PUZZLES WITH SEVERAL CRITICAL POINTS

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Koslovski-vanStrien and Qiu-Yin proved this conjecture.

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Dynamics

If $f : U \to V$ is a ramified covering of degree d and $A \subset U$, $B \subset V$ are closed disks, $mod(U \setminus A) \geq \frac{1}{d} mod(V \setminus B)$

Property of bounded degree

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Definition

The point x has property $(\star)$ if :

$$\exists z, (k_n)_{n\geq 0} \mid f^{k_n} : P_{k_n+k_0}(x) \longrightarrow P_{k_0}(z),\ \forall n \geq 1, \quad \text{has bounded degree.}$$

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For a graph Γ, say that z *accumulates* y if for every $n \geq 0$ there exists $k > 0$ such that $f^k(z) \in P_n(y)$. Write $y \in \omega(z)$.

Lemma

Every point x falls in one of the cases:

- If $\omega(x) \cap Crit = \emptyset$ then $(\star)$ is satisfied ;

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- For $\forall c \in \omega(x) \cap Crit$ and $\forall c' \in \omega(c) \cap Crit$ we have that $c \notin \omega(c')$ we call it the recurrent case.

First entrance

Let U, V be puzzle pieces. For $y \in U$, let r be the smallest integer such that $f^r(y) \in V$, then $deg(f^r : U \to V) \leq \delta^b$.

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$f^{n-i} : P_n(c) = P_i(c)$ and every critical point appears at most twice in $\{f^j(P_n(c)) \mid 0 \leq j < n-i\}$.

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Of course if z is non-persistently recurrent then $(\star)$ is satisfied

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Theorem :

There exists a sequence of puzzle pieces (K_n) in the nest $(P_j(c))$ defined by the operator Γ and two operators $\mathcal{A}$ and $\mathcal{B}$ of bounded degree : $K_n := \mathcal{A}(\Gamma^{b+1}(K_{n-1})$ and $K_n' := \mathcal{B}(\Gamma^{b+1}(K_{n-1})$ with the property that $K_n' \setminus K_n$ does not intersect the postcritical set.

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It is called the enhanced nest by Kozlovski, Shen, van Strien KSS

Corollary (from the construction of the nest) :

- $f^{p_n}(K_n) = K_{n-1}$, $p_{n+1} \geq 2p_n$, $deg(f^{p_n} : K_n \to K_{n-1}) \leq C(b, \delta)$.

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Using "Kahn-Lyubich covering Lemma", we can prove that $\liminf \mu_n$ is bounded from below. K-L

Hence $Imp(c) = \{c\}.\square$

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From KSS nest to Kahn-Lyubich Lemma

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Take the first return of z in K_m : $f^l(z) \in K_m$.

For any m consider $f^{\xi} : K'_{m+2} \to K_m$ and denote by y the point $f^{\xi}(c)$.

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Pull back by f^l the pieces $K_m \subset K'_m$ to $B \subset B'$ containing z,
and then by f^M to puzzle pieces $A \subset A'$ containing y.

Claim

- $f^{\xi}(K_{m+2}) \subset A$
- the degree of $f^{\xi}_{|K'_{m+2}}$ is bounded by C_1 independently of Z, m;
- the degree of $f^{M}_{|A'}$ is bounded by C_2 independently of Z, m;
- the degree of $f^{M}_{|U}$ is bounded by C independent of m;
- the degree of $f^{I}_{|B'}$ is bounded by C_3 independently of Z, m.

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Hence

- $mod(U \setminus A) \leq mod(U \setminus f^{\xi}(K_{m+2})) \leq \frac{1}{C_1}\mu_{m+2}$.
- $mod(B' \setminus B) \geq \frac{1}{C_3}\mu_m$
- Choose m so that $\mu_{m+2} \leq \mu_k$ for $k \leq m+2$.

Hence we get the condition of the covering Lemma:

$$mod(B' \setminus B) \geq \frac{1}{C_1 C_3} mod(U \setminus A)$$

So that either $mod(U \setminus A) > \epsilon(C_1 C_3, D)$ or

$$mod(U \setminus A) \geq \frac{1}{C_1 C_3 2 (C_2)^2} mod(V \setminus B).$$

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Looking at the first entrance of z in the annuli $K'_i \setminus \overline{K}_i$ for $m \leq i \leq m - Z + 1$:

$$mod(V \setminus B) \geq \mu_m + \mu_{m-1} + \ldots + \mu_{m-Z+1} \geq Z\mu_{m+2}.$$

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So that the first inequality implies that

$$\frac{1}{C_1}\mu_{m+2} \geq \frac{Z}{C_1 C_3 2 (C_2)^2}\mu_{m+2}.$$

which is not possible for large Z.

Theorem

Let $f : U \to V$ *be a degree* D *ramified covering. For any* $\eta > 0$*, there exists* $\epsilon = \epsilon(\eta, D) > 0$ *such that :*

- *if* $A \subset\subset A' \subset\subset U$ *and* $B \subset\subset B' \subset\subset V$ *are sequences of disks ;*
- *if* f *is a proper map from* A *to* B*, and from* A' *to* B' *with degree* d *;*
- *if* $mod(B' \setminus B) \geq \eta mod(U \setminus A)$ *;*

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Then

- *$mod(U \setminus A) > \epsilon$*
- *or $mod(U \setminus A) > \frac{\eta}{2d^2} mod(V \setminus B)$.*

retour

Given a puzzle piece I containing c there exist puzzle pieces $\mathcal{A}(I)$ and $\mathcal{B}(I)$ containing c such that

- they are pullback of I;
- $\mathcal{A}(I) \subset \mathcal{B}(I)$;
- the degrees $\mathcal{A}(I) \to I$ and $\mathcal{B}(I) \to I$ are bounded by $C(b, \delta)$ and one meets c at most $b+1$,resp. b times;
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Then define $K'_{n+1} = \mathcal{B}(I_{n+1})$. retour