VARIATIONAL METHODS FOR A SINGULAR SPDE YIELDING THE UNIVERSALITY OF THE MAGNETIZATION RIPPLE

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ABSTRACT. The magnetization ripple is a microstructure formed in thin ferromagnetic films. It can be described by minimizers of a nonconvex energy functional leading to a nonlocal and nonlinear elliptic SPDE in two dimensions driven by white noise, which is singular. We address the universal character of the magnetization ripple using variational methods based on Γ -convergence. Due to the infinite energy of the system, the (random) energy functional has to be renormalized. Using the topology of Γ -convergence, we give a sense to the law of the renormalized functional that is independent of the way white noise is approximated. More precisely, this universality holds in the class of (not necessarily Gaussian) approximations to white noise satisfying the spectral gap inequality, which allows us to obtain sharp stochastic estimates. As a corollary, we obtain the existence of minimizers with optimal regularity.

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1. INTRODUCTION

We study minimizers of the energy functional

$$E_{tot}(u) := \int_{\mathbb{T}^2} (\partial_1 u)^2 \, \mathrm{d}x + \int_{\mathbb{T}^2} (|\partial_1|^{-\frac{1}{2}} (\partial_2 u - \partial_1 \frac{1}{2} u^2))^2 \, \mathrm{d}x - 2\sigma \int_{\mathbb{T}^2} \xi u \, \mathrm{d}x \tag{1.1}$$

where ξ is (periodic) white noise, $\sigma \in \mathbb{R}$, $\mathbb{T}^2 = [0, 1)^2$ is the two-dimensional torus and $u : \mathbb{T}^2 \to \mathbb{R}$ is a periodic function with vanishing average in x_1 , i.e.,

$$\int_0^1 u \, \mathrm{d} x_1 = 0 \quad \text{for all} \quad x_2 \in [0, 1).$$

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The energy functional E_{tot} was considered in [IO19] as a reduced model describing the magnetization ripple, a microstructure formed by the magnetization in a thin ferromagnetic film, which is a result of the polycrystallinity of the material. In thin films, the magnetization can approximately be described by a two-dimensional unit-length vector field (in the film plane); the ripple is a perturbation of small amplitude of the constant state, say (1, 0). In this context, the function *u* corresponds to the transversal component of the magnetization, after a suitable rescaling. The theoretical treatment in the physics literature [Hof68, Har68] takes it for granted that the ripple is universal, in the sense that it does not depend on the precise composition and geometry of the polycrystalline material. Our main result gives a rigorous justification of the universal behavior of the ripple (see Remark 1.6).

The first term in E_{tot} can be interpreted as the exchange energy, an attractive short-range interaction of the spins. The second term is the energy of the stray field generated by the magnetization; the fractional structure is due to the scaling of the stray field in the thin film regime. On the scales that are relevant for the description of the magnetization ripple, the noise acts like a random transversal field of white-noise character. It comes via the crystalline anisotropy from the fact that the material is made up of randomly oriented grains that are smaller than the ripple scale, which is set to unity in the above model. In view of its origin, it is reasonable to assume that this noise, which is quenched as opposed to thermal in character, is isotropic, nevertheless the nonlocal interaction given by the stray field energy leads to an *anisotropic* response of the magnetization. For a more in-depth description and formal derivation of the energy E_{tot} we refer to the discussion in [IO19, Section 2], which in turn follows [SSWMO12].

Formally, critical points of E_{tot} are solutions to the Euler-Lagrange equation

$$(-\partial_1^2 - |\partial_1|^{-1}\partial_2^2)u + P\left(uR_1\partial_2 u - \frac{1}{2}uR_1\partial_1 u^2\right) + \frac{1}{2}R_1\partial_2 u^2 = \sigma P\xi \quad \text{in } \mathbb{T}^2,$$
(1.2)

where $R_1 = |\partial_1|^{-1}\partial_1$ is the Hilbert transform acting on the x_1 variable, see (1.15), and P is the L^2 -orthogonal projection on functions of zero average in x_1 (extended to periodic Schwartz distributions in the natural way). One of the main challenges of this equation is that the right-hand side of (1.2) is too irregular to make sense of the nonlinear terms, even though the nonlocal elliptic operator

$$\mathcal{L} \coloneqq -\partial_1^2 - |\partial_1|^{-1}\partial_2^2$$

has the expected regularizing properties.

If we endow our space with a Carnot–Carathéodory metric which respects the natural scaling induced by \mathcal{L} , that is, one derivative in the x_1 direction costs as much as $\frac{2}{3}$ derivatives in the x_2 direction, the effective dimension in terms of scaling is given by dim = $\frac{5}{2}$. It is well-known that in this case ξ is a Schwartz distribution of regularity just below $-\frac{\dim}{2}$, i.e., a Schwartz distribution of order $-\frac{5}{4}$ – (measured in a scale of Hölder spaces C^{α} associated to this Carnot–Carathéodory metric; see Section 1.1.3 below for the definition), where for $\alpha \in \mathbb{R}$, we use the notation α – to denote $\alpha - \varepsilon$ for any $\varepsilon > 0$ (suitably small).

We now argue that the nonlinear term $uR_1\partial_2 u$ on the left-hand side of (1.2) is ill-defined: On the one hand, Schauder theory for the operator \mathcal{L} improves regularity by 2 degrees on the Hölder scale, indicating that the expected regularity of a solution u is $(2 - \frac{5}{4}) - = \frac{3}{4} -$. On the other hand, in our anisotropic scaling, one derivative in the x_2 direction reduces regularity by $\frac{3}{2}$, while the Hilbert transform has a negligible effect on the regularity. Hence the regularity of the Schwartz distribution $R_1\partial_2 u$ is $-\frac{3}{4}$ -. It is well-known that the product of a function and a Schwartz distribution can be classically and unambiguously defined only if the regularities of the individual terms sum up to a strictly positive number. In the case of the product $uR_1\partial_2 u$ of the function u and the Schwartz distribution $R_1\partial_2 u$, the sum of regularities is 0-, not allowing its treatment by means of classical analysis.

This is a common problem in the theory of *singular* Stochastic Partial Differential Equations (SPDEs), which has become a very active field in the recent years. Here the word singular relates

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to the fact that the driving noise of these equations is so irregular that their nonlinear terms (which usually involve products of the solution and its derivatives) are not classically defined. We refer the reader to [Hai14] for a more detailed exposition of the theory of singular SPDEs.

In [IO19] the well-posedness of (1.2) for noise strength $|\sigma|$ below a – random – threshold was studied based on Banach's fixed point argument. The ill-defined product $uR_1\partial_2 u$ was treated via a more direct renormalization technique (in contrast to the more general one appearing in the framework of Regularity Structures), known as Wick renormalization. In fact, a similar technique had been introduced by Da Prato and Debussche in their work [DD03] on the stochastic quantization equations of the $\mathcal{P}(\varphi)_2$ -Euclidean Quantum Field theory.

One of the goals of this paper is to get rid of the smallness condition from [IO19]. Without loss of generality we may therefore assume that the parameter $\sigma = 1$, which we will always do in the following.¹ This means in particular that we have to give up the use of a fixed point theorem on the level of the Euler-Lagrange equation, and use instead the direct method of the calculus of variations on the level of the functional. The functional E_{tot} is in need of a renormalization. This is indicated by the fact that if v is the unique solution with zero average in x_1 to the linearized Euler-Lagrange equation²

$$(-\partial_1^2 - |\partial_1|^{-1}\partial_2^2)v = P\xi \quad \text{in } \mathbb{T}^2,$$
(1.3)

which is explicit on the level of its Fourier transform, one has that $E_{tot}(v) = -\infty$ almost surely, see Proposition C.1. As in [DDo3] and [IO19], we decompose any admissible configuration u in (1.1) into u = v + w, where the remainder w is a periodic function with vanishing average in x_1 .

As is usual in renormalization, we may approximate white noise by a probability measure that is supported on smooth ξ 's. This allows for a pathwise approach: For smooth (and periodic) ξ , the solution v of (1.3) is smooth, too (see Lemma D.3). In view of the almost sure divergence of $E_{tot}(v)$ in case of the white noise, we consider the renormalized functional in w = u - v,

$$E_{ren} := E_{tot}(v + \cdot) - E_{tot}(v) \quad \text{with} \quad v = \mathcal{L}^{-1}P\xi.$$
(1.4)

It follows from Lemma D.3 that for smooth ξ , E_{ren} is well-defined (with values in \mathbb{R}) on the space

$$\mathcal{W} := \left\{ w \in L^2(\mathbb{T}^2) : \int_0^1 w \, \mathrm{d}x_1 = 0 \text{ for every } x_2 \in [0,1), \, \mathcal{H}(w) < \infty \right\},\tag{1.5}$$

endowed with the strong L^2 -topology, where \mathcal{H} denotes the harmonic energy, i.e., the quadratic part of E_{tot} given by

$$\mathcal{H}(w) \coloneqq \int_{\mathbb{T}^2} (\partial_1 w)^2 \,\mathrm{d}x + \int_{\mathbb{T}^2} (|\partial_1|^{-\frac{1}{2}} \partial_2 w)^2 \,\mathrm{d}x. \tag{1.6}$$

Loosely speaking, the task now is to show that E_{ren} can still be given a sense as we approximate the white noise. We will consider approximations that belong to the following class of probability measures:³

Assumption 1.1. We consider the class of probability measures $\langle \cdot \rangle$ on the space of periodic Schwartz distributions⁴ ξ (endowed with the Schwartz topology), satisfying the following:

(i) $\langle \cdot \rangle$ is centered: $\langle \xi \rangle = 0$, that is, $\langle |\xi(\varphi)| \rangle < \infty$ and $\langle \xi(\varphi) \rangle = 0$ for all $\varphi \in C^{\infty}(\mathbb{T}^2)$.

¹Note however, that all our results also hold for $\sigma \neq 1$ by considering σv instead of v given in (1.3).

²From now on, given ξ , we denote by v the unique solution with zero average in x_1 to the linearized equation $\mathcal{L}v = P\xi$ from (1.3).

³In the following we specify the topology on the space where we define a (probability) measure, the σ -algebra always being given by the Borel σ -algebra associated to that topology.

⁴In our notation, we do not distinguish between the probability measure and its expectation, and use $\langle \cdot \rangle$ to denote in particular the latter. In the probability jargon, ξ plays the role of a dummy variable like the popular ω . We prefer to adopt this point of view, but sometimes it is convenient to also think of ξ as a random variable taking values in the space of periodic Schwartz distributions by identifying it with the canonical evaluation $\xi \mapsto \mathbf{ev}(\xi)$, where $\mathbf{ev}(\xi)(\varphi) := \xi(\varphi)$ for all $\varphi \in C^{\infty}(\mathbb{T}^2)$. In our notation, when we refer to the law of ξ , we mean the law of the random variable \mathbf{ev} or rather the probability measure $\langle \cdot \rangle$.

(ii) $\langle \cdot \rangle$ is stationary, that is, for every shift vector $h \in \mathbb{R}^2$, ξ and $\xi(\cdot + h)$ have the same law.⁵ (iii) $\langle \cdot \rangle$ is invariant under reflection in x_1 , that is, ξ and $x \mapsto \xi(-x_1, x_2)$ have the same law.⁶ (iv) $\langle \cdot \rangle$ satisfies the spectral gap inequality (SGI), meaning that⁷

$$\left\langle \left| G(\xi) - \left\langle G(\xi) \right\rangle \right|^2 \right\rangle^{\frac{1}{2}} \le \left\langle \left\| \frac{\partial}{\partial \xi} G(\xi) \right\|_{L^2}^2 \right\rangle^{\frac{1}{2}},\tag{1.7}$$

for every functional G on the space of Schwartz distributions such that $\langle |G(\xi)| \rangle < \infty$ and G is well-approximated by cylindrical functionals. More precisely, for cylindrical functionals $G(\xi) = g(\xi(\varphi_1), \ldots, \xi(\varphi_n))$ with $g \in C^{\infty}(\mathbb{R}^n)$ which itself and all its derivatives have at most polynomial growth, $\varphi_1, \ldots, \varphi_n \in C^{\infty}(\mathbb{T}^2)$, and $n \in \mathbb{N}$, we define the Malliavin derivative as the random field

$$\frac{\partial}{\partial\xi}G(\xi) := \sum_{i=1}^n \partial_i g(\xi(\varphi_1), \dots, \xi(\varphi_n))\varphi_i$$

and complete the space of all cylindrical functionals with respect to the norm $\langle |G(\xi)|^2 \rangle^{\frac{1}{2}} + \langle \left\| \frac{\partial}{\partial \xi} G(\xi) \right\|_{L^2}^2 \rangle^{\frac{1}{2}}$. Assuming closability of $\frac{\partial}{\partial \xi}$, i.e., the closure of the graph of $G(\xi) \mapsto \frac{\partial}{\partial \xi} G(\xi)$ on the space of cylindrical functionals with respect to the product topology of $\langle |\cdot|^2 \rangle^{\frac{1}{2}}$ and $\langle \|\cdot\|^2 \rangle^{\frac{1}{2}}$ remains a graph⁸, the resulting space can be identified with a subspace of $L^2_{\langle \cdot \rangle}$ for which (1.7) is well-defined⁹.

Remark 1.2. Note that for a finite dimensional Gaussian probability measure on \mathbb{R}^n with covariance matrix $A^{-1} \in \mathbb{R}^{n \times n}$, the spectral gap inequality holds in the form

$$\left\langle |G - \langle G \rangle|^2 \right\rangle^{\frac{1}{2}} \le \left\langle \nabla G \cdot A^{-1} \nabla G \right\rangle^{\frac{1}{2}}$$

for every bounded continuously differentiable $G : \mathbb{R}^n \to \mathbb{R}$ (see for example [Hel98, Theorem 2.1]). In the case of white noise this translates into (1.7). Furthermore, it is easy to check that if we convolve white noise with a smooth mollifier, the resulting random field satisfies (1.7).

Remark 1.3. For a linear functional *G*, i.e. *G* of the form $G(\xi) = \xi(\varphi)$, for some $\varphi \in C^{\infty}(\mathbb{T}^2)$, the spectral gap inequality (1.7) turns into $\langle \xi(\varphi)^2 \rangle \leq \int_{\mathbb{T}^2} \varphi^2 dx$, which is a defining property of white noise turned into an inequality. Note that this allows us to extend $\xi(\varphi)$ to $\varphi \in L^2(\mathbb{T}^2)$ as a centered random variable in $L^2_{\langle \cdot \rangle}$ which is admissible in (1.7). In our application, the spectral

gap inequality implies that $\langle \cdot \rangle$ is supported on the Hölder space $C^{-\frac{5}{4}-}$ (see Proposition 1.9 below), yielding the same regularity as in the case of white noise.

The merit of SGI is that it also applies to nonlinear *G* (in this paper, we need it for quadratic G)¹⁰. In addition, it allows us to obtain sharp stochastic estimates for non-Gaussian measures (see Proposition 5.3 and Proposition 5.6) by providing a substitute for Nelson's hyper-contractivity. To

⁷Without loss of generality we have set the constant equal to one.

⁹Incidentally, for cylindrical functionals *G* we also have the relation

$$\left\|\frac{\partial}{\partial\xi}G(\xi)\right\|_{L^2}^2 = \sup_{\substack{\delta\xi \in L^2(\mathbb{T}^2) \\ \|\delta\xi\|_{L^2} \le 1}} \lim_{t \to 0} \left|\frac{G(\xi + t\delta\xi) - G(\xi)}{t}\right|^2.$$

¹⁰Actually, in order to obtain the L^p version of SGI in Proposition 5.1 we need (1.7) for more general *G*, but the main application concerns the quadratic functional in Lemma 5.5.

⁵Namely, for any test function $\varphi \in C^{\infty}(\mathbb{T}^2)$ and shift vector $h \in \mathbb{R}^2$, $\xi(\varphi) = \xi(\varphi(\cdot - h))$ in law.

⁶That is, for any test function $\varphi \in C^{\infty}(\mathbb{T}^2)$, denoting $\tilde{\varphi}(x) = \varphi(-x_1, x_2)$, there holds $\xi(\varphi) = \xi(\tilde{\varphi})$ in law. We note that for our results to hold one could also ask for invariance under reflection in x_2 .

⁸Closability implies that the definition of the Malliavin derivative for a functional *G* does not depend on the specific approximation of *G* from cylindrical functionals. Note that closability in $L^2_{\langle \cdot \rangle} \times L^2_{\langle \cdot \rangle} L^2_x$ implies closability in $L^p_{\langle \cdot \rangle} \times L^p_{\langle \cdot \rangle} L^2_x$ for every $p \ge 2$.

the best of our knowledge, the use of the spectral gap inequality is new in the context of singular SPDEs.

The second, and more subtle, goal of this paper, is to establish universality of the ripple. By this we mean that the limiting law of the renormalized energy functional E_{ren} in (1.4) is independent of the way white noise is approximated, provided the natural symmetry condition in form of Assumption 1.1 (iii), is satisfied. In view of its physical origin, $\langle \cdot \rangle$ derives from the random orientation of the grains. Such a model could be based on random tessellations, which suggests a modelling through a non-Gaussian process.¹¹ This motivates our interest in non-Gaussian approximations of white noise. Our substitute for Gaussian calculus is the spectral gap inequality¹² (1.7), see Assumption 1.1 (iv).

Since we cannot expect almost-sure uniqueness of the absolute minimizer w due to nonconvexity of E_{ren} , this universality is better expressed on the level of the variational problems E_{ren} themselves. Hence, rather than considering the (ill-defined) random fields w of minimal configurations, we consider the random functionals E_{ren} . The latter notion calls for a topology on the space of variational problems, that is, of lower semicontinuous functionals E on \mathcal{W} (taking values in $\mathbb{R} \cup \{+\infty\}$) that have compact sublevel sets (with respect to the strong L^2 -topology). The appropriate topology is the one generated by Γ -convergence¹³; this is tautological since Γ -convergence of functionals in this topology is essentially equivalent to convergence of the minimizers, which do exist provided the functionals have compact sublevel sets. Hence we are led to consider probability measures on the space of lower semicontinuous functionals E on \mathcal{W} endowed with the topology of Γ -convergence¹⁴. From this point of view, the universality of the ripple takes the following form:

Theorem 1.4. Every probability measure $\langle \cdot \rangle$ on the space of periodic Schwartz distributions ξ satisfying Definition 1.1 gives rise to a probability measure $\langle \cdot \rangle^{\text{ext}}$ on the product space of periodic Schwartz distributions ξ in the Hölder space $C^{-\frac{5}{4}-}$ and lower semicontinuous functionals E on W endowed with the topology of Γ -convergence with the following three properties:

- (i) The law of ξ under $\langle \cdot \rangle^{\text{ext}}$ is $\langle \cdot \rangle$, that is, for every continuous and bounded functional G on the space of periodic Schwartz distributions, $\langle G(\xi) \rangle^{\text{ext}} = \langle G(\xi) \rangle$.
- (ii) If ξ is smooth $\langle \cdot \rangle$ -almost surely, then $E = E_{ren}$ for $\langle \cdot \rangle^{\text{ext}}$ -almost every (ξ, E) , where E_{ren} is given by (1.4).
- (iii) If a sequence $\{\langle \cdot \rangle_{\ell}\}_{\ell \downarrow 0}$ of probability measures that satisfy Assumption 1.1 converges weakly to $\langle \cdot \rangle$ (which automatically satisfies Assumption 1.1), then $\{\langle \cdot \rangle_{\ell}^{\text{ext}}\}_{\ell \downarrow 0}$ converges weakly to $\langle \cdot \rangle^{\text{ext}}$.

Remark 1.5. As we mentioned in Remark 1.3 (see also Proposition 1.9 below), Assumption 1.1 implies that the original measure $\langle \cdot \rangle$ in Theorem 1.4 is supported on the Hölder space $C^{-\frac{5}{4}-}$. From this point of view, the measure $\langle \cdot \rangle^{\text{ext}}$ in Theorem 1.4 can be seen as an extension to the product space of the Hölder space $C^{-\frac{5}{4}-}$ and the space of lower semicontinuous functionals on W.

Remark 1.6. Let us explain why Theorem 1.4 expresses the desired universality of the ripple. We are given a sequence $\{\langle \cdot \rangle_{\ell}\}_{\ell \downarrow 0}$ which converges weakly to white noise $\langle \cdot \rangle$, such that $\langle \cdot \rangle_{\ell}$ satisfies Assumption 1.1, and such that for $\ell > 0$, ξ is smooth $\langle \cdot \rangle_{\ell}$ -almost surely. In view of Theorem 1.4 (ii), as long as $\ell > 0$, the pathwise defined E_{ren} , see (1.4), can be identified with the random functional E associated to $\langle \cdot \rangle_{\ell}^{\text{ext}}$. According to Theorem 1.4 (iii), as $\ell \downarrow 0$, the law of (ξ, E_{ren}) under $\langle \cdot \rangle_{\ell}^{\text{ext}}$ converges weakly to $\langle \cdot \rangle_{\ell}^{\text{ext}}$ associated to the law of white noise $\langle \cdot \rangle$.

¹¹Incidentally, random tessellations based on Poisson point processes are known to satisfy a variant of the spectral gap inequality, see [DG20].

¹²In a Gaussian setting the right-hand side of (1.7) would correspond to having L^2 as the Cameron–Martin space.

¹³ Γ -convergence with respect to the strong L^2 -topology in the domain W on which the functionals E are defined.

¹⁴The general framework of Γ -convergence (w.r.t. the convergence of minimizers and their existence for functionals with compact sublevel sets) is explained in the book of Dal Maso [Dal93, Theorem 7.8]. Recall that the space of lower semicontinuous functionals $E: \mathcal{W} \to \mathbb{R} \cup \{+\infty\}$ is a compact space, see [Dal93, Theorem 8.5 and Theorem 10.6].

As a corollary of our results we have the following stronger statement.

Corollary 1.7. Assume that the probability measure $\langle \cdot \rangle$ satisfies Assumption 1.1 and consider its extension $\langle \cdot \rangle^{\text{ext}}$ to the product space of periodic Schwartz distributions ξ in the Hölder space $C^{-\frac{5}{4}-}$ and lower semicontinuous functionals E on W endowed with the topology of Γ -convergence. Then minimizers of E exist in W for $\langle \cdot \rangle^{\text{ext}}$ -almost every (ξ, E) . Moreover, for every $1 \leq p < \infty$, the following estimate holds,¹⁵

$$\left\langle \inf_{w \in \operatorname{argmin} E} [w]_{\frac{5}{4}}^{p} \right\rangle^{\operatorname{ext}} \leq C, \tag{1.8}$$

for a constant C that only depends on p, uniformly in the class of probability measures $\langle \cdot \rangle$ satisfying Assumption 1.1.

Remark 1.8. Since under $\langle \cdot \rangle^{\text{ext}}$ functionals *E* are non-convex (see the discussion below Proposition 1.11) we do not expect uniqueness of minimizers. In that sense, Corollary 1.7 shows the existence of minimizers $w \in W$ with finite $C^{\frac{5}{4}-}$ -norm. However, we do not know if all minimizers have this regularity. The $C^{\frac{5}{4}-}$ -regularity in (1.8) relies on the Euler–Lagrange equation (1.13) (see the discussion below Proposition 1.11). If minimizers were unique, (1.8) would imply tightness of their law in the class of probability measures satisfying Assumption 1.1. This in turn would imply that if a sequence $\{\langle \cdot \rangle_{\ell}\}_{\ell \downarrow 0}$ satisfying Assumption 1.1 converges weakly to $\langle \cdot \rangle$, then the law of the unique minimizer under $\langle \cdot \rangle_{\ell}^{\text{ext}}$ converges weakly to the law of the unique minimizer under $\langle \cdot \rangle_{\ell}^{\text{ext}}$.

