(ii). In the same fashion, there is a ~~largest~~ smallest $k_0 > 0$ such that $v_s \leq k u_s$. And we may prove $k_0 = 1$. Hence $v_s \leq u_s$, Thus $v_s = u_s$. $\boxtimes$

Up to now, we have not really seen any drastic effect of the fractional diffusion on reaction-diffusion models. This is why we are going to look more closely at a particular case of (1).

III.1. Spreading velocity in KPP-type models

As said before, let us particularize the population dynamics model that we were investigating, and let us make more restrictive assumptions.

Model under study:

$$u_t + \mathcal{A}u = \mu(x)u - u^2 \quad (x \in \mathbb{R}^N).$$

$$u(0,x) = u_0(x) \quad \left(\text{compactly supported}\right). \quad \geq 0$$

Assumption on μ - $\mu(x) \in [\mu_-, \mu_+]$,

$$1 < \mu_- < \mu_+ < +\infty.$$

Clearly, $\lambda_1^{per}(\mathcal{A} - \mu(x) I) < 0$. And thus, there is $u_s(x) > 0$, 1-periodic, such that:

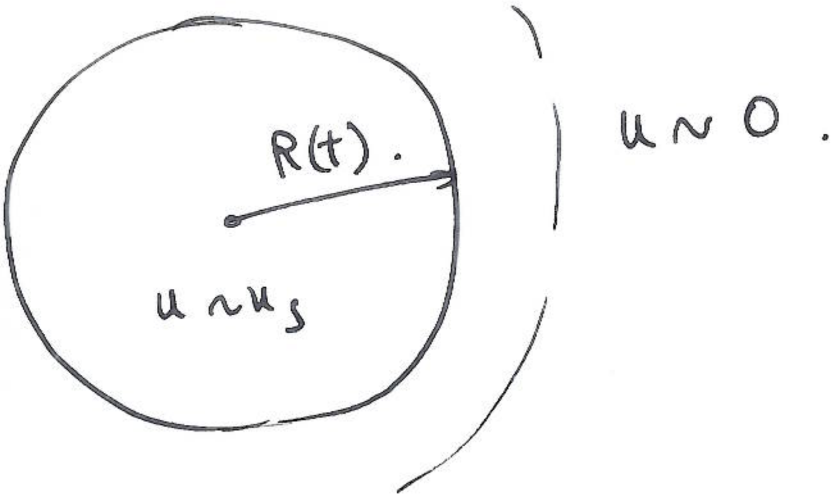
$$\lim_{t \to +\infty} u(t,x) = u_s(x)$$

uniformly on compact subsets of $\mathbb{R}^N$. Hence, as time goes to $+\infty$, the stable state u_s invades the unstable state 0. The population spreads.

<u>Question</u>: At what speed? Can we find balls, with radii $R(t)$, such that:

$u(t,x)$ close to u_s in $B_{R(t)}$.

$u(t,x)$ still close to 0 well outside $R(t)$?



We say that there is linear propagation in time if $\frac{R(t)}{t}$ is bdd from above and below. And we say that it is super-linear if $\frac{R(t)}{t} \to +\infty$.

It occurs that we may answer very precisely to these questions, and even find exact rates when $\mathcal{A} = (-\Delta)$. When $\mathcal{A} = (-\Delta)^\alpha$ we may find upper and lower bounds, and a very natural question would be to understand exact asymptotic rates of propagation. In order to keep this session to a ~~Noti~~ reasonable length, we will only be interested in crude bounds for $R(t)$.

We will prove that:

- for $\mathcal{A} = -\Delta$, there is linear propagation.
- for $\mathcal{A} = (-\Delta)^\alpha$, there is <u>exponential</u> propagation: $e^{\lambda_- t} \leq R(t) \leq e^{\lambda_+ t}$,

$$0 < \lambda_- < \lambda_+ < +\infty .$$

We will see how the fundamental solutions of the heat operator can explain that big difference.

1°). <u>Usual diffusion and linear in time propagation</u>.

We have:

$$\mu_- u - u^2 \leq u_t - \Delta u \leq \mu_+ u - u^2 .$$

Hence $u(t,x) \in [u_-(t,x), \; u_+(t,x)]$:

$$(\partial_t + A) u_\pm = \mu_\pm u_\pm - u_\pm^2 .$$

Hence it really suffices to investigate what happens with $\mu_+ = \mu_- = \mu$. And the theorem is

Th. (Aronson - Weinberger, 1976). We have:

$$\lim_{t \to +\infty} \frac{R(t)}{t} = 2\sqrt{\mu} .$$

Hence, for the model with variable μ, we have $2\sqrt{\mu_-} + o(1) \leq \frac{R(t)}{t} \leq 2\sqrt{\mu_+} + o(1)$.

Aronson - Weinberger is a very classical result, that one may prove in a lot of different ways. A very fruitful point of view is to notice that the 1D problem

$$u_t - u_{xx} = \mu u - u^2$$

has a family of travelling waves $\phi_c(x)$:

$$-\phi'' + c\phi' = \cancel{\mu\phi(1-\phi)} \; \mu\phi - \phi^2 .$$

$$\phi(-\infty) = 0, \qquad \phi(+\infty) = \mu .$$

for all $c \geq 2\sqrt{\mu}$. Actually, this is a very famous quiz by Arnold...

