Puzzles
an overview of some results

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The origin

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A puzzle is a partition that allows to study long-term dynamics on a space of sequences of abstract symbols

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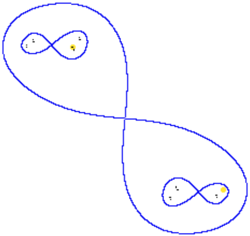

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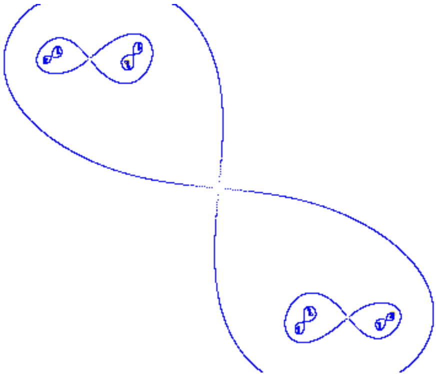

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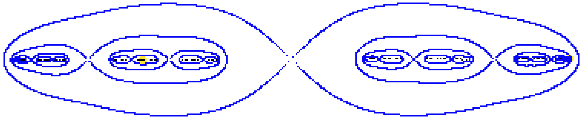

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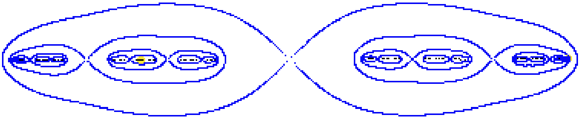

P is conjugated to the shift σ on two symbols $\Sigma = \{0,1\}^n$.

P. Fatou and G. Julia proved the following theorem.

Theorem A

1. The Julia set of a complex polynomial f is connected if and only if $K(f)$ contains all critical points of f.
2. The Julia set of a complex polynomial f is a Cantor set if $K(f)$ contains no critical points of f.

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Using combinatorial system of tableaux, Branner and Hubbard completely settled the question of when the Julia set of a cubic polynomial is a Cantor set.

Emergence of the puzzles.

Theorem (B-H)

For a cubic polynomial f with one critical point in $K(f)$, the Julia set $J(f)$ is a Cantor set if and only if the critical components of $K(f)$ are aperiodic.

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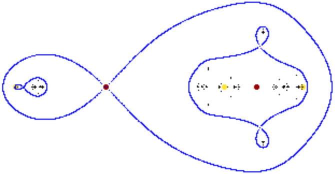

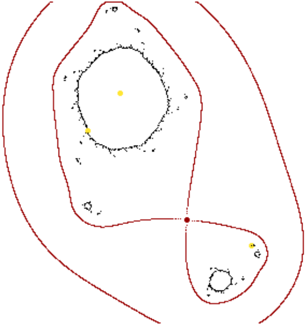

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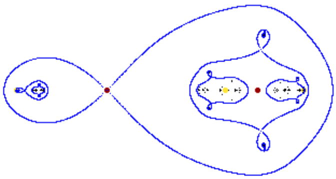

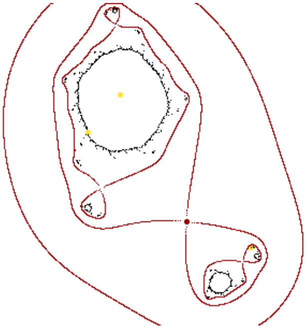

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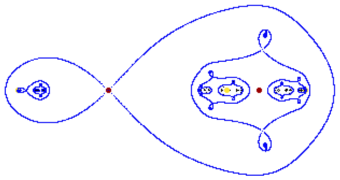

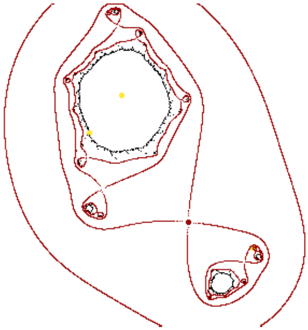

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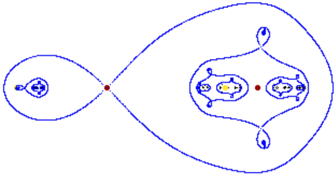

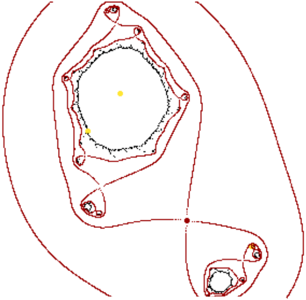

Theorem (McMullen)

For a cubic polynomial f with Cantor Julia set, the Lebesgue measure of $J(f)$ is zero.

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- Let $P_n(x)$ be the puzzle piece containing x
- $f(P_{n+1}(x)) = P_n(f(x))$
- $f : P_{n+1}(x) \to P_n(f(x))$ is a covering of degree at most 2

Branner-Hubbard Puzzle

Every point $x \in K(f)$ defines a "nest" of puzzle pieces

$$x \in P_{n+1}(x) \subset P_n(x) \subset \cdots \subset P_1(x) \subset P_0(x)$$

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Remark

- $K(f)$ is a Cantor set $\iff$ $diam(P_n(x)) \to 0$ for every x.
- K_x is k-periodic $\iff$ the nest is k-periodic :
  $f^k(P_{n+k}(x)) = P_n(x)$ for $n \geq n_0$.

Branner-Hubbard Tableaux

The dynamics can be read on the diagonal of the tableaux

$P_0(x)$

$P_1(x)$

$P_2(x)$

$P_3(x)$

$\vdots$

$\vdots$

$P_n(x)$

$P_{n+1}(x)$

$\vdots$

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$$
\begin{array}{ll}
P_0(x) & \\
 & f\nearrow \\
P_1(x) & \\
 & f\nearrow \\
P_2(x) & \\
 & f\nearrow \\
P_3(x) & \\
\vdots & \vdots \\
\vdots & \vdots \\
 & f\nearrow \\
P_n(x) & \\
 & f\nearrow \\
P_{n+1}(x) & \\
\vdots &
\end{array}
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$$
\begin{array}{ccc}
P_0(x) & & P_0(f(x)) \\
 & f\nearrow & \\
P_1(x) & & P_1(f(x)) \\
 & f\nearrow & \\
P_2(x) & & P_2(f(x)) \\
 & f\nearrow & \vdots \\
P_3(x) & & \vdots \\
\vdots & \vdots & \\
\vdots & \vdots & P_{n-1}(f(x)) \\
 & f\nearrow & \\
P_n(x) & & P_n(f(x)) \\
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 & f\nearrow & & f\nearrow \\
P_2(x) & & P_2(f(x)) & \\
 & f\nearrow & \vdots & \vdots \\
P_3(x) & & \vdots & \vdots \\
\vdots & \vdots & & \\
\vdots & \vdots & P_{n-1}(f(x)) & f\nearrow \\
 & f\nearrow & & f\nearrow \\
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\end{array}
$$

