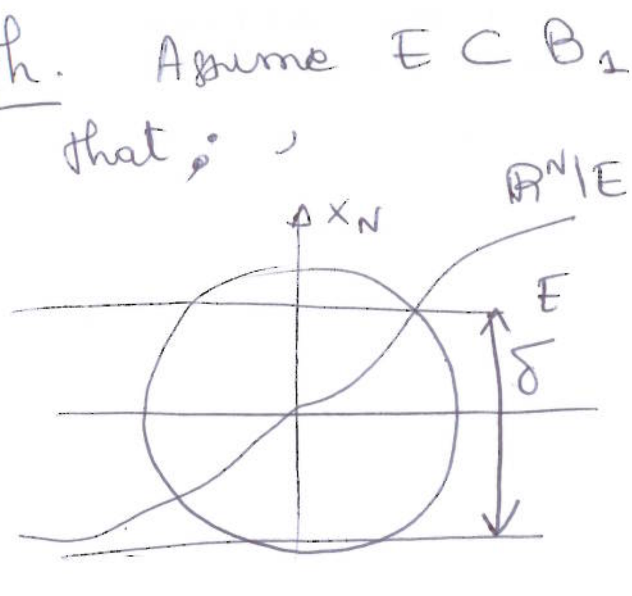
$E \subset \mathbb{R}^N$ is said to be minimal in B_1 if $\mathbb{1}_E \in BV(B_1)$ and if : for all set F coinciding with E on ∂B_1, we have

$$\int_{B_1} |D\mathbb{1}_E| \leq \int_{B_1} |D\mathbb{1}_F|.$$

Exercise. Check that this coincides w. the definition of a minimal ~~surf~~ $\mathbb{C}^1$ surface when $\partial E \cap B_1$ is C^1.

3. de Giorgi's theorem -

Th. Assume $E \subset B_1$, δ - flat. This means that ;



$$\{x_N \leq -\tfrac{\delta}{2}\} \cap B_1 \subset E \subset \{x_N \leq \tfrac{\delta}{2}\}.$$

If $\delta > 0$ is small enough, then $\partial E \cap B_{1/2}$ is a $C^{1,\alpha}$ graph in the x_N - direction.

From that it follows the analyticity of $\partial E \cap B_{1/4}$.

The story does not end here. With the aid of some geometric measure theory, and the fact that the Simons cone in dimensions $2m$: $\left\{ \sum_{i \leq m} x_i^2 = \sum_{i \geq m+1} x_i^2 \right\}$ is minimal, ~~one~~ for $N \geq 8$, one can prove:

Th. The dimension of the singular set is $\leq N-8$.

Now, the game that we wish to play is the following: what happens when we replace the BV norm by the H^α norm? Why $\alpha < \frac{1}{2}$?

2°). Minimising H^α norms.

E: set of $\mathbb{R}^N$,

$$\| \mathbb{1}_E \|^2_{H^\alpha} = \int \frac{(\mathbb{1}_E(x) - \mathbb{1}_E(y))^2}{|x-y|^{N+2\alpha}} \, dy$$

$$= \int_E \int_{\mathbb{R}^N \setminus E} \frac{dx\,dy}{|x-y|^{N+2\alpha}}.$$

We 1^{st} see that

- For $\alpha \geq \frac{1}{2}$, $\mathbb{1}_E \notin H^\alpha$ unless $E = \emptyset$.

Take $N=1$, $\mathbb{1}_E(x) = H(x)$ (or a step), and see that the singularity $\frac{1}{|x-y|^{\cancel{2\alpha}}}$ is too strong $(\alpha = \frac{1}{2})$.

Now, let us try and devise a theory.

1. Admissible sets.

E admissible in B_1

$$\iff \int_{E\cap B_1} \int_{\mathbb{R}^N\setminus E} \frac{dx\,dy}{|x-y|^{N+2\alpha}} + \int_{\mathbb{R}^N\setminus(E\cup B_1)} \int_{E\setminus B_1} \frac{dx\,dy}{|x-y|^{N+2\alpha}} < +\infty$$

Why such a barbarian expression?

We want to take into account possibly unbded sets. And, if E is unbounded, we have

$$\int_{\mathbb{R}^N\setminus E} \int_E \frac{dx\,dy}{|x-y|^{N+2\alpha}} = +\infty$$

no matter how nice ∂E is. We remove the nonconvergent part of the integral.

2. Minimisation.

E defined in $\mathbb{R}^N$. We say that E is minimal in B_1 iff: $\forall F$ subset of E admissible in B_1,

$E = F$ outside B_1

$$\Rightarrow \int_{E\cap B_1}\int_{\mathbb{R}^N\setminus E} \le \int_{F\setminus B_1}\int_{\mathbb{R}^N\setminus(F\cup B_1)}$$

Rk. If $E = F$ outside B_1, ~~we have removed~~ the nonconvergent parts coincide (in some sense).