Unfortunately the TW point of view will be of no help in the fractional case. Hence we discard it and try to find something else.

Proof of the upper bound:

We make the very wise remark that

$$\mu u - u^2 \leq \mu u$$

and thus $u(t,x) \leq \bar{u}(t,x)$:

$$\begin{cases} (\partial_t - \Delta)\bar{u} = \mu \bar{u}. \\ u(0,x) = u_0(x). \end{cases}$$

$$\bar{u}(t,x) = \frac{e^{\mu t}}{(4\pi t)^{N/2}} \int e^{-\frac{|x-y|^2}{4t}} u_0(y)\, dy.$$

Actually, taking u_0 compactly supported and $u_0 = \delta_0$ (the Dirac mass at $x=0$) does not really change things for large t... So, we take $u_0 = \delta_0$

exercise: Check it!

and we have:

$$\bar{u}(t,x) = \frac{e^{\mu t - \frac{|x|^2}{4t}}}{(4\pi t)^{N/2}}.$$

level set $\bar{u}(t,x) = \frac{1}{2}$.

$$\mu t - \frac{|x|^2}{4t} - \frac{N}{2}\log t = O(1), \text{ hence :}$$

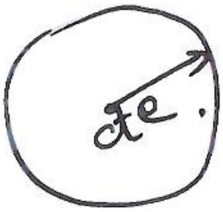
$$|x|^2 = 4\mu t^2 - 2N t \log t + O(1).$$

$$\frac{|x|}{t} \sim 2\sqrt{\mu}.$$

Hence the level sets of u do not move faster than $R(t) \sim 2\sqrt{\mu}\, t$.

Proof of the lower bound.

This is a very crude upper bound, so there is no reason to believe that this will give a lower bound. But, actually, it will.



Wish to prove : take e any direction. If we run with speed $c < 2\sqrt{\mu}$ in the direction e, we will see the solution remain bdd from below as $t \to +\infty$. ($u(t) \to 1\!\!1_s$ is true but requires an additional effort).

WLOG we may take : $e = -e_1$, we wish to prove an asymptotic lower bound for

$$v(t,x) = u(t, x_1 - ct, \underbrace{x_2, \ldots, x_N}_{x'}).$$

Equation for v :

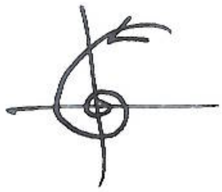
$$v_t - \Delta v + c\,\partial_{x_1} v = \mu v - v^2.$$

For this we will construct a family of sub-solutions of arbitrarily small size, w. compact support.

<u>Building block.</u> If $c < 2\sqrt{\mu}$, the equation

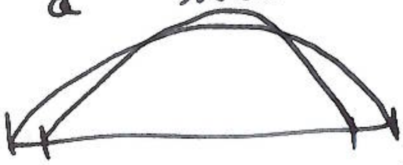
$$-\phi'' + c\phi' = \mu\phi \qquad (*)$$

has a spiral point (as $x \to -\infty$).



Hence there is $a > 0$ such that $(*)$ has a solution $\phi_0 > 0$ on $[-a, a]$, $\phi_0(\pm a) = 0$.

For $\delta > 0$ small enough there is $a_\delta \to a$ and a solution $\phi_\delta \to \phi_0$, > 0 in $[-a_\delta, a_\delta]$, $\phi_\delta(\pm a_\delta) = 0$.



$-a \quad -a_\delta \qquad\qquad a_\delta \quad a$

of: $-\phi'' + c\phi' = (\mu - \delta)\phi$.

<u>First attempt for a sub-solution</u>: fix $\delta > 0$ and consider $\varepsilon\phi_\delta(x_1)$. Set

$$NL[u] = -\Delta u + c\partial_{x_1} u - \mu u + u^2;$$

$$NL[\varepsilon\phi_\delta] = \varepsilon\left(-\delta\phi_\delta + \varepsilon\phi_\delta^2\right) < 0 \text{ if } \varepsilon \text{ is small enough.}$$

Are we happy? Yes, in 1 dimension.

<u>Problem.</u> Transverse directions! Solution not compactly supported.

However this is easy: if $\mu_1(R)$ is the $1^{\underline{st}}$ eigenvalue of ~~to~~ the Dirichlet Laplacian in B_R, we have

$$\lim_{R\to+\infty} \mu_1(R) = 0.$$

<u>Choose</u>: $R>0$ such that $\mu_1(R) < \frac{\delta}{3}$.

<u>Second attempt</u>. $\psi_R(x')$: $1^{\underline{st}}$ eigenfunction.

Set $\underline{u}(x_1, x') = \varepsilon\, \phi_\delta(x_1)\, \psi_R(x')$.

$$NL[\underline{u}] = \varepsilon\left[(-\delta + \mu_1(R)) + \varepsilon\, \underline{u}\right]\underline{u}$$
< 0 if ε is small enough.

For $\varepsilon > 0$ small enough, one may put $\underline{u}$ below $v(1,\cdot)$. Hence $\underline{\lim}_{t\to+\infty} u(t,x) \geq 1$ on every compact set of $\mathbb{R}^N$.