In establishing Theorem 1.4, we follow very much the spirit of rough path theory of a clear separation between a stochastic and a deterministic (pathwise) ingredient. The genuinely stochastic ingredient is formulated in Proposition 1.9, where we extend the probability distribution of ξ 's to a (joint) law of (ξ , v, F). Compared to rough paths, v is the analogue of (multi-dimensional) Brownian motion B, and F similar to the iterated "integrand"¹⁶ $B\frac{dB}{dt}$, see Proposition 1.9 (iii), which relates F to $vR_1\partial_2 v$. The degree of indeterminacy reflected by the difference between Stratonovich (midpoint rule) and Itô (explicit) is suppressed by the symmetry condition in Assumption 1.1 (iii), which feeds into the characterizing property given by (1.9). The crucial stability of this construction is provided by Proposition 1.9 (iv).

Like for rough paths this unfolding into several random building blocks allows for a pathwise solution theory, i.e. the construction of a continuous solution map on this augmented space. In our variational case this turns into a continuous map from the space of (ξ, v, F) 's into the space of functionals, with the above-advertised topology of Γ -convergence, see Proposition 1.11).

Proposition 1.9. Every probability measure $\langle \cdot \rangle$ satisfying Assumption 1.1 is supported on the Hölder space $C^{-\frac{5}{4}-}$ and lifts to a probability measure $\langle \cdot \rangle^{\text{lift}}$ on the space of triples (ξ, v, F) in $C^{-\frac{5}{4}-} \times C^{\frac{3}{4}-} \times C^{-\frac{3}{4}-}$ with the following properties:

- (i) The law of ξ under $\langle \cdot \rangle^{\text{lift}}$ is $\langle \cdot \rangle$.
- (ii) $v = \mathcal{L}^{-1}P\xi \langle \cdot \rangle^{\text{lift}}$ -almost surely.
- (iii) The law of F under $\langle \cdot \rangle^{\text{lift}}$ is characterized by

$$\lim_{t=2^{-n}\downarrow 0} \left\langle \left[F - vR_1 \partial_2 v_t \right]_{-\frac{3}{4}}^p \right\rangle^{\text{lift}} = 0, \tag{1.9}$$

for every $1 \le p < \infty$, where v_t denotes the convolution of v with a suitable semigroup ψ_t .¹⁷ Moreover, if ξ is smooth $\langle \cdot \rangle$ -almost surely, we have that $F = vR_1 \partial_2 v \langle \cdot \rangle^{\text{lift}}$ -almost surely.

(iv) Finally, if a sequence $\{\langle \cdot \rangle_{\ell}\}_{\ell \downarrow 0}$ of probability measures that satisfy Assumption 1.1 converges weakly to a probability measure $\langle \cdot \rangle$, then also $\{\langle \cdot \rangle_{\ell}^{\text{lift}}\}_{\ell \downarrow 0}$ converges weakly to $\langle \cdot \rangle_{\ell}^{\text{lift}}$.

¹⁵With the convention that the infimum in (1.8) is $+\infty$ if argmin $E = \emptyset$.

¹⁶As opposed to its integral $\int B \, dB$, which is called the iterated integral. We refer to [FH14, Chapter 3] for details. ¹⁷Actually, ψ_t is the semigroup associated to the operator $|\partial_1|^3 - \partial_2^2$ (see also (1.16)).

Remark 1.10. Let us point out that (1.9) implies that *F* is actually a $\langle \cdot \rangle$ -measurable function of ξ . Indeed, by (i), (ii), and the triangle inequality for $s, t \in (0, 1]$ dyadic (i.e. $t = 2^{-n}, s = 2^{-m}$ for $m, n \in \mathbb{N}$) we have

$$\left\langle \left[vR_{1}\partial_{2}v_{s} - vR_{1}\partial_{2}v_{t} \right]_{-\frac{3}{4}-}^{p} \right\rangle^{\frac{1}{p}} = \left(\left\langle \left[vR_{1}\partial_{2}v_{s} - vR_{1}\partial_{2}v_{t} \right]_{-\frac{3}{4}-}^{p} \right\rangle^{\text{lift}} \right)^{\frac{1}{p}} \\ \leq \left(\left\langle \left[F - vR_{1}\partial_{2}v_{s} \right]_{-\frac{3}{4}-}^{p} \right\rangle^{\text{lift}} \right)^{\frac{1}{p}} + \left(\left\langle \left[F - vR_{1}\partial_{2}v_{t} \right]_{-\frac{3}{4}-}^{p} \right\rangle^{\text{lift}} \right)^{\frac{1}{p}},$$

which in turn implies that the sequence $\{vR_1\partial_2 v_t\}_{t=2^{-n}\downarrow 0}$ is Cauchy in $L^p_{\langle\cdot\rangle}C^{-\frac{3}{4}-}$. Hence it converges to a random variable $F(\xi) \in L^p_{\langle\cdot\rangle}C^{-\frac{3}{4}-}$ and it is easy to check that $F = F(\xi) \langle\cdot\rangle^{\text{lift}}$ -almost surely. This allows us to identify the lift measure $\langle\cdot\rangle^{\text{lift}}$ as the joint law of $(\xi, \mathcal{L}^{-1}P\xi, F(\xi))$ under $\langle\cdot\rangle$.

The main idea of the deterministic ingredient, Proposition 1.11, is to extend the definition (1.4) of E_{ren} from only depending on $(\xi, v)^{18}$ to depending on (ξ, v, F) , in such a way that the definitions (formally) coincide for $F = vR_1\partial_2 v$. This is achieved by¹⁹

$$E_{ren}(v,F;w) := \mathcal{E}(w) + \mathcal{G}(v,F;w), \tag{1.10}$$

where the anharmonic energy \mathcal{E} is given by the first two contributions of E_{tot} ,

$$\mathcal{E}(w) := \int_{\mathbb{T}^2} (\partial_1 w)^2 \, \mathrm{d}x + \int_{\mathbb{T}^2} \left(|\partial_1|^{-\frac{1}{2}} (\partial_2 w - \partial_1 \frac{1}{2} w^2) \right)^2 \, \mathrm{d}x. \tag{1.11}$$

Note that \mathcal{E} contains the Burgers nonlinearity $\eta_w := \partial_2 w - \partial_1 \frac{1}{2} w^2$, which will play an important role in our analysis. Only the remainder \mathcal{G} depends on v and F, and is given by

$$\mathcal{G}(v,F;w) := \int_{\mathbb{T}^2} \left(w^2 R_1 \partial_2 v + v^2 R_1 \eta_w + 2v w R_1 \eta_w + 2w F - w v R_1 \partial_1 v^2 + (R_1 |\partial_1|^{\frac{1}{2}} (vw))^2 \right) \mathrm{d}x.$$
 (1.12)

Equipped with these definitions, we now may state the main deterministic ingredient.

Proposition 1.11. The application $(\xi, v, F) \mapsto E_{ren}$ described through (1.10) is well-defined and continuous from the space of (ξ, v, F) endowed with the norm $C^{-\frac{5}{4}-} \times C^{\frac{3}{4}-} \times C^{-\frac{3}{4}-}$ into the space of lower semicontinuous functionals E_{ren} on W endowed with the topology of Γ -convergence (with respect to the L^2 -topology on W).

On the level of the Euler–Lagrange equation, minimizers w of $E_{ren}(v,F;\cdot)$ are weak solutions of

$$\mathcal{L}w + P\left(F + wR_1\partial_2 v + vR_1\partial_2 w + wR_1\partial_2 w - \frac{1}{2}(v+w)R_1\partial_1(v+w)^2\right) + \frac{1}{2}R_1\partial_2(v+w)^2 = 0,$$
(1.13)

whose existence is established by Theorem 1.4 (see also Theorem 1.15 (iv) for the validity of (1.13) in the sense of Schwartz distributions). By a simple power counting the expected regularity of solutions *w* to (1.13) is $\frac{5}{4}$ -, which justifies the existence of minimizers $w \in W$ with finite $C^{\frac{5}{4}-}$

$$\int_{\mathbb{T}^2} \left(\partial_1 w \, \partial_1 v + \partial_2 w \, |\partial_1|^{-1} \partial_2 v - w \, \xi \right) \, \mathrm{d}x = 0,$$

which follows from testing (1.3) with w.

¹⁸In fact, E_{ren} depends effectively only on ξ , as $v = \mathcal{L}^{-1}P\xi$. However, we drop ξ in the notation for E_{ren} and keep the dependence on v.

¹⁹If u = v + w, this can be seen by the identity $\eta_u = \eta_v + \eta_w - \partial_1(vw)$ for the Burgers operator $\eta_u = \partial_2 u - \partial_1 \frac{1}{2}u^2$, as well as the equality



FIGURE 1. Construction of the extension measure $\langle \cdot \rangle^{\text{ext}}$. The vertical arrow corresponds to the probabilistic step (Proposition 1.9), while the horizontal arrow is the deterministic step (Proposition 1.11). For smooth ξ 's, *F* is given by $vR_1\partial_2 v$.

norm proved in Corollary 1.7. This generalizes the existence of solutions to (1.13) in [IO19] which was shown for small values of the noise strength $|\sigma|$.

From a variational point of view, the main challenge is to establish the coercivity of the renormalized energy $E_{ren}(v, F; \cdot)$. Ideally, one would like to control the remainder $\mathcal{G}(v, F; \cdot)$ by the "good" term of the renormalized energy, namely, the anharmonic energy $\mathcal{E}(w)$ given in (1.11). At first sight, this is not obvious since the remainder $\mathcal{G}(v, F; \cdot)$ contains quadratic and cubic terms in w, and it is not immediate that the anharmonic energy $\mathcal{E}(w)$ provides higher than quadratic control to absorb these terms. Hence, we need to exploit the control on the nonlinear part coming from the Burgers operator η_w .²⁰ We do this using tools from fluid mechanics, more precisely, the Howarth–Kármán–Monin identities (2.5) and (2.6), following [GJO15]. Based on these identities we can prove that the anharmonic energy $\mathcal{E}(w)$ grows cubically in suitable Besov spaces (see Proposition 2.4). This allows us to absorb $\mathcal{G}(v, F; \cdot)$ and obtain the coercivity of the renormalized energy (see Theorem 1.15 (i)).

Let us point out that here, we prove existence of solutions for any value of the noise strength σ using the coercivity of the renormalized energy functional through the direct method of the calculus of variations. Recent works on the dynamic Φ^4 model (or stochastic Ginzburg–Landau model), where the system is favoured by the "good" sign of the cubic nonlinearity, have used coercivity on the level of the Euler–Lagrange equation. For example, in [MW17a, TW18, MW17b] energy estimates have been used to obtain global-in-time existence in the parabolic case, while in [GH19] both the parabolic and the elliptic cases have been treated based on a different approach that uses coercivity through a maximum principle. A maximum principle has been used also in [CMW19] where the parabolic model is considered in the full subcritical regime.

A further challenge, which turns out to be more on the technical side, comes from the fact that \mathcal{L} is nonlocal. We recall that this feature arises completely naturally from the magnetostatic energy in the thin-film limit (see [IO19, Section 2]), but resonates well with the recent surge in activity on nonlocal operators. It was worked out in [IO19, Lemma 5] that the robust approach of [OW19] to negative (parabolic) Hölder spaces and Schauder theory extends to this situation. This approach involves a suitable convolution semigroup ψ_t ; the fact that it extends from the smooth parabolic symbol $k_1^2 + ik_2$ to our nonsmooth symbol $k_1^2 + |k_1|^{-1}k_2^2$ is not obvious due to the poor decay properties of the corresponding convolution kernel.

Variational problems that in a singular limit require subtraction of a divergent term are wellknown in deterministic settings. A famous example concerns S^1 -valued harmonic maps defined in a two-dimensional smooth bounded simply-connected domain D. The aim there is to minimize the Dirichlet energy of maps $u : D \to S^1$ that satisfy a smooth boundary condition $g : \partial D \to S^1$. When g carries a nontrivial winding number $N > 0^{21}$, the problem is singular, that is, every configuration u has infinite energy as they generate vortex point singularities. The question is

²⁰Incidentally, despite different physical origins, the inviscid Burgers part η_w arises as in the KPZ equation from expanding a square root nonlinearity. Not unlike there, the coercivity comes from the interaction between the first and second term in $\mathcal{E}(w)$, the first term being the analogue to the viscosity in KPZ.

²¹For simplicity, we assume N > 0; the case N < 0 follows by complex conjugation.

to determine the least "infinite" Dirichlet energy of a harmonic S^1 -valued map satisfying the boundary condition g on ∂D . The seminal book of Bethuel–Brezis–Hélein [BBH94] presents two methods to achieve this goal, both reaching the same renormalized energy associated to the problem.

FIRST APPROACH: One prescribes N > 0 vortex points a_1, \ldots, a_N in D and determines the unique harmonic \mathbb{S}^1 -valued map u_* with $u_* = g$ on ∂D that has the prescribed singularities a_1, \ldots, a_N in D, each one carrying a winding number equal to one²². Then one cuts-off disks $B(a_k, r)$ centered at a_k of small radius r > 0 carrying the diverging logarithmic energy of u_* and introduces the renormalized energy

$$W(a_1,\ldots,a_N) = \lim_{r\to 0} \left(\int_{D\setminus \cup_k B(a_k,r)} |\nabla u_*(x)|^2 \,\mathrm{d}x - 2\pi N\log\frac{1}{r} \right)$$

The minimum of the renormalized energy

$$\min_{a_1,\dots,a_N \in D} W(a_1,\dots,a_N) \tag{1.14}$$

represents the minimal second order term in the expansion of the Dirichlet energy and yields optimal positions of the N vortex point singularities (which might not be unique in general).

Second Approach: One considers a nonlinear approximation of the harmonic map problem given by the Ginzburg-Landau model for a small parameter $\varepsilon > 0$:

$$E_{\varepsilon}(u) = \int_D |\nabla u|^2 + \frac{1}{\varepsilon^2} (1 - |u|^2)^2 \, \mathrm{d}x, \quad u : D \to \mathbb{R}^2, \quad u = g \text{ on } \partial D.$$

Note that the maps u are no longer with values into \mathbb{S}^1 , but their distance to \mathbb{S}^1 is strongly penalized as $\varepsilon \to 0$. It is proved in [BBH94, Theorem X.1] that if u_{ε} is a minimizer of the above Ginzburg–Landau problem, then for a subsequence, $u_{\varepsilon} \to u_*$ weakly in $W^{1,1}(D)$ as $\varepsilon \to 0$ where u_* is an \mathbb{S}^1 valued harmonic map whose N vortex points of winding number one correspond to a minimizer of the renormalized energy (1.14). Moreover,

$$E_{\varepsilon}(u_{\varepsilon}) = 2\pi N \log \frac{1}{\varepsilon} + \min_{a_1,\dots,a_N \in D} W(a_1,\dots,a_N) + N\gamma + o(1), \quad \text{as } \varepsilon \to 0,$$

where γ is a constant coming from the nonlinear penalization in E_{ε} .

We also refer to [SSo7, SS15], [Kuro6], [IM16], and [IJ19] for similar renormalized energies.

1.1. Notation. For a periodic function $f : \mathbb{T}^2 \to \mathbb{R}$ we define its Fourier coefficients by

$$\widehat{f}(k) = \int_{\mathbb{T}^2} e^{-ik \cdot x} f(x) \, dx \quad \text{for} \quad k \in (2\pi\mathbb{Z})^2,$$

which extends to periodic Schwartz distributions in the natural way. We also denote by P the L^2 -orthogonal projection onto the set of functions of vanishing average in x_1 , extended in the natural way to periodic Schwartz distributions.

For $p \in [1, \infty]$ we write $\|\cdot\|_{L^p}$ to denote the usual L^p norm on \mathbb{T}^2 , unless indicated otherwise. For example, we write $\|\cdot\|_{L^p(\mathbb{R}^2)}$ for the L^p norm of a function defined on \mathbb{R}^2 . We sometimes write L_x^p (respectively $L_{x_j}^p$, j = 1, 2) to denote the L^p space with respect to the x (respectively x_j) variable. We also write $L_{(...)}^p$ to denote the usual L^p space with respect to the measure $\langle \cdot \rangle$.

We will often make use of the notation $a \leq b$ meaning that there exists a constant C > 0 such that $a \leq Cb$. Moreover, for $\kappa \in \mathbb{R}$, the notation \leq_{κ} will be used to stress the dependence of the implicit constant C on κ , i.e., $C \equiv C(\kappa)$. Similarly, $a \sim b$ means $a \leq b$ and $b \leq a$.

²²In fact, u_* belongs to the Sobolev space $W^{1,1}(D, \mathbb{S}^1)$ and the nonlinear PDE satisfied by u_* , i.e., $-\Delta u_* = |\nabla u_*|^2 u_*$ in D, can be written in a "linear" way in terms of the current $j(u_*) = u_* \times \nabla u_* \in L^1(D)$ of u_* that satisfies the system $\nabla \times j(u_*) = 2\pi \sum_k \delta_{a_k}$ in D, and $\nabla \cdot j(u_*) = 0$ in D. In terms of the so-called conjugate harmonic function ϕ given by $\nabla^{\perp} \phi = j(u_*)$, the problem becomes $-\Delta \phi = 2\pi \sum_k \delta_{a_k}$ in D and $\partial_v \phi = g \times \partial_\tau g$ on ∂D . One could think of ϕ as playing the role of our solution v to the linearized Euler-Lagrange equation (1.3) that carries the "infinite" part of the energy.

1.1.1. *Hilbert transform.* We will frequently make use of the *Hilbert transform* R_j for j = 1, 2, acting on periodic functions $f : \mathbb{T}^2 \to \mathbb{R}$ in x_j as

$$R_j := \frac{\partial_j}{|\partial_j|}, \quad \text{i.e.,} \quad \widehat{R_j f}(k) = \begin{cases} \operatorname{i} \operatorname{sgn}(k_j) \widehat{f}(k) & \operatorname{if} k_j \in 2\pi \mathbb{Z} \setminus \{0\}, \\ 0 & \operatorname{if} k_j = 0, \end{cases}$$
(1.15)

where sgn is the sign function. In particular, $R_j P = PR_j = R_j$.

1.1.2. Anisotropic metric and kernel. The leading-order operator $\mathcal{L} = -\partial_1^2 - |\partial_1|^{-1}\partial_2^2$ suggests to endow the space \mathbb{T}^2 with a Carnot–Carathéodory metric that is homogeneous with respect to the scaling $(x_1, x_2) = (\ell \hat{x}_1, \ell^{\frac{3}{2}} \hat{x}_2)$. The simplest expression is given by

$$d(x,y) := |x_1 - y_1| + |x_2 - y_2|^{\frac{2}{3}}, \quad x,y \in \mathbb{T}^2,$$

which in particular means that we take the x_1 variable as a reference.

We now introduce the convolution semigroup used in [IO19]. This is the "heat kernel" $\{\psi_T\}_{T>0}$ of the operator

$$\mathcal{A} := |\partial_1|^3 - \partial_2^2 = |\partial_1|\mathcal{L},$$

which, in Fourier space \mathbb{R}^2 , is given by

$$\widehat{\psi}_T(k) = \exp(-T(|k_1|^3 + k_2^2)), \text{ for all } k \in \mathbb{R}^2.$$
 (1.16)

It is easy to check that the kernel has scaling properties in line with the metric *d*, that is,

$$\psi_T(x_1, x_2) = \frac{1}{(T^{\frac{1}{3}})^{1+\frac{3}{2}}} \psi\left(\frac{x_1}{T^{\frac{1}{3}}}, \frac{x_2}{(T^{\frac{1}{3}})^{\frac{3}{2}}}\right), \quad \text{for all} \quad x \in \mathbb{R}^2,$$
(1.17)

where for simplicity we write $\psi := \psi_1$. Note that ψ is a symmetric smooth function with integrable derivatives and we have for every $p \in [1, \infty]$ (see [IO19, Proof of Lemma 10]),

$$\|D_1^{\alpha} D_2^{\beta} \psi_T\|_{L^p(\mathbb{R}^2)} \sim (T^{\frac{1}{3}})^{-\alpha - \frac{3}{2}\beta - \frac{5}{2}(1 - \frac{1}{p})},\tag{1.18}$$

for every T > 0, $D_j \in \{\partial_j, |\partial_j|\}$, j = 1, 2, and $\alpha, \beta \ge 0$. For a periodic Schwartz distribution f, we denote by f_T its convolution with ψ_T , i.e., $f_T = \psi_T * f$, which yields a smooth periodic function. Notice that $\{\psi_T\}_{T>0}$ is a convolution semigroup, so that

$$(f_t)_T = f_{t+T}$$
 for all $t, T > 0$.

Remark 1.12. By the space of periodic Schwartz distributions f we understand the (topological) dual of the space of C^{∞} -functions φ on the torus (endowed with the family of seminorms $\{\|\partial_1^j \partial_2^l \varphi\|_{L^{\infty}}\}_{j,l\geq 0}$).

For a C^{∞} -function ψ on \mathbb{R}^2 with integrable derivatives, i.e. $\int_{\mathbb{R}^2} |\partial_1^j \partial_2^l \psi| \, dx < \infty$ for all $j, l \ge 0$, and a periodic Schwartz distribution f we write $(f * \psi)(x)$ to denote $f(\Psi(x - \cdot))$, where $\Psi := \sum_{z \in \mathbb{Z}^2} \psi(\cdot - z)$ is the periodization of ψ , which is well-defined and belongs to $C^{\infty}(\mathbb{T}^2)$. In particular, if Ψ_T denotes the periodization of our "heat kernel" ψ_T , then Ψ_T is a smooth semigroup whose Fourier coefficients are given by $\hat{\psi}_T(k)$ in (1.16) for $k \in (2\pi\mathbb{Z})^2$, yielding for any $T \in (0, 1]$, $D_i \in \{\partial_i, |\partial_i|\}, j = 1, 2,$ and $\alpha, \beta \ge 0$: ²³

$$\|D_1^{\alpha} D_2^{\beta} \Psi_T\|_{L^p} \lesssim \|D_1^{\alpha} D_2^{\beta} \psi_T\|_{L^p(\mathbb{R}^2)} \stackrel{(1.18)}{\lesssim} (T^{\frac{1}{3}})^{-\alpha - \frac{3}{2}\beta - \frac{5}{2}(1 - \frac{1}{p})}.$$
(1.19)

²³Indeed, for p = 1, we have

$$\|D_1^{\alpha} D_2^{\beta} \Psi_T\|_{L^1} \leq \sum_{z \in \mathbb{Z}^2} \int_{\mathbb{T}^2} |D_1^{\alpha} D_2^{\beta} \psi_T(x-z)| \, \mathrm{d}x = \|D_1^{\alpha} D_2^{\beta} \psi_T\|_{L^1(\mathbb{R}^2)}$$

while for $p = \infty$,

$$\begin{split} \|D_1^{\alpha} D_2^{\beta} \Psi_T\|_{L^{\infty}} &\leq 1 + \sum_{k \in (2\pi\mathbb{Z})^2 \setminus \{(0,0)\}} |k_1|^{\alpha} |k_2|^{\beta} |\hat{\psi}_T(k)| \leq 1 + \int_{\mathbb{R}^2} |\xi_1|^{\alpha} |\xi_2|^{\beta} \exp(-T(|\xi_1|^3 + \xi_2^2)) \, \mathrm{d}\xi \\ &\leq (T^{\frac{1}{3}})^{-\alpha - \frac{3}{2}\beta - \frac{5}{2}} \overset{(1.18)}{\lesssim} \|D_1^{\alpha} D_2^{\beta} \psi_T\|_{L^{\infty}(\mathbb{R}^2)}. \end{split}$$

For $p \in (1, \infty)$, one argues by interpolation.

