Invasions in periodic media: the Freidlin-Gärtner formula

Model under study:

$$u_t - \Delta u = \mu(x)\, u - u^2, \quad t > 0, \; x \in \mathbb{R}^N. \qquad (1)$$

$\mu(x)$: a smooth, 1-periodic function in all its variables.

<u>Question</u>: what happens to an initially compactly supported solution?

The final result is the following: assume that

$$\mu_1^{per}(-\Delta - \mu I) < 0,$$

then:

(i). the steady equation

$$-\Delta u = \mu(x) u - u^2, \quad x \in \mathbb{R}^N$$

has a unique positive bounded solution u_+,

(ii). any nonnegative, initially compactly supported solution $u(t,x)$ of (1) will tend to μ_+, as $t \to +\infty$, <u>on every compact subset of $\mathbb{R}^N$</u>,

(iii). For each unit vector e, let $c_*(e)$ be the smallest $c>0$ such that the linear equation

$$v_t - \Delta v = \mu(x) v \qquad \begin{array}{l} t \in \mathbb{R} \\ x \in \mathbb{R}^N \end{array}$$

has solutions of the form

$$v(t,x) = e^{\lambda(x.e - ct)} \phi(x)$$

with $\phi > 0$, 1-periodic. Then set:

$$w_*(e) = \inf_{\substack{|e'|=1 \\ e.e'>0}} \frac{c_*(e')}{e.e'}.$$

We have the following.

- For each $w \in (0, w_*(e))$,

$$\lim_{t \to +\infty} \sup_{x \in [0, wte]} |u(t,x) - u_+(x)| = 0.$$

- For each $w \in (w_*(e), +\infty)$,

$$\lim_{t \to +\infty} \sup_{r \geq wt} u(t, re) = 0.$$

The main result is of course (iii), and the expression of w_*, together with the limits, constitutes the Freidlin-Gärtner formula. It was, not surprisingly, discovered by Freidlin and Gärtner [GF], and proved with probabilistic tools. Since then, its scope was considerably extended and at least 4 methods of proofs are known:

- Probabilistic proofs studying sample trajectories (Freidlin [F]),
- Viscosity solutions methods (Evans-Souganidis [ES]),
- dynamical systems methods (monotone dynamical systems, Weinberger [W]),
- PDE methods (Berestycki et al, [BHN]).

The goal of these notes is to explain, in a pedestrian fashion, the Freidlin-Gärtner formula, as well as its underlying biological implications. So, we will successively examine:

I). Some elementary modelling issues,

II). Steady solutions to (1),

III). Exponential solutions to (2),

IV). the nonlinear equation and the formula.

I). Modelling issues

Although Freidlin and Gärtner rather refer to (1) as a model for the propagation of concentration waves in a periodic medium, equation (1) has a nice interpretation in terms of population dynamics. Let therefore a population (of animals, bacteria, even vegetals) be described in terms of its local density $u(t,x)$ (roughly speaking: $u(t,x)$ is the "number" of individuals present, at time t, in the cube centred at x and infinitesimal volume dx, so that the number of individuals in a given region Ω is $\int u(t,x)\,dx$.)

This, by the way, is an assumption that the population is not too sparse – we are not describing animals in the desert this way. The individuals are subject to

– growth and disappearance. In other words, in the absence of displacement, we assume that the population density evolves as

$$\frac{du}{dt}(t,x) = \mu(x)\,u - u^2 = (\mu(x) - u)\,u .$$

(Here x is still a dummy variable).

We are saying that, in the absence of displacement, the growth rate of the population is limited by a function of u (too many individuals prevent the growth of the population) and, in this context, we simply say that the limiting term is $-u$.

The threshold value for which the growth rate becomes negative is $u = +\mu(x)$. It is called the <u>carrying capacity</u> of the population. See for instance a much more detailed explanation in Murray [Mu].

- Migrations and displacements. Assume this time that growth does not occur but that the population ~~does not~~ may move. One way to count how many individuals enter and leave the cube centred at x and of volume dx is to write

$$\frac{\partial u}{\partial t}(t,x) = \int k(x,y)\, u(t,y)\, dy - u(t,x) \int k(y,x)\, dy$$

where $k(x,y)$ is the proportion of individuals which travel from point y to point x.
Assuming an expression of the form

$$k(x,y) = \frac{1}{\varepsilon^N} \rho\left(\frac{|x-y|}{\varepsilon}\right), \left(\rho \geq 0, \text{ compactly supported}, \int \rho = 1\right)$$

we obtain, neglecting the ε^3 terms:

$$\frac{\partial u}{\partial t} = D\varepsilon^2 \Delta u, \quad D = \int |x|^2 \rho(x)\, dx.$$

Such an expression is easily justified by saying that the migrations occur, preferably, in a small neighbourhood of the point considered.

- Putting everything together (and changing the time scale) yields (1).

However, one problem appears: what happens to the carrying capacity (which has a true biological significance)? This is exactly what is explained in the next section.

II). Steady solutions.