One may also notice that the pf is totally independent of e. ∎

Now we will turn to the fractional Laplacian.

Before that, one exercise (trivial).

<u>Exercise</u>. $N=1$, $u(0,x) = H(x)$ (Heaviside).

How does the level set $\{u = \frac{1}{2}\}$ behave?

2°). The fractional heat kernel.

I will not make the suspense last too much. What makes the drastic change between linear and exponential propagation is the heat kernel.

For $\mathcal{A} = -\Delta$, the heat kernel has a Gaussian tail, it decays very quickly to 0. On the other hand you want to balance it with $e^{\mu t}$. Thus we only require $|x|^2 \sim 4\mu t$.

For $\mathcal{A} = (-\Delta)^\alpha$, $\alpha \in (0,1)$, we will see that it has a <u>power-like</u> decay. So, trying to match the heat kernel with $e^{\mu t}$ requires much larger $|x|$ (typically exponentially large).

<u>Definition of the heat kernel</u>: the solution of

$$G_t + (-\Delta)^\alpha G = 0 \qquad t > 0, \ x \in \mathbb{R}^N.$$

$$G(0,x) = \delta_0(x) \longrightarrow \text{Dirac mass at } 0.$$

(For those who want sth rigorous:

$$\lim_{t\to 0} \int G(t,x)\,\phi(x)\,dx = \phi(0)$$ for all

$\phi \in C_0^\infty(\mathbb{R}^N)$).

<u>Th.</u> $G(t,x) = \dfrac{1}{t^{N/2\alpha}}\, p\left(\dfrac{x}{t^{1/2\alpha}}\right)$, and there is

$C>0$ such that

$$\frac{1}{C(1+|x|^{N+2\alpha})} \le p(x) \le \frac{C}{1+|x|^{N+2\alpha}}.$$

I will not give a full proof of this (very well known by probabilists) result, but I wish to dwell a little bit on it.

<u>First, let us check that I am not lying</u>: $N=1$, $\alpha = \frac{1}{2}$.

<u>Fourier transform in x</u>:

$$\begin{cases} \partial_t \hat{G} + |\xi|\,\hat{G} = 0. \\ \hat{G}(0,x) = 1. \end{cases}$$

$$\hat{G}(t,\xi) = e^{-t|\xi|}$$

So, $G(t,x) = \displaystyle\int_{-\infty}^{+\infty} e^{i x.\xi - t|\xi|}\, d\xi$

(there is a $\frac{1}{2\pi}$ that I want to ignore).

$$G(t,x) = 2\int_0^{+\infty} e^{-t\xi}\cos(x\xi)\,d\xi$$

$$= c^{te} \cdot \frac{t}{t^2+x^2}$$

$$= \frac{c}{t} \cdot \frac{1}{1+\left(\frac{x}{t}\right)^2} \;; \qquad \frac{1}{2\alpha} = 1. \quad N+2\alpha = 1+1 = 2.$$

How do we retrieve the exact constant? Well, write the preservation of the L^1 norm.

Let us look at the general case.

Fourier in x: $\hat{G}(t,\xi) = \int e^{-ix.\xi}\, G(t,x)\,dx$

$$\begin{cases} \partial_t \hat{G} + |\xi|^{2\alpha}\hat{G} = 0. \\ \hat{G}(0,\xi) = 1. \end{cases}$$

$$\hat{G}(t,\xi) = e^{-t|\xi|^{2\alpha}}.$$

Already, the reason why we have a not so strong decay for $\alpha < 1$ is clear: $\hat{G}(t,\cdot)$ is not smooth at 0 (although it decays very well). As opposed to that, for $\alpha = 1$, $\hat{G}$ is in the Schwartz class. Then, its Fourier transform is in the Schwartz class. Let us go a little further:

$$G(t,x) = \int e^{ix.\xi - t|\xi|^2}$$

(ignore the $\frac{1}{(2\pi)^{N/2}}$, we are not going to retrieve an exact expression anyway).

$$= \frac{1}{t^{N/2\alpha}} \int e^{i \frac{x}{t^{1/2\alpha}} . \xi - |\xi|^{2\alpha}} d\xi .$$

$(\xi := t^{1/2\alpha} \xi)$

$$= \frac{1}{t^{N/2\alpha}} p\left(\frac{x}{t^{1/2\alpha}}\right).$$ Now,

$$p(x) = \int e^{ix.\xi - |\xi|^{2\alpha}} d\xi,$$ $|x| \gg 1$. We are not interested in $|x| = O(1)$, we already know all.