Therefore, for a periodic function $f \in L^q$, we will often use Young's inequality for convolution with $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ in the form

$$\|f * D_1^{\alpha} D_2^{\beta} \psi_T\|_{L^r} \le \|f\|_{L^q} \|D_1^{\alpha} D_2^{\beta} \Psi_T\|_{L^p} \le (T^{\frac{1}{3}})^{-\alpha - \frac{3}{2}\beta - \frac{5}{2}(1 - \frac{1}{p})} \|f\|_{L^q}.$$

We sometimes write Γ for the integral kernel of $\mathcal{L}^{-1}P$, given by

$$\widehat{\Gamma}(k) = \frac{1}{k_1^2 + |k_1|^{-1}k_2^2}, \quad \text{for } k_1 \neq 0, \quad \text{and} \quad \widehat{\Gamma}(k) = 0 \quad \text{for } k_1 = 0.$$
 (1.20)

Note that

$$\|\Gamma\|_{L^2}^2 = \sum_{k_1 \neq 0} \frac{1}{(k_1^2 + |k_1|^{-1}k_2^2)^2} \lesssim \sum_{k_1 \neq 0} \frac{1}{d(0,k)^4} < \infty.$$
(1.21)

1.1.3. *Definition of Hölder spaces.* We now introduce the scale of Hölder seminorms based on the distance function *d*, where we restrict ourselves to the range $\alpha \in (0, \frac{3}{2})$ needed in this work (see [IO19, Definition 1]).

Definition 1.13. For a function $f : \mathbb{T}^2 \to \mathbb{R}$ and $\alpha \in (0, \frac{3}{2})$, we define

$$[f]_{\alpha} \coloneqq \begin{cases} \sup_{x \neq y} \frac{|f(y) - f(x)|}{d^{\alpha}(y,x)} & \text{for } \alpha \in (0,1], \\ \sup_{x \neq y} \frac{|f(y) - f(x) - \partial_1 f(x)(y - x)_1|}{d^{\alpha}(y,x)} & \text{for } \alpha \in (1,\frac{3}{2}). \end{cases}$$

We denote by C^{α} the closure of periodic C^{∞} -functions $f : \mathbb{T}^2 \to \mathbb{R}$ with respect to the norm $\|f\|_{L^{\infty}} + [f]_{\alpha}$.

We will also need the following Hölder spaces of negative exponents. We will restrict to the range required in this work, namely $\beta \in (-\frac{3}{2}, 0)$ (see [IO19, Definition 3]).

Definition 1.14. Let *f* be a periodic Schwartz distribution on \mathbb{T}^2 . For $\beta \in (-1, 0)$ we define

$$[f]_{\beta} := \inf\{|c| + [g]_{\beta+1} + [h]_{\beta+\frac{3}{2}} : f = c + \partial_1 g + \partial_2 h\}$$

and for $\beta \in (-\frac{3}{2}, -1]$ we define

$$[f]_{\beta} := \inf\{|c| + [g]_{\beta+2} + [h]_{\beta+\frac{3}{2}} : f = c + \partial_1^2 g + \partial_2 h\}.$$

We denote by C^{β} the closure of periodic C^{∞} -functions $f : \mathbb{T}^2 \to \mathbb{R}$ with respect to the norm $[f]_{\beta}$.

In Appendix A we provide all the necessary estimates on Hölder spaces needed in this work.

1.2. Strategy of the proofs. Recall the set \mathcal{W} defined in (1.5), endowed with the strong topology in $L^2(\mathbb{T}^2)$. We will show that the harmonic energy $\mathcal{H}(w)$ defined in (1.6) controls the anharmonic part $\mathcal{E}(w)$ defined in (1.11) of the total energy, that is,

$$\mathcal{E}(w) \leq 1 + \mathcal{H}(w)^2,$$

for every $w \in W$, and vice-versa, the anharmonic energy controls the harmonic part, that is, for every $\kappa > 0$ we have

$$\mathcal{H}(w) \lesssim_{\kappa} 1 + \mathcal{E}(w)^{\frac{3}{2} + \kappa},$$

for any $w \in W$, see Proposition 2.5 below. By standard embedding theorems (see Lemma B.5), any sublevel set of \mathcal{H} (respectively \mathcal{E}) over \mathcal{W} is relatively compact in L^2 and \mathcal{H} (respectively \mathcal{E}) is lower semicontinuous with respect to the L^2 -norm (see (3.3)).

In the following, for $\varepsilon > 0$ sufficiently small, we will also write

$$\mathcal{T} = \left\{ (\xi, v, F) \in C^{-\frac{5}{4} - \varepsilon} \times C^{\frac{3}{4} - \varepsilon} \times C^{-\frac{3}{4} - \varepsilon} : \mathcal{L}v = P\xi \right\}.$$

Note that \mathcal{T} is a closed subspace of $C^{-\frac{5}{4}-\varepsilon} \times C^{\frac{3}{4}-\varepsilon} \times C^{-\frac{3}{4}-\varepsilon}$ endowed with the norm given by $\max\{[\cdot]_{-\frac{5}{4}-\varepsilon}, [\cdot]_{\frac{3}{4}-\varepsilon}, [\cdot]_{-\frac{3}{4}-\varepsilon}\}.$

The deterministic ingredient in the proof of Theorem 1.4, that is Proposition 1.11, is essentially a consequence of the following theorem.

Theorem 1.15.

(i) (Coercivity) For every $\lambda \in (0, 1)$ and M > 0, there exists a constant C > 0 which depends on λ and polynomially on M^{24} such that for every $\varepsilon \in (0, \frac{1}{100})$ and every $(\xi, v, F) \in \mathcal{T}$ with $[\xi]_{-\frac{5}{4}-\varepsilon}, [v]_{\frac{3}{4}-\varepsilon}, [F]_{-\frac{3}{4}-\varepsilon} \leq M$, the functional \mathcal{G} defined in (1.12) satisfies

 $|\mathcal{G}(v, F; w)| \leq \lambda \mathcal{E}(w) + C$, for every $w \in \mathcal{W}$.

In particular, $E_{ren}(v, F; \cdot)$ defined in (1.10) is coercive.

(ii) (Continuity) Let $\varepsilon \in (0, \frac{1}{100})$ and $(\xi_{\ell}, v_{\ell}, F_{\ell}) \to (\xi, v, F)$ in \mathcal{T} and $w_{\ell} \to w$ in \mathcal{W} with the property that $\limsup_{\ell \to 0} \mathcal{E}(w_{\ell}) < \infty$. Then

$$\mathcal{G}(v_{\ell}, F_{\ell}; w_{\ell}) \to \mathcal{G}(v, F; w) \quad as \quad \ell \to 0.$$

(iii) (Compactness) Let $\varepsilon \in (0, \frac{1}{100})$ and $(\xi, v, F) \in \mathcal{T}$ be fixed. Then for any $M \in \mathbb{R}$ the sublevel sets of $E_{ren}(v, F; \cdot)$ defined in (1.10), given by

$$\left\{ w \in L^2 : \int_0^1 w \, \mathrm{d}x_1 = 0, \, E_{ren}(v, F; w) \leq M \right\},\,$$

are compact in the L^2 -norm.

(iv) (Existence of minimizers) If $\varepsilon \in (0, \frac{1}{100})$ and $(\xi, v, F) \in \mathcal{T}$, then there exists a minimizer $w \in W$ of the renormalized energy $E_{ren}(v, F; \cdot)$ which is a weak solution of (1.13).

Note that $E_{ren}(v, F; 0) = 0$, therefore every minimizer of E_{ren} belongs to the sublevel set M = 0 of E_{ren} . Using Theorem 1.15, we obtain the following Γ -convergence result.

Corollary 1.16 (Γ -convergence). Let $\varepsilon \in (0, \frac{1}{100})$ and $(\xi_{\ell}, v_{\ell}, F_{\ell}) \to (\xi, v, F)$ in \mathcal{T} . Then

 $E_{ren}(v_{\ell}, F_{\ell}; w) \to E_{ren}(v, F; w)$ for every $w \in \mathcal{W}$ as $\ell \to 0$.

Also, $E_{ren}(v_{\ell}, F_{\ell}; \cdot) \rightarrow E_{ren}(v, F; \cdot)$ in the sense of Γ -convergence over W, that is,

(i) $(\Gamma - \liminf)$ For all sequences $\{w_\ell\}_{\ell \downarrow 0} \subset W$ with $w_\ell \to w$ strongly in L^2 , we have

 $\liminf_{\ell \to 0} E_{ren}(v_{\ell}, F_{\ell}; w_{\ell}) \ge E_{ren}(v, F; w).$

(ii) $(\Gamma - \limsup)$ For every $w \in W$, there exists a sequence $\{w_\ell\}_{\ell \downarrow 0} \subset W$ with $w_\ell \to w$ strongly in L^2 such that

$$\lim_{\ell \to \infty} E_{ren}(v_{\ell}, F_{\ell}; w_{\ell}) = E_{ren}(v, F; w).$$

Proof of Proposition 1.11. Corollary 1.16 establishes the continuity of the map that associates to each $(\xi, v, F) \in \mathcal{T}$ the functional $E_{ren}(v, F; \cdot)$, when the space of (lower semicontinuous) functionals over \mathcal{W} is equipped with the topology of Γ-convergence (with respect to the L^2 -topology on \mathcal{W}). In particular, this map is Borel measurable when \mathcal{T} is endowed with its Borel σ -algebra. ■

Taking the main stochastic ingredient from Proposition 1.9 for granted (which we prove in Section 5), we can now give the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $\langle \cdot \rangle$ be a probability measure on the space of periodic Schwartz distributions ξ that satisfies Assumption 1.1. By Proposition 1.9 $\langle \cdot \rangle$ lifts to a probability measure $\langle \cdot \rangle^{\text{lift}}$ on the space of triples $(\xi, v, F) \in C^{-\frac{5}{4}-} \times C^{\frac{3}{4}-} \times C^{-\frac{3}{4}-}$.

By Proposition 1.11 the mapping $(\xi, v, F) \mapsto E_{ren}(v, F; \cdot)$ is continuous. Hence, the push-forward $(E_{ren})_{\#} \langle \cdot \rangle^{\text{lift}}$ is well-defined as a probability measure on the space of lower semicontinuous functionals equipped with the Borel σ -algebra corresponding to the topology of Γ -convergence (based on the strong L^2 -topology). We now define $\langle \cdot \rangle^{\text{ext}}$ as the joint law of ξ and $E_{ren}(v, F; \cdot)$.²⁵

²⁴We say that a constant C > 0 depends polynomially on M if there exist c > 0 and $N \ge 1$ such that $C \le c(1 + M^N)$.

²⁵Here we understand $E_{ren}(v, F; \cdot)$ as a measurable function of ξ , which is a composition of the measurable function $\xi \mapsto (\xi, v, F)$ and the continuous function $(\xi, v, F) \mapsto E_{ren}(v, F; \cdot)$.

- (i) This is immediate by the definition of $\langle \cdot \rangle^{\text{ext}}$ and the fact that by Proposition 1.9, $\langle \cdot \rangle$ is supported on the Hölder space $C^{-\frac{5}{4}-}$.
- (ii) If ξ is smooth $\langle \cdot \rangle$ -almost surely, by Proposition 1.9 (iii) we have that $F = v\partial_2 R_1 v \langle \cdot \rangle$ -almost surely. In this case, $E_{ren}(v, F; \cdot) = E_{ren}(v, v\partial_2 R_1 v; \cdot) \langle \cdot \rangle$ -almost surely and agrees with the definition given in (1.4).
- (iii) Let $\{\langle \cdot \rangle_{\ell}\}_{\ell \downarrow 0}$ be a sequence of probability measures that satisfy Assumption 1.1 and converges weakly to $\langle \cdot \rangle$, which then automatically satisfies Assumption 1.1. Then by Proposition 1.9 (iv), the sequence $\{\langle \cdot \rangle_{\ell}^{\text{lift}}\}_{\ell \downarrow 0}$ converges weakly to $\langle \cdot \rangle^{\text{lift}}$. Given a bounded continuous function $G : (\xi, E) \mapsto G(\xi, E) \in \mathbb{R}$ we have that

$$\langle G \rangle_{\ell}^{\text{ext}} = \langle G(\xi, E_{ren}(v, F; \cdot)) \rangle_{\ell}^{\text{lift}} \xrightarrow{\ell \downarrow 0} \langle G(\xi, E_{ren}(v, F; \cdot)) \rangle^{\text{lift}} = \langle G \rangle^{\text{ext}},$$

which in turn implies that $\langle \cdot \rangle_{\ell}^{\text{ext}} \to \langle \cdot \rangle^{\text{ext}}$ weakly as $\ell \downarrow 0$.

Finally, we have an a priori estimate for the $C^{\frac{5}{4}-}$ norm of minimizers of $E_{ren}(v, F; \cdot)$, which we prove in Section 4.

Proposition 1.17 (Hölder regularity). For any M > 0 and $\varepsilon \in (0, \frac{1}{100})$, there exists a constant C > 0 which depends on ε and polynomially on M such that

$$[w]_{\frac{5}{4}-2\varepsilon} \le C,$$

for every minimizer $w \in W \cap C^{\frac{5}{4}-2\epsilon}$ of $E_{ren}(v,F;\cdot)$ with $(\xi,v,F) \in \mathcal{T}$ satisfying the bound $[\xi]_{-\frac{5}{4}-\epsilon}, [v]_{\frac{3}{4}-\epsilon}, [F]_{-\frac{3}{4}-\epsilon} \leq M.$

Combined with an approximation argument, this is the main ingredient in the proof of Corollary 1.7, which we give now.

Proof of Corollary 1.7. For $\varepsilon \in (0, \frac{1}{100})$, we define the functional $g : \mathcal{W} \to \mathbb{R} \cup \{+\infty\}$ given by

$$g(w) := \begin{cases} [w]_{\frac{5}{4}-2\varepsilon}, & \text{if } w \in C^{\frac{5}{4}-2\varepsilon} \\ +\infty, & \text{otherwise.} \end{cases}$$

By [IO19, Lemma 13] we know that g is lower semicontinuous on W endowed with the strong topology in $L^2(\mathbb{T}^2)$. We define the non-negative functional $G : E \mapsto G(E)$ on the space of lower semicontinuous functionals E on W by

$$G(E) := \begin{cases} \inf_{w \in \operatorname{argmin} E} g(w), & \text{if } \operatorname{argmin} E \neq \emptyset, \\ +\infty, & \text{otherwise.} \end{cases}$$

We claim that *G* is lower semicontinuous, that is, if $E_{\ell} \to E$ as $\ell \downarrow 0$ in the sense of Γ -convergence, then

$$G(E) \leq \liminf_{\ell \downarrow 0} G(E_{\ell}).$$

Indeed, without loss of generality we may assume that $G(E_{\ell}) \rightarrow \liminf_{\ell \downarrow 0} G(E_{\ell}) < \infty$ by possibly extracting a subsequence. This implies that $\sup_{\ell} G(E_{\ell}) < \infty$, hence by the definition of *G* there exists a sequence of minimizers w_{ℓ} of E_{ℓ} such that

$$[w_{\ell}]_{\frac{5}{4}-2\varepsilon} \leq G(E_{\ell}) + \ell \leq \sup_{\ell} G(E_{\ell}) + 1.$$

By Lemma A.6 there exists $w \in C^{\frac{5}{4}-2\varepsilon}$ such that $w_{\ell} \to w$ in $C^{\frac{5}{4}-3\varepsilon}$ along a subsequence, and

$$[w]_{\frac{5}{4}-2\varepsilon} \leq \liminf_{\ell \mid 0} [w_{\ell}]_{\frac{5}{4}-2\varepsilon}.$$

This, in particular, implies that $w_{\ell} \to w$ strongly in $L^2(\mathbb{T}^2)$ and since $E_{\ell} \to E$ in the sense of Γ -convergence, w is a minimizer of E. Thus, we have the estimate

$$G(E) \le [w]_{\frac{5}{4}-2\varepsilon} \le \liminf_{\ell \downarrow 0} [w_{\ell}]_{\frac{5}{4}-2\varepsilon} \le \liminf_{\ell \downarrow 0} G(E_{\ell}).$$

which proves the desired claim.

Let now $\{\langle \cdot \rangle_{\ell}\}_{\ell \downarrow 0}$ be a sequence of probability measures such that $\langle \cdot \rangle_{\ell} \rightarrow \langle \cdot \rangle$ weakly and for every $\ell \in (0, 1]$, ξ is smooth $\langle \cdot \rangle_{\ell}$ -almost surely. Since under $\langle \cdot \rangle_{\ell}$, ξ is smooth, by Lemma D.3 v is smooth. By Theorem 1.15 (iv) there exists a minimizer $w \in \mathcal{W}$ of $E_{ren}(v, F; \cdot)$, which is a weak solution to (1.13). If we let u = v + w, then $u \in \mathcal{W}$ and since $F = vR_1\partial_1 v \langle \cdot \rangle_{\ell}$ -almost surely (see Proposition 1.9 (iii)), u is a weak solution to (1.2). By Proposition E.2 we know that $\||\partial_1|^s u\|_2 + \||\partial_2|^{\frac{2}{3}s} u\|_2 \leq 1$ for every s < 3, hence by Lemma B.8 $u \in C^{\frac{5}{4}-2\varepsilon}$, which in turn implies that $w = u - v \in C^{\frac{5}{4}-2\varepsilon}$. By Proposition 1.17 we have the estimate

$$[w]_{\frac{5}{2}-2\varepsilon} \leq C$$

where the constant *C* depends polynomially on $\max\{[\xi]_{-\frac{5}{4}-\varepsilon}, [v]_{\frac{3}{4}-\varepsilon}, [F]_{-\frac{3}{4}-\varepsilon}\}$. In particular, this implies that

$$G(E_{ren}(v,F;\cdot)) \le C.$$

By Corollaries 5.4 and 5.7 we know that for every $1 \le p < \infty$,

$$\sup_{\ell} \langle C^p \rangle_{\ell}^{\text{lift}} \lesssim_p 1$$

Hence, for the functional G we have that

$$\sup_{\ell} \langle G(E)^p \rangle_{\ell}^{\text{ext}} = \sup_{\ell} \langle G(E_{ren}(v, F; \cdot))^p \rangle_{\ell}^{\text{lift}} \le \sup_{\ell} \langle C^p \rangle_{\ell}^{\text{lift}} \lesssim_p 1.$$

Since by Theorem 1.4 (iii) $\langle \cdot \rangle_{\ell}^{\text{ext}} \to \langle \cdot \rangle^{\text{ext}}$ weakly and *G* is lower semicontinuous we have that

$$\langle G(E)^p \rangle^{\text{ext}} \leq \liminf_{\ell \downarrow 0} \langle G^p \rangle_{\ell}^{\text{ext}} \lesssim_p 1,$$

which completes the proof.

1.3. **Outline.** In Section 2 we show how the Howarth–Kármán–Monin identities can be used to control certain Besov and L^p norms by the anharmonic energy \mathcal{E} .

In Section 3 we prove Theorem 1.15 and the Γ -convergence result for the renormalized energy, see Corollary 1.16.

In Section 4 we prove the optimal Hölder regularity $\frac{5}{4}$ – of minimizers of the renormalized energy, see Proposition 1.17.

In Section 5, based on the spectral gap inequality (1.7), we provide the stochastic arguments to prove Proposition 1.9.

Last, in the Appendix we include some technical results and proofs.

2. Estimates for the Burgers equation

In this section we bound certain Besov and L^p norms of a function $w \in W$ by the anharmonic energy $\mathcal{E}(w)$. These bounds will be used in later sections to study the Γ -convergence of the renormalized energy (1.10) and regularity properties of its minimizers (see Sections 3 and 4 below). The proof of these estimates is based on the application of the Howarth–Kármán–Monin identity for the Burgers operator.

We first need to introduce (directional) Besov spaces. These spaces appear naturally through the application of the Howarth–Kármán–Monin identity (see Proposition 2.3).

Throughout this section, for a function $f : \mathbb{T}^2 \to \mathbb{R}$ we write

$$\partial_i^h f(x) \coloneqq f(x + he_i) - f(x)$$

where $x \in \mathbb{T}^2$, $j \in \{1, 2\}$, $h \in \mathbb{R}$, $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

Definition 2.1. For a function $f : \mathbb{T}^2 \to \mathbb{R}$, $j \in \{1, 2\}$, $s \in (0, 1]$ and $p \in [1, \infty)$ we define the following (directional) Besov seminorm²⁶

$$\|f\|_{\dot{\mathcal{B}}^{s}_{p;j}} \coloneqq \sup_{h \in (0,1]} \frac{1}{h^{s}} \left(\int_{\mathbb{T}^{2}} |\partial_{j}^{h} f(x)|^{p} \, \mathrm{d}x \right)^{\frac{1}{p}}.$$
(2.1)

Notice that in comparison to standard Besov spaces our definition measures regularity in x_1 and x_2 separately. We have also omitted the second lower index which usually appears in standard Besov spaces since in our case it is always ∞ (corresponding to $\dot{\mathcal{B}}_{p,\infty}^s$).

Remark 2.2. For $s \ge 0$, given a periodic function $f : \mathbb{T}^2 \to \mathbb{R}$, we define $|\partial_j|^s f$ in Fourier space via

$$\widehat{|\partial_j|^s f}(k) := |k_j|^s \widehat{f}(k), \quad k \in (2\pi\mathbb{Z})^2.$$

For s < 0 and a periodic distribution f of vanishing average in x_j^{27} , we can define $|\partial_j|^s f$ in the same way for $k_j \neq 0$. For p = 2, $s \in (0, 1)$ and $s' \in (s, 1)$, the Parseval identity implies the equivalence 2^{28}

$$\int_{\mathbb{T}^2} \left| |\partial_j|^s f \right|^2 \, \mathrm{d}x = \sum_{k \in (2\pi\mathbb{Z})^2} |k_j|^{2s} |\widehat{f}(k)|^2 = c_s \int_{\mathbb{R}} \frac{1}{|h|^{2s}} \int_{\mathbb{T}^2} |\partial_j^h f(x)|^2 \, \mathrm{d}x \frac{\mathrm{d}h}{|h|} \lesssim C(s, s') ||f||^2_{\dot{\mathcal{B}}^{s'}_{2;j}} \tag{2.2}$$

for some positive constant C(s, s') depending only on s and s', where we used (B.4) below.