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$$
\begin{array}{ccccc}
P_0(x) & & P_0(f(x)) & & P_0(f^2(x)) \\
 & f\nearrow & & f\nearrow & \\
P_1(x) & & P_1(f(x)) & & P_1(f^2(x)) \\
 & f\nearrow & & f\nearrow & \vdots \\
P_2(x) & & P_2(f(x)) & & \vdots \\
 & f\nearrow & \vdots & \vdots & \\
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\vdots & \vdots & P_{n-1}(f(x)) & f\nearrow & \\
 & & & & P_{n-1}(f^2(x)) \\
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\vdots & & \vdots & &
\end{array}
$$

Some Analysis

To prove that $K_x = \bigcap P_n(x)$ is reduced to $\{x\}$ one needs to understand this combinatorics and the following analysis.

1. The modulus of an annulus A estimates its "size", it is a conformal invariant and $\text{mod}\,(D_R \setminus \overline{D_1}) = \frac{1}{2\pi}\log(R)$;

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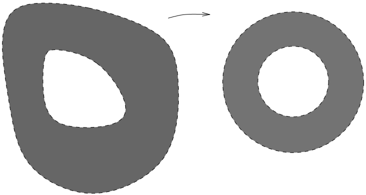

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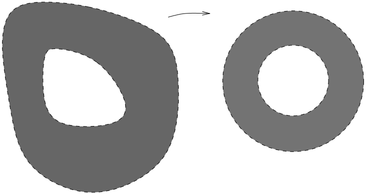

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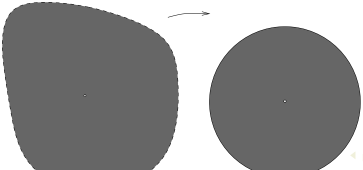

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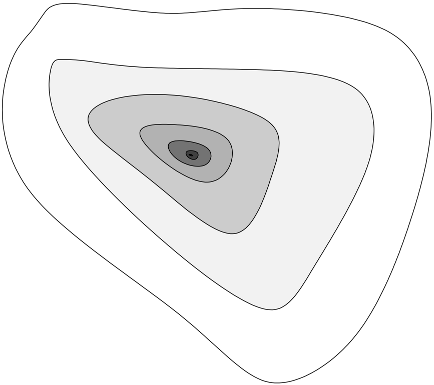
3. it is enough to prove that $\sum mod(A_n(x)) = \infty$.



Generally $f(A_{n+1}(x)) \neq A_n(f(x))$ for $A_n(x) = P_n(x) \setminus \overline{P_{n+1}(x)}$ but

- It is critical if $P_{n+1}(x)$ contains the critical point and $\mod(A_n(x)) = \frac{1}{2} \mod(A_{n-1}(f(x)))$.

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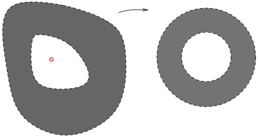

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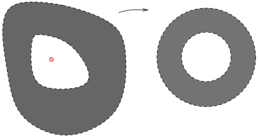

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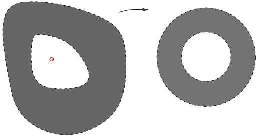

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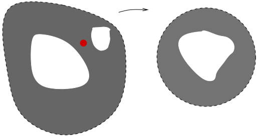

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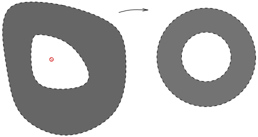

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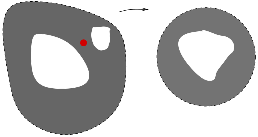
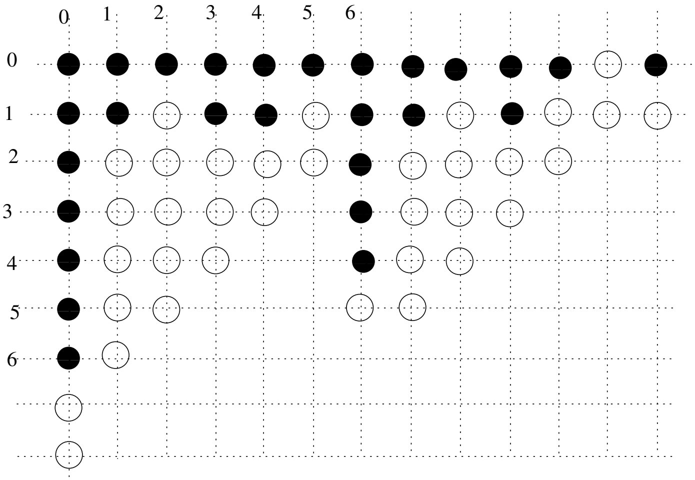


- It is non critical if $P_n(x)$ contains no critical point and $\mathrm{mod}\,(A_n(x)) = \ \mathrm{mod}\,(A_{n-1}(f(x)))$.