ϕ: cut-off function, $\phi(\xi) = \begin{cases} 1 & |\xi| \leq \frac{1}{|x|^\delta} \\ 0 & |\xi| \geq \frac{2}{|x|^\delta} \end{cases}$.

$$p(x) = \int \phi(\xi) e^{ix.\xi - |\xi|^{2\alpha}} d\xi + \underbrace{\int e^{ix.\xi - |\xi|^{2\alpha}} (1-\phi(\xi)) d\xi}_{\text{Schwartz class}}.$$

So, we only have to worry about

$$p_1(x) = \frac{1}{|x|^N} \int_{|\xi| \leq 2|x|^{1-\delta}} \phi\left(\frac{\xi}{|x|}\right) e^{i \xi \frac{x}{|x|} - \frac{|\xi|^{2\alpha}}{|x|^{2\alpha}}} d\xi$$

Clearly, P_1 is rotationally invariant. Thus we may choose $\frac{x}{|x|} = e_1$,

$$P_1(x) = \frac{1}{|x|^N} \int_{|\xi| \leq 2|x|^{1-\delta}} \phi\left(\frac{\xi}{|x|}\right) e^{i\xi_1} e^{-\frac{|\xi|^{2\alpha}}{|x|^{2\alpha}}} d\xi .$$

The idea is now that $\frac{|\xi|}{|x|}$ is very small,

thus

$$P_1(x) \sim \frac{1}{|x|^{N+2\alpha}} \int_{|\xi| \leq 2|x|^{1-\delta}} \phi\left(\frac{\xi}{|x|}\right) |\xi|^{2\alpha} d\xi \, e^{i\xi_1}$$

(May take $\phi = \phi'(\xi') \phi_1(\xi_1)$)

$$= \frac{1}{|x|^{N+2\alpha}} \int_{|\xi'| \in [2|x|^{1-\delta}, 2|x|^{1-\delta}]^{N-1}} d\xi' \, \phi'\left(\frac{\xi'}{|x|}\right) \int_0^{2|x|^{1-\delta}} |\xi|^{2\alpha} \phi_1\left(\frac{\xi_1}{|x|}\right) \cos \xi_1 \, d\xi_1$$

And thus we more or less have a 1D integral:

$$\int_0^{2|x|^{1-\delta}} \xi_1^{2\alpha} \phi\left(\frac{\xi_1}{|x|}\right) \cos \xi_1 \, d\xi_1 .$$

Integrate by parts twice: this is like

$$\int_0^{2|x|^{1-\delta}} \xi_1^{2\alpha-2} \phi\left(\frac{\xi_1}{|x|}\right) \cos \xi_1 \, d\xi_1 \text{ and}$$

the integral $\int \xi_1^{2\alpha-2} \cos \xi_1 \, d\xi_1$ is convergent and >0.

3°). Exponential propagation with fractional diffusion

Let us recall the equation:

$$\begin{cases} u_t + (-\Delta)^\alpha u = \mu u - u^2 & (x \in \mathbb{R}^N) \\ & t > 0 \\ u(0,x) = u_0(x) \end{cases} \quad (1)$$

u_0 : $u_0 \geq 0$, compactly supported.

<u>Th.</u> (Cabré, R.). Let $c_\alpha = \dfrac{\mu}{N+2\alpha}$. Then:

- if $c < c_*$, $\liminf\limits_{\substack{t \to +\infty \\ |x| \leq \exp(ct)}} u(t,x) = \mu$.
- if $c > c_*$, $\limsup\limits_{\substack{t \to +\infty \\ |x| \geq \exp(ct)}} u(t,x) = 0$.

<u>Upper bound</u>. We have everything to prove it. Same as in the classical case. $u \leq \bar{u}$,

$$\bar{u}_t + (-\Delta)^\alpha \bar{u} = \mu \bar{u}$$
$$\bar{u}(0,x) = u_0(x).$$

$$\bar{u}(t,x) \leq \frac{C_\alpha e^{\mu t}}{t^{N/2\alpha}} \int \frac{u_0(y)}{1 + \left|\frac{x-y}{t^{1/2\alpha}}\right|^{N+2\alpha}} dy.$$

We do not cheat much if we take $u_0 = \delta_0$:

$$\bar{u}(t,x) \leq \frac{Ce^{\mu t}}{t^{N/2\alpha}\left(1+\left|\frac{x}{t^{1/2\alpha}}\right|^{N+2\alpha}\right)}$$

$$\leq \frac{Ct\,e^{\mu t}}{|x|^{N+2\alpha}}$$ for $|x| \gg t^{1/2\alpha}$, which will be the case.

Take now $c > \frac{\mu}{N+2\alpha}$ and $|x| = e^{ct}$:

$$u(t,x) \leq Ct\, e^{(N+2\alpha)\left(\frac{\mu}{N+2\alpha}-c\right)t} \xrightarrow[t\to+\infty]{} 0.$$

This is the upper bound, which is always the one displayed in the physics papers.

<u>lower bound.</u> Consider $c < c_\alpha$ and $\delta > 0$ such that $\delta < \mu$. A solution $\underline{u}(t,x)$ of

$$\underline{u}_t + (-\Delta)^\alpha \underline{u} = (\mu-\delta)\underline{u}$$

is a subsolution to (1) as soon as $(\mu-\delta)\underline{u} \leq \mu\underline{u} - \underline{u}^2$, i.e. $\underline{u} \leq \delta$.

From now on we take $N=1$ for simplicity, $N>1$ is only a matter of technical details.

* Consider $\varepsilon \ll \delta$ and $\underline{u}_0 = \mathrm{int}(u_0, \varepsilon)$.