In the next proposition we prove two core estimates based on the Howarth–Kármán–Monin identities [GJO15, Lemma 4.1] for the Burgers operator. In [GJO15, Lemma 4.1] the authors deal with the operator $w \mapsto \partial_2 w + \partial_1 \frac{1}{2} w^2$, but the same proof extends to our setting.

Proposition 2.3. There exists C > 0 such that for every $w \in L^2(\mathbb{T}^2)$ with vanishing average in x_1 and for every $h \in (0, 1)$ we have

$$\int_{\mathbb{T}^2} |\partial_1^h w|^3 \,\mathrm{d}x \le Ch^{\frac{3}{2}} \mathcal{E}(w), \tag{2.3}$$

$$\sup_{x_2 \in [0,1)} \frac{1}{h} \int_0^h \int_0^1 |\partial_1^{h'} w|^2 \, \mathrm{d}x_1 \mathrm{d}h' \le C h^{\frac{1}{2}} \mathcal{E}(w). \tag{2.4}$$

Proof. By the Howarth–Kármán–Monin identities [GJO15, Lemma 4.1] for the Burgers operator we know that for every $h' \in (0, 1)$

$$\partial_2 \frac{1}{2} \int_0^1 |\partial_1^{h'} w| \partial_1^{h'} w \, \mathrm{d}x_1 - \partial_{h'} \frac{1}{6} \int_0^1 |\partial_1^{h'} w|^3 \, \mathrm{d}x_1 = \int_0^1 \partial_1^{h'} \eta_w |\partial_1^{h'} w| \, \mathrm{d}x_1, \tag{2.5}$$

$$\partial_2 \frac{1}{2} \int_0^1 |\partial_1^{h'} w|^2 \, \mathrm{d}x_1 - \partial_{h'} \frac{1}{6} \int_0^1 (\partial_1^{h'} w)^3 \, \mathrm{d}x_1 = \int_0^1 \partial_1^{h'} \eta_w \partial_1^{h'} w \, \mathrm{d}x_1.$$
(2.6)

To prove (2.3), we integrate (2.5) over x_2 and use the periodicity of w to obtain,

$$\partial_{h'} \int_{\mathbb{T}^2} |\partial_1^{h'} w|^3 \, \mathrm{d}x = -6 \int_{\mathbb{T}^2} \eta_w \partial_1^{-h'} |\partial_1^{h'} w| \, \mathrm{d}x.$$
(2.7)

²⁶Note that one can take the supremum over all $h \in \mathbb{R}$ in (2.1) by replacing h^s with $|h|^s$. Indeed, if $h \in [-1, 0)$ the quantity on the right-hand side of (2.1) does not change by symmetry. If $h \in \mathbb{R} \setminus [-1, 1]$, one writes $h = h_{\text{fr}} + h_{\text{int}}$ with $h_{\text{fr}} \in (0, 1]$ and $h_{\text{int}} \in \mathbb{Z}$ and uses that $\|\partial_j^h f\|_{L^p} = \|\partial_j^{h_{\text{fr}}} f\|_{L^p}$ (by periodicity) while $\frac{1}{|h|^s} \leq \frac{1}{h_{\text{fr}}^s}$.

²⁷We say that a periodic distribution f has vanishing average in x_1 if $f(e^{-ik_2}) = 0$ for all $k_2 \in 2\pi\mathbb{Z}$, and analogously for f with vanishing average in x_2 .

²⁸Note that $C(s, s') \to \infty$ as $s' \searrow s$.

The last term is estimated as follows,

$$\begin{split} \left(\int_{\mathbb{T}^2} \eta_w \partial_1^{-h'} |\partial_1^{h'} w| \, \mathrm{d}x\right)^2 &\leq \int_{\mathbb{T}^2} (|\partial_1|^{-\frac{1}{2}} \eta_w)^2 \, \mathrm{d}x \int_{\mathbb{T}^2} (|\partial_1|^{\frac{1}{2}} \partial_1^{-h'} |\partial_1^{h'} w|)^2 \, \mathrm{d}x \\ &\lesssim |h'| \int_{\mathbb{T}^2} (|\partial_1|^{-\frac{1}{2}} \eta_w)^2 \, \mathrm{d}x \int_{\mathbb{T}^2} (\partial_1 w)^2 \, \mathrm{d}x, \end{split}$$

where we use that

$$\begin{split} \int_{\mathbb{T}^2} (|\partial_1|^{\frac{1}{2}} \partial_1^{-h'} |\partial_1^{h'} w|)^2 \, \mathrm{d}x &\lesssim \left(\int_{\mathbb{T}^2} (\partial_1^{-h'} |\partial_1^{h'} w|)^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^2} (\partial_1 |\partial_1^{h'} w|)^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \\ &\lesssim |h'| \int_{\mathbb{T}^2} (\partial_1 |\partial_1^{h'} w|)^2 \, \mathrm{d}x. \end{split}$$

Integrating (2.7) over $h' \in (0, h)$, we obtain that

$$\int_{\mathbb{T}^2} |\partial_1^h w(x)|^3 \, \mathrm{d}x \lesssim h^{\frac{3}{2}} \left(\int_{\mathbb{T}^2} (\partial_1 w)^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^2} (|\partial_1|^{-\frac{1}{2}} \eta_w)^2 \, \mathrm{d}x \right)^{\frac{1}{2}}$$

which in turns implies (2.3).

To prove (2.4), we integrate (2.6) over $h' \in (0, h)$ to obtain with $\partial_1^0 w = 0$

$$\partial_2 \frac{1}{2} \int_0^h \int_0^1 |\partial_1^{h'} w|^2 \, \mathrm{d}x_1 \mathrm{d}h' - \frac{1}{6} \int_0^1 (\partial_1^h w)^3 \, \mathrm{d}x_1 = \int_0^h \int_0^1 \partial_1^{h'} \eta_w \partial_1^{h'} w \, \mathrm{d}x_1 \, \mathrm{d}h'.$$

By the Sobolev embedding $W^{1,1}(\mathbb{T}) \subset L^{\infty}(\mathbb{T})$ on the torus $\mathbb{T} = [0, 1)$ in the form

$$\sup_{z\in\mathbb{T}}|f(z)|\leq \int_{\mathbb{T}}|f(z)|\,\mathrm{d} z+\int_{\mathbb{T}}|f'(z)|\,\mathrm{d} z,$$

we can therefore estimate

$$\sup_{x_{2} \in [0,1)} \int_{0}^{h} \int_{0}^{1} |\partial_{1}^{h'} w|^{2} dx_{1} dh' \lesssim \int_{0}^{h} \int_{\mathbb{T}^{2}} |\partial_{1}^{h'} w|^{2} dx dh' + \int_{\mathbb{T}^{2}} |\partial_{1}^{h} w|^{3} dx + \int_{0}^{h} \int_{0}^{1} \left| \int_{0}^{1} \partial_{1}^{h'} \eta_{w} \partial_{1}^{h'} w dx_{1} \right| dx_{2} dh'.$$

The first term on the right-hand side can be bounded using

$$\int_{\mathbb{T}^2} |\partial_1^{h'} w|^2 \, \mathrm{d} x \le (h')^2 \int_{\mathbb{T}^2} |\partial_1 w|^2 \, \mathrm{d} x \le (h')^2 \mathcal{E}(w).$$

For the second term we use (2.3). Last, for the third term, the same argument used to estimate the right-hand side of (2.7) leads to

$$\int_{0}^{h} \int_{0}^{1} \left| \int_{0}^{1} \partial_{1}^{h'} \eta_{w} \partial_{1}^{h'} w \, \mathrm{d}x_{1} \right| \mathrm{d}x_{2} \mathrm{d}h' \lesssim \||\partial_{1}|^{-\frac{1}{2}} \eta_{w}\|_{L^{2}} \|\partial_{1}w\|_{L^{2}} \int_{0}^{h} (h')^{\frac{1}{2}} \, \mathrm{d}h' \lesssim h^{\frac{3}{2}} \mathcal{E}(w).$$

Hence, we can bound

$$\sup_{x_2 \in [0,1)} \int_0^h \int_0^1 |\partial_1^{h'} w|^2 \, \mathrm{d}x_1 \mathrm{d}h' \lesssim (h^3 + h^{\frac{3}{2}}) \mathcal{E}(w) \lesssim h^{\frac{3}{2}} \mathcal{E}(w)$$

for all $h \in (0, 1)$.

We are now ready to prove the main result of this section.

Proposition 2.4. We have the following estimates:

(i)
$$\|w\|_{\dot{\mathcal{B}}^{s}_{3,1}} \le C\mathcal{E}(w)^{\frac{1}{3}}, \text{ for every } s \in (0, \frac{1}{2}],$$
 (2.8)

(*ii*)
$$\|w\|_{\dot{\mathcal{B}}^{s}_{2,1}} \le C\mathcal{E}(w)^{\frac{2s+1}{6}}, \text{ for every } s \in [\frac{1}{2}, 1],$$
 (2.9)

(iii)
$$\|w\|_{L^p} \le C(p)\mathcal{E}(w)^{\frac{p-1}{2p}}, \text{ for every } p \in [3,7),$$
 (2.10)

(iv)
$$\|w^2\|_{\dot{\mathcal{B}}^{\frac{2p-6}{2p}}_{2,1}} \le C(p)\mathcal{E}(w)^{\frac{2p-3}{2p}}, \text{ for every } p \in [6,7),$$
 (2.11)

for every $w \in L^2(\mathbb{T}^2)$ with vanishing average in x_1 , where C > 0 is a universal constant and C(p) > 0 depends on p.

Note that a result similar to (2.8) was obtained in [OS10, Lemma 4] using different techniques.

Proof. (i) This is immediate from (2.3), Definition 2.1 and (B.3).

(ii) By interpolation we have for $s \in [\frac{1}{2}, 1]$:

$$\|w\|_{\dot{\mathcal{B}}^{s}_{2;1}} \leq \|w\|^{2(1-s)}_{\dot{\mathcal{B}}^{\frac{1}{2}}_{2;1}} \|w\|^{2s-1}_{\dot{\mathcal{B}}^{\frac{1}{2}}_{2;1}}.$$

Using (B.3) and (2.8) we get

$$\|w\|_{\dot{\mathcal{B}}^{\frac{1}{2}}_{2;1}} \leq \|w\|_{\dot{\mathcal{B}}^{\frac{1}{2}}_{3;1}} \lesssim \mathcal{E}(w)^{\frac{1}{3}},$$

with an implicit constant independent of *s*. We also have the bound

$$\|w\|_{\dot{\mathcal{B}}^{1}_{2,1}} \leq \|\partial_{1}w\|_{L^{2}} \leq \mathcal{E}(w)^{\frac{1}{2}}.$$

Combining these estimates implies (2.9).

(iii) We divide the proof into several steps.

STEP 1: We first prove that

$$\sup_{x_2 \in [0,1)} \left(\int_0^1 |w(x)|^p \, \mathrm{d} x_1 \right)^{\frac{1}{p}} \lesssim_p \mathcal{E}(w)^{\frac{1}{2}} \quad \text{for all} \quad 2 \le p < 4.$$

Indeed, by (B.12) for $p \in [2, 4)$, q = 2 and $f = w(\cdot, x_2)$ (with $x_2 \in [0, 1)$ fixed) we know that

$$\sup_{x_{2}\in[0,1)} \left(\int_{0}^{1} |w(x)|^{p} dx_{1}\right)^{\frac{1}{p}} \leq_{p} \sup_{x_{2}\in[0,1)} \int_{0}^{1} \left(\int_{0}^{1} |\partial_{1}^{h}w(x)|^{2} dx_{1}\right)^{\frac{1}{2}} \frac{1}{h^{\frac{1}{2}-\frac{1}{p}}} \frac{dh}{h}.$$

Since p < 4 we have that

$$\sup_{x_{2}\in[0,1)}\int_{0}^{1}\left(\int_{0}^{1}|\partial_{1}^{h}w(x)|^{2} dx_{1}\right)^{\frac{1}{2}}\frac{1}{h^{\frac{1}{2}-\frac{1}{p}}}\frac{dh}{h} \leq_{p} \sup_{x_{2}\in[0,1)}\sup_{h\in(0,1]}\frac{1}{h^{\frac{1}{4}}}\left(\int_{0}^{1}|\partial_{1}^{h}w(x)|^{2} dx_{1}\right)^{\frac{1}{2}}$$

By (B.18) and (2.4) we also know that

$$\sup_{x_{2}\in[0,1)}\sup_{h\in(0,1]}\frac{1}{h^{\frac{1}{4}}}\left(\int_{0}^{1}|\partial_{1}^{h}w(x)|^{2} dx_{1}\right)^{\frac{1}{2}}$$

$$\lesssim \sup_{x_{2}\in[0,1)}\sup_{h\in(0,1]}\frac{1}{h^{\frac{1}{4}}}\left(\frac{1}{h}\int_{0}^{h}\int_{0}^{1}|\partial_{1}^{h'}w(x)|^{2} dx_{1} dh'\right)^{\frac{1}{2}} \lesssim \mathcal{E}(w)^{\frac{1}{2}},$$

which combined with the previous estimates implies the desired estimate. STEP 2: We prove that

$$\left(\int_0^1 \sup_{x_1 \in [0,1)} |w(x)|^3 \, \mathrm{d}x_2\right)^{\frac{1}{3}} \lesssim \mathcal{E}(w)^{\frac{1}{3}}.$$

.

By (B.12) for $p = \infty$, q = 3 and $f = w(\cdot, x_2)$, we know that

$$\sup_{x_1 \in [0,1)} |w(x)| \lesssim \int_0^1 \left(\frac{1}{h} \int_0^1 |\partial_1^h w(x)|^3 \, \mathrm{d}x_1\right)^{\frac{1}{3}} \frac{\mathrm{d}h}{h}$$

for every $x_2 \in [0, 1)$. Using Minkowski's inequality we obtain the bound

$$\left(\int_{0}^{1} \sup_{x_{1} \in [0,1)} |w(x)|^{3} dx_{2}\right)^{\frac{1}{3}} \lesssim \int_{0}^{1} \left(\frac{1}{h} \int_{\mathbb{T}^{2}} |\partial_{1}^{h} w(x)|^{3} dx\right)^{\frac{1}{3}} \frac{dh}{h}.$$

Using (2.3), the last term in the above inequality is bounded by

$$\int_{0}^{1} \left(\int_{\mathbb{T}^{2}} |\partial_{1}^{h} w(x)|^{3} \mathrm{d}x \frac{1}{h} \right)^{\frac{1}{3}} \frac{\mathrm{d}h}{h} \lesssim \int_{0}^{1} \frac{1}{h^{\frac{5}{6}}} \mathcal{E}(w)^{\frac{1}{3}} \mathrm{d}h$$

which implies the desired estimate.

STEP 3: We are now ready to prove (2.10). For $5 \le p < 7$ this is immediate from STEP 1 and STEP 2, since we have that

$$\left(\int_{\mathbb{T}^2} |w(x)|^p \, \mathrm{d}x\right)^{\frac{1}{p}} \lesssim \left(\int_0^1 \sup_{x_1 \in [0,1)} |w(x)|^3 \, \mathrm{d}x_2 \sup_{x_2 \in [0,1)} \int_0^1 |w(x)|^{p-3} \, \mathrm{d}x_1\right)^{\frac{1}{p}} \lesssim_p \mathcal{E}(w)^{\frac{p-1}{2p}}.$$

STEP 2 also implies that $||w||_{L^3} \leq \mathcal{E}(w)^{\frac{1}{3}}$ which proves the bound for p = 3, so it remains to prove the bound for $p \in (3,5] \subset [3,6]$. We proceed using interpolation for $\frac{1}{p} = \frac{1}{3}\frac{6-p}{p} + \frac{1}{6}(2-\frac{6}{p})$ to bound

$$\|w\|_{L^{p}} \leq \|w\|_{L^{3}}^{\frac{6-p}{p}} \|w\|_{L^{6}}^{2-\frac{6}{p}} \leq \mathcal{E}(w)^{\frac{6-p}{3p}} \mathcal{E}(w)^{(2-\frac{6}{p})\frac{5}{12}} = \mathcal{E}(w)^{\frac{p-1}{2p}}$$

(iv) We first notice that by Hölder's inequality, with exponents $\frac{p-2}{p} + \frac{2}{p} = 1$, translation invariance and Minkowski's inequality, we have that

$$\int_{\mathbb{T}^{2}} \left(\partial_{1}^{h} w^{2}(x)\right)^{2} dx \leq \left(\int_{\mathbb{T}^{2}} |\partial_{1}^{h} w(x)|^{\frac{2p}{p-2}} dx\right)^{\frac{p-2}{p}} \left(\int_{\mathbb{T}^{2}} |w(x+he_{1})+w(x)|^{p} dx\right)^{\frac{2}{p}} \leq 4 \left(\int_{\mathbb{T}^{2}} |\partial_{1}^{h} w(x)|^{\frac{2p}{p-2}} dx\right)^{\frac{p-2}{p}} \left(\int_{\mathbb{T}^{2}} |w(x)|^{p} dx\right)^{\frac{2}{p}}.$$
(2.12)

As $p \in [6,7)$, we have $\frac{2p}{p-2} \in (2,3]$, and by interpolation we obtain the bound

$$\int_{\mathbb{T}^2} |\partial_1^h w(x)|^{\frac{2p}{p-2}} \, \mathrm{d}x \le \left(\int_{\mathbb{T}^2} |\partial_1^h w(x)|^2 \, \mathrm{d}x \right)^{\frac{p-6}{p-2}} \left(\int_{\mathbb{T}^2} |\partial_1^h w(x)|^3 \, \mathrm{d}x \right)^{\frac{4}{p-2}}.$$

Using that $\int_{\mathbb{T}^2} |\partial_1^h w(x)|^2 dx \leq h^2 \mathcal{E}(w)$ and (2.3), the last inequality implies that

$$\int_{\mathbb{T}^2} |\partial_1^h w(x)|^{\frac{2p}{p-2}} \, \mathrm{d}x \lesssim h^{2\left(\frac{p-6}{p-2}\right) + \frac{3}{2}\left(\frac{4}{p-2}\right)} \mathcal{E}(w).$$
(2.13)

Combining (2.12), (2.13) and (2.10) we get

$$\int_{\mathbb{T}^2} \left(\partial_1^h w^2(x) \right)^2 \, \mathrm{d}x \, \leq_p \left(h^{2\left(\frac{p-6}{p-2}\right) + \frac{3}{2}\left(\frac{4}{p-2}\right)} \mathcal{E}(w) \right)^{\frac{p-2}{p}} \mathcal{E}(w)^{\frac{p-1}{p}} = h^{\frac{2p-6}{p}} \mathcal{E}(w)^{\frac{2p-3}{p}}$$

which implies (2.11).

As (B.4) and (2.11) imply that $\mathcal{E}(w)$ (to some power) controls the quantity $|||\partial_1|^{\frac{1}{2}}w^2||_2$, it follows that the harmonic part $\mathcal{H}(w)$ given in (1.6) of the energy $\mathcal{E}(w)$ is also controlled by $\mathcal{E}(w)$. Moreover, $\mathcal{E}(w)$ controls the L^2 norm of the $\frac{2}{3}$ -fractional derivative in x_2 because the harmonic part $\mathcal{H}(w)$ does. We summarize this in the next proposition, where we also prove that $\mathcal{E}(w)$ is controlled by $\mathcal{H}(w)$.

Proposition 2.5.

(i) For every $\kappa \in (0, \frac{1}{14})$, there exists a constant $C(\kappa) > 0$ such that

$$\mathcal{H}(w) \le C(\kappa) \left(1 + \mathcal{E}(w)^{\frac{3}{2} + \kappa} \right), \tag{2.14}$$

for every $w \in L^2(\mathbb{T}^2)$ with vanishing average in x_1 . In addition, there exists a constant C > 0 such that

$$\int_{\mathbb{T}^2} |\partial_1 w|^2 \, \mathrm{d}x + \int_{\mathbb{T}^2} ||\partial_2|^{\frac{2}{3}} w|^2 \, \mathrm{d}x \le C \mathcal{H}(w), \tag{2.15}$$

for every $w \in L^2(\mathbb{T}^2)$ with vanishing average in x_1 . In particular, $[w]_{-\frac{1}{4}} \leq \mathcal{H}(w)^{\frac{1}{2}}$.

(ii) There exists a constant C > 0 such that

$$\mathcal{E}(w) \leq 1 + \mathcal{H}(w)^2 \quad \text{for every } w \in \mathcal{W}.$$
 (2.16)

Proof. (i) Fix $\kappa \in (0, \frac{1}{14})$ and choose $p = p(\kappa) \in (6, 7)$ such that $\frac{2p-3}{p} = \frac{3}{2} + \kappa$. Recalling that $\eta_w = \partial_2 w - \partial_1 \frac{1}{2} w^2$, by (B.4) and the fact that $\frac{2p-6}{2p} \in (\frac{1}{2}, 1)$ we have

$$\mathcal{H}(w) \lesssim \int_{\mathbb{T}^2} |\partial_1 w|^2 \, \mathrm{d}x + \int_{\mathbb{T}^2} ||\partial_1|^{-\frac{1}{2}} \eta_w|^2 \, \mathrm{d}x + \int_{\mathbb{T}^2} ||\partial_1|^{\frac{1}{2}} w^2|^2 \, \mathrm{d}x \lesssim_{\kappa} \mathcal{E}(w) + ||w^2||^2_{\dot{\mathcal{B}}_{2,1}^{\frac{2p-6}{2p}}}$$

By (2.11) we know that $\|w^2\|_{\dot{\mathcal{B}}_{2,1}^{\frac{2p-6}{2p}}} \lesssim_{\kappa} \mathcal{E}(w)^{\frac{2p-3}{p}}$, thus (2.14) follows by Young's inequality.

Inequality (2.15) is proved in a more general context in Lemma B.6 and the last statement follows from Lemma B.8.

(ii) We have

$$\begin{split} \mathcal{E}(w) &= \mathcal{H}(w) - \int_{\mathbb{T}^2} \left(|\partial_1|^{-\frac{1}{2}} \partial_2 w |\partial_1|^{-\frac{1}{2}} \partial_1 w^2 \right) \, \mathrm{d}x + \frac{1}{4} \int_{\mathbb{T}^2} \left(|\partial_1|^{-\frac{1}{2}} \partial_1 w^2 \right)^2 \, \mathrm{d}x \\ &\leq \mathcal{H}(w) + \mathcal{H}(w)^{\frac{1}{2}} \left(\int_{\mathbb{T}^2} \left(|\partial_1|^{\frac{1}{2}} w^2 \right)^2 \, \mathrm{d}x \right)^{\frac{1}{2}} + \frac{1}{4} \int_{\mathbb{T}^2} \left(|\partial_1|^{\frac{1}{2}} w^2 \right)^2 \, \mathrm{d}x. \end{split}$$

By Lemma B.7 the claimed inequality (2.16) follows.

3. Γ -convergence of the renormalized energy

In this section we study the Γ -convergence of the renormalized energy as the regularization of white noise is removed, i.e., the limit $\ell \downarrow 0$. As a consequence we will get the existence of minimizers of the limiting "renormalized energy", in particular, the existence of weak solutions of the Euler-Lagrange equation in (1.13).