A possible alternative notion is a space-dependent state that would attract all solutions for large times. This indeed encompasses the notion of constant carrying capacity: if $t \to +\infty$, the solution of $\dot{N} = (\mu - N)N$ obviously tends to μ. And so, ~~the~~ what we are looking for is a positive solution to

$$\begin{cases} -\Delta p = \mu(x)p - p^2 \quad (x \in \mathbb{R}^N) \\ p > 0, \quad p \text{ bounded.} \end{cases} \quad (3).$$

Th. (3) has a unique solution if

$$\lambda_1^{\mathbb{R}^N}(-\Delta - \mu I) < 0 . \quad (4)$$

Call it u_+. In the opposite case, (3) has no solution.

Moreover:

- if (4) holds, any solution of (1) goes to u_+ as $t \to +\infty$, uniformly on

compact sets.
- If not, all bounded solutions go to 0 as $t \to +\infty$.

This result has been known for quite a while (more than 30 years) but the uniqueness is slightly more recent and due to [BHR].

Recall that $\mu_1^{per}(-\Delta - \mu(x) I)$ is the 1^{st} periodic eigenvalue of $-\Delta - \mu(x) I$ which, by virtue of Krein-Rutman, exists and is unique. For instance, it is equal to -1 if $\mu(x) \equiv -1$.

The proof of the theorem goes along the following lines.

1. If $\mu_1(-\Delta - \mu(x) I) > 0$. (call it μ_1) Then $Me^{-\mu_1 t}$ is a super-solution to (1).

2. If $\mu_1(-\Delta - \mu(x) I) = 0$. Then set $m_n = \|u(n, \cdot)\|_\infty$; By the strong maximum principle, $(m_n)_n$ is a strictly decreasing sequence;

Exercise. Prove that $(m_n)_n$ goes to 0.

3. The interesting case $\mu_1(-\Delta - \mu(x) I) < 0$.

Let $\varphi_1(x)$ be a positive periodic eigenfunction, then (exercise) $\varepsilon\varphi_1$ is a subsolution to (3) for ε small enough and, conversely, $M > 0$ is a super-solution for M large enough. So, there is a solution u_+ between $\varepsilon\varphi_1$ and M, one may also take it periodic (this is allowed by the sub/super-solution method). To prove uniqueness, let $p(x) > 0$ be another solution. We claim the existence of $\lambda_0 > 0$ such that ~~with with false inequality~~
$p(x) \leq \lambda_0 u_+(x)$,
λ_0 being the smallest $\lambda > 0$ such that $p(x) \leq \lambda u_+(x)$. Let us prove $\lambda_0 \leq 1$:

if $\lambda_0 > 1$, we have:

$$\begin{cases} -\Delta v + (-\mu + \lambda_0 u_+ + p) v = \lambda_0(\lambda_0 - 1) u_+^2 > 0. \\ v \geq 0. \end{cases}$$

If v ~~has a~~ vanishes somewhere, this contradicts the strong maximum principle.

<u>Exercise</u>. If there is a sequence $(x_n)_n$ such that
- $|x_n| \to +\infty$.
- $\lim_{n \to +\infty} v(x_n) = 0$

this is also a contradiction to the strong maximum principle.

We would like to reverse inequalities and contradict an inequality of the form $u_+(x) \leq \lambda p(x)$ for some $\lambda > 0$, but nothing tells us that u_+ and p are comparable. This needs to be proved, and this is why we need the

<u>Lemma</u>. $\inf_{x \in \mathbb{R}^N} p(x) > 0$.

Of course it would be trivial if p was periodic, and would close the theorem.
The idea would be to compare p to a continuous family of subsolutions, possible candidates being $\varepsilon \varphi_1(x)$ ($1^{\underline{st}}$ eigenfunctions of $-\Delta - \mu I$).
But φ_1 is periodic, so this might cause problems at infinity. So, we need compactly supported functions, and a first victim would be $\varepsilon \varphi_1^R(x)$, where φ_1^R is the $1^{\underline{st}}$ eigenfunction of $-\Delta - \mu I$ in B_R, with Dirichlet conditions. But then we need to know that the $1^{\underline{st}}$ eigenvalue does not diverge too far from the periodic one – to keep subsolutions subsolutions.
So, everything reduces to the

<u>Lemma</u>. If λ_1^R is the $1^{\underline{st}}$ Dirichlet

eigenvalue of $-\Delta - \mu(x) I$ in B_R, then

$$\lim_{R \to +\infty} \lambda_1^R = \mu_1^{per}(-\Delta - \mu(x) I) := \lambda_1^{per}.$$

This result can also be viewed with the homogenisation glasses: if λ_1^ε is the 1st ~~period~~ Dirichlet eigenvalue of $(-\Delta - \mu(\frac{x}{\varepsilon}) I)$ in Ω, it converges to $\mu_1^{per}(-\Delta - \mu(x) I)$. And, as a matter of fact, was proved in [Cap] for the 1st time.