If ε is small enough and $\underline{u}$ solves

$$\begin{cases} \underline{u}_t + (-\Delta)^\alpha \underline{u} = (\mu - \delta)\underline{u} \\ \underline{u}(0,x) = \underline{u}_0(x) \end{cases}$$

then $\underline{u}(1,x) \leq \delta$. Thus $\underline{u} \leq u$ and we have

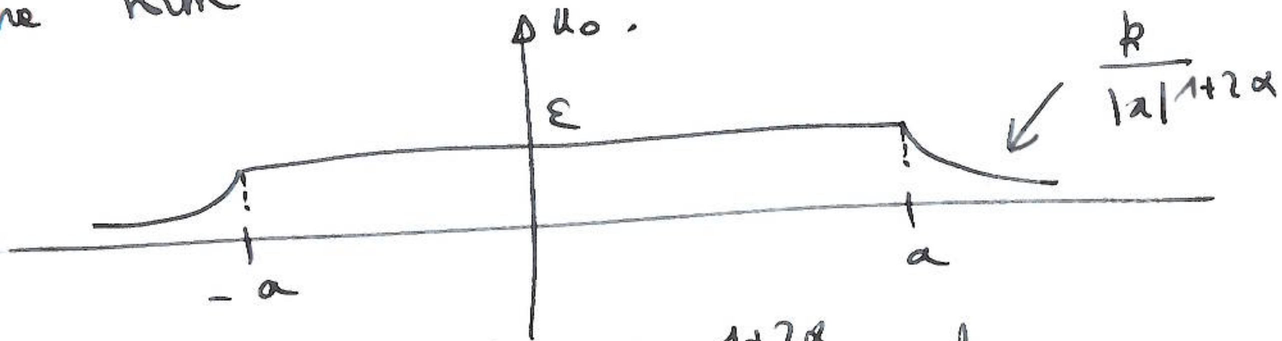
$$\underline{u}(1,x) \geq \int_{\mathrm{supp}(u_0)} \frac{C}{1+|x-y|^{1+2\alpha}}$$

$$\geq \frac{c}{|x|^{1+2\alpha}} \quad \text{for } |x| \text{ large.}$$

Thus we may assume $u(0,x) \geq \frac{c}{|x|^{1+2\alpha}}$ for $|x|$ large.

<u>The idea is to perform an iteration scheme.</u>

Here is the basic step. Assume that $\underline{u}_0$ has the form



Then we have $k = \varepsilon a^{1+2\alpha}$ and

$$u_0(x) = \varepsilon \left(\frac{a}{|x|}\right)^{1+2\alpha} \quad \text{for } |x| \geq a.$$

Assume $\varepsilon \ll \delta$. The 1^{st} ~~place~~ where u hits the value δ is at $x=0$,

$$\frac{C}{t^{N/2\alpha}} e^{(\mu-\delta)t}\varepsilon \int_{-a}^{a} \frac{dy}{1+|y|^{1+2\alpha}} \le \underline{u}(t,0) \le \frac{C e^{(\mu-\delta)t}\varepsilon}{t^{N/2\alpha}} \left(\int_{-a}^{a} + \int_{|x|\geq a} \right)$$

Hence, if $\varepsilon > 0$ is small enough:

$$t \simeq \frac{1}{\mu-\delta} \log \frac{\delta}{\varepsilon} > 1.$$

Value of u for large $|x|$, ex. $x \ll a$.

$$\underline{u}(t,x) \simeq \frac{C e^{(\mu-\delta)t}}{t^{N/2\alpha}} \int \frac{a^{1+2\alpha}\varepsilon}{|y|^{1+2\alpha}} \cdot \frac{1}{1+|x-y|^{1+2\alpha}}$$

$$\simeq \frac{e^{(\mu-\delta)t} a^{1+2\alpha}\varepsilon}{|x|^{1+2\alpha}}.$$

Thus $u(t,x) = \varepsilon \iff |x| = a \left(\frac{\delta}{\varepsilon}\right)^{\frac{1}{1+2\alpha}}$. The level set $\{u=\varepsilon\}$ has jumped from a to $\left(\frac{\delta}{\varepsilon}\right)^{\frac{1}{1+2\alpha}} a$.

Iteration. $t_n = \frac{n}{(\mu-\delta)} \log \frac{\delta}{\varepsilon}$. Assume that at step n, $u \geq \underline{u}_n$ which has the features of the basic iteration w. $a = a_n$. A time t_{n+1},

- compute $u(t_{n+1}, \cdot)$.
- Remove everything above $\varepsilon \longrightarrow \underline{u}_{n+1}$

Then $a_{n+1} \sim \left(\frac{\delta}{\varepsilon}\right)^{\frac{1}{1+2\alpha}} a_n$ and thus:

$$a_n \sim \left(\frac{\delta}{\varepsilon}\right)^{\frac{n}{1+2\alpha}} \sim e^{(\mu-\delta)\frac{t_n}{1+2\alpha}}.$$

<u>Thus</u>. For every $\delta > 0$,

$$\liminf_{t\to+\infty} \inf_{|x| = e^{\frac{\mu-\delta}{1+2\alpha}t}} u(t,x) > 0.$$

Once we have that we need another lemma that says: if $\varepsilon \ll 1$,

$$\lim_{t\to+\infty} \inf_{|x| = e^{\frac{(\mu-\delta-\varepsilon)t}{1+2\alpha}}} u(t,x) = 1.$$

This is done by examining the "back" of the solution, and will not be treated here.