3.1. Proof of Theorem 1.15. We begin with the proof of coercivity statement (i) in Theorem 1.15.

Proof of Theorem 1.15 (*i*). Let $(\xi, v, F) \in \mathcal{T}$ be fixed. Since v has vanishing average in x_1 we can estimate $||v||_{L^{\infty}} \leq [v]_{\frac{3}{4}-\varepsilon}$ (see e.g., [IO19, Lemma 12]), where the implicit constant is universal for small ε (e.g., $\varepsilon \in (0, \frac{1}{100})$). We will use this estimate several times in what follows. We split

$$\mathcal{G}(v,F;w) = \int_{\mathbb{T}^2} \left(w^2 R_1 \partial_2 v + v^2 R_1 \eta_w + 2v w R_1 \eta_w + 2w F - w v R_1 \partial_1 v^2 + (R_1 |\partial_1|^{\frac{1}{2}} (vw))^2 \right) dx$$

=: $\sum_{k=1}^6 \mathcal{G}_k(v,F;w),$ (3.1)

and bound each term separately:

(T1) Notice that setting $g := |\partial_1|^{-1} \partial_2 v$ we have

$$\partial_1 g = R_1 \partial_2 v$$

and $g \in C^{\frac{1}{4}-\varepsilon}$, with $[g]_{\frac{1}{4}-\varepsilon} \lesssim [\xi]_{-\frac{5}{4}-\varepsilon}$, (see Lemma D.1). We can therefore integrate by parts

$$\int_{\mathbb{T}^2} w^2 R_1 \partial_2 v \, \mathrm{d}x = \int_{\mathbb{T}^2} w^2 \partial_1 g \, \mathrm{d}x = -2 \int_{\mathbb{T}^2} w \partial_1 w \, g \, \mathrm{d}x$$

and obtain the bound

$$\begin{aligned} |\mathcal{G}_{1}(v,F;w)| &\leq \left| \int_{\mathbb{T}^{2}} w^{2} R_{1} \partial_{2} v \, \mathrm{d}x \right| \leq 2 ||g||_{L^{\infty}} ||w||_{L^{2}} ||\partial_{1}w||_{L^{2}} \leq [g]_{\frac{1}{4}-\varepsilon} ||w||_{L^{3}} ||\partial_{1}w||_{L^{2}} \\ &\leq [g]_{\frac{1}{4}-\varepsilon} \mathcal{E}(w)^{\frac{1}{3}+\frac{1}{2}} \leq [\xi]_{-\frac{5}{4}-\varepsilon} \mathcal{E}(w)^{\frac{5}{6}}, \end{aligned}$$

where we used Hölder's and Jensen's inequality, together with (2.10), as well as $||g||_{L^{\infty}} \leq [g]_{\frac{1}{4}-\varepsilon}$ because g has zero average. By Young's inequality, it follows for any $\lambda \in (0, 1)$

$$|\mathcal{G}_1(v,F;w)| \le \lambda \mathcal{E}(w) + C_{\lambda}^{(1)}[\xi]_{-\frac{5}{4}-\varepsilon}^6$$

(T2) For the term \mathcal{G}_2 we have that

$$\begin{aligned} |\mathcal{G}_{2}(v,F;w)| &= \left| \int_{\mathbb{T}^{2}} (|\partial_{1}|^{\frac{1}{2}} v^{2}) \left(R_{1} |\partial_{1}|^{-\frac{1}{2}} \eta_{w} \right) dx \right| \leq \||\partial_{1}|^{\frac{1}{2}} v^{2}\|_{L^{2}} \||\partial_{1}|^{-\frac{1}{2}} \eta_{w}\|_{L^{2}} \\ &\lesssim \|v^{2}\|_{\dot{\mathcal{B}}^{\frac{2}{3}}_{2;1}} \mathcal{E}(w)^{\frac{1}{2}} \lesssim [v^{2}]_{\frac{2}{3}} \mathcal{E}(w)^{\frac{1}{2}} \lesssim [v]_{\frac{3}{4}-\varepsilon}^{2} \mathcal{E}(w)^{\frac{1}{2}}, \end{aligned}$$

where we used the Cauchy–Schwarz inequality, boundedness of R_1 on L^2 , the estimates (B.4), (B.1) and [IO19, Lemma 12]. Hence, by Young's inequality, for any $\lambda \in (0, 1)$,

$$|\mathcal{G}_2(v,F;w)| \le \lambda \mathcal{E}(w) + C_{\lambda}^{(2)}[v]_{\frac{3}{4}-\varepsilon}^4.$$

(T₃) We estimate \mathcal{G}_3 using Cauchy–Schwarz, the boundedness of R_1 on L^2 , and (B.4) by

$$\begin{aligned} |\mathcal{G}_{3}(v,F;w)| &= \left| 2 \int_{\mathbb{T}^{2}} (|\partial_{1}|^{\frac{1}{2}}(vw))(R_{1}|\partial_{1}|^{-\frac{1}{2}}\eta_{w}) \, \mathrm{d}x \right| \\ &\lesssim \||\partial_{1}|^{\frac{1}{2}}(vw)\|_{L^{2}} \||\partial_{1}|^{-\frac{1}{2}}\eta_{w}\|_{L^{2}} \lesssim \|vw\|_{\dot{\mathcal{B}}^{\frac{2}{3}}_{2;1}} \mathcal{E}(w)^{\frac{1}{2}}. \end{aligned}$$

By the fractional Leibniz rule (Lemma B.2 (i)) we can further bound

$$\|vw\|_{\dot{\mathcal{B}}^{\frac{2}{3}}_{2;1}} \lesssim \|v\|_{L^{\infty}} \|w\|_{\dot{\mathcal{B}}^{\frac{2}{3}}_{2;1}} + [v]_{\frac{2}{3}} \|w\|_{L^{2}} \lesssim [v]_{\frac{3}{4}-\varepsilon} \left(\|w\|_{\dot{\mathcal{B}}^{\frac{2}{3}}_{2;1}} + \|w\|_{L^{3}} \right),$$

where we also used Jensen's inequality. Combined with (2.9) and (2.10), this gives

$$|\mathcal{G}_3(v,F;w)| \leq [v]_{\frac{3}{4}-\varepsilon} \left(\mathcal{E}(w)^{\frac{7}{18}} + \mathcal{E}(w)^{\frac{1}{3}}\right) \mathcal{E}(w)^{\frac{1}{2}}$$

so that Young's inequality yields for any $\lambda \in (0, 1)$,

$$|\mathcal{G}_3(v,F;w)| \leq \lambda \mathcal{E}(w) + C_{\lambda}^{(3)}\left([v]_{\frac{3}{4}-\varepsilon}^6 + [v]_{\frac{3}{4}-\varepsilon}^9\right).$$

(T4) By the duality Lemma B.3, $\mathcal{G}_4(v, F; w) = 2 \int_{\mathbb{T}^2} wF \, dx$ can be bounded by

$$\begin{aligned} |\mathcal{G}_{4}(v,F;w)| &\lesssim \left(\|\partial_{1}w\|_{L^{2}} + \|\partial_{2}\|_{3}^{\frac{2}{3}}w\|_{L^{2}} + \|w\|_{L^{1}} \right) [F]_{-\frac{8}{9}} \\ &\lesssim \left(\|\partial_{1}w\|_{L^{2}} + \|\partial_{2}\|_{3}^{\frac{2}{3}}w\|_{L^{2}} + \|w\|_{L^{3}} \right) [F]_{-\frac{3}{4}-\varepsilon} \end{aligned}$$

with a uniform implicit constant for every $\varepsilon \in (0, \frac{1}{100})$, where in the second step we used Jensen's inequality and [IO19, Remark 2]. With (2.14), (2.15) and (2.10) we obtain that

$$\|\partial_1 w\|_{L^2} + \|\partial_2\|^{\frac{2}{3}} w\|_{L^2} + \|w\|_{L^3} \leq 1 + \mathcal{E}(w)^{\frac{3}{4}+\kappa} + \mathcal{E}(w)^{\frac{1}{3}}$$

where $\kappa > 0$ can be chosen arbitrarily small (e.g., $\kappa = \frac{1}{100}$). This yields the estimate

$$|\mathcal{G}_4(v,F;w)| \lesssim \left(1 + \mathcal{E}(w)^{\frac{3}{4}+\kappa} + \mathcal{E}(w)^{\frac{1}{3}}\right) [F]_{-\frac{3}{4}-\epsilon}.$$

It follows for any $\lambda \in (0, 1)$,

$$|\mathcal{G}_4(v,F;w)| \le \lambda \mathcal{E}(w) + C^{(4)}_{\lambda,\kappa} \left([F]_{-\frac{3}{4}-\varepsilon} + [F]_{-\frac{3}{4}-\varepsilon}^{\frac{3}{2}} + [F]_{-\frac{3}{4}-\varepsilon}^{\frac{4}{1-4\kappa}} \right)$$

(T5) For $\mathcal{G}_5(v, F; w) = -\int_{\mathbb{T}^2} w \, v R_1 \partial_1 v^2 \, dx$, we use again the duality estimate Lemma B.3,

$$|\mathcal{G}_{5}(v,F;w)| \leq \left(\|\partial_{1}w\|_{L^{2}} + \||\partial_{2}|^{\frac{2}{3}}w\|_{L^{2}} + \|w\|_{L^{1}} \right) \left[vR_{1}\partial_{1}v^{2} \right]_{-\frac{2}{5}}$$

By [IO19, Lemmata 6 and 12] together with (A.13), we have the uniform bound for any $\varepsilon \in (0, \frac{1}{100})$

$$[vR_1\partial_1 v^2]_{-\frac{2}{5}} \lesssim [v]_{\frac{1}{2}} [R_1\partial_1 v^2]_{-\frac{2}{5}} \lesssim [v]_{\frac{3}{4}-\varepsilon} [\partial_1 v^2]_{-\frac{1}{3}} \lesssim [v]_{\frac{3}{4}-\varepsilon}^3$$

Hence, as in (T4), we can bound $\mathcal{G}_5(v, F; w)$ for some small $\kappa > 0$ (e.g., $\kappa = \frac{1}{100}$) by

$$|\mathcal{G}_5(v,F;w)| \lesssim \left(1 + \mathcal{E}(w)^{\frac{3}{4}+\kappa} + \mathcal{E}(w)^{\frac{1}{3}}\right) [v]_{\frac{3}{4}-\varepsilon}^3.$$

So, for any $\lambda \in (0, 1)$, by Young's inequality,

$$|\mathcal{G}_5(v,F;w)| \leq \lambda \mathcal{E}(w) + C_{\lambda,\kappa}^{(5)} \left(\left[v \right]_{\frac{3}{4}-\varepsilon}^{\frac{9}{2}} + \left[v \right]_{\frac{3}{4}-\varepsilon}^{3} + \left[v \right]_{\frac{3}{4}-\varepsilon}^{\frac{12}{1-4\kappa}} \right).$$

(T6) For the term $\mathcal{G}_6(v, F; w) = \int_{\mathbb{T}^2} (R_1 |\partial_1|^{\frac{1}{2}} (vw))^2 dx$, we first notice that by boundedness of R_1 on L^2 and the basic estimate (B.4),

$$\mathcal{G}_{6}(v,F;w) = \||\partial_{1}|^{\frac{1}{2}}(vw)\|_{L^{2}}^{2} \leq \|vw\|_{\dot{\mathcal{B}}^{\frac{2}{3}}_{2;1}}^{2}$$

Hence, by Lemma B.2 and Jensen's inequality,

$$|\mathcal{G}_{6}(v,F;w)| \leq \|v\|_{L^{\infty}}^{2} \|w\|_{\dot{\mathcal{B}}_{2;1}^{\frac{2}{3}}}^{2} + [v]_{\frac{2}{3}}^{2} \|w\|_{L^{2}}^{2} \leq [v]_{\frac{3}{4}-\varepsilon}^{2} \left(\|w\|_{\dot{\mathcal{B}}_{2;1}^{\frac{2}{3}}}^{2} + \|w\|_{L^{3}}^{2}\right).$$

Together with (2.11) and (2.10) we can therefore estimate

$$|\mathcal{G}_6(v,F;w)| \lesssim [v]_{\frac{3}{4}-\varepsilon}^2 \left(\mathcal{E}(w)^{\frac{7}{9}} + \mathcal{E}(w)^{\frac{2}{3}}\right).$$

Young's inequality then yields for any $\lambda \in (0, 1)$

$$|\mathcal{G}_6(v,F;w)| \leq \lambda \mathcal{E}(w) + C_{\lambda}^{(6)} \left([v]_{\frac{3}{4}-\varepsilon}^6 + [v]_{\frac{3}{4}-\varepsilon}^9 \right).$$

In the proof of the continuity statement Theorem 1.15 (ii), we need the following lemma.

Lemma 3.1. Let $\{w_\ell\}_{\ell \downarrow 0} \subset W$ with uniformly bounded energy $\sup_\ell \mathcal{E}(w_\ell) < \infty$, and assume that $w_\ell \to w$ strongly in L^2 as $\ell \to 0$ for some $w \in W$. Then as $\ell \to 0$,

$$\partial_1 w_\ell \rightharpoonup \partial_1 w, \quad |\partial_1|^{-\frac{1}{2}} \partial_2 w_\ell \rightharpoonup |\partial_1|^{-\frac{1}{2}} \partial_2 w, \quad |\partial_1|^{-\frac{1}{2}} \eta_{w_\ell} \rightharpoonup |\partial_1|^{-\frac{1}{2}} \eta_w, \quad |\partial_2|^{\frac{2}{3}} w_\ell \rightharpoonup |\partial_2|^{\frac{2}{3}} w_\ell$$

weakly in L^2 , and for any $s_1 \in (0, 1)$ and $s_2 \in (0, \frac{2}{3})$,

$$|\partial_1|^{s_1} w_\ell \to |\partial_1|^{s_1} w, \qquad |\partial_2|^{s_2} w_\ell \to |\partial_2|^{s_2} w \qquad strongly in L^2.$$

Proof. For the first part, we use the fact that a uniformly bounded sequence in L^2 converging in the distributional sense converges weakly in L^2 . By Proposition 2.5, we have that $\sup_{\ell} \mathcal{H}(w_{\ell}) < \infty$. Therefore, $\{\partial_1 w_{\ell}\}_{\ell}, \{|\partial_1|^{-\frac{1}{2}}\partial_2 w_{\ell}\}_{\ell}, \{|\partial_1|^{-\frac{1}{2}}\eta_{w_{\ell}}\}_{\ell}, \text{and }\{|\partial_2|^{\frac{2}{3}}w_{\ell}\}_{\ell}$ are uniformly bounded in L^2 . They also converge in the distributional sense since $w_{\ell} \to w$ strongly in L^2 (in particular, $(w_{\ell})^2 \to w^2$ strongly in L^1 , so $\eta_{w_{\ell}} \to \eta_w$ in the distributional sense). For the second part, by Lemma B.5, for any $s_1 \in (0, 1), s_2 \in (0, \frac{2}{3})$, we have that $\{|\partial_1|^{s_1}w_{\ell}\}_{\ell}$ is uniformly bounded in the homogeneous

Sobolev space $\dot{H}^{\frac{2}{3}(1-s_1)}$, and $\{|\partial_2|^{s_2}w_\ell\}_\ell$ is uniformly bounded in $\dot{H}^{\frac{2}{3}-s_2}$. The compact embedding $\dot{H}^{\min\{\frac{2}{3}(1-s_1),\frac{2}{3}-s_2\}} \hookrightarrow L^2$ (of periodic functions with vanishing average) yields the conclusion.

Proof of Theorem 1.15 (*ii*). First, all the convergence statements from Lemma 3.1 hold for the sequence $\{w_\ell\}_\ell$. As $(\xi_\ell, v_\ell, F_\ell) \rightarrow (\xi, v, F)$ in \mathcal{T} , by Lemma D.1 we also have that $g_\ell := |\partial_1|^{-1} \partial_2 v_\ell \rightarrow |\partial_1|^{-1} \partial_2 v =: g$ in $C^{\frac{1}{4}-\varepsilon}$. We will prove the continuity of \mathcal{G} using the decomposition $\mathcal{G} = \sum_{k=1}^6 \mathcal{G}_k$ in (3.1) and study each term \mathcal{G}_k separately.

(T'1) For the term $\mathcal{G}_1(v_\ell, F_\ell; w_\ell) = \int_{\mathbb{T}^2} (w_\ell)^2 R_1 \partial_2 v_\ell \, dx$, we use integration by parts, and that $w_\ell \to w$ strongly in L^2 , $\partial_1 w_\ell \to \partial_1 w$ weakly in L^2 , and $g_\ell \to g$ uniformly on \mathbb{T}^2 ,

$$\mathcal{G}_1(v_\ell, F_\ell; w_\ell) = \int_{\mathbb{T}^2} (w_\ell)^2 \partial_1 g_\ell \, \mathrm{d}x = -2 \int_{\mathbb{T}^2} w_\ell \partial_1 w_\ell \, g_\ell \, \mathrm{d}x$$
$$\to -2 \int_{\mathbb{T}^2} w \partial_1 w \, g \, \mathrm{d}x = \int_{\mathbb{T}^2} w^2 \partial_1 g \, \mathrm{d}x = \int_{\mathbb{T}^2} w^2 R_1 \partial_2 v \, \mathrm{d}x = \mathcal{G}_1(v, F; w).$$

(**T'2**) For the term $\mathcal{G}_2(v_\ell, F_\ell; w_\ell) = \int_{\mathbb{T}^2} v_\ell^2 R_1 \eta_{w_\ell} \, dx$ we use that $|\partial_1|^{-\frac{1}{2}} \eta_{w_\ell} \rightarrow |\partial_1|^{-\frac{1}{2}} \eta_w$ weakly in L^2 (hence also $R_1 |\partial_1|^{-\frac{1}{2}} \eta_{w_\ell} \rightarrow R_1 |\partial_1|^{-\frac{1}{2}} \eta_w$ weakly in L^2), and $|\partial_1|^{\frac{1}{2}} v_\ell^2 \rightarrow |\partial_1|^{\frac{1}{2}} v^2$ strongly in $C^{\frac{1}{4}-\varepsilon}$ (see Lemma A.5),

$$\mathcal{G}_{2}(v_{\ell}, F_{\ell}; w_{\ell}) = \int_{\mathbb{T}^{2}} (|\partial_{1}|^{\frac{1}{2}} v_{\ell}^{2}) (R_{1}|\partial_{1}|^{-\frac{1}{2}} \eta_{w_{\ell}}) \, \mathrm{d}x \to \int_{\mathbb{T}^{2}} (|\partial_{1}|^{\frac{1}{2}} v^{2}) (R_{1}|\partial_{1}|^{-\frac{1}{2}} \eta_{w}) \, \mathrm{d}x = \mathcal{G}_{2}(v, F; w).$$

(**T'3**) Since $\mathcal{G}_3(v_\ell, F_\ell; w_\ell) = 2 \int_{\mathbb{T}^2} v_\ell w_\ell R_1 \eta_{w_\ell} dx = 2 \int_{\mathbb{T}^2} |\partial_1|^{\frac{1}{2}} (v_\ell w_\ell) R_1 |\partial_1|^{-\frac{1}{2}} \eta_{w_\ell} dx$ and, as in (**T'2**), $R_1 |\partial_1|^{-\frac{1}{2}} \eta_{w_\ell} \rightarrow R_1 |\partial_1|^{-\frac{1}{2}} \eta_w$ weakly in L^2 , the claimed convergence follows if we show that $|\partial_1|^{\frac{1}{2}} (v_\ell w_\ell) \rightarrow |\partial_1|^{\frac{1}{2}} (vw)$ strongly in L^2 . For this, we use the triangle inequality, Lemma B.2 (ii), and that $w_\ell \rightarrow w$, $|\partial_1|^{\frac{1}{2}} w_\ell \rightarrow |\partial_1|^{\frac{1}{2}} w$ strongly in L^2 as well as $v_\ell \rightarrow v$ in $C^{\frac{3}{4}-\varepsilon} \subset C^{\frac{2}{3}}$ which yield as $\ell \rightarrow 0$,

$$\begin{split} \||\partial_{1}|^{\frac{1}{2}}(v_{\ell}w_{\ell}) - |\partial_{1}|^{\frac{1}{2}}(vw)\|_{L^{2}} &\leq \||\partial_{1}|^{\frac{1}{2}}((v_{\ell} - v)w_{\ell})\|_{L^{2}} + \||\partial_{1}|^{\frac{1}{2}}(v(w_{\ell} - w))\|_{L^{2}} \\ &\leq \|v_{\ell} - v\|_{L^{\infty}}\||\partial_{1}|^{\frac{1}{2}}w_{\ell}\|_{L^{2}} + [v_{\ell} - v]_{\frac{2}{3}}\|w_{\ell}\|_{L^{2}} \\ &+ \|v\|_{L^{\infty}}\||\partial_{1}|^{\frac{1}{2}}(w_{\ell} - w)\|_{L^{2}} + [v]_{\frac{2}{3}}\|w_{\ell} - w\|_{L^{2}} \\ &\leq [v_{\ell} - v]_{\frac{3}{4} - \varepsilon}\left(\||\partial_{1}|^{\frac{1}{2}}w_{\ell}\|_{L^{2}} + \|w_{\ell}\|_{L^{2}}\right) \\ &+ [v]_{\frac{3}{4} - \varepsilon}\left(\||\partial_{1}|^{\frac{1}{2}}(w_{\ell} - w)\|_{L^{2}} + \|w_{\ell} - w\|_{L^{2}}\right) \to 0. \end{split}$$

(**T'4**) The term $\mathcal{G}_4(v_\ell, F_\ell; w_\ell) = 2 \int_{\mathbb{T}^2} w_\ell F_\ell \, dx$ is treated by duality. Since $F_\ell \to F$ in $C^{-\frac{3}{4}-\varepsilon} \subset C^{-\frac{4}{5}}$ (see e.g., [IO19, Remark 2]), by Lemmata 3.1 and B.3 we have for $\ell \to 0$,

$$\begin{aligned} |\mathcal{G}_{4}(v_{\ell}, F_{\ell}; w_{\ell}) - \mathcal{G}_{4}(v, F; w)| \\ \lesssim \left| \int_{\mathbb{T}^{2}} (w_{\ell} - w) F_{\ell} \, dx \right| + \left| \int_{\mathbb{T}^{2}} w(F_{\ell} - F) \, dx \right| \\ \lesssim [F_{\ell}]_{-\frac{4}{5}} \left(\||\partial_{1}|^{\frac{5}{6}} (w_{\ell} - w)\|_{L^{2}} + \||\partial_{2}|^{\frac{2}{3} \cdot \frac{5}{6}} (w_{\ell} - w)\|_{L^{2}} + \|w_{\ell} - w\|_{L^{2}} \right) \\ + [F_{\ell} - F]_{-\frac{4}{5}} \left(\||\partial_{1}|^{\frac{5}{6}} w\|_{L^{2}} + \||\partial_{2}|^{\frac{2}{3} \cdot \frac{5}{6}} w\|_{L^{2}} + \|w\|_{L^{2}} \right) \to 0. \end{aligned}$$

(T'5) For the continuity of the term $\mathcal{G}_5(v_\ell, F_\ell; w_\ell) = -\int_{\mathbb{T}^2} w_\ell v_\ell R_1 \partial_1 v_\ell^2 dx$ we again use the duality Lemma B.3. Here, the situation is even easier than in **(T'4)**, as $v_\ell R_1 \partial_1 v_\ell^2$ converges to the

nonsingular product $vR_1\partial_1 v^2$ in $C^{-\frac{1}{4}-2\epsilon}$. This convergence follows by

$$\begin{split} & [v_{\ell}R_{1}\partial_{1}v_{\ell}^{2} - vR_{1}\partial_{1}v^{2}]_{-\frac{1}{4}-2\varepsilon} \\ & = [(v_{\ell} - v)R_{1}\partial_{1}v^{2} + v_{\ell}R_{1}\partial_{1}((v_{\ell} - v)(v_{\ell} + v))]_{-\frac{1}{4}-2\varepsilon} \\ & \leq [v_{\ell} - v]_{\frac{3}{4}-\varepsilon}[R_{1}\partial_{1}v^{2}]_{-\frac{1}{4}-2\varepsilon} + [v_{\ell}]_{\frac{3}{4}-\varepsilon}[R_{1}\partial_{1}((v_{\ell} - v)(v_{\ell} + v))]_{-\frac{1}{4}-2\varepsilon} \\ & \leq [v_{\ell} - v]_{\frac{3}{4}-\varepsilon}[v^{2}]_{\frac{3}{4}-\varepsilon} + [v_{\ell}]_{\frac{3}{4}-\varepsilon}[(v_{\ell} - v)(v_{\ell} + v)]_{\frac{3}{4}-\varepsilon} \\ & \leq [v_{\ell} - v]_{\frac{3}{4}-\varepsilon}([v]_{\frac{3}{4}-\varepsilon}^{2} + [v_{\ell}]_{\frac{3}{4}-\varepsilon}^{2}) \to 0, \end{split}$$

where we used that $v_{\ell} \to v$ in $C^{\frac{3}{4}-\varepsilon}$ and [IO19, Lemmata 6, 7, and 12]. We conclude as for \mathcal{G}_4 (with $v_{\ell}R_1\partial_1 v_{\ell}^2$ corresponding to F_{ℓ} and $vR_1\partial_1 v^2$ to F, using also that $C^{-\frac{1}{4}-2\varepsilon} \subset C^{-\frac{3}{4}-\varepsilon}$).