<u>Proof of the lemma.</u> The Rayleigh formula yields

$$\lambda_1^{per} = \inf_{v \in H^1_{per}} \frac{\int_{[0,1]^N} |\nabla v|^2 - \mu(x) v^2}{\int_{[0,1)^N} v^2}$$

$$= \inf_{\underbrace{v \in H^1_{per,n}}_{n\text{-periodic}}} \frac{\int_{[0,n]^N} |\nabla v|^2 - \mu(x) v^2}{\int_{[0,n]^N} v^2}.$$

the last inequality being true by the uniqueness of the 1st eigenfunction. So, take $n \geq 2R$, and $v_{R,n}$ the n-periodic function which, in the cell $[0,n]^N$, equals $\varphi_R(x-\frac{n}{2}e)$ in $B(\frac{n}{2}e, R)$ $(e=(1,\dots,1))$ and 0 everywhere else. But the Rayleigh quotient of this object is exactly λ_1^R, so $\lambda_1^{per} \leq \lambda_1^R$.

The converse inequality is obtained by examining the Rayleigh quotient of

$$\tilde{v}_R(x) = \chi_R(x)\varphi_1(x),$$

φ_1 : 1^{st} eigenfunction of $-\Delta-\mu_1 I$ w. periodic conditions,

$$\chi_R(x) = \begin{cases} 1 & \text{in } B_{R-1} \\ 0 & \text{outside } B_R \end{cases} \quad \text{and } \geq 0, \text{ smooth.}$$

We have: $\displaystyle\int_{B_R} |\nabla\tilde{v}_R(x)|^2 = \int_{B_R} |v_R \nabla\chi_R + \chi_R \nabla v_R|^2$.

All the integrands are bounded. Moreover, there are at ~~least~~ most $O(R^{N-1})$ cubes of size 1 where $\nabla\chi_R \neq 0$ or $\chi_R \neq 1$. they are all clustered in $B_R \setminus B_{R-1}$. So, we have $\displaystyle\int_{B_R} |\nabla\tilde{v}_R|^2 = \int_{B_R} |\nabla\varphi_1|^2 + O(R^{N-1})$.

Similarly, $\displaystyle\int \mu\tilde{v}_R^2 = \int \mu\varphi_1^2 + O(R^{N-1})$

$$\int_{B_R} v_R^2 = \int_{B_R} \varphi_1^2 + O(R^{N-1}).$$

Finally : $\left| \int_{B_R} |\nabla\varphi_1|^2 - \mu\varphi_1^2 \right| \geq CR^N$.

Indeed $\displaystyle\int_{[0,1]^N} |\nabla\varphi_1|^2 - \mu\varphi_1^2 = \underbrace{\lambda_1^{per}}_{<0} \int \varphi_1^2$;

counting the cubes of size 1 that are inside B_R we obtain

$$\int_{B_R} |\nabla\varphi_1|^2 - \mu\varphi_1^2 = \lambda_1^{per} R^N \int \varphi_1^2 + O(R^{N-1}).$$

And, in the end :

$$\lambda_1^R \leq \lambda_1^{per} + O\left(\frac{1}{R}\right). \quad \boxtimes$$

To finish the proof of the theorem it remains to see that every compactly supported solution goes to u_+, but this is easy : one may put below the initial datum a compactly supported subsolution of the form $\varepsilon\varphi_1^R$ (Exercise : by the way, check that it is a subsolution) ; the solution starting from $\varepsilon\varphi_1^R$ will go to the smallest solution which stays above ; by uniqueness this has to be u_+.

Now one must understand how fast the stable state u_+ invades the unstable state 0. If $\mu \equiv 1$, there is a very simple way to see it. For simplicity, set $N = 1$ and change the nonlinearity $u - u^2$ into ~~...~~ $f(u)$ with

$$f(u) = \begin{cases} u & \text{if } u \leq \theta . \\ (1-u)\,u & \text{if } u \text{ close to } 1 \end{cases}$$

and smooth > 0 in-between θ and 1, with $f'(u) \leq 1$. (this is an important condition as we will see).

So, we have $f(u) \leq u$.

- <u>Estimates of the level set from above.</u>

If $u(t, x)$ solves

$$u_t - u_{xx} = f(u) \qquad \begin{matrix}(t > 0) \\ (x \in \mathbb{R})\end{matrix} \qquad (5)$$

$$(6)$$

we have $u_t - u_{xx} \leq u$.

let us look for solutions of (5) under the form $\bar{u}(t,x) = e^{\lambda(x - ct)}$. We should have $-\lambda^2 - c\lambda - 1 = 0$, so $c \geqslant 2$ if we wish positive solutions. Conversely, we have solutions of the form $e^{\lambda(x + ct)}$ that are > 0 iff $c \geqslant 2$. So,

$$M \min\left(e^{-(x - 2t)}, e^{x + 2t}\right)$$

is a super-solution to (5) and this yields

$$\lim_{t \to +\infty} \sup_{|x| \geq ct} u(t,x) = 0$$

as soon as $c > 2$.

- <u>Estimates of the level sets from below.</u>

If $u \leq \delta$ and u solves (5), then

$$u_t - u_{xx} = u.$$

So, we may still look for exponential solutions, but we should be more careful. However, for $c < 2$, there are still solutions of the form $e^{\lambda(x \pm ct)}$ but this time we have $\lambda \in \mathbb{C} \setminus \mathbb{R}$. A compactly supported sub-solution can therefore be easily devised as

$$\underline{u}(t,x) = \begin{cases} e^{\operatorname{Re}\lambda (x \pm ct)} q_\delta \cos(\operatorname{Im}\lambda)(x \pm ct) \\ \quad \text{if } (x \pm ct) \in \left[-\frac{\pi}{2\operatorname{Im}\lambda}, \frac{\pi}{2\operatorname{Im}\lambda}\right]. \\ 0 \text{ everywhere else}. \end{cases}$$

Where q_δ is devised so that $\underline{u} \leq \delta$.