<u>Exercise</u>. What happens if $u_0(x) = H(x)$?

<u>Questions on this part</u>.

- $\alpha \to 1$? Can we describe the whole dynamics.
- μ variable: can we find a c_α?

These questions do not seem to be very trivial and seem quite interesting.

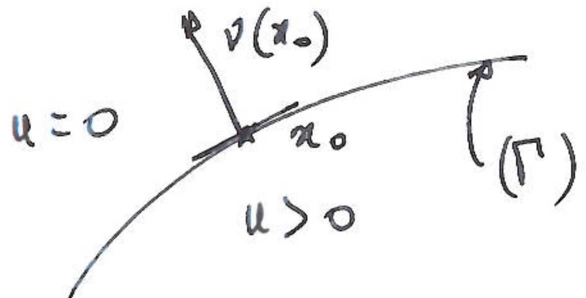
From then on we look at more geometric pbs.

IV). A free boundary problem for the fractional laplacian.

We switch in this part to questions of a more geometric character. Here we start the investigation of a free boundary problem where the jump of the Hölder quotient is imposed.

Here is the strong formulation. Consider $u \in C(\mathbb{R}^N)$ which satisfies:

$$
(S)_\alpha \begin{cases} (-\Delta)^\alpha u = 0 & \text{in } \{u>0\} \cap \Omega . \\ u = 0 & \text{outside } \{u>0\} \cap \Omega . \\ [u] = 0 \text{ across } \Gamma = (\partial\{u>0\}) \cap \Omega . \\ u(x) \underset{x\to x_0}{\sim} A\left[(x-x_0)\cdot \nu(x_0)\right]^\alpha & \text{for all } x_0 \in \Gamma . \end{cases}
$$



Why free boundary problem? Because the given quantities are $\alpha \in (0,1)$ and $A>0$. This makes (S) over-determined, unless Γ is allowed to vary. And, indeed, what we do is let Γ adjust to fulfill the Hölder quotient jump condition.

One may ask several questions.

1. Why should we look at such problems?
2. Is such a problem non void?
3. What results could be expected?

Let us start with 3: what is of interest to us are qualitative properties of the free boundary. Is it smooth? Does it have a special shape? Does it develop singularities? The mathematical questions are thus well identified.

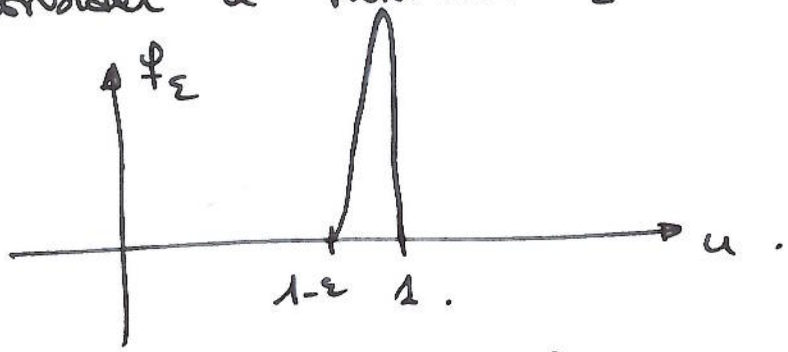
On the mathematical side, $(S)_\alpha$ is the generalisation to $\alpha < 1$ of the classical FBP: in a given domain Ω, study the properties of u such that

$$
(S_\alpha)_1 \quad \begin{cases} -\Delta u = 0 & (\Omega \cap \{u > 0\}). \\ u = 0 & \text{outside.} \\ [u] = 0 & \text{across } \Gamma = \partial\{u > 0\}. \\ \dfrac{\partial u}{\partial \nu} = -A & \text{on } \Gamma. \ (\nu: \text{outer normal to } \{u>0\}) \end{cases}
$$

A lot is known about question 3.

- If Γ is "flat" in B_1, then it is smooth in $B_{1/2}$ (Caffarelli). Analogous to the minimal surface de Giorgi theorem.
- In large dimensions, Γ may develop cone-like singularities (~~[illegible]~~). (de Silva - Jerison).

As for model $(S)_1$, question (1) is still pending but we may indeed answer it. One of its interpretation is the limit of a reaction-diffusion model. To be more specific, let us consider a function f_ε like



$\text{supp } f_\varepsilon = [1-\varepsilon, 1)$, $\int f_\varepsilon = M$

Example: $f_\varepsilon(u) = \frac{1}{\varepsilon}\phi\left(\frac{u-1}{\varepsilon}\right)$, ϕ supported in $(-1, 0)$

Consider the reaction-diffusion equation:

$$u_t - \Delta u = \frac{1}{\varepsilon} f_\varepsilon(u). \quad (S)_{1\varepsilon}$$

One can show, at least at the formal level, that a solution of $(S)_{1\varepsilon}$ will converge to a solution of

$$u_t - \Delta u = 0. \qquad u = 1.$$

$$[u] = 0.$$

$$\left[\frac{\partial u}{\partial \nu}\right] = \sqrt{2M}.$$

This is exactly $(S)_1$. A mathematically rigorous convergence proof can be very hard, depending on how much smoothness we have on the solution of the FBP.