Tableaux



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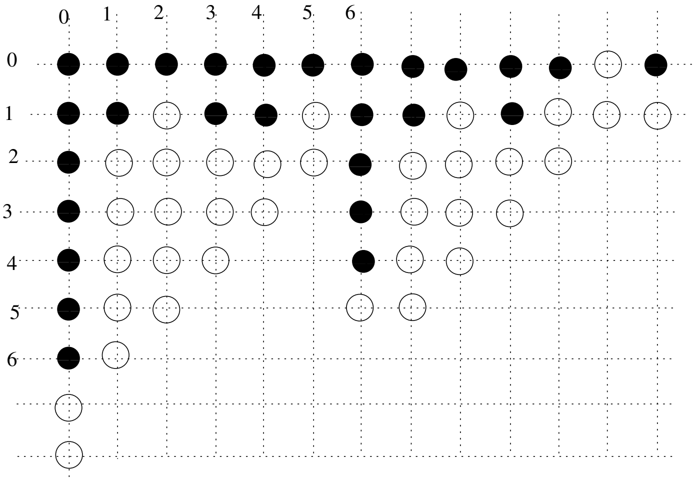

Rule 1: $T(i_0, j) = \bullet \implies T(i, j) = \bullet$ for $i \leq i_0$

$T(i_0, j) = \circ \implies T(i, j) = \circ$ for $i \geq i_0$

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If critical levels are bounded outside the first column then $K_x = x$:

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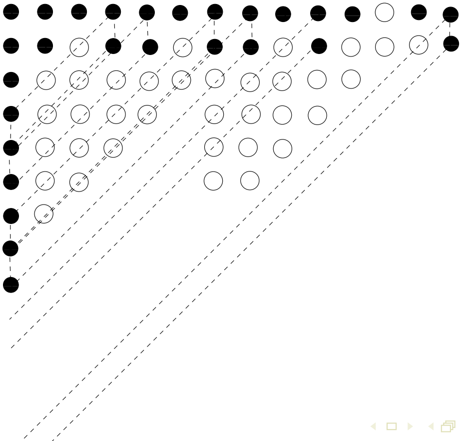

Basic Remarks

If critical levels are bounded outside the first column then $K_x = x$:

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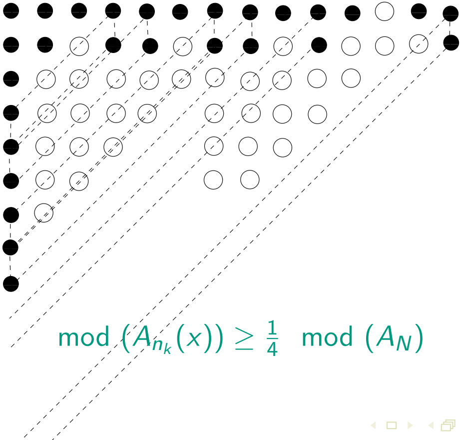

$$\text{mod } (A_{n_k}(x)) \geq \frac{1}{4} \ \text{ mod } (A_N)$$

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If there exists N and (n_k) such that
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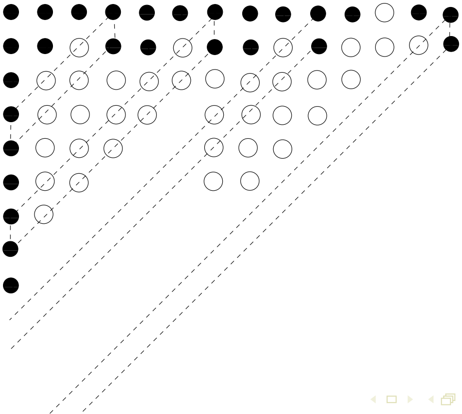
$$\sum \text{mod } (A_n(x)) = \infty$$



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Critical nest

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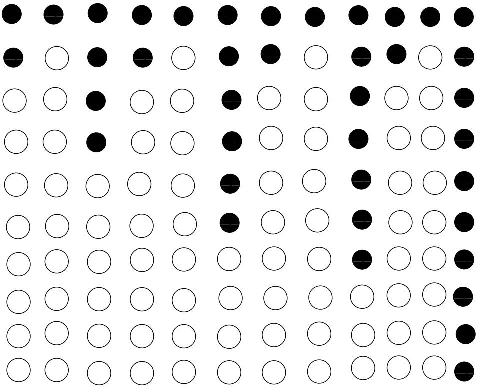

The annulus $A_N(c)$ has infinitely many children

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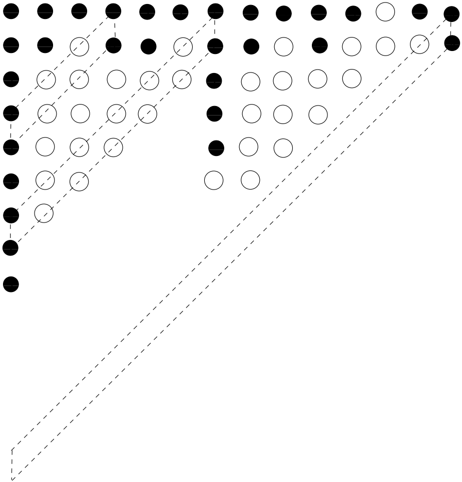

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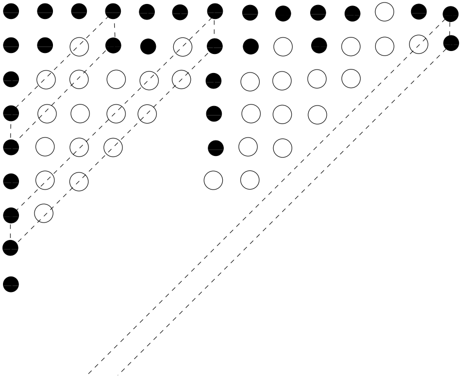

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An annulus A' is a child of an annulus A if there exists some $k > 0$ such that $f^k : A' \to A$ is a degree 2 non ramified covering.

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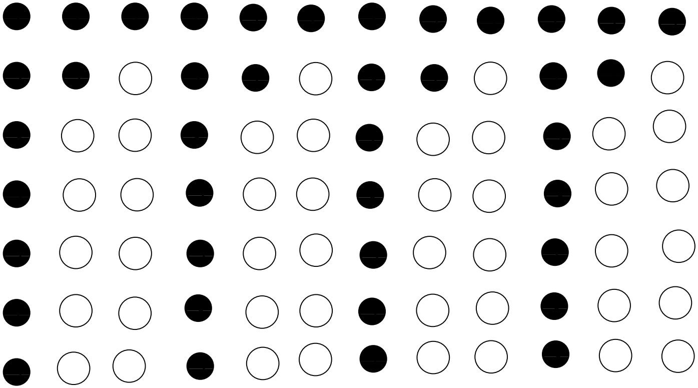