(**T'6**) Noting that $\mathcal{G}_6(v_\ell, F_\ell; w_\ell) = \int_{\mathbb{T}^2} (R_1 |\partial_1|^{\frac{1}{2}} (v_\ell w_\ell))^2 \, dx = \||\partial_1|^{\frac{1}{2}} (v_\ell w_\ell)\|_{L^2}^2$, continuity follows since $|\partial_1|^{\frac{1}{2}} (v_\ell w_\ell) \to |\partial_1|^{\frac{1}{2}} (vw)$ in L^2 used in (**T'3**).

We now prove the compactness Theorem 1.15 (iii) of the sublevel sets of E_{ren} with respect to the strong topology in L^2 .

Proof of Theorem 1.15 (iii). By the coercivity Theorem 1.15 (i) for $\lambda = \frac{1}{2}$, it follows that

$$E_{ren}(v,F;w) = \mathcal{E}(w) + \mathcal{G}(v,F;w) \ge \frac{1}{2}\mathcal{E}(w) - C \ge -C.$$
(3.2)

Thus $E_{ren}(v, F; \cdot)$ is bounded from below and the sublevel set $\{E_{ren}(v, F; \cdot) \leq M\}$ over \mathcal{W} is included in a sublevel set of \mathcal{E} over \mathcal{W} which is relatively compact in L^2 by Lemma B.5. It remains to prove that the sublevel set $\{E_{ren}(v, F; \cdot) \leq M\}$ over \mathcal{W} is closed in L^2 . By the continuity of $\mathcal{G}(v, F; \cdot)$ (Theorem 1.15 (ii)), it suffices to show that \mathcal{E} is lower semicontinuous in \mathcal{W} , i.e., for every $\{w^\ell\}_{\ell \mid 0} \subset \mathcal{W}$ with $w^\ell \to w$ in L^2 , there holds

$$\liminf_{\ell \downarrow 0} \mathcal{E}(w^{\ell}) \ge \mathcal{E}(w). \tag{3.3}$$

Indeed, since $a^2 \ge b^2 + 2(a - b)b$, it follows that

$$\mathcal{E}(w^{\ell}) \geq \mathcal{E}(w) + 2 \int_{\mathbb{T}^2} \left(\partial_1 w^{\ell} - \partial_1 w \right) \partial_1 w \, \mathrm{d}x + 2 \int_{\mathbb{T}^2} \left(|\partial_1|^{-\frac{1}{2}} \eta_{w^{\ell}} - |\partial_1|^{-\frac{1}{2}} \eta_w \right) |\partial_1|^{-\frac{1}{2}} \eta_w \, \mathrm{d}x.$$

Without loss of generality, we may assume that $\liminf_{\ell \downarrow 0} \mathcal{E}(w^{\ell}) = \limsup_{\ell \downarrow 0} \mathcal{E}(w^{\ell}) < \infty$. Hence, by Lemma 3.1, $\partial_1 w^{\ell} \rightharpoonup \partial_1 w$ and $|\partial_1|^{-\frac{1}{2}} \eta_{w^{\ell}} \rightharpoonup |\partial_1|^{-\frac{1}{2}} \eta_w$ weakly in L^2 , and thus, (3.3) follows. The same argument shows that \mathcal{H} is lower semicontinuous in \mathcal{W} .

We are now ready to prove the existence of minimizers Theorem 1.15 (iv).

Proof of Theorem 1.15 (*iv*). Note that $E_{ren}(v, F; 0) = 0$ and recall that $E_{ren}(v, F; \cdot)$ is bounded from below (see (3.2)) and lower semicontinuous in L^2 over its zero sublevel set (due to (1.10), $\mathcal{G}(v, F; \cdot)$ being continuous over any sublevel set of \mathcal{E} and \mathcal{E} being lower semicontinuous in L^2). By the L^2 -compactness of the zero sublevel set of $E_{ren}(v, F; \cdot)$ over \mathcal{W} (Theorem 1.15 (iii)), the direct method in the calculus of variations yields the existence of minimizers.

In order to show that a minimizer is a distributional solution of the Euler–Lagrange equation (1.13), we do the following splitting:

$$E_{ren}(v,F;w) = \mathcal{H}(w) + (\mathcal{E}(w) - \mathcal{H}(w)) + \mathcal{G}(v,F;w) = \mathcal{H}(w) + \sum_{j=1}^{4} L_j(w),$$

where

$$\begin{split} L_1(w) &= \int_{\mathbb{T}^2} \left(2wF - wvR_1\partial_1 v^2 + v^2R_1\partial_2 w \right) \, \mathrm{d}x \\ L_2(w) &= \int_{\mathbb{T}^2} \left(w^2R_1\partial_2 v - \frac{1}{2}v^2R_1\partial_1 w^2 + 2vwR_1\partial_2 w + (R_1|\partial_1|^{\frac{1}{2}}(vw))^2 \right) \, \mathrm{d}x \\ L_3(w) &= \int_{\mathbb{T}^2} \left(-R_1|\partial_1|^{\frac{1}{2}}w^2|\partial_1|^{-\frac{1}{2}}\partial_2 w - vwR_1\partial_1 w^2 \right) \, \mathrm{d}x \\ L_4(w) &= \int_{\mathbb{T}^2} \frac{1}{4} \left(|\partial_1|^{\frac{1}{2}}w^2 \right)^2. \end{split}$$

We will show that, given $(\xi, v, F) \in \mathcal{T}$, the functional $E_{ren}(v, F; \cdot)$ is C^{∞} on the space \mathcal{W} endowed with the norm $\mathcal{H}^{\frac{1}{2}}$, denoted by $(\mathcal{W}, \mathcal{H}^{\frac{1}{2}})$.

STEP 1 (Estimating the linear functional L_1). We claim that L_1 is a continuous linear functional on $(\mathcal{W}, \mathcal{H}^{\frac{1}{2}})$, i.e.,

$$|L_1(w)| \le C\mathcal{H}(w)^{\frac{1}{2}},$$
 (3.4)

where *C* depends polynomially on $[v]_{\frac{3}{4}-\epsilon}$, $[F]_{-\frac{3}{4}-\epsilon}$. Indeed, as in (T₄) in the proof of (i), by the duality Lemma B.₃ and Poincaré's inequality, we may bound

$$\left| \int_{\mathbb{T}^2} 2wF \, \mathrm{d}x \right| \lesssim \left(\|\partial_1 w\|_{L^2} + \|\partial_2\|^{\frac{2}{3}} w\|_{L^2} + \|w\|_{L^2} \right) [F]_{-\frac{8}{9}} \lesssim [F]_{-\frac{3}{4}-\epsilon} \mathcal{H}(w)^{\frac{1}{2}}.$$

By the same argument, see also (T5) above, we have

$$\left| \int_{\mathbb{T}^2} w v R_1 \partial_1 v^2 \, \mathrm{d}x \right| \lesssim \left(\|\partial_1 w\|_{L^2} + \||\partial_2|^{\frac{2}{3}} w\|_{L^2} + \|w\|_{L^2} \right) \left[v R_1 \partial_1 v^2 \right]_{-\frac{2}{5}} \lesssim \left[v \right]_{\frac{3}{4} - \epsilon}^3 \mathcal{H}(w)^{\frac{1}{2}}.$$

As in (T2), the last term of L_1 is estimated using Cauchy–Schwarz, (B.4), and (B.1), by

$$\left| \int_{\mathbb{T}^2} v^2 R_1 \partial_2 w \, \mathrm{d}x \right| \le \||\partial_1|^{\frac{1}{2}} v^2\|_{L^2} \||\partial_1|^{-\frac{1}{2}} \partial_2 w\|_{L^2} \le [v]_{\frac{3}{4}-\epsilon}^2 \mathcal{H}(w)^{\frac{1}{2}}$$

STEP 2 (Estimating the quadratic functional L_2). We claim that L_2 is a continuous quadratic functional on $(\mathcal{W}, \mathcal{H}^{\frac{1}{2}})$, i.e., there exists a continuous bilinear functional M_2 given by

$$M_{2}(w_{1}, w_{2}) = \int_{\mathbb{T}^{2}} \left(w_{1}w_{2}R_{1}\partial_{2}v - \frac{1}{2}v^{2}R_{1}\partial_{1}(w_{1}w_{2}) + 2vw_{1}R_{1}\partial_{2}w_{2} + (|\partial_{1}|^{\frac{1}{2}}(vw_{1}))(|\partial_{1}|^{\frac{1}{2}}(vw_{2})) \right) dx$$

such that $L_2(w) = M_2(w, w)$, and satisfying the inequality

$$|M_2(w_1, w_2)| \le C\mathcal{H}(w_1)^{\frac{1}{2}} \mathcal{H}(w_2)^{\frac{1}{2}},$$
(3.5)

where *C* depends polynomially on $[\xi]_{-\frac{5}{4}-\epsilon}$, $[v]_{\frac{3}{4}-\epsilon}$, $[F]_{-\frac{3}{4}-\epsilon}$. To prove (3.5), we again treat each term separately. Similarly to (T1), let $g := |\partial_1|^{-1}\partial_2 v$, such that $R_1\partial_2 v = \partial_1 g$. Recall that by Lemma D.1, $[g]_{\frac{1}{4}-\epsilon} \leq [\xi]_{-\frac{5}{4}-\epsilon}$. Then integration by parts and Cauchy–Schwarz, together with Poincaré's inequality, gives

$$\left| \int_{\mathbb{T}^2} w_1 w_2 R_1 \partial_2 v \, \mathrm{d}x \right| = \left| \int_{\mathbb{T}^2} \partial_1 (w_1 w_2) g \, \mathrm{d}x \right| \lesssim [g]_{\frac{1}{4} - \epsilon} \int_{\mathbb{T}^2} |w_1 \partial_1 w_2 + w_2 \partial_1 w_1| \, \mathrm{d}x$$
$$\lesssim [\xi]_{-\frac{5}{4} - \epsilon} \mathcal{H}(w_1)^{\frac{1}{2}} \mathcal{H}(w_2)^{\frac{1}{2}}.$$

Similarly, the second term can be estimated by

$$\left| \int_{\mathbb{T}^2} \frac{1}{2} v^2 R_1 \partial_1(w_1 w_2) \, \mathrm{d}x \right| = \left| \int_{\mathbb{T}^2} \frac{1}{2} R_1 v^2(w_1 \partial_1 w_2 + w_2 \partial_1 w_1) \, \mathrm{d}x \right| \lesssim [R_1 v^2]_{\frac{3}{4} - 2\epsilon} \mathcal{H}(w_1)^{\frac{1}{2}} \mathcal{H}(w_2)^{\frac{1}{2}} \\ \lesssim [v]_{\frac{3}{4} - \epsilon}^2 \mathcal{H}(w_1)^{\frac{1}{2}} \mathcal{H}(w_2)^{\frac{1}{2}}.$$

By Cauchy–Schwarz, Lemma B.2 (i), interpolation and Poincaré's inequality, we can bound the third term by

$$\begin{aligned} \left| \int_{\mathbb{T}^2} 2v w_1 R_1 \partial_2 w_2 \, \mathrm{d}x \right| &= 2 \left| \int_{\mathbb{T}^2} |\partial_1|^{\frac{1}{2}} (v w_1) R_1 |\partial_1|^{-\frac{1}{2}} (\partial_2 w_2) \, \mathrm{d}x \right| \le 2 \||\partial_1|^{\frac{1}{2}} (v w_1)\|_{L^2} \||\partial_1|^{-\frac{1}{2}} \partial_2 w\|_{L^2} \\ &\lesssim \left(\||\partial_1|^{\frac{1}{2}} w_1\|_{L^2} \|v\|_{L^\infty} + \|w_1\|_{L^2} [v]_{\frac{1}{2}+\epsilon} \right) \mathcal{H}(w_2)^{\frac{1}{2}} \lesssim [v]_{\frac{3}{4}-\epsilon} \mathcal{H}(w_1)^{\frac{1}{2}} \mathcal{H}(w_2)^{\frac{1}{2}}. \end{aligned}$$

Analogously, the fourth term is estimated by

$$\left| \int_{\mathbb{T}^2} (|\partial_1|^{\frac{1}{2}}(vw_1)) (|\partial_1|^{\frac{1}{2}}(vw_2)) \, \mathrm{d}x \right| \leq \||\partial_1|^{\frac{1}{2}}(vw_1)\|_{L^2} \||\partial_1|^{\frac{1}{2}}(vw_2)\|_{L^2} \lesssim [v]_{\frac{3}{4}-\epsilon}^2 \mathcal{H}(w_1)^{\frac{1}{2}} \mathcal{H}(w_2)^{\frac{1}{2}}.$$

STEP 3 (Estimating the cubic functional L_3). We claim that L_3 is a continuous cubic functional on $(\mathcal{W}, \mathcal{H}^{\frac{1}{2}})$, i.e., there exists a continuous three-linear functional M_3 given by

$$M_3(w_1, w_2, w_3) = -\int_{\mathbb{T}^2} \left(R_1 |\partial_1|^{\frac{1}{2}} (w_1 w_2) |\partial_1|^{-\frac{1}{2}} \partial_2 w_3 + v w_1 R_1 \partial_1 (w_2 w_3) \right) \, \mathrm{d}x,$$

such that $L_3(w) = M_3(w, w, w)$, and M_3 is controlled by

$$|M_3(w_1, w_2, w_3)| \le C\mathcal{H}(w_1)^{\frac{1}{2}} \mathcal{H}(w_2)^{\frac{1}{2}} \mathcal{H}(w_3)^{\frac{1}{2}},$$
(3.6)

where *C* depends polynomially on $[v]_{\frac{3}{4}-\epsilon}, [F]_{-\frac{3}{4}-\epsilon}$. Indeed, the first term is estimated using Cauchy–Schwarz and Lemma B.7,

$$\begin{aligned} \left| \int_{\mathbb{T}^2} R_1 |\partial_1|^{\frac{1}{2}} (w_1 w_2) |\partial_1|^{-\frac{1}{2}} \partial_2 w_3 \, \mathrm{d}x \right| &\leq \| |\partial_1|^{\frac{1}{2}} (w_1 w_2)\|_{L^2} \| |\partial_1|^{-\frac{1}{2}} \partial_2 w_3\|_{L^2} \\ &\lesssim \mathcal{H}(w_1)^{\frac{1}{2}} \mathcal{H}(w_2)^{\frac{1}{2}} \mathcal{H}(w_3)^{\frac{1}{2}}. \end{aligned}$$

Similarly, Lemma B.2 (i) and Lemma B.7 imply that

$$\left| \int_{\mathbb{T}^2} v w_1 R_1 \partial_1(w_2 w_3) \, \mathrm{d}x \right| \le \||\partial_1|^{\frac{1}{2}} (v w_1)\|_{L^2} \||\partial_1|^{\frac{1}{2}} (w_2 w_3)\|_{L^2} \lesssim [v]_{\frac{3}{4} - \epsilon} \mathcal{H}(w_1)^{\frac{1}{2}} \mathcal{H}(w_2)^{\frac{1}{2}} \mathcal{H}(w_3)^{\frac{1}{2}}.$$

STEP 4 (Estimating the quartic functional L_4). We claim that L_4 is a continuous quartic functional on $(\mathcal{W}, \mathcal{H}^{\frac{1}{2}})$, i.e., there exists a continuous four-linear functional M_4 given by

$$M_4(w_1, w_2, w_3, w_4) = \int_{\mathbb{T}^2} \frac{1}{4} \left(|\partial_1|^{\frac{1}{2}} (w_1 w_2) |\partial_1|^{\frac{1}{2}} (w_3 w_4) \right) \, \mathrm{d}x$$

such that $L_4(w) = M_4(w, w, w, w)$. Indeed, Cauchy–Schwarz and Lemma B.7 implies that

$$|M_4(w_1, w_2, w_3, w_4)| \le C\mathcal{H}(w_1)^{\frac{1}{2}} \mathcal{H}(w_2)^{\frac{1}{2}} \mathcal{H}(w_3)^{\frac{1}{2}} \mathcal{H}(w_4)^{\frac{1}{2}}, \qquad (3.7)$$

where C depends polynomially on $[v]_{\frac{3}{4}-\epsilon}, [F]_{-\frac{3}{4}-\epsilon}$.

Therefore, the gradient $\nabla_{w} E_{ren}(v, F; w)$ belongs to the dual space of $(\mathcal{W}, \mathcal{H}^{\frac{1}{2}})$, in particular, it is a distribution, so that

$$\nabla_{w} E_{ren}(v, F; w) = 0$$

is the Euler–Lagrange equation (1.13).

3.2. Γ -convergence. In view of Theorem 1.15, we give the proof of Γ -convergence of the renormalized energy for sequences $(\xi_{\ell}, v_{\ell}, F_{\ell}) \rightarrow (\xi, v, F)$ in \mathcal{T} as $\ell \rightarrow 0$.

Proof of Corollary 1.16. Assume that $(\xi_{\ell}, v_{\ell}, F_{\ell}) \rightarrow (\xi, v, F)$ in \mathcal{T} as $\ell \rightarrow 0$. By the decomposition of E_{ren} in (1.10) and the continuity of \mathcal{G} in Theorem 1.15 (ii), the pointwise convergence $E_{ren}(v_{\ell}, F_{\ell}; \cdot) \rightarrow E_{ren}(v, F; \cdot)$ over \mathcal{W} is immediate. We proceed with the proof of the remaining statements.

(i) $(\Gamma - \lim \inf)$: Without loss of generality, we may assume that

 $\liminf_{\ell \to 0} E_{ren}(v_{\ell}, F_{\ell}; w^{\ell}) = \limsup_{\ell \to 0} E_{ren}(v_{\ell}, F_{\ell}; w^{\ell}) < \infty.$

As $\{(\xi_{\ell}, v_{\ell}, F_{\ell})\}_{\ell}$ is uniformly bounded in \mathcal{T} , the coercivity Theorem 1.15 (i) implies via (3.2) the existence of a constant C > 0 (uniform in ℓ) such that $E_{ren}(v_{\ell}, F_{\ell}; w^{\ell}) \geq \frac{1}{2}\mathcal{E}(w^{\ell}) - C$, i.e., $\limsup_{\ell \to 0} \mathcal{E}(w^{\ell}) < \infty$. The desired inequality is a consequence of (1.10) combined with the continuity of \mathcal{G} (Theorem 1.15 (ii)) and the lower semicontinuity of \mathcal{E} over \mathcal{W} in (3.3).

- (ii) $(\Gamma \limsup)$: For $w \in W$, one sets $w_{\ell} = w$ for all $\ell \in (0, 1]$ and the conclusion follows by the pointwise convergence of $E_{ren}(v_{\ell}, F_{\ell}; \cdot)$ to $E_{ren}(v, F; \cdot)$.
- (iii) (Convergence of minimizers): Let $\{w_\ell\}_{\ell\downarrow 0} \subset W$ be a sequence of minimizers of the sequence of functionals $\{E_{ren}(v_\ell, F_\ell; \cdot)\}_{\ell\downarrow 0}$ (the existence of minimizers follows from Theorem 1.15 (iv)). As $\{(\xi_\ell, v_\ell, F_\ell)\}_\ell$ is uniformly bounded in \mathcal{T} as $\ell \to 0$, the coercivity Theorem 1.15 (i) implies via (3.2) the existence of a constant C > 0 (uniform in ℓ) such that for all ℓ

$$0 = E_{ren}(v_{\ell}, F_{\ell}, 0) \ge E_{ren}(v_{\ell}, F_{\ell}; w^{\ell}) \ge \frac{1}{2} \mathcal{E}(w^{\ell}) - C.$$

This implies that $\{w_\ell\}_{\ell\downarrow 0}$ belongs to the sublevel set 2*C* of the energy \mathcal{E} . Hence, by Lemma B.5, there exists $w \in \mathcal{W}$ such that, upon a subsequence, $w_\ell \to w$ strongly in L^2 . Moreover, w is a minimizer of $E_{ren}(v, F; \cdot)$ over \mathcal{W} because for every $w_0 \in \mathcal{W}$, by the Γ – lim inf inequality and the pointwise convergence of $E_{ren}(v_\ell, F_\ell; \cdot)$ to $E_{ren}(v, F; \cdot)$, we have

$$E_{ren}(v, F; w) \leq \liminf_{\ell \to 0} E_{ren}(v_{\ell}, F_{\ell}; w^{\ell}) \leq \limsup_{\ell \to 0} E_{ren}(v_{\ell}, F_{\ell}; w^{\ell})$$
$$\leq \limsup_{\ell \to 0} E_{ren}(v_{\ell}, F_{\ell}; w_{0}) = E_{ren}(v, F; w_{0}).$$

Choosing $w_0 = w$ in the above relation, we deduce that

$$E_{ren}(v,F;w) = \lim_{\ell \to 0} E_{ren}(v_\ell,F_\ell;w^\ell).$$

4. A priori estimate for minimizers in Hölder spaces

In this section we prove an a priori estimate for minimizers of the renormalized energy E_{ren} , as stated in Proposition 1.17. We first need the following proposition.