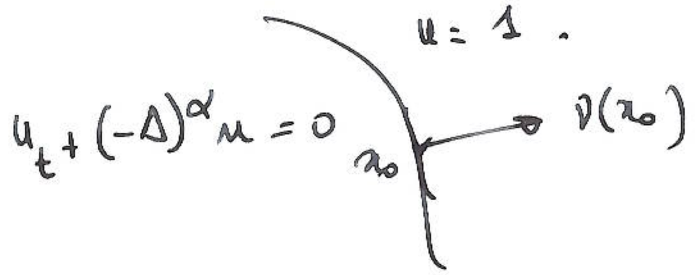
~~Now~~,
<u>Exercise.</u> Do the formal derivation, it is nothing more than a boundary layer analysis.

<u>What about our problem $(S)_\alpha$</u>? Well, remember that $(S)_\alpha$ and (S), come from a common ancestor, i.e. a common integro-differential model. Thus it is legitimate to look at the model

$$u_t + (-\Delta)^\alpha u = \frac{1}{\varepsilon} f_\varepsilon(u). \qquad (S)_{\alpha\varepsilon}$$

This time $f_\varepsilon(u) = \frac{1}{\varepsilon^{\alpha}} \phi\left(\frac{u-1}{\varepsilon}\right)$.

And we may prove, at least at the formal level (a rigorous proof is an open question), that the limiting equation for $(S)_{\alpha\varepsilon}$ is



$$u(t,x) \sim q_\alpha\left[(x-x_0).\nu(x_0)\right.$$

By the way, nonlinearities like f_ε are of common use in flame propagation modelling.

Now that we have answered question 1 for our model $(S)_1$, let us turn to question 2: do the objects that we are studying make sense?

BTW, one definition.

<u>def</u>. Ω open subset of $\mathbb{R}^N$, $u \in C^2(\Omega)$,

|| u is fractional harmonic in Ω iff $(-\Delta)^\alpha u = 0$ in Ω.

Remember that, although we are still looking at u in Ω, we need to know u everywhere in order to compute its fractional laplacian! So, question 2 is the subject of the 1st subsection.

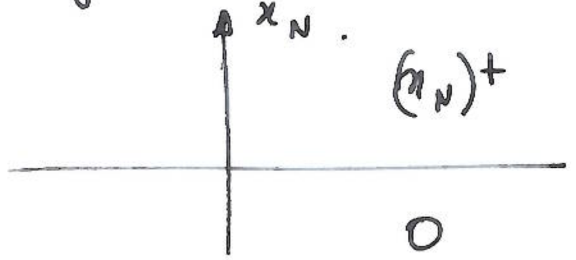
1°). <u>Boundary behaviour of fractional harmonic functions</u>.

Let us look, for a last time, at Model $(S)_1$. What makes it work is that, if a function $u \in C^2(\Omega) \cap C(\bar{\Omega})$, $\partial\Omega$ smooth is such that

$$-\Delta u = 0 \quad (\Omega) \qquad u = 0.$$

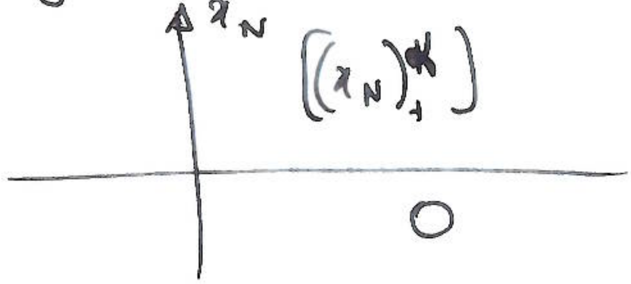
Then, at each point of $\partial\Omega$: $\dfrac{\partial u}{\partial \nu} < 0$.

Of course, if $\partial\Omega$ is given, $\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega}$ varies from point to point. In $(S)_1$, we impose $\frac{\partial u}{\partial \nu}$ and hope that the ~~bo~~ part of the boundary where it is imposed will adjust to this constraint. If we really look to the heart of the proof of the Hopf lemma, we realise that what really makes it work is that



is a solution of $(S)_1$ with $\Gamma = \mathbb{R}^{N-1}$. Of course, the regularity of Γ is easy to study, but these functions provide blow-up solutions to $(S)_1$. We will come back to this later.