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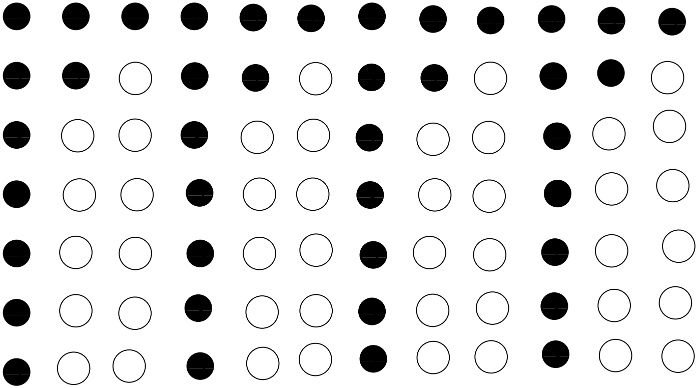
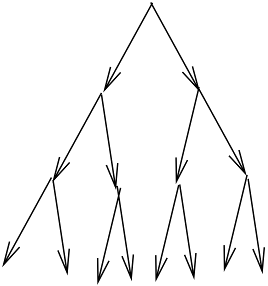


Assume now that the nest $(P_n(c))$ is not periodic

Two is enough



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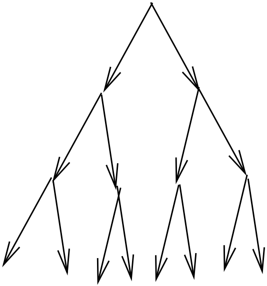

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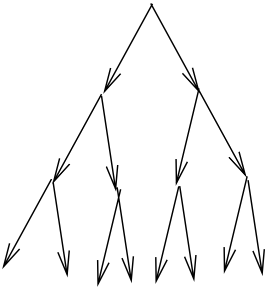

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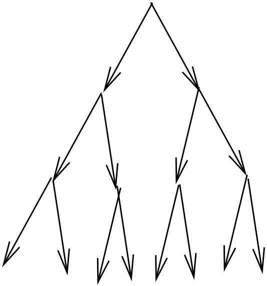

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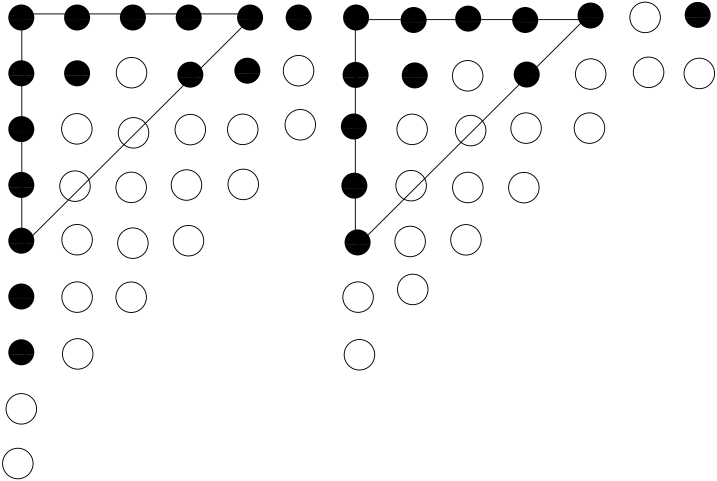
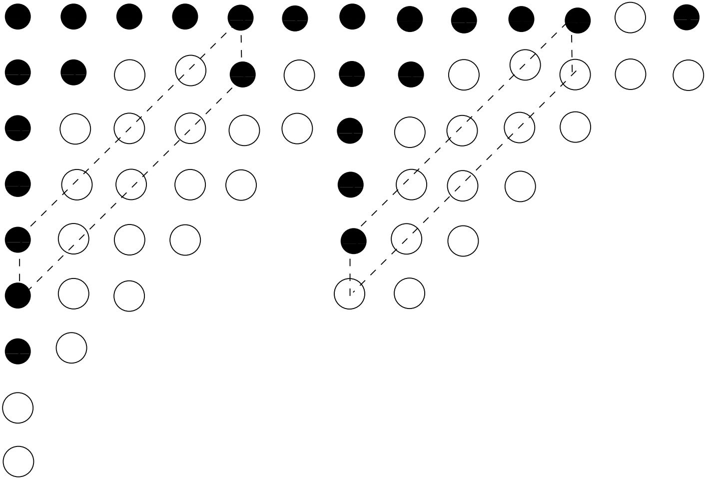
Rule 2



Rule 3



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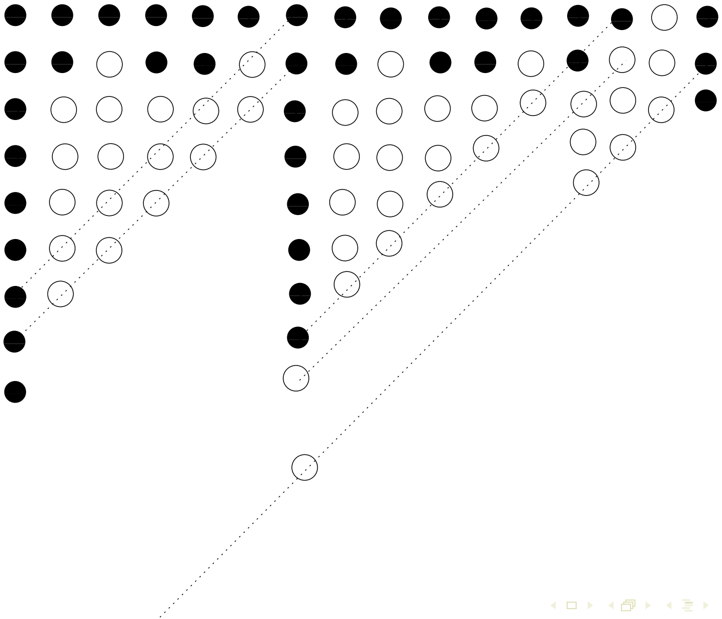