<u>Exercise</u>. Can one devise such an easy compactly supported subsolution for $N > 1$?

<u>Hint</u>: if e is a direction of propagation, try $u(t, x\cdot e)\, \varphi_1^R(x)$.

This quick and easy argument tells 2 things:

- the Freidlin - Gärtner formula has a chance to be true.
- one should now have a serious look at exponential solutions.

III). Exponential solutions.

Consider the equation

$$v_t - \Delta v = \mu(x) v. \quad t \in \mathbb{R},\ x \in \mathbb{R}^N. \quad (7).$$

An exponential solution to (7) is a function $v(t, x)$ of the form:

$$v(t, x) = e^{-\lambda(x.e - ct)} \phi(x) \quad (8)$$

ϕ: 1-periodic

e: a given direction. Such an exponential solution will, by the way, be called "exponential solution in the direction e". The equation for ϕ is

$$-\Delta \phi + 2\lambda \partial_e \phi + (-\lambda^2 + c\lambda - \mu)\phi = 0. \quad (9)$$

ϕ: 1-periodic.

And, because we have in mind that v

could serve as a super-solution, we also require $\phi > 0$. But this rings a bell: (9) is an eigen-value problem but, because ϕ is positive, it can only be a 1^{st} eigenvalue problem.
To rewrite (9), we use a slight artefact (that it is useful will become clear in the sequel): set, in (8):

$$v(t,x) = \varphi_1(x)\, w(t,x)$$

where $\varphi_1(x)$ is the 1^{st} periodic eigenfunction of $-\Delta - \mu(x) I$. And so, w solves

$$\lambda_1 w + w_t - \Delta w - 2\,\frac{\nabla \varphi_1}{\varphi_1}\cdot \nabla w = 0 .$$

And $w(t,x)$ is sought for under the form

$$w(t,x) = e^{-\lambda(x\cdot e - ct)}\,\phi(x).$$

But now, we write the equation for ϕ as:

$$-\Delta\left(e^{-\lambda x\cdot e}\phi\right) - 2\,\frac{\nabla \varphi_1}{\varphi_1}\cdot\nabla\left(e^{-\lambda x\cdot e}\phi\right)$$

$$= -\left(\lambda_1 + c\lambda\right)\phi .$$

Recall that $\phi > 0$ and ϕ 1-periodic. Let $\tilde{L}_\lambda$ be defined as:

$$e^{\lambda x.e}\left(-\Delta\left(e^{-\lambda x.e}\,\phi\right) - 2\,\frac{\nabla\varphi_1}{\varphi_1}\cdot\nabla\left(e^{-\lambda x.e}\,\phi\right)\right)$$

$$= \widetilde{L}_\lambda\,\phi\,.$$

The abstract form chosen for (9) will be:

$$c\lambda = -\lambda_1 - \mu_1^{per}(\widetilde{L}_\lambda). \qquad (10)$$

And the result is

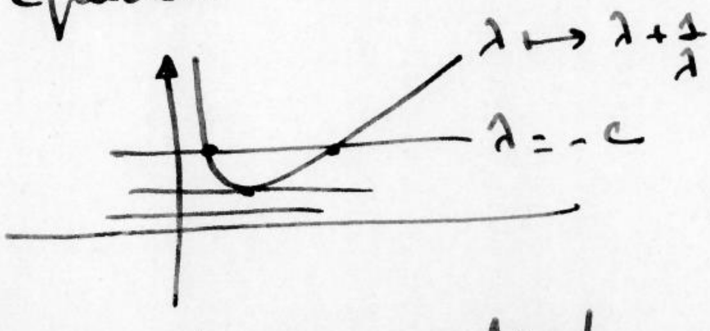
<u>Th</u>. There is $c_*(e) > 0$ such that

- If $c < c_*(e)$, (10) has no solution,
- If $c > c_*(e)$, (10) has 2 solutions,
- If $c = c_*(e)$, (10) has one solution exactly.

Let us pause to evaluate the situation for $N = 1$, $\mu = 1$. Here the problem is simply written as

$$c\lambda = -\lambda^2 - 1, \quad \text{or}: \quad c = -\lambda - \frac{1}{\lambda}.$$

The equation is solved graphically as



and it looks like this very simple situation can be generalised.

By the Kato - Rellich theorem, the function

$$\lambda \mapsto \mu_1^{per}(\widetilde{L}_\lambda)$$

is analytic. The main step in the proof of the theorem is that

<u>Lemma</u>. $\lambda \longmapsto \mu_1^{per}(L_\lambda)$ is concave.