Now, look at $(S)_\alpha$. The starting point of the study is that

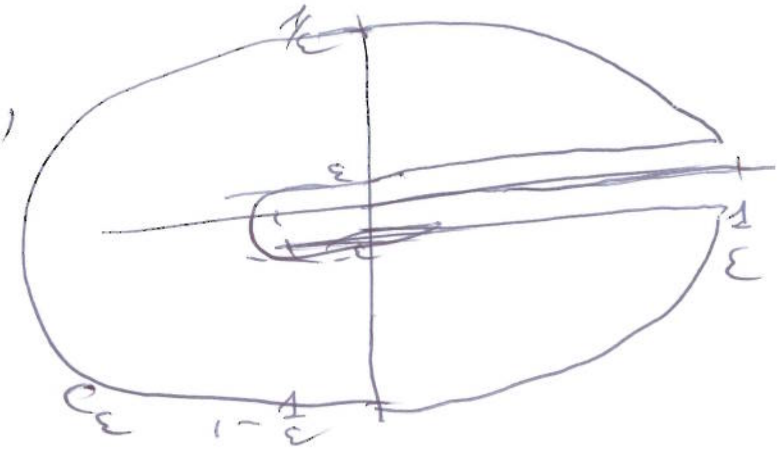


is α-harmonic in $\mathbb{R}^N_+$. To see this one just has to prove that

$$x \in \mathbb{R} \longmapsto (-\partial_{xx})^\alpha [x_+^\alpha] = 0 \text{ on } \mathbb{R}_+.$$ And, by the way, $(-\partial_{xx})[x_+^\alpha] \neq 0$ on $\mathbb{R}_-$!

Exercise. Prove that fact directly. It is not, however, entirely trivial. For $\alpha < \frac{1}{2}$ the suggested approach is to use that

$$\int_{C_\varepsilon} \frac{1-(1+z)^\alpha}{z^{1+2\alpha}}\, dz = 0,$$



For $\alpha > \frac{1}{2}$ we may use that

$$\int_{C_\varepsilon} \frac{1-(1+z)^\alpha - \alpha z}{z^{1+2\alpha}}\, dz = 0 .$$

Here is a quick argument for $\alpha = \frac{1}{2}$. Let me not make the suspense last too much, everyone knows that the fractional Lapl. will have an extension in the upper half space. This fact was discovered by Caffarelli and Silvestre. However, the case $\alpha = \frac{1}{2}$ owes nothing to them, it is the Poisson formula: if $u \in C^{1/2}(\mathbb{R})$, let $v(x,y)$ solve

$$\begin{cases} -\Delta v = 0 \quad (x \in \mathbb{R},\ y > 0). \\ v(x,0) = u(x). \end{cases}$$

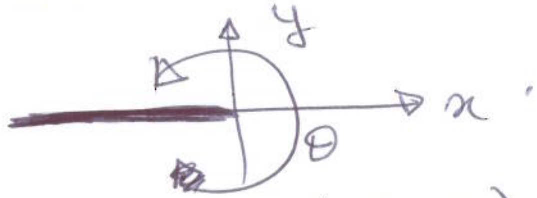
Then $(-\partial_{xx})^{1/2} u(x) = -\frac{\partial v}{\partial y}(0,y)$.

To see this, write down the Poisson formula

$$v(x,y) = \int_{\mathbb{R}^2_+} \frac{y\,u(x')}{(x-x')^2+y^2}\,dx'.$$

let $y \to 0$, this is exactly (up to the constant) $(-\partial_{xx})^{1/2} u(x)$.

Now that we have this information in hand, define $v(z) = \sqrt{z}$ in the complex plane from which we have removed the axis $\mathbb{R}_-$.



Hence, if $z = |z| e^{i\theta}$ w. $\theta \in (-\pi, \pi)$ we have $V(z) = |z|^{1/2} e^{i\theta/2}$. In particular, if $x > 0$: $V(x,0) = \sqrt{x}$ and, if $x < 0$: $V(x,0^{\pm}) = \sqrt{|x|}\, e^{\pm i\pi/2}$. But, in particular, if $v(x,y) = \mathrm{Re}\, V(z)$: $-\Delta v = 0$ in $\mathbb{C} \setminus \mathbb{R}_-$ and $u(x) := v(x,0) = \sqrt{x_+}$. Hence $\frac{\partial v}{\partial y} = (-\Delta)^{1/2} u$, but $v(x,y) = v(x,-y)$ if $x > 0$. Hence $(-\partial_{xx})^{1/2} \sqrt{x_+} =$

on $\mathbb{R}_-$.

And so, we may state the following result:

<u>Th.</u> Ω: open smooth subset of $\mathbb{R}^N$,
$u \in C(\mathbb{R}^N)$: $u \geq 0$ in $\mathbb{R}^N$, $(-\Delta)^\alpha u \geq 0$
in Ω, $u \in C^2(\Omega) \cap C(\overline{\Omega})$.
If x_0 is such that $u(x_0) = 0$ and $x_0 \in \partial\Omega$, then $\exists\, q > 0$ s.t.

$$u(x) \geq q\,|(x - x_0).\nu(x_0)|^\alpha.$$

Two options to prove this result:

- either we know of the Riesz potentials. If B_r is the ball of radius r,
$$G_r(x,y) = \frac{1}{|x-y|^N}\left(\frac{r^2 - |x|^2}{|y|^2 - r^2}\right)^{\alpha/2}$$
(Green's fct. of the Lapl. w. pole y outside)
- Or we make a barehand (but instructive) computation.

<u>Exercise</u>. 1. Aided with $(-\partial_{xx})^\alpha (x_+)^\alpha = 0$, show that $(-\Delta)^\alpha (1 - |x|)_+^\alpha = O(1)$ in the vicinity of $|x| = 1$.

2. Show that $(-\partial_{xx})^\alpha (x_+ e^{-\varepsilon x}) > 0$ for $\varepsilon > 0$ large enough.