Proposition 4.1. There exists C > 0 such that for every $w \in W$ and periodic distribution f,

$$[wf]_{-\frac{3}{4}} \le C\mathcal{H}(w)^{\frac{1}{2}}[f]_{-\frac{1}{4}}$$

Proof. CASE 1 $(f \in L^2 \cap C^{-\frac{1}{2}}(\mathbb{T}^2))$. Since $w \in W$, the product wf belongs to $L^1(\mathbb{T}^2)$. We estimate $[wf]_{-\frac{3}{4}}$ via (A.1) by studying the blow-up of $||(wf)_T||_{L^{\infty}}$ for $T \in (0, 1]$. We use the "telescopic" decomposition

$$(wf)_{T} = (wf_{\frac{T}{2}})_{\frac{T}{2}} + \sum_{k \ge 2, \ t = \frac{T}{2^{k}}} \left((wf_{t})_{T-t} - (wf_{2t})_{T-2t} \right).$$

$$(4.1)$$

STEP 1 (Bound on $\|(wf_{\frac{T}{2}})_{\frac{T}{2}}\|_{L^{\infty}}$): For p = 10, Young's inequality for convolution in Remark 1.12, Lemma B.4, (2.15) and (A.2) yield for every $T \in (0, 1]$,

$$\begin{split} \|(wf_{\frac{T}{2}})_{\frac{T}{2}}\|_{L^{\infty}} &\lesssim \left(T^{\frac{1}{3}}\right)^{-\frac{5}{2p}} \|wf_{\frac{T}{2}}\|_{L^{p}} \lesssim \left(T^{\frac{1}{3}}\right)^{-\frac{1}{4}} \|w\|_{L^{10}} \|f_{\frac{T}{2}}\|_{L^{\infty}} \\ &\lesssim \left(T^{\frac{1}{3}}\right)^{-\frac{1}{4}-\frac{1}{2}} \mathcal{H}(w)^{\frac{1}{2}}[f]_{-\frac{1}{2}} = \left(T^{\frac{1}{3}}\right)^{-\frac{3}{4}} \mathcal{H}(w)^{\frac{1}{2}}[f]_{-\frac{1}{2}}. \end{split}$$

STEP 2 (Bound on the telescopic sum): By Young's inequality for convolution in Remark 1.12 and Lemma 4.2 (see below), we obtain via (A.2) for every $T \in (0, 1]$,

$$\begin{split} \left\| \sum_{k \ge 2, t = \frac{T}{2^{k}}} \left((wf_{t})_{T-t} - (wf_{2t})_{T-2t} \right) \right\|_{L^{\infty}} \\ & \le \sum_{k \ge 2, t = \frac{T}{2^{k}}} \left\| \left((wf_{t})_{t} - wf_{2t} \right)_{T-2t} \right\|_{L^{\infty}} \lesssim \sum_{k \ge 2, t = \frac{T}{2^{k}}} \left((T-2t)^{\frac{1}{3}} \right)^{-\frac{5}{4}} \| (wf_{t})_{t} - wf_{2t} \|_{L^{2}} \\ & \lesssim \left(T^{\frac{1}{3}} \right)^{-\frac{5}{4}} \sum_{k \ge 2, t = \frac{T}{2^{k}}} t^{\frac{1}{3}} \mathcal{H}(w)^{\frac{1}{2}} \| f_{t} \|_{L^{\infty}} \lesssim \left(T^{\frac{1}{3}} \right)^{-\frac{5}{4}} \sum_{k \ge 2, t = \frac{T}{2^{k}}} (t^{\frac{1}{3}})^{\frac{1}{2}} \mathcal{H}(w)^{\frac{1}{2}} [f]_{-\frac{1}{2}} \\ & \lesssim \left(T^{\frac{1}{3}} \right)^{-\frac{3}{4}} \mathcal{H}(w)^{\frac{1}{2}} [f]_{-\frac{1}{2}}. \end{split}$$

STEP 3 (Hölder regularity): By STEP 1 and STEP 2 we know that for every $T \in (0, 1]$,

$$\|(wf)_T\|_{L^{\infty}} \lesssim \left(T^{\frac{1}{3}}\right)^{-\frac{3}{4}} \mathcal{H}(w)^{\frac{1}{2}}[f]_{-\frac{1}{2}},$$

which combined with (A.1) completes the proof.

CASE 2 $(f \in C^{-\frac{1}{2}}(\mathbb{T}^2))$. We consider an arbitrary approximation $f_{\ell} \in L^2 \cap C^{-\frac{1}{2}}(\mathbb{T}^2)$ of f with respect to $[\cdot]_{-\frac{1}{2}}$. By CASE 1 we deduce that wf_{ℓ} is a Cauchy sequence in $C^{-\frac{3}{4}}$, therefore it converges to the product wf by the same argument as in [IO19, Lemma 6].

Lemma 4.2. There exists a constant C > 0 such that for every $t \in (0, 1]$, $w \in W$ and periodic distribution f,

$$\|(wf_t)_t - wf_{2t}\|_{L^2} \le Ct^{\frac{1}{3}}\mathcal{H}(w)^{\frac{1}{2}}\|f_t\|_{L^{\infty}}.$$

Proof. We start with the identity

$$((wf_t)_t - wf_{2t})(x) = \int_{\mathbb{R}^2} \psi_t(y)(w(x-y) - w(x))f_t(x-y)\,dy, \quad x \in \mathbb{T}^2.$$

By Minkowski's inequality, we deduce

$$\begin{split} \|(wf_t)_t - wf_{2t}\|_{L^2} &\leq \|f_t\|_{L^{\infty}} \int_{\mathbb{R}^2} |\psi_t(y)| \|w(\cdot - y) - w(\cdot)\|_{L^2} \, \mathrm{d}y \\ &\leq \|f_t\|_{L^{\infty}} \int_{\mathbb{R}^2} |\psi_t(y)| \|\partial_1^{-y_1} w(x_1, x_2 - y_2)\|_{L^2_x} \, \mathrm{d}y \\ &+ \|f_t\|_{L^{\infty}} \int_{\mathbb{R}^2} |\psi_t(y)| \|\partial_2^{-y_2} w\|_{L^2} \, \mathrm{d}y, \end{split}$$

where we used that $w(x - y) - w(x) = \partial_1^{-y_1} w(x_1, x_2 - y_2) + \partial_2^{-y_2} w(x_1, x_2)$ for every $x \in \mathbb{T}^2$ and $y \in \mathbb{R}^2$.

The first integral can be estimated using the mean value theorem and translation invariance of the torus by

$$\int_{\mathbb{R}^2} |\psi_t(y)| \left(\int_{\mathbb{T}^2} |\partial_1^{-y_1} w(x_1, x_2 - y_2)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \, \mathrm{d}y \le \int_{\mathbb{R}^2} |y_1 \psi_t(y)| \mathrm{d}y \, \|\partial_1 w\|_{L^2} \lesssim t^{\frac{1}{3}} \, \|\partial_1 w\|_{L^2},$$

since by Step 1 in [IO19, proof of Lemma 10] we know that $y \mapsto y_1 \psi(y) \in L^1(\mathbb{R}^2)$.

The second integral can be estimated using the Cauchy-Schwarz inequality and (2.2) by

$$\int_{\mathbb{R}^2} |\psi_t(y)| \left(\int_{\mathbb{T}^2} |\partial_2^{-y_2} w(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \, \mathrm{d}y$$

$$= (t^{\frac{1}{3}})^{\frac{3}{2}\left(\frac{2}{3}+\frac{1}{2}\right)} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{y_2}{(t^{\frac{1}{3}})^{\frac{3}{2}}} \right|^{\frac{2}{3}+\frac{1}{2}} |\psi_t(y)| \, \mathrm{d}y_1 \left(\int_{\mathbb{T}^2} \frac{|\partial_2^{-y_2}w(x)|^2}{|y_2|^{\frac{4}{3}}} \, \mathrm{d}x \right)^{\frac{1}{2}} \frac{\mathrm{d}y_2}{|y_2|^{\frac{1}{2}}} \\ \lesssim (t^{\frac{1}{3}})^{\frac{3}{2}\left(\frac{2}{3}+\frac{1}{2}\right)} \left\| \int_{\mathbb{R}} \left| \frac{y_2}{(t^{\frac{1}{3}})^{\frac{3}{2}}} \right|^{\frac{2}{3}+\frac{1}{2}} |\psi_t(y)| \, \mathrm{d}y_1 \right\|_{L^2_{y_2}(\mathbb{R})} \left\| |\partial_2|^{\frac{2}{3}} w \|_{L^2} \lesssim t^{\frac{1}{3}} \| |\partial_2|^{\frac{2}{3}} w \|_{L^2},$$

where we also used Minkowski's inequality and a change of variables to deduce that

$$\left\| \int_{\mathbb{R}} \left| \frac{y_2}{(t^{\frac{1}{3}})^{\frac{3}{2}}} \right|^{\frac{2}{3} + \frac{1}{2}} |\psi_t(y)| \, \mathrm{d}y_1 \right\|_{L^2_{y_2}(\mathbb{R})} \le \frac{1}{(t^{\frac{1}{3}})^{\frac{3}{4}}} \left\| \||y_2|^{\frac{2}{3} + \frac{1}{2}} \psi(y_1, y_2)\|_{L^2_{y_2}(\mathbb{R})} \right\|_{L^1_{y_1}(\mathbb{R})}, \tag{4.2}$$

along with the fact that $y_1 \mapsto ||y_2|^{\frac{2}{3} + \frac{1}{2}} \psi(y_1, y_2)||_{L^2_{y_2}(\mathbb{R})} \in L^1_{y_1}(\mathbb{R})$. ²⁹

Combining the previous estimates with (2.15) implies the desired bound.

We are now ready to prove Proposition 1.17.

Proof of Proposition 1.17. By Theorem 1.15 (iv) if $w \in W$ is a minimizer of $E_{ren}(v, F; \cdot)$, then w is a weak solution to (1.13). By the Schauder theory for the operator $\mathcal L$ (see [IO19, Lemma 5]), if $w \in \mathcal{W} \cap C^{\frac{5}{4}-2\epsilon}$ satisfies (1.13), we have that

$$[w]_{\frac{5}{4}-2\varepsilon} \lesssim \left[P(F+wR_1\partial_2 v+vR_1\partial_2 w+wR_1\partial_2 w) -\frac{1}{2}(v+w)R_1\partial_1(v+w)^2 +\frac{1}{2}\partial_2 R_1(v+w)^2 \right]_{-\frac{3}{4}-2\varepsilon}.$$
(4.3)

We estimate each term on the right-hand side of (4.3) separately. The idea is to bound any term containing *w* in the seminorm $[\cdot]_{-\frac{3}{4}-2\varepsilon}$ by a product $\mathcal{H}(w)^{\gamma(\theta)}[w]_{\frac{5}{4}-2\varepsilon}^{\theta}$ with $\theta \in (0, 1)$ and $\gamma(\theta) > 0$. To prove the statement, the main tools are

- the interpolation inequality in Lemma A.2,
- Lemma B.8 which yields that $\mathcal{H}(w)$ controls $[w]_{-\frac{1}{2}}^2$,
- [IO19, Lemmata 6 and 12] stating that for a distribution $f \in C^{\beta}$, $\beta \in (-\frac{3}{2}, 0) \setminus \{-1, -\frac{1}{2}\}$, and two functions $g \in C^{\gamma}$, $\tilde{g} \in C^{\tilde{\gamma}}$ with $\gamma, \tilde{\gamma} \in (0, \frac{3}{2})$ both of vanishing average, provided that $\beta + \gamma > 0$ and $\tilde{\gamma} \geq \gamma$, the following estimates hold

$$[fg]_{\beta} \leq [f]_{\beta}[g]_{\gamma} \text{ and } [g\tilde{g}]_{\gamma} \leq [g]_{\gamma}[\tilde{g}]_{\tilde{\gamma}}.$$

Also, we use that $C^{\alpha} \subset C^{\beta}$ for any $-\frac{3}{2} < \beta < \alpha < \frac{3}{2}$ with $\alpha, \beta \neq 0$, see [IO19, Remark 2], and Lemma A.4 which implies that the Hilbert transform reduces the regularity by ε on Hölder spaces.

For $[\xi]_{-\frac{5}{4}-\epsilon}, [v]_{\frac{3}{4}-\epsilon}, [F]_{-\frac{3}{4}-\epsilon} \leq M$, the following estimates hold with an implicit constant depending on M and ε .

1. Terms independent of w: First, we notice that $[PF]_{-\frac{3}{4}-2\varepsilon} \leq [F]_{-\frac{3}{4}-2\varepsilon} \leq [F]_{-\frac{3}{4}-\varepsilon} \leq 1$. Also, by Definition 1.14, we have that

$$\begin{split} & [\partial_2 R_1 v^2]_{-\frac{3}{4} - 2\epsilon} \lesssim [R_1 v^2]_{\frac{3}{4} - 2\epsilon} \lesssim [v^2]_{\frac{3}{4} - \epsilon} \lesssim [v]_{\frac{3}{4} - \epsilon}^2 \lesssim 1, \\ & [v\partial_1 R_1 v^2]_{-\frac{3}{4} - 2\epsilon} \lesssim [v\partial_1 R_1 v^2]_{-\frac{1}{4} - 2\epsilon} \lesssim [v]_{\frac{3}{4} - \epsilon} [\partial_1 R_1 v^2]_{-\frac{1}{4} - 2\epsilon} \lesssim [v]_{\frac{3}{4} - \epsilon} [v^2]_{\frac{3}{4} - \epsilon} \lesssim [v]_{\frac{3}{4} - \epsilon}^3 \lesssim 1. \end{split}$$

²⁹ This follows easily from the bound

$$\left\| \| \|y_2\|^{\frac{2}{3} + \frac{1}{2}} \psi(y_1, y_2) \|_{L^2_{y_2}(\mathbb{R})} \right\|_{L^1_{y_1}(\mathbb{R})} \lesssim \|(1 + |y_1|)(1 + |y_2|^2) \psi(y)\|_{L^2_y(\mathbb{R}^2)}$$

and Plancherel's identity, using that $\widehat{\psi}(k) = e^{-|k_1|^3 - k_2^2}$, see also Step 1 in [IO19, proof of Lemma 10] and Footnote 34.

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2. Linear terms in w: By the interpolation estimate in Lemma A.2 and Lemma B.8, we have

$$[w]_{\frac{3}{4}+3\varepsilon} \lesssim [w]_{-\frac{1}{4}}^{\frac{1}{3}-2\kappa_{1}} [w]_{\frac{5}{4}-2\varepsilon}^{\frac{2}{3}+2\kappa_{1}} \lesssim \mathcal{H}(w)^{\frac{1}{6}-\kappa_{1}} [w]_{\frac{5}{4}-2\varepsilon}^{\frac{2}{3}+2\kappa_{1}},$$
(4.4)

$$[w]_{\frac{1}{4}+3\varepsilon} \lesssim [w]_{-\frac{1}{4}}^{\frac{2}{3}-2\kappa_2} [w]_{\frac{5}{4}-2\varepsilon}^{\frac{1}{3}+2\kappa_2} \lesssim \mathcal{H}(w)^{\frac{1}{3}-\kappa_2} [w]_{\frac{5}{4}-2\varepsilon}^{\frac{1}{3}+2\kappa_2},$$
(4.5)

where $\kappa_1, \kappa_2 > 0$ are small (as functions of ε) for $\varepsilon > 0$ small enough. This yields

$$\begin{split} \left[w\partial_{2}R_{1}v\right]_{-\frac{3}{4}-2\varepsilon} &\leq \left[w\right]_{\frac{3}{4}+3\varepsilon} \left[\partial_{2}R_{1}v\right]_{-\frac{3}{4}-2\varepsilon} &\leq \mathcal{H}(w)^{\frac{1}{6}-\kappa_{1}} \left[w\right]_{\frac{5}{4}-2\varepsilon}^{\frac{2}{3}+2\kappa_{1}} \left[v\right]_{\frac{3}{4}-\varepsilon} \\ &\leq \mathcal{H}(w)^{\frac{1}{6}-\kappa_{1}} \left[w\right]_{\frac{5}{4}-2\varepsilon}^{\frac{2}{3}+2\kappa_{1}}, \\ \left[w\partial_{1}R_{1}v^{2}\right]_{-\frac{3}{4}-2\varepsilon} &\leq \left[w\partial_{1}R_{1}v^{2}\right]_{-\frac{1}{4}-2\varepsilon} &\leq \left[w\right]_{\frac{1}{4}+3\varepsilon} \left[\partial_{1}R_{1}v^{2}\right]_{-\frac{1}{4}-2\varepsilon} \\ &\leq \mathcal{H}(w)^{\frac{1}{3}-\kappa_{2}} \left[w\right]_{\frac{5}{4}-2\varepsilon}^{\frac{1}{3}+2\kappa_{2}} \left[v\right]_{\frac{3}{4}-\varepsilon}^{2} &\leq \mathcal{H}(w)^{\frac{1}{3}-\kappa_{2}} \left[w\right]_{\frac{5}{4}-2\varepsilon}^{\frac{1}{3}+2\kappa_{2}}, \\ \left[v\partial_{2}R_{1}w\right]_{-\frac{3}{4}-2\varepsilon} &\leq \left[v\partial_{2}R_{1}w\right]_{-\frac{3}{4}+2\varepsilon} &\leq \left[v\right]_{\frac{3}{4}-\varepsilon} \left[\partial_{2}R_{1}w\right]_{-\frac{3}{4}+2\varepsilon} &\leq \left[w\right]_{\frac{3}{4}+3\varepsilon} \\ &\leq \mathcal{H}(w)^{\frac{1}{6}-\kappa_{1}} \left[w\right]_{\frac{5}{4}-2\varepsilon}^{\frac{2}{3}+2\kappa_{1}}, \\ \left[\partial_{2}R_{1}(vw)\right]_{-\frac{3}{4}-2\varepsilon} &\leq \left[vw\right]_{\frac{3}{4}-\varepsilon} &\leq \left[v\right]_{\frac{3}{4}-\varepsilon} \left[w\right]_{\frac{3}{4}+3\varepsilon} &\leq \mathcal{H}(w)^{\frac{1}{6}-\kappa_{1}} \left[w\right]_{\frac{5}{4}-2\varepsilon}^{\frac{2}{3}+2\kappa_{1}}, \\ \left[v\partial_{1}R_{1}(vw)\right]_{-\frac{3}{4}-2\varepsilon} &\leq \left[v\partial_{1}R_{1}(vw)\right]_{-\frac{1}{4}-2\varepsilon} &\leq \left[v\right]_{\frac{3}{4}-\varepsilon} \left[\partial_{1}R_{1}(vw)\right]_{-\frac{1}{4}-2\varepsilon} &\leq \left[vw\right]_{\frac{3}{4}-\varepsilon} \\ &\leq \left[v\right]_{\frac{3}{4}-\varepsilon} \left[w\right]_{\frac{3}{4}+3\varepsilon} &\leq \mathcal{H}(w)^{\frac{1}{6}-\kappa_{1}} \left[w\right]_{\frac{5}{4}-2\varepsilon}^{\frac{2}{3}+2\kappa_{1}}. \end{split}$$

3. Quadratic terms in w:

(a) We start with the term $\partial_2 R_1 w^2$. By the interpolation estimate in Lemma A.2 for $\gamma = 0$ and Lemma B.8, we have

$$\|w\|_{L^{\infty}} \lesssim [w]_{-\frac{1}{4}}^{\frac{5}{6}-2\kappa_{3}} [w]_{\frac{5}{4}-2\varepsilon}^{\frac{1}{6}+2\kappa_{3}} \lesssim \mathcal{H}(w)^{\frac{5}{12}-\kappa_{3}} [w]_{\frac{5}{4}-2\varepsilon}^{\frac{1}{6}+2\kappa_{3}},$$
(4.6)

where $\kappa_3 > 0$ is small for $\varepsilon > 0$ small enough. Together with (4.4), it follows that

$$\begin{split} [\partial_2 R_1 w^2]_{-\frac{3}{4}-2\varepsilon} &\lesssim [\partial_2 R_1 w^2]_{-\frac{3}{4}+2\varepsilon} \lesssim [w^2]_{\frac{3}{4}+3\varepsilon} \lesssim \|w\|_{L^{\infty}} [w]_{\frac{3}{4}+3\varepsilon} \\ &\lesssim \mathcal{H}(w)^{\frac{7}{12}-(\kappa_1+\kappa_3)} [w]_{\frac{5}{4}-2\varepsilon}^{\frac{5}{4}+2(\kappa_1+\kappa_3)}. \end{split}$$

Similarly, we estimate

$$\begin{split} \left[v\partial_1 R_1 w^2 \right]_{-\frac{3}{4} - 2\varepsilon} &\lesssim \left[v\partial_1 R_1 w^2 \right]_{-\frac{1}{4} + 2\varepsilon} \lesssim \left[v \right]_{\frac{3}{4} - \varepsilon} \left[\partial_1 R_1 w^2 \right]_{-\frac{1}{4} + 2\varepsilon} \lesssim \left[w^2 \right]_{\frac{3}{4} + 3\varepsilon} \\ &\lesssim \mathcal{H}(w)^{\frac{7}{12} - (\kappa_1 + \kappa_3)} \left[w \right]_{\frac{5}{4} - 2\varepsilon}^{\frac{5}{4} - 2\varepsilon} \end{split}$$

(b) The term $w \partial_2 R_1 w$ is treated via Proposition 4.1,

$$[w\partial_{2}R_{1}w]_{-\frac{3}{4}} \lesssim \mathcal{H}(w)^{\frac{1}{2}}[\partial_{2}R_{1}w]_{-\frac{1}{2}} \lesssim \mathcal{H}(w)^{\frac{1}{2}}[R_{1}w]_{1} \lesssim \mathcal{H}(w)^{\frac{1}{2}}[w]_{1+\varepsilon}.$$

Then, Lemma A.2 and Lemma B.8 yield

$$[w]_{1+\varepsilon} \lesssim [w]_{-\frac{1}{4}}^{\frac{1}{6}-2\kappa_4} [w]_{\frac{5}{4}-2\varepsilon}^{\frac{5}{6}+2\kappa_4} \lesssim \mathcal{H}(w)^{\frac{1}{12}-\kappa_4} [w]_{\frac{5}{4}-2\varepsilon}^{\frac{5}{6}+2\kappa_4},$$

where $\kappa_4 > 0$ is small for $\varepsilon > 0$ small enough. Hence we have

$$[w\partial_2 R_1w]_{-\frac{3}{4}-2\varepsilon} \lesssim [w\partial_2 R_1w]_{-\frac{3}{4}} \lesssim \mathcal{H}(w)^{\frac{7}{12}-\kappa_4}[w]_{\frac{5}{4}-2\varepsilon}^{\frac{5}{6}+2\kappa_4}.$$