<u>Exercise</u>. Set

$$E_\lambda = \left\{ \phi \in C^2(\mathbb{R}^N) : e^{\lambda x.e}\phi \text{ is 1-periodic, } \phi > 0 \right\}.$$

Then

$$\mu_1^{per}(L_\lambda) = \max_{\phi \in E_\lambda} \inf_{x \in \mathbb{R}^N} \frac{L\phi(x)}{\phi(x)} := k(\lambda).$$

with $L = -\Delta - 2\frac{\nabla \varphi_1}{\varphi_1}.\nabla$.

<u>Proof of the lemma</u>. For $t \in [0,1]$, we wish to prove that:

$$k(t\lambda_1 + (1-t)\lambda_2) \geq t\,k(\lambda_1) + (1-t)\,k(\lambda_2).$$

So, let ψ_1 (resp ψ_2) a 1^{st} eigenfunction of L_{λ_1} (resp. L_{λ_2}). Set: $\phi_i(x) = e^{-\lambda_i x.e}\psi_i(x)$,

$$\phi(x) = \phi_1^t(x)\phi_2^{1-t}(x).$$ <u>Note</u>: $\phi \in E_\lambda$.

~~And we have~~ with $\lambda = t\lambda_1 + (1-t)\lambda_2$. We have:

$$\frac{\nabla\phi}{\phi} = t\frac{\nabla\phi_1}{\phi_1} + (1-t)\frac{\nabla\phi_2}{\phi_2}.$$

$$\frac{\Delta\phi}{\phi} = t\frac{\Delta\phi_1}{\phi_1} + (1-t)\frac{\Delta\phi_2}{\phi_2} + t(t-1)\sum_{i=1}^N\left(\frac{\partial_{x_i}\phi_1}{\phi_1} - \frac{\partial_{x_i}\phi_2}{\phi_2}\right)$$

So : $\dfrac{L\phi(x)}{\phi(x)} \geq t\,\dfrac{L\phi_1(x)}{\phi_1(x)} + (1-t)\,\dfrac{L\phi_2(x)}{\phi_2(x)}$,

and so :

$$\inf_{x\in\mathbb{R}^N} \frac{L\phi(x)}{\phi(x)} \geq t \inf_{x\in\mathbb{R}^N} \frac{L\phi_1(x)}{\phi_1(x)} + (1-t) \inf_{x\in\mathbb{R}^N} \frac{L\phi_2(x)}{\phi_2(x)},$$

But now, ϕ_1 and ϕ_2 are independent, so :

$$\max_{\phi\in E_\lambda} \inf_{x\in\mathbb{R}^N} \frac{L\phi(x)}{\phi(x)} \geq t \max_{\phi_1\in E_{\lambda_1}} \inf_{x\in\mathbb{R}^N} \frac{L\phi_1(x)}{\phi_1(x)}$$

$$+ (1-t) \max_{\phi_2\in E_{\lambda_2}} \inf_{x\in\mathbb{R}^N} \frac{L\phi_2(x)}{\phi_2(x)}.$$

This is exactly the concavity of $\lambda \mapsto -k(\lambda)$. ⊠

<u>Proof of the theorem</u>. The function

$$\lambda \mapsto \lambda_1 - \mu_1^{per}(\overline{L}_\lambda) - c\lambda$$

- is equal to $-\lambda_1$ for $\lambda = 0$ (recall $-\lambda_1 > 0$).
- is convex (this is the lemma)
- and tends to $+\infty$ as $\lambda \to +\infty$ (exercise: $\lambda \mapsto -\mu_1(\overline{L}_\lambda) \sim \lambda^2$ as $\lambda \to +\infty$).
- and has a positive slope at $\lambda = 0$. ~~For small $c>0$, the function obviously never vanishes.~~

This last fact can be checked as follows:

let $\tilde{\Phi}_\lambda$ be a 1^{st} eigenfunction to $\tilde{L}_\lambda$; we have

$$-\Delta\tilde{\Phi}_\lambda + 2\lambda\partial_e\tilde{\Phi}_\lambda + \left(-\lambda^2 - 2\frac{\partial_e\varphi_1}{\varphi_1}\right)\tilde{\Phi}_\lambda - 2\frac{\nabla\varphi_1}{\varphi_1}\cdot\nabla\tilde{\Phi}_\lambda$$

$$= k(\lambda)\tilde{\Phi}_\lambda .$$

By analyticity, ~~we~~ $\tilde{\Psi}_\lambda := \dfrac{d\tilde{\Phi}_\lambda}{d\lambda}$ and $\dfrac{dk}{d\lambda}(\lambda)$ exist and we have, differentiating the whole equation and taking $\lambda = 0$:

$$\tilde{L}_0\tilde{\Psi}_0 + 2\partial_e\tilde{\Phi}_0 = k(0)\tilde{\Psi}_0 + \frac{dk}{d\lambda}(0)\tilde{\Phi}_0 .$$

Obviously : $\tilde{\Phi}_0 = 1$, $k(0) = 0$. Integration of the equation over the unit cell yields $\dfrac{dk}{d\lambda}(0) = 0$.

So :
- for small $c>0$, the equation has no solution,
- for very large $c>0$, there are two solutions.

Moreover, $c \longmapsto -\mu_1(\tilde{L}_\lambda) + \lambda_1 - c\lambda$ is decreasing for each λ. This implies the existence of the threshold c_*, that we now call $c_*(e)$.