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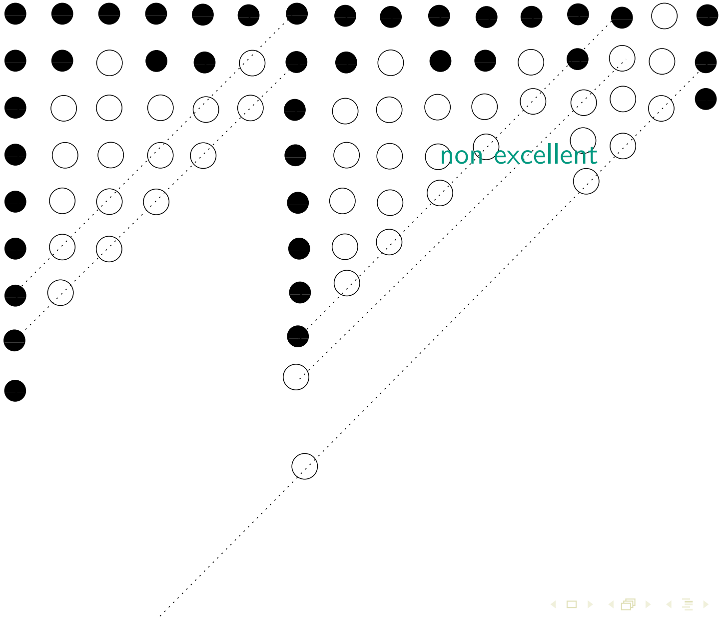

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Proposition :

If the nest $(P_n(c))$ is periodic then K_c is (quasi-conformally) homeomorphic to the filled-in Julia set of a quadratic polynomial.

(by Douady-Hubbard straightening Theorem)

Generalization?

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OF PUZZLES

Key Remark :

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WHAT FOR???

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- Get semi-conjugacy to a rational map and obtain that it is a mating.

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5. get the conjugacy since there are no other identifications than the ones given by the nests shrinking to points.

Quadratic polynomials.

Yoccoz puzzle

Consider quadratic polynomials

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Fatou conjecture:

are hyperbolic polynomials dense?

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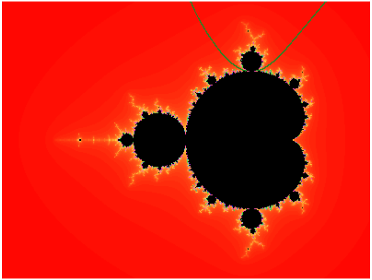