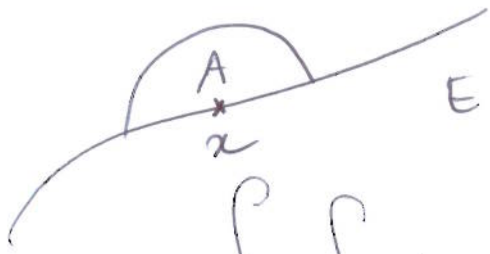
3. Exercise. Impose E outside B_1. ~~We~~ We may choose E inside B_1 minimal in B_1.

What can we do about such a minimization?

- Try to write an Euler-Lagrange equation.

To do this, the best way is to take "an infinitesimal" perturbation of E: ~~for~~

$\mathbb{R}^N\setminus E$. Forget about nonconvergent part:



$$\int_E\int_{\mathbb{R}^N\setminus E} \le \int_{E\cup A}\int_{\mathbb{R}^N\setminus(E\cup A)}$$

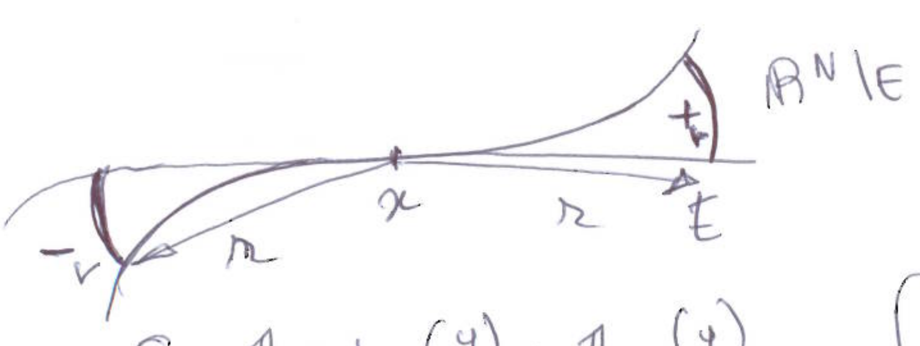
Expand: $0 \le \int_A\left(\int_{\mathbb{R}^N\setminus E} - \int_E\right) - \int_A\int_A$.

Essentially false, but philosophically true...
Now, suppose that A is any very small set containing x, and inside $\mathbb{R}^N \setminus E$. Remove it, and get:

$$\int \frac{\mathbb{1}_{\mathbb{R}^N \setminus E}(y) - \mathbb{1}_E(y)}{|x-y|^{N+2\alpha}}\, dy \geq 0 .$$

Same if we subtract a small blimp from E.
So, we are left with this strange curvature eq:

$$\int_{\mathbb{R}^N} \frac{\mathbb{1}_{\mathbb{R}^N \setminus E}(y) - \mathbb{1}_E(y)}{|x-y|^{N+2\alpha}}\, dy = 0 .$$ Let us try to interpret it and assume that ∂E is smooth.



$$\int_{\mathbb{R}^N} \frac{\mathbb{1}_{\mathbb{R}^N \setminus E}(y) - \mathbb{1}_E(y)}{|x-y|^{N+2\alpha}} = \int_0^{+\infty} \frac{dr}{r^{1+2\alpha}} \frac{\text{area}(+_r) - \text{area}(-r)}{r^{N-1}}$$

Rk. For a corner: $\frac{\text{area}(+_r) - \text{area}(-r)}{r^{N-1}} \sim k_0$ as $r \to 0$.
Divergent integral.

For a smooth piece, $\dfrac{\text{area}(+_r) - \text{area}(-r)}{r^{N-1}} = O(r)$

Convergent integral.

Exercise. $\displaystyle\lim_{\alpha \to \frac{1}{2}^-} \left(\frac{1}{2} - \alpha\right) \int \frac{\mathbb{1}_{\mathbb{R}^N \setminus E}(y) - \mathbb{1}_E(y)}{|x-y|^{N+2\alpha}}\, dy$

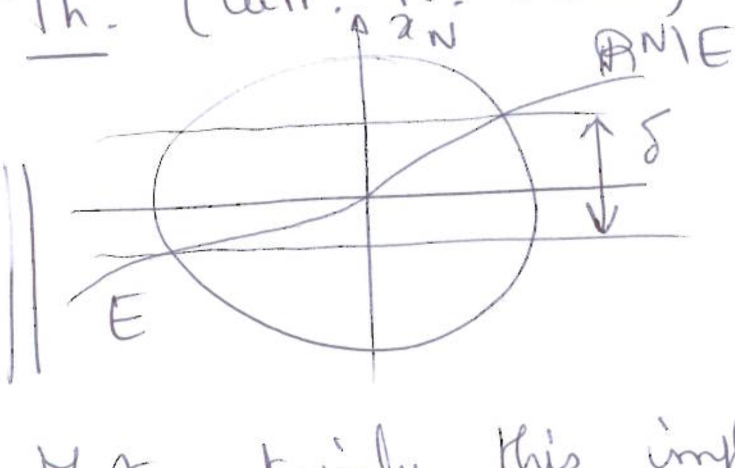
$= k(x)$ (mean curvature of ∂E at x).