The question is now: what do we do with these exponential solutions?

Before answering this question, let us slightly extend the notion of exponential solution to 7 by that of γ-exponential solution ($\gamma > 0$):

a γ-exponential solution $w(t,x)$ to 7 is a solution of (7) of the form

$$v(t,x) = e^{\gamma t} e^{-\lambda(x.e - ct)} \cdot \psi(\xi, x)$$

$\psi(\xi, x)$: 1-periodic in x, $\psi > 0$.

The exponential solutions we have just examined are 0-exponential solutions.

A γ-exponential solution forces the following equations on its parameters c, λ and γ:

$$c\lambda = -\mu_1^{\mu}(L_\lambda) - \lambda_1 - \gamma. \qquad (10)_{bis}$$

And so, it is easy to prove the following

Corollary. For $\gamma > 0$ small enough, there is $c_*^{\gamma}(e) \in (0, c_*(e))$ such that $(10)_{bis}$ has solutions if and only if $c \geq c_*^{\gamma}(e)$.

In other words, lowering c a bit gives a little more room for γ-exponential solutions. This trivial remark will be useful at the end of the proof of the Freidlin-Gärtner formula.

IV). The formula.

The quantity $c_*(e)$ is now born. To understand why the stable state will invade the whole space at velocity $w_*(e)$ instead of $c_*(e)$, recall now that every exponential solution

$$v_e(t,x) = e^{-\lambda_*(e)\,(x\cdot e - c_*(e)\,t)}\,\phi_e(x)$$

(with obvious notation: $\lambda_*(e)$ is the corresponding nonlinear eigenvalue, and $\phi_e(x) := \phi_{\lambda_*(e)}(x)$) is a supersolution to (1). And so,

$$\bar{v}(t,x) = \inf_{|e|=1} e^{-\lambda_*(e)\,(x\cdot e - c_*(e)\,t)}\,\phi_e(x)$$

is also a super-solution, which is >0 initially. So, every initially compactly supported solution to (1) is below a large multiple of $\bar{v}$.
To understand when $\bar{v}$ is nonzero on the line $\{re,\ r>0\}$ with $|e|=1$, we only need to understand when

$$\inf_{|e'|=1} e^{-\lambda_*(e')\,(x\cdot e' - c_*(e')\,t)}$$

deviates from an exponentially small quantity, which occurs when

$$x \cdot e' \leq c_*(e')t$$

for all $|e'| = 1$. If $x = r e$, this occurs when $r\, e \cdot e' \leq c_*(e')t$, and the constraint is active only if $e \cdot e' > 0$. This implies

$$r \leq \frac{c_*(e')t}{e \cdot e'} \quad \text{for all } e' \text{ s.t. } e \cdot e' > 0,$$

which is exactly the first part of the Freidlin-Gärtner formula. It now remains to prove that it is sharp.

So, let e be a chosen direction; we wish to prove that, if $c < c_*(e)$ and $r \leq ct$, then

$$\liminf_{t \to +\infty} u(t, re) > 0 .$$

This is sufficient to imply the convergence to u_+ on every radius of the form cte, $c < c_*(e)$. So, let us choose $c < c_*(e)$ and set

$$v(t, x) = u(t, x + cte)$$

so that v solves

should be modified at 2 places.

1. The definition of $c_*(R)$.

The notion of periodic eigenvalue does not hold anymore, but we may define $c_*(R)$ as the first positive c such that any positive solution $w(t,x)$ of (12) satisfies

$$\liminf_{t\to+\infty} \sup_{B_R} w(t,x) > 0 . \qquad (14)$$

And, in fact, the only item that has to be checked is that any solution of (12) satisfies (14) for small $c>0$. To see this we always may do the change of functions reducing (12) to (13); for $c=0$ the (time-dependent) operator

$$L(t) = -\Delta + \frac{c^2}{4} - \mu(x-ct)I$$

has a negative eigenvalue λ_1 ~~by analytic~~

the operator

$$\tilde{L}(t) = -\Delta - \mu(x-ct)I$$

still has λ_1 as an eigenvalue, but with $\varphi_1(x-ct)$ as a periodic eigenfunction. Let $\lambda_1^R(t)$ be the 1st eigenvalue of $\tilde{L}(t)$ with Dirichlet boundary

$$v_t - \Delta v - c\partial_e v = \mu(x - cte)v - v^2 \quad (11)$$

And all boils down to finding a compactly supported subsolution to (11) that does not vanish as $t \to +\infty$.

<u>Assume first $e \in \mathbb{Q}^N$.</u> Then (11) is a parabolic equations with time-dependent coefficients, with period

$$T = \frac{\mathrm{scm}(e_i)}{c}, \quad e = (e_i)_{1 \le i \le N}.$$

To slightly alleviate the notations, set

$$a(t,x) = \mu(x - cte).$$

The problem

$$w_t - \Delta w - c\partial_e w = a(t,x)w \qquad (12)$$

has a first T-periodic eigenvalue, denoted by $\lambda_1(c)$. Similarly, when posed in B_R with Dirichlet conditions, it has a first periodic eigenvalue denoted by $\lambda_1(c,R)$. To even alleviate the notations, assume

$$e = e_1.$$

We lose the 1-periodicity in x, but

still keep some periodicity, which will be sufficient for our purpose. We claim the existence of $c_*(R)$ such that

$$\lambda_1(c_*(R), R) = 0 .$$

Indeed, because $\mu_1^{per}(-\Delta - \mu(x)\, I) < 0$ and the analysis of Section 1, we have

$$\mu_1^R(-\Delta - \mu(x)\, I) = \lambda_1(0, R) < 0 .$$

This, by analyticity of $c \mapsto \lambda_1(c, R)$, is kept for small c.