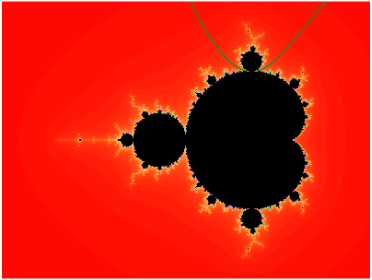

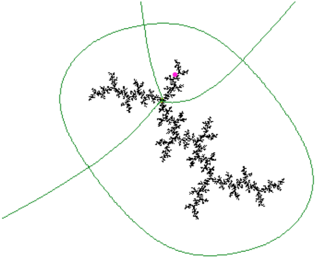

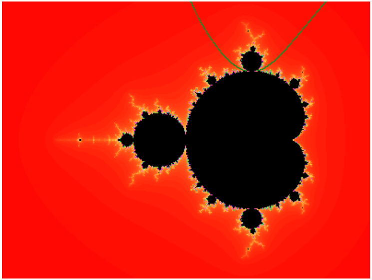

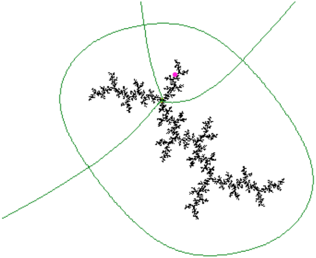

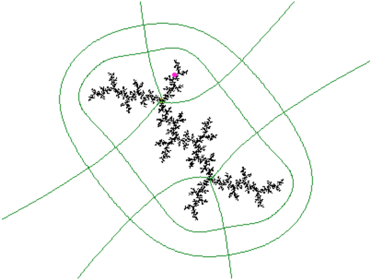

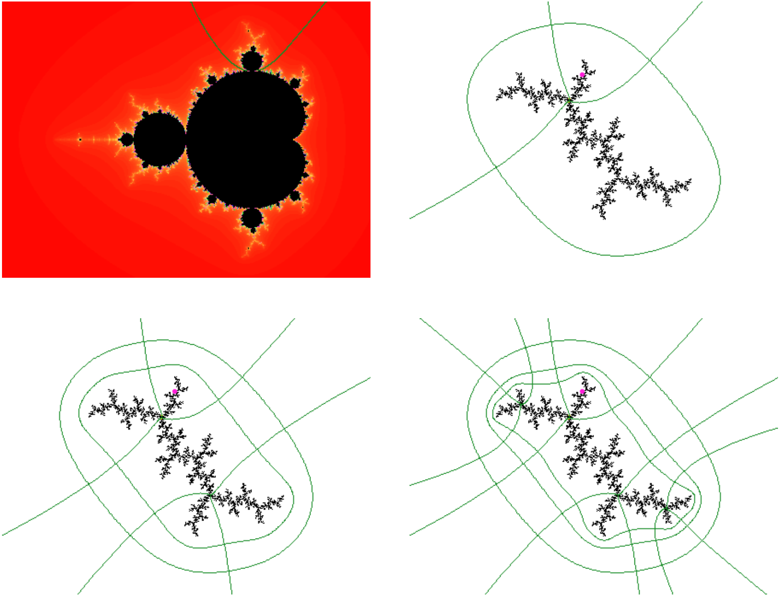

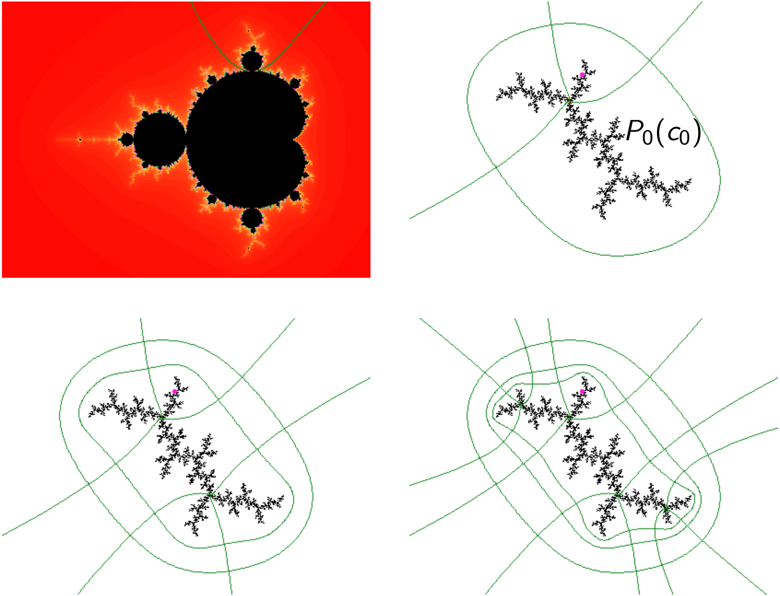
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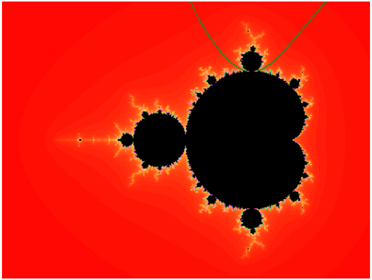

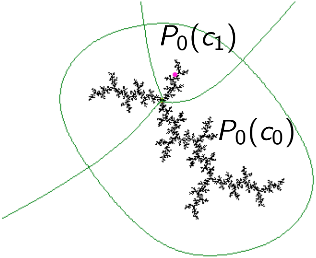
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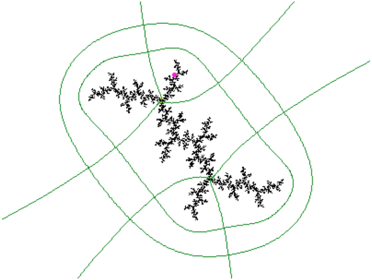

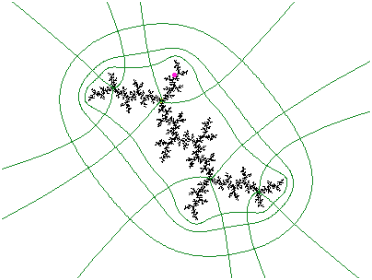

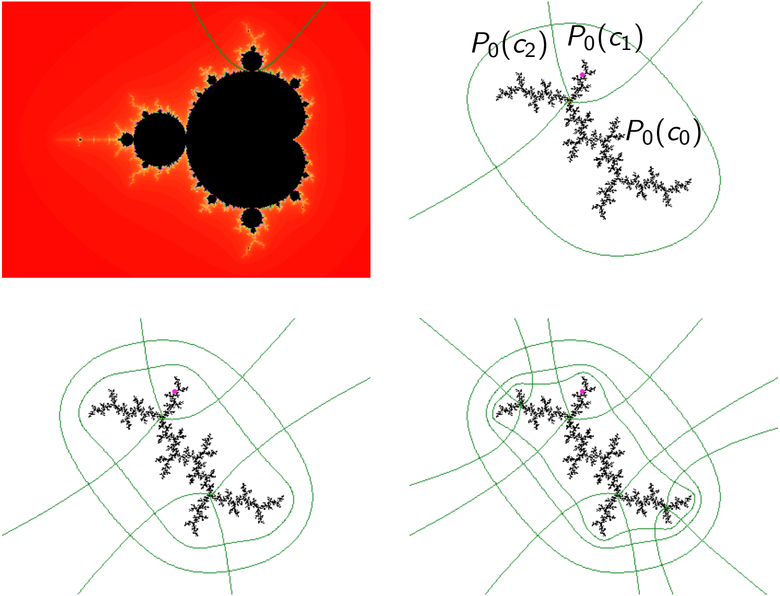
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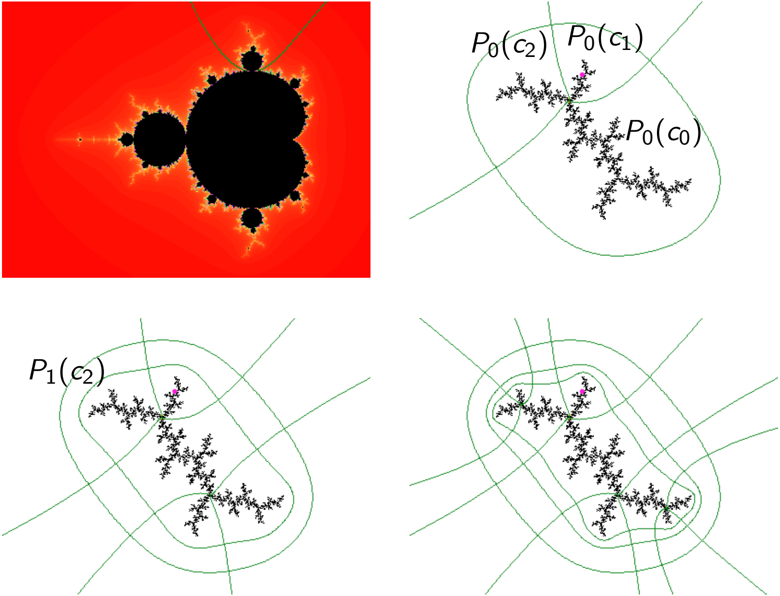
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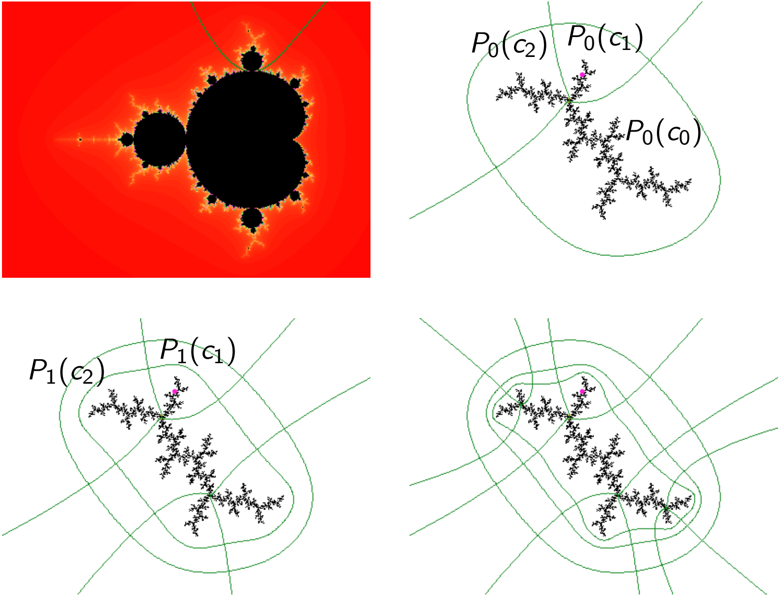
$P_0(c_0)$
$P_1(c_2)$