Exercise. Assume that E is bounded, $\mathbb{1}_E \in BV(\mathbb{R}^N)$. Then ~~limit~~ $\displaystyle\lim_{\alpha \to \frac{1}{2}^-} \left(\frac{1}{2} - \alpha\right) \|\mathbb{1}_E\|_{H^\alpha} = \int |D\mathbb{1}_E|$

So, we retrieve minimal surface theory as $\alpha \to \frac{1}{2}$.

The theorem that we are able to prove is the following:

Th. (Caff. R. Savin).



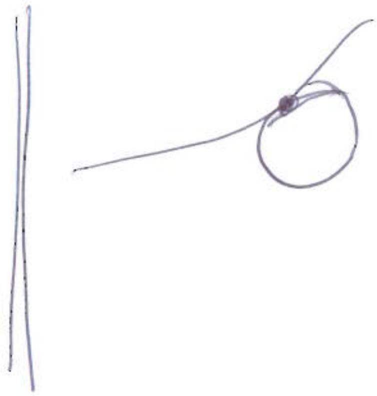
E defined in $\mathbb{R}^N$.
E δ-flat in B_1.
If δ is small enough, then ~~E~~ $\partial E \cap B_{1/2}$ is a $C^{1,\gamma}$ ~~hypersurface~~ graph.

Most certainly, this implies C^∞. I do not know about analyticity.

The proof goes through the following steps.

1. <u>A viscosity relation</u> (in the spirit of Caffarelli–Cordoba).

<u>Th.</u> E minimal in B_1, $0 \in \partial E$. Assume a ball touches ∂E from below at 0:



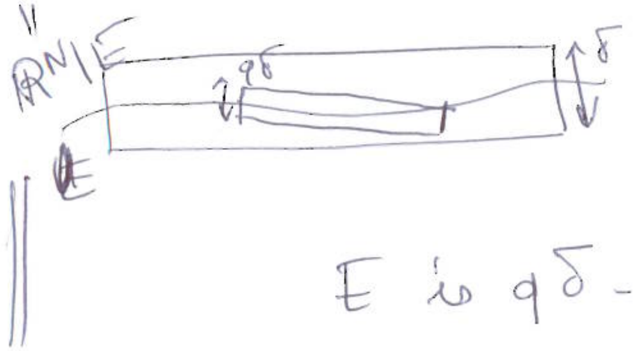
Then $\displaystyle\int \frac{\mathbb{1}_{\mathbb{R}^N\setminus E}(y) - \mathbb{1}_E(y)}{|y|^{N+2\alpha}}\,dy \geq 0.$

<u>Forbidden.</u>

2. <u>A compactness argument.</u>

The basic step that we would like to perform is the following result:

<u>Th.</u> Assume E to be δ-flat in B_1. Then there is $q>0$, universal, such that: there is $\delta_0 > 0$ for which, if $\delta \leq \delta_0$, then



$$E \text{ is } q\delta\text{-flat in } B_{1/2}.$$

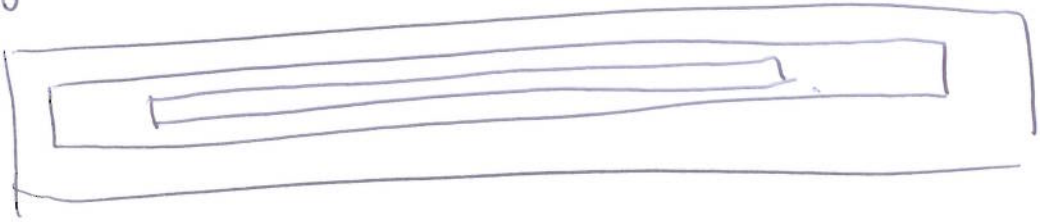
This, by iteration, implies $C^{1,\gamma}$ regularity at 0. The base point can of course be changed.

To prove this we argue as follows:

a). Prove a (rough) Harnack inequality.

Th. Suppose that E is δ-flat in $B_{1/4}$. There is $\delta_0 > 0$ such that, if $\delta \le \delta_0$, then $\partial E \cap B_{1/2}$ can be put in a ~~ct~~ cylinder of height $q\delta$ $(q < 1)$.

In fact, one would need to do it in a scale of cylinders.



b). Assume the thm is false. There is a sequence $(E_n)_n$ of δ_n-flat minimal sets in B_1, for which improvement of flatness does not hold. Rescale in the x_N-direction. If a) is done in a clever fashion, the limit is a global graph satisfying $(-\Delta)^{\frac{1+\alpha}{2}}\varphi = 0$ in $\mathbb{R}^{N-1}$. Hence φ is $C^{1,\gamma}$, improvement of flatness holds for a subsequence of E_n's, a contradiction.

There are very few things that we know about these surfaces... ~~any~~ In particular we would like to classify the minimal cones. The story is therefore far from finished.

This ends the course. We hope that this selection of nonlocal problems has conveyed the idea that there is still a vast and interesting domain to explore.