Conversely, for $c > 0$ large, we have $\lambda_1(c,R) > 0$.

Indeed, the change of unknowns

$$w(t,x) = e^{-\frac{c}{2}x}\, \tilde{w}(t,x) \qquad (13)$$

brings (12) to the equation

$$(\partial_t - \Delta)\tilde{w} + \left(\frac{c^2}{4} - a(t,x)\right)\hat{w} = 0$$

still with Dirichlet boundary conditions. So, 0 is stable as soon as

$$c > \sqrt{1 + 4\|a\|_\infty}$$

and so the first periodic eigenvalue is > 0.

This implies the existence of $c_*(R)$, and we might as well take the lowest. Notice that it is bounded from above and below, uniformly with respect to R. Not so unexpectedly, the main lemma is now:

<u>Lemma</u>. There holds $\liminf\limits_{R\to+\infty} c_*(R) \geq w_*(e)$.

<u>Proof</u>. let $\varphi_R(t,x)$ be a first eigenfunction, it is T-periodic in t. Assume without loss of generality: $\varphi_R(0,0)=1$; because of the uniform boundedness of $c_*(R)$ w.r.t. R the sequence of functions $(\varphi_R)_R$ converges, locally uniformly, to a solution $\varphi(t,x)$ of (12) (now we do not need to assume $e=1$ anymore) which

- is T-periodic.
- satisfies $\varphi(0,0)=1$.

We would like to prove that φ is an exponential solution to (2), which would prove

Such a property is probably true, but certainly nontrivial. Another way around that is to prove that φ has an exponential solution in its guts. To see this, observe that the Harnack inequality, applied in cubes of size 2, yields (no time shift because φ is a global solution):

$$m\,\varphi(t, x+e_1) \leq \varphi(t,x) \leq M\,\varphi(t, x+e_1),$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. Let, for instance, M_1 be the smallest M such that the above holds; if it is attained at finite distance the strong maximum principle implies

$$\varphi(t,x) = M_1\,\varphi(t, x+e_1).$$

If not, there is a limiting value of φ, of the form $\lim_{n\to+\infty} \varphi([x_n]+x)$, which satisfies the equality. In other words we have found a solution of (12) which additionally satisfies

$$\varphi(t,x) = e^{\lambda_1 x_1}\,\varphi_1(t,x)$$

with φ_1 1-periodic in x_1. Iterating the process we eventually find a solution of (12) of the form
$\varphi(t,x)$

$$\varphi(t,x) = e^{\sum \lambda_i x_i} \varphi_N(t,x)$$

where φ_N is T-periodic in t and 1-periodic in x. Reverting to the original coordinates we have

$$\psi(t,x) := \varphi(t, x - cte) = e^{\sum \lambda_i (x_i - cte_i)} \varphi_N(t, x - cte)$$
$$:= e^{\sum \lambda_i (x_i - cte_i)} \psi_N(t,x)$$

with ψ_N T-periodic in t and 1-periodic in x, and >0, and ψ solving $\psi_t - \Delta\psi = \mu(x)\psi$.

This implies that ψ does not depend on t and that it is an exponential solution. In other words, if we write $e' = \left(\frac{\lambda_i}{|\lambda|}\right)$ we have

$$\sum \lambda_i (x_i - cte_i) = |\lambda| \sum (x_i e'_i - ct e'_i . e_i)$$
$$= |\lambda| (x.e' - ct\, e.e').$$

<u>Exercise</u>. check that* we may always choose $e.e' > 0$ in the above process.

And so, in other words, $c\, e.e'$ is a $c_*(e')$. This implies the lemma. ▨

<u>Assume now that $e \notin \mathbb{Q}^N$</u>. The above proof

shows that any limiting value of $c_*(e, R)$ $(e \in Q)$ generates an exponential solution. So, if $e \notin Q$, let $(e_n)_n \in Q^N$ such that $e_n \to e$ and $|e_n| = 1$.
If $T_n = \dfrac{\mathrm{scm}(e_{1,n}, \dots, e_{N,n})}{c_*(e_n)}$, ~~normalise~~ and going to the original coordinates, normalise the exponential solution

$$\psi_{e_n}(t,x) = e^{\lambda(x \cdot e_n - (e'_n \cdot e_n) c_*(e'_n) t)} \phi_{e_n}(x)$$

such that $\phi_e(0) = 1$. So, up to a subsequence,

$$\bullet\ e'_n \to e'_\infty$$
$$\bullet\ c_*(e'_n) \to c_{*,\infty}$$
$$\bullet\ e_n \to e$$
$$\bullet\ \phi_{e_n} \to \phi_e$$

so that $\psi_e(t,x) = e^{\lambda(x \cdot e - (e'_\infty \cdot e) c_{*,\infty} t)} \phi_e(x)$ is an exponential solution. ~~Because of the super-sol~~

<u>Exercise</u>. Prove that $e \mapsto c_*(e)$ is continuous.