$P_0(c_2)$
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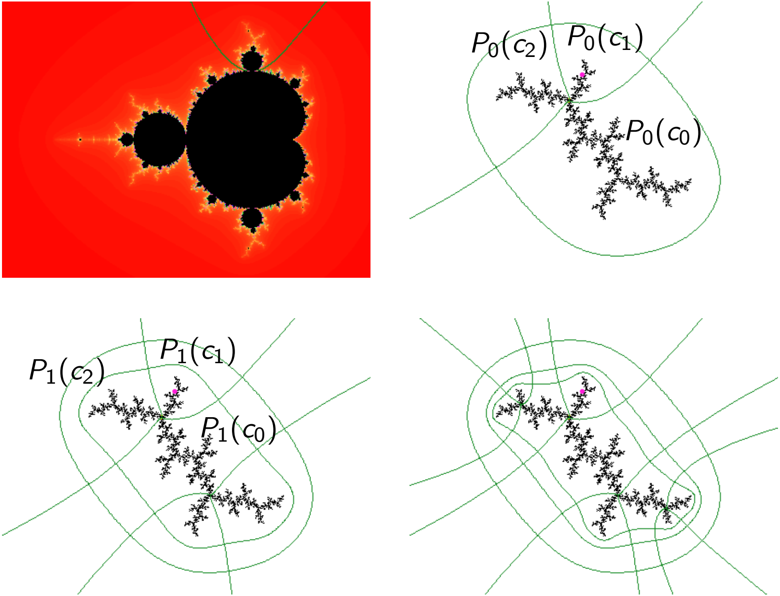
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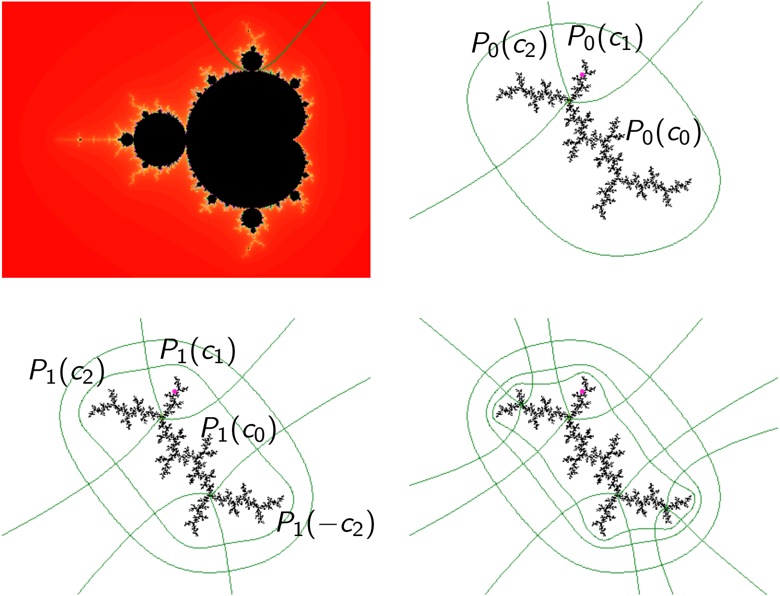
$P_0(c_2)$
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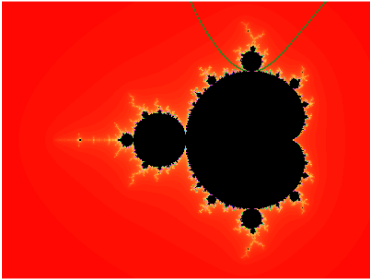
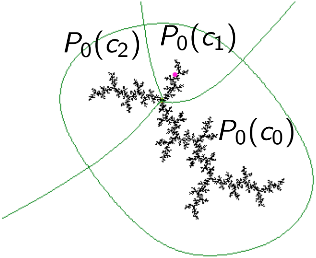
$P_1(c_0)$
$P_1(-c_2)$





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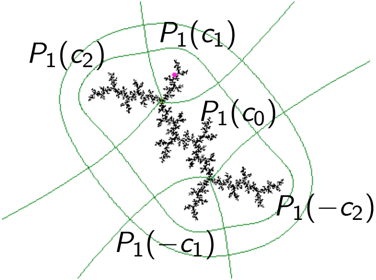

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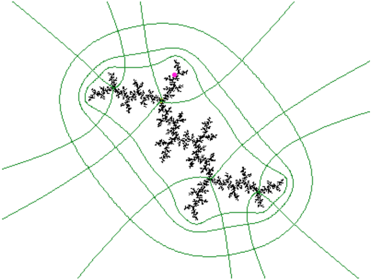
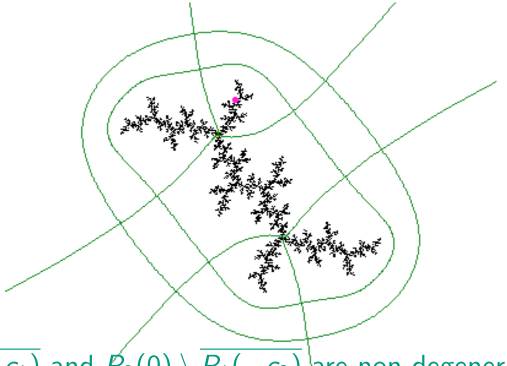
$P_1(-c_2)$
$P_1(-c_1)$



Problem of annuli

A new problem arise here:

TO FIND ENOUGH NON DEGENERATE ANNULI



$P_0(0) \setminus \overline{P_1(-c_1)}$ and $P_0(0) \setminus \overline{P_1(-c_2)}$ are non degenerate.

As before we focus on the critical nest (to be defined we need to)
Assume that the critical orbit does not contain the fixed point α

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2. *if the critical orbit stays in $P_1(0) \cup P_1(c_1) \cup P_1(c_2)$ then the map is renormalizable.*

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Remark

In the tableau non degeneracy is stable along diagonals :

$$P_n(f^k(x)) \setminus \overline{P_{n+1}(f^k(x))} \text{ is non degenerate}$$

then

$$P_{n+k}(x) \setminus \overline{P_{n+k+1}(x)} \text{ also.}$$

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The Mandelbrot set is locally connected at such polynomials.

Generalization?

Definition

A map $f : X' \to X$ is (weakly) *rational like* if

- X and X' are connected open sets of $\widehat{\mathbf{C}}$ with smooth boundary, X contains the closure $\overline{X}'$ of X' ($X \supset X'$ and $\partial X \cap \partial X'$ is finite) and ∂X has finitely many connected components ;
- $f : X' \to X$ is a proper holomorphic map with a finite number of critical point and extends continuously to $\overline{X}' \to \overline{X}$.

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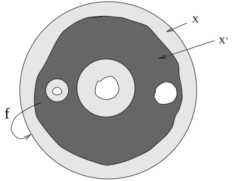

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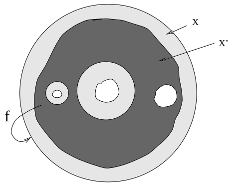

If X, X' are discs it is polynomial-like.

Definition

Let $f : X' \to X$ a (weakly) rational like map with only one critical point, which is simple. A graph Γ is said *admissible* if it satisfies the following conditions:

- Γ is a connected finite graph included in $\overline{X}$ and containing ∂X;
- Γ is *stable* meaning $f^{-1}(\Gamma)$ contains $\Gamma \cap X'$;
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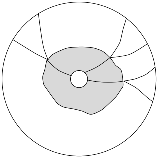

Admissible graph in a basin of attraction

Given an admissible graph Γ for $f : X' \to X$, the *puzzle pieces of level n*, are the connected components of

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Theorem

Let Γ be an admissible graph the that there exists a non degenerate annulus surrounding the critical point c.
Let x be a point of $K(f) = \cap f^{-n}(X)$ that is surrounded by infinitely many annuli of Γ, then

- *if the critical nest is not periodic, then the puzzles pieces shrink to the point (c or x);*
- *if the critical nest is periodic, then the map is renormalizable.*

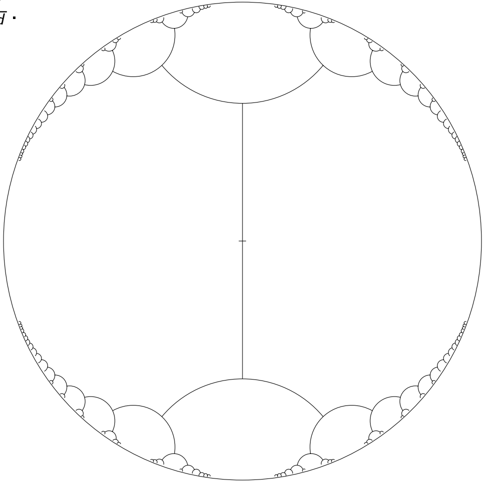
Generalization?

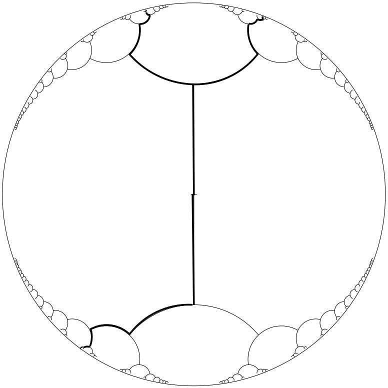
Generalization?

to others kind of graphs

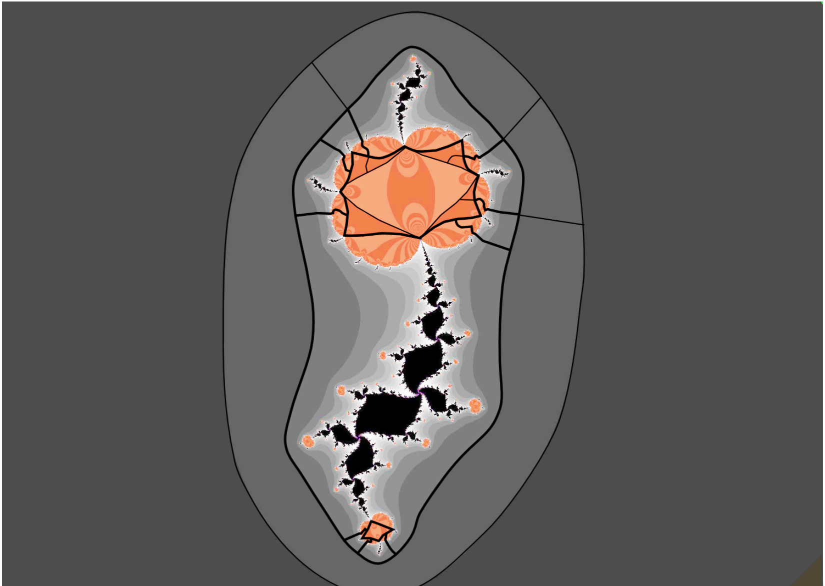
Parabolic rays

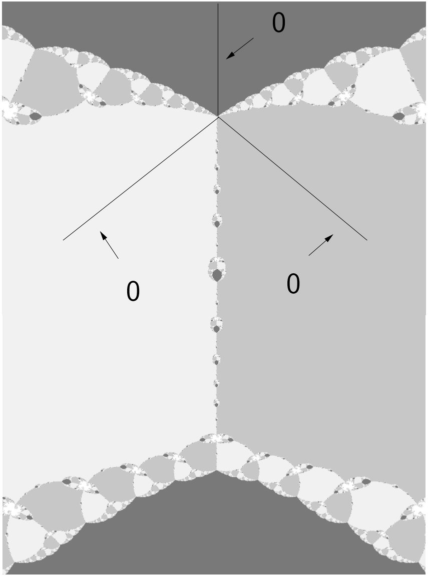
Path of good itinerary in the tree $\mathcal{T} = \cup_n B^{-n}([0, v])$, where $B : z \mapsto \frac{z^d + v}{1 + vz^d}$.





Puzzles

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