So, $c_{*,\infty} = c_*(e'_\infty)$.
Now, for all $\varepsilon > 0$ and all $n \in \mathbb{N}$, there is $R_n > 0$ (with $\lim_{n \to +\infty} R_n = +\infty$) such that $|c_*(e_n, R_n) - c_*(e_n)| \leq \varepsilon$. So, for n large enough:

$$|c_*(e) - c_*(e_n, R_n)| \leq \varepsilon .$$

Fix $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the above is true. For simplicity we set $R_0 := R_{n_0}$.
Consider the Dirichlet problem for (12) in B_{R_0} with $a(t,x) = \mu(x - cte)$. We claim that, if n_0 is chosen large enough, there is a time-global solution $w_e(t,x)$ such that

- $w_e(t,x) > 0$.
- there is $\gamma > 0$ such that $\liminf_{t \to +\infty} \sup_{B_{R_0}} e^{-\gamma t} w_e(t,x) > 0$

If not, there is a sequence $\gamma_n \to +\infty$ such that the Dirichlet problem (12), with

$$a(t,x) = \mu(x - cte_n)$$

has a ~~$T(e_n)$ periodic~~ solution of the form

$$e^{\gamma_n t} \psi_n(t,x) \qquad (14)$$

with $\psi_n(t,x)$ $T(e_n)$-periodic. And the same analysis ~~of~~ as in the periodic case yields an exponential solution to (7) with speed c, a contradiction.
So, there is $\gamma > 0$ such that the Dirichlet problem (12), with $a(t,x) = \mu(x - cte_n)$, has a solution of the form (14) with $\gamma_n \geq \gamma$. Passing to the limit $n \to +\infty$ yields the result.

As a conclusion we have proved the following important step:

<u>Theorem</u>. Choose $|e| = 1$ and $c < w_+^*(e)$. There is $R_0 > 0$ large enough and a function $w_e(t,x) > 0$ such that there is $\gamma > 0$ for which

$$\liminf_{t \to +\infty} \sup_{x \in B_{R_0}} w_e(t,x) e^{-\gamma t} > 0.$$

This ends the proof of the Freidlin – Gärtner formula in a very easy fashion. Recall that, if we manage to prove the existence of $R > 0$ such that any positive solution $u(t,x)$ of

$$\begin{aligned} u_t - \Delta u - c\partial_e u &= \mu(x - cte)u - u^2 \quad (B_R) \\ u &= 0 \quad (\partial B_R) \end{aligned}$$

satisfies $\liminf_{t \to +\infty} \sup_{x \in B_R} u(t,x) > 0$, we are done.

~~Notice that, if we perturb $\mu(x-cte)$ into $\mu(x-cte)-\delta$, for small $\delta > 0$, the theorem remains true.~~

Assume this is not true, there is a sequence $(t_n)_n$ going to infinity such that, by the Harnack inequality: $\lim_{n \to +\infty} \|u(t_n, \cdot)\|_\infty = 0$. Notice that

$\varepsilon w_e(t,x)$ is a sub-solution, as soon as $w_e(t,x) \leq 1$ and ε is small. This implies the existence of a time $s_n > 0$ such that $\varepsilon w(s_n, x)$ touches $u(s_n, x)$ from below. This is a contradiction with the strong maximum principle.

And the formula is proved! Let us summarise the main argument: any unstable solution of

$$\begin{aligned} u_t - \Delta u - c\partial_e u &= \mu(x - cte)\,u && (B_R) \\ u &= 0 && (\partial B_R) \end{aligned}$$

gives raise, as $R \to +\infty$, to an exponential solution of

$$u_t - \Delta u = \mu(x)\,u$$

in some direction. This is essentially why the estimate $w_*(e) \leqslant \inf\limits_{\substack{|e'|=1 \\ e.e'=0}} \dfrac{c_*(e')}{e.e'}$ is in fact sharp.

Also note that, in the course of this proof, we have encountered quite a lot of different ideas in analysis:

- homogenisation,
- heat kernels [N]
- global solutions of linear equations [P]

all these ideas being of course intertwined. Of course, lurking around all that are large deviations ideas and viscosity solutions. This is one of the reasons why this formula is so interesting.

References.

[BHN]. H. Berestycki, F. Hamel, G. Nadin, JFA 2008.

[F]. M. Freidlin, Functional Integration and PDE's, Princeton Univ. Press, 1985.

[GF]. J. Gärtner, M. Freidlin, Doklady Akadem. Nauk. USSR, 1979.

[ES]. L. Evans, P. Souganidis, Indiana Univ Math. J, 19

[KPP]. A.N. Kolmogorov, I.G. Petrovskii, S.N. Piskunov, Soviet. Math. Doklady, 1937.

[N]. J. Norris, Arch. Rat. Mech. Anal., 1997.

[P]. P. Polacik, I. Terescak, J. Dyn. Diff. Eq, 1993.