(c) We decompose $wR_1\partial_1(vw)$ into $wR_1(w\partial_1v) + wR_1(v\partial_1w)$ and treat each term separately. By Proposition 4.1, we have

$$\begin{split} \left[wR_{1}(v\partial_{1}w)\right]_{-\frac{3}{4}} &\lesssim \mathcal{H}(w)^{\frac{1}{2}}\left[R_{1}(v\partial_{1}w)\right]_{-\frac{1}{2}} \lesssim \mathcal{H}(w)^{\frac{1}{2}}\left[R_{1}(v\partial_{1}w)\right]_{-\frac{1}{2}+\varepsilon} \\ &\lesssim \mathcal{H}(w)^{\frac{1}{2}}\left[v\partial_{1}w\right]_{-\frac{1}{2}+2\varepsilon} \lesssim \mathcal{H}(w)^{\frac{1}{2}}\left[v\right]_{\frac{3}{4}-\varepsilon}\left[\partial_{1}w\right]_{-\frac{1}{2}+2\varepsilon} \lesssim \mathcal{H}(w)^{\frac{1}{2}}\left[w\right]_{\frac{1}{2}+2\varepsilon}. \end{split}$$

Again, Lemma A.2 and Lemma B.8 yield

$$[w]_{\frac{1}{2}+2\varepsilon} \lesssim [w]_{-\frac{1}{4}}^{\frac{1}{2}-2\kappa_{5}} [w]_{\frac{5}{4}-2\varepsilon}^{\frac{1}{2}+2\kappa_{5}} \lesssim \mathcal{H}(w)^{\frac{1}{4}-\kappa_{5}} [w]_{\frac{5}{4}-2\varepsilon}^{\frac{1}{2}+2\kappa_{5}},$$
(4.7)

where $\kappa_5 > 0$ is small for $\varepsilon > 0$ small enough. Hence we obtain that

$$[wR_1(v\partial_1w)]_{-\frac{3}{4}-2\varepsilon} \lesssim [wR_1(v\partial_1w)]_{-\frac{3}{4}} \lesssim \mathcal{H}(w)^{\frac{3}{4}-\kappa_5}[w]_{\frac{5}{4}-2\varepsilon}^{\frac{1}{2}+2\kappa_5}.$$

The term $wR_1(w\partial_1 v)$ can be estimated using (4.5) as

$$\begin{split} [wR_1(w\partial_1 v)]_{-\frac{3}{4}-2\varepsilon} &\lesssim [wR_1(w\partial_1 v)]_{-\frac{1}{4}-2\varepsilon} \lesssim [w]_{\frac{1}{4}+3\varepsilon} [R_1(w\partial_1 v)]_{-\frac{1}{4}-2\varepsilon} \\ &\lesssim [w]_{\frac{1}{4}+3\varepsilon} [w\partial_1 v]_{-\frac{1}{4}-\varepsilon} \lesssim [w]_{\frac{1}{4}+3\varepsilon}^2 [\partial_1 v]_{-\frac{1}{4}-\varepsilon} \\ &\lesssim \mathcal{H}(w)^{\frac{2}{3}-2\kappa_2} [w]_{\frac{5}{4}-2\varepsilon}^{\frac{2}{3}+4\kappa_2} [v]_{\frac{3}{4}-\varepsilon} \lesssim \mathcal{H}(w)^{\frac{2}{3}-2\kappa_2} [w]_{\frac{5}{4}-2\varepsilon}^{\frac{2}{3}+4\kappa_2} \end{split}$$

4. Cubic term in w: The cubic term $w \partial_1 R_1 w^2$ is treated by Proposition 4.1 which yields

$$\begin{split} [w\partial_{1}R_{1}w^{2}]_{-\frac{3}{4}} &\lesssim \mathcal{H}(w)^{\frac{1}{2}}[R_{1}\partial_{1}w^{2}]_{-\frac{1}{2}} \lesssim \mathcal{H}(w)^{\frac{1}{2}}[R_{1}\partial_{1}w^{2}]_{-\frac{1}{2}+\varepsilon} \lesssim \mathcal{H}(w)^{\frac{1}{2}}[\partial_{1}w^{2}]_{-\frac{1}{2}+2\varepsilon} \\ &\lesssim \mathcal{H}(w)^{\frac{1}{2}}[w^{2}]_{\frac{1}{2}+2\varepsilon} \lesssim \mathcal{H}(w)^{\frac{1}{2}} \|w\|_{L^{\infty}} [w]_{\frac{1}{2}+2\varepsilon}. \end{split}$$

By (4.6) and (4.7) we have that

$$\|w\|_{L^{\infty}}[w]_{\frac{1}{2}+2\varepsilon} \lesssim \mathcal{H}(w)^{\frac{2}{3}-(\kappa_{3}+\kappa_{5})}[w]_{\frac{5}{4}-2\varepsilon}^{\frac{2}{3}+2(\kappa_{3}+\kappa_{5})},$$

which in turn implies that

$$[w\partial_{1}R_{1}w^{2}]_{-\frac{3}{4}-2\varepsilon} \lesssim [w\partial_{1}R_{1}w^{2}]_{-\frac{3}{4}} \lesssim \mathcal{H}(w)^{\frac{3}{2}-(\kappa_{3}+\kappa_{5})}[w]_{\frac{5}{4}-2\varepsilon}^{\frac{2}{3}+2(\kappa_{3}+\kappa_{5})}.$$

Summing up, Young's inequality yields the bound

$$[w]_{\frac{5}{4}-2\varepsilon} \le C(1+\mathcal{H}(w))^N,\tag{4.8}$$

for some $N \ge 1$, and by our estimates it is clear that the constant *C* depends polynomially on *M*. To conclude, using the fact that *w* is a minimizer of $E_{ren}(v, F; \cdot)$, we have that $E_{ren}(v, F; w) \le E_{ren}(v, F; 0) = 0$. Since $E_{ren}(v, F; \cdot) = \mathcal{E} + \mathcal{G}(v, F; \cdot)$ and by Theorem 1.15 (i) we know that $|\mathcal{G}(v, F; w)| \le \frac{1}{2}\mathcal{E}(w) + C$ for some constant *C* which also depends polynomially on *M*, we obtain that $\mathcal{E}(w) \le 2C$. By (2.14), this implies that $H(w) \le C$ for some constant *C* which depends polynomially on *M* and combining with (4.8) we obtain the desired bound.

5. Approximations to white noise under the spectral gap assumption

The main goal of this section is to prove Proposition 1.9. Given a probability measure $\langle \cdot \rangle$ which satisfies Assumption 1.1, we prove that $\langle \cdot \rangle$ is concentrated on $C^{-\frac{5}{4}-}$, thus, by Schauder theory for the operator $\mathcal{L}, v := \mathcal{L}^{-1}P\xi \in C^{\frac{3}{4}-} \langle \cdot \rangle$ -almost surely, and we construct F as the $L^p_{\langle \cdot \rangle}C^{-\frac{3}{4}-}$ -limit of the sequence $\{v\partial_2 R_1 v_t\}_{t=2^{-n}\downarrow 0}$. This will allow us to lift $\langle \cdot \rangle$ to a probability measure $\langle \cdot \rangle^{\text{lift}}$ on $C^{-\frac{5}{4}-} \times C^{\frac{3}{4}-} \times C^{-\frac{3}{4}-}$ in a continuous way, i.e., given a sequence of probability measures $\{\langle \cdot \rangle_\ell\}_{\ell\downarrow 0}$ which satisfy Assumption 1.1 and converge weakly to a limit $\langle \cdot \rangle$ as $\ell \downarrow 0$, then $\{\langle \cdot \rangle_\ell^{\text{lift}}\}_{\ell\downarrow 0}$ converges weakly to $\langle \cdot \rangle^{\text{lift}}$.

The proof of Proposition 1.9 is based on suitable estimates on the *p*-moments of multilinear expressions in the corresponding stochastic objects. In the case of Gaussian approximations, it is enough to bound the second moments, since we can use Nelson's hypercontractivity estimate

to bound the *p*-moments in a finite Wiener chaos by the second moments for every p > 2 (see [IO19, Lemmata 4 and 8]). On the other hand, for non-Gaussian approximations one has to find alternative methods to estimate the *p*-moments. This has been achieved with great success in the last few years under very mild assumptions on the random field, see for example [HS17, CH16]. In these works a direct computation of the *p*-moments is made through explicit formulas in terms of the cumulant functions of the random field and the final bounds are obtained by combinatorial arguments.

Here we are interested in approximations of white noise that satisfy the spectral gap inequality (1.7), uniformly in the approximation parameter ℓ . This covers the Gaussian case, but it allows for more general random fields. The basic observation is that one can bound the *p*-moments directly by estimating the derivative with respect to the noise, based on the following consequence of the spectral gap assumption (1.7).

Proposition 5.1. The spectral gap inequality (1.7) implies

$$\left\langle |G(\xi) - \langle G(\xi) \rangle|^{2p} \right\rangle^{\frac{1}{2p}} \le C(p) \left\langle \left\| \frac{\partial}{\partial \xi} G(\xi) \right\|_{L^2}^{2p} \right\rangle^{\frac{1}{2p}}, \tag{5.1}$$

for every $1 \le p < \infty$ and every functional *G* on periodic Schwartz distributions which can be approximated by cylindrical functionals with respect to the norm $\langle |G(\xi)|^{2p} \rangle^{\frac{1}{2p}} + \langle \| \frac{\partial}{\partial \xi} G(\xi) \|_{L^2}^{2p} \rangle^{\frac{1}{2p}}$, where the constant C(p) > 0 depends only on *p*.

Remark 5.2. As in Remark 1.3, using (5.1) we can extend $\xi(\varphi)$ for $\varphi \in L^2(\mathbb{T}^2)$ as a centered random variable in $L^{2p}_{(\cdot)}$, admissible in (5.1) for any $1 \le p < \infty$.

Proof. The proof follows [JO20, Lemma 3.1]. Let p > 1. We assume that *G* is a cylindrical functional on periodic Schwartz distributions of finite norm $\langle |G(\xi)|^{2p} \rangle^{\frac{1}{2p}} + \langle ||\frac{\partial}{\partial \xi}G(\xi)||_{L^2}^{2p} \rangle^{\frac{1}{2p}}$. The general case follows by approximation. Without loss of generality we can assume that $\langle G(\xi) \rangle = 0$. For $\lambda \in (0, 1]$ we consider the functional

$$F_{\lambda}(\xi) := (G(\xi)^2 + \lambda^2)^{\frac{P}{2}}.$$

Noting that $\frac{\partial}{\partial\xi}F_{\lambda}(\xi) = pF_{\lambda}(\xi)^{\frac{p-2}{p}}G(\xi)\frac{\partial}{\partial\xi}G(\xi)$ and using Hölder's inequality, (1.7) applied to F_{λ} yields

$$\left\langle \left| F_{\lambda}(\xi) - \left\langle F_{\lambda}(\xi) \right\rangle \right|^{2} \right\rangle \leq p^{2} \left\langle F_{\lambda}(\xi)^{\frac{2(p-2)}{p}} G(\xi)^{2} \left\| \frac{\partial}{\partial \xi} G(\xi) \right\|_{L^{2}}^{2} \right\rangle \leq p^{2} \left\langle F_{\lambda}(\xi)^{\frac{2(p-1)}{p}} \left\| \frac{\partial}{\partial \xi} G(\xi) \right\|_{L^{2}}^{2} \right\rangle$$

$$\leq p^{2} \left\langle F_{\lambda}(\xi)^{2} \right\rangle^{\frac{p-1}{p}} \left\langle \left\| \frac{\partial}{\partial \xi} G(\xi) \right\|_{L^{2}}^{2p} \right\rangle^{\frac{1}{p}}.$$

$$(5.2)$$

By Cauchy–Schwarz we see that

$$\langle F_{\lambda}(\xi) \rangle^2 \leq \left\langle (G(\xi)^2 + \lambda^2)^{p-1} \right\rangle \left\langle G(\xi)^2 + \lambda^2 \right\rangle$$

By Hölder's inequality and (1.7) applied to $G(\xi)$ (recall that $\langle G(\xi) \rangle = 0$) the two terms on the right hand side can be estimated as

$$\left\langle (G(\xi)^2 + \lambda^2)^{p-1} \right\rangle \leq \left\langle F_{\lambda}(\xi)^2 \right\rangle^{\frac{p-1}{p}},$$

$$\left\langle G(\xi)^2 + \lambda^2 \right\rangle \leq \left\langle \left\| \frac{\partial}{\partial \xi} G(\xi) \right\|_{L^2}^2 \right\rangle + \lambda^2 \leq \left\langle \left\| \frac{\partial}{\partial \xi} G(\xi) \right\|_{L^2}^{2p} \right\rangle^{\frac{1}{p}} + \lambda^2.$$

Combining with (5.2) we get

$$\left\langle F_{\lambda}(\xi)^{2} \right\rangle \lesssim \left\langle |F_{\lambda}(\xi) - \langle F_{\lambda}(\xi) \rangle_{\ell}|^{2} \right\rangle + \left\langle F_{\lambda}(\xi) \right\rangle^{2}$$

$$\lesssim_p \left\langle F_{\lambda}(\xi)^2 \right\rangle^{\frac{p-1}{p}} \left\langle \left\| \frac{\partial}{\partial \xi} G(\xi) \right\|_{L^2}^{2p} \right\rangle^{\frac{1}{p}} + \lambda^2 \left\langle F_{\lambda}(\xi)^2 \right\rangle^{\frac{p-1}{p}}.$$

By Young's inequality we finally obtain that

$$\left\langle (G(\xi)^2 + \lambda^2)^p \right\rangle = \left\langle F_{\lambda}(\xi)^2 \right\rangle \lesssim_p \left\langle \left\| \frac{\partial}{\partial \xi} G(\xi) \right\|_{L^2}^{2p} \right\rangle + \lambda^{2p}.$$

The conclusion follows by the monotone convergence theorem as $\lambda \searrow 0$.

Proposition 5.1 allows us to estimate the *p*-moments of multilinear expressions in ξ (shifted by their expectation) by the operator norm of (random) linear functionals on $L^2(\mathbb{T}^2)$ (in the case of white noise this is the Cameron–Martin space), after taking one derivative with respect to ξ . To estimate the operator norm of these linear functionals we use the regularizing properties of \mathcal{L}^{-1} in Sobolev spaces.

5.1. Estimates on ξ and v. In this section, we prove several stochastic estimates for ξ and $v := \mathcal{L}^{-1}P\xi$ which are uniform in the class of probability measures satisfying Assumption 1.1. As a corollary, we obtain that the law of (ξ, v) is concentrated on $C^{-\frac{5}{4}-\varepsilon} \times C^{\frac{3}{4}-\varepsilon}$ (see Corollary 5.4). A similar result was proved in [IO19, Lemma 4] for $\langle \cdot \rangle$ being the law of white noise. Here we consider more general probability measures which are not necessarily Gaussian. Some of the results of this section will be used in Section 5.2 below in the construction of F.

We start with the following proposition.

Proposition 5.3. Let $\langle \cdot \rangle$ satisfy Assumption 1.1 and let $v := \mathcal{L}^{-1}P\xi$. For every $1 \le p < \infty, T \in (0, 1]$, and $y \in \mathbb{T}^2$ we have that

$$\sup_{x \in \mathbb{T}^2} \left\langle |\xi_T(x)|^p \right\rangle^{\frac{1}{p}} \le C \left(T^{\frac{1}{3}} \right)^{-\frac{3}{4}}, \tag{5.3}$$

$$\sup_{x \in \mathbb{T}^2} \left\langle |v(x-y) - v(x)|^p \right\rangle^{\frac{1}{p}} \le Cd(0,y)^{\frac{3}{4}}, \tag{5.4}$$

$$\sup_{x \in \mathbb{T}^2} \left\langle |(\partial_2 R_1 v)_T(x)|^p \right\rangle^{\frac{1}{p}} \le C \left(T^{\frac{1}{3}}\right)^{-\frac{3}{4}},\tag{5.5}$$

where the constant *C* depends only on *p*.

Proof. The proof is based on a direct application of the spectral gap inequality (5.1). In the following, for every $\delta \xi \in L^2(\mathbb{T}^2)$ we consider δv the unique solution of zero average in x_1 of $\mathcal{L}\delta v = P\delta\xi$. *Proof of* (5.3): Let $x_0 \in \mathbb{T}^2$ be fixed. We consider the linear functional $G(\xi) = \xi_T(x_0)$. Since $\xi_T = \xi * \psi_T$, by Assumption 1.1 (i) we have that $\langle \xi_T(x_0) \rangle = 0$. Then, by (5.1) applied to *G* (which is a cylindrical functional), we get that for every $q \ge 1$

$$\left\langle |\xi_T(x_0)|^{2q} \right\rangle^{\frac{1}{2q}} \lesssim \left\langle \left\| \frac{\partial}{\partial \xi} \xi_T(x_0) \right\|_{L^2}^{2q} \right\rangle^{\frac{1}{2q}}$$

It is easy to check that $\frac{\partial}{\partial\xi}\xi_T(x_0) = \Psi_T(x_0 - \cdot) \in L^2(\mathbb{T}^2)$ (with Ψ_T the periodization of ψ_T), and by Remark 1.12

$$\left\|\frac{\partial}{\partial\xi}\xi_T(x_0)\right\|_{L^2} = \|\Psi_T(x_0-\cdot)\|_{L^2} = \|\Psi_T\|_{L^2} \lesssim \left(T^{\frac{1}{3}}\right)^{-\frac{5}{4}}$$

which proves (5.3) for every $2 \le p < \infty$, as the implicit constant above does not depend on x_0 . For $p \in [1, 2)$, the conclusion then follows by Jensen's inequality.

Proof of (5.4): Let $x_0 \in \mathbb{T}^2$ be fixed. For every $y \in \mathbb{T}^2$, recalling the kernel Γ in (1.20), we consider the linear functional $G(\xi) = v(x_0 - y) - v(x_0) = \xi(\Gamma(x_0 - y - \cdot)) - \xi(\Gamma(x_0 - \cdot))$, which is well-defined

as a centered random variable in $L^{2q}_{\langle \cdot \rangle}$ for any $q \ge 1$ by Remark 5.2 and (1.21) and is admissible in (5.1). Then

$$\frac{\partial}{\partial \xi} G(\xi) : \delta \xi \mapsto \delta v(x_0 - y) - \delta v(x_0).$$

By (D.1) we know that

$$|\delta v(x_0 - y) - \delta v(x_0)| \le [\delta v]_{\frac{3}{4}} d(0, y)^{\frac{3}{4}} \le d(0, y)^{\frac{3}{4}} \|\delta \xi\|_{L^2},$$

which in turn implies that

$$\left\|\frac{\partial}{\partial\xi}G(\xi)\right\|_{L^2} \lesssim d(0,y)^{\frac{3}{4}}.$$

Thus, by (5.1) applied to *G*, we have for every $q \ge 1$,

$$\left< |v(x_0 - y) - v(x_0)|^{2q} \right>^{\frac{1}{2q}} \leq d(0, y)^{\frac{3}{4}},$$

which proves (5.4) for $2 \le p < \infty$, since the implicit constant does not depend on x_0 . For $p \in [1, 2)$, the conclusion follows by Jensen's inequality.

Proof of (5.5): Let $x_0 \in \mathbb{T}^2$ be fixed. We consider the linear functional $G(\xi) = (\partial_2 R_1 v)_T(x_0)$ (which is cylindrical). We have that

$$\frac{\partial}{\partial \xi} G(\xi) : \delta \xi \mapsto (\partial_2 R_1 \delta v)_T(x_0).$$

By Young's inequality for convolution (see Remark 1.12), (D.3), and the fact that R_1 is bounded on $L^{\frac{10}{3}}(\mathbb{T}^2)$, we get that

$$|(\partial_2 R_1 \delta v)_T(x_0)| \lesssim \left(T^{\frac{1}{3}}\right)^{-\frac{3}{4}} \|\partial_2 R_1 \delta v\|_{L^{\frac{10}{3}}} \lesssim \left(T^{\frac{1}{3}}\right)^{-\frac{3}{4}} \|\partial_2 \delta v\|_{L^{\frac{10}{3}}} \lesssim \left(T^{\frac{1}{3}}\right)^{-\frac{3}{4}} \|\delta \xi\|_{L^2},$$

yielding the estimate

$$\left\|\frac{\partial}{\partial\xi}G(\xi)\right\|_{L^2} \lesssim \left(T^{\frac{1}{3}}\right)^{-\frac{3}{4}}$$

We also note that $\langle G(\xi) \rangle = 0$. Indeed, since $\langle v(x) \rangle = \langle \xi(\Gamma(x - \cdot)) \rangle = 0$, for every $x \in \mathbb{T}^2$, we get that $\langle (\partial_2 R_1 v)_T(x_0) \rangle = 0$. The conclusion then follows as above.

As a corollary of (5.3) and the Schauder theory for the operator \mathcal{L} , we prove that the laws of ξ and v are concentrated on $C^{-\frac{5}{4}-\varepsilon}$ and $C^{\frac{3}{4}-\varepsilon}$.

Corollary 5.4. Let $\langle \cdot \rangle$ satisfy Assumption 1.1. For every $\varepsilon \in (0, \frac{1}{100})$ and $1 \le p < \infty$ there holds

$$\left\langle \left[\xi\right]_{-\frac{5}{4}-\varepsilon}^{p}\right\rangle^{\frac{1}{p}} \leq C,\tag{5.6}$$

$$\left\langle \left[v\right]_{\frac{3}{4}-\varepsilon}^{p}\right\rangle^{\frac{1}{p}} \leq C,\tag{5.7}$$

where the constant *C* depends only on *p* and ε . Moreover, $\langle v(x) \rangle = 0$ for every $x \in \mathbb{T}^2$.

Proof. The proof of (5.6) follows by (5.3) and Lemma C.2. To prove (5.7), we use Schauder theory for the operator \mathcal{L} (see [IO19, Lemma 5]), which implies

$$[v]_{\frac{3}{4}-\varepsilon} = [\mathcal{L}^{-1}P\xi]_{\frac{3}{4}-\varepsilon} \lesssim [P\xi]_{-\frac{5}{4}-\varepsilon} \lesssim [\xi]_{-\frac{5}{4}-\varepsilon}.$$

As in the proof of Proposition 5.3, $\langle v(x) \rangle = \langle \xi(\Gamma(x - \cdot)) \rangle = 0$ for every $x \in \mathbb{T}^2$, by Remark 5.2 and (1.21).

5.2. **Estimates on** *F*. In this section we use the spectral gap inequality (5.1) and (5.4) and (5.5) to construct *F* as the $L_{\langle \cdot \rangle}^{p} C^{-\frac{3}{4}-\varepsilon}$ -limit of the sequence of random variables $\{v\partial_{2}R_{1}v_{t}\}_{t\downarrow0}$. A similar result was proved in [IO19, Lemma 8] for $\langle \cdot \rangle$ being the law of white noise (but instead considering approximations $\{v_{\ell}\partial_{2}R_{1}v_{\ell}\}_{\ell\downarrow0}$, where $v_{\ell} := \mathcal{L}^{-1}P\xi_{\ell}$ and $\xi_{\ell} := \phi_{\ell} * \xi$, for a suitable mollifier ϕ_{ℓ}) using Nelson's hypercontractivity estimate. As in Section 5.1, our estimate holds for more general probability measures which are not necessarily Gaussian.

In what follows, we use the convolution-commutator

 $[v, (\cdot)_s](\partial_2 R_1 v)_s := v(\partial_2 R_1 v)_{2s} - (v(\partial_2 R_1 v)_s)_s.$

We will need the following lemma, based on (5.4) and (5.5).

Lemma 5.5. Let $\langle \cdot \rangle$ satisfy Assumption 1.1 and for a periodic Schwartz distribution ξ let $v = \mathcal{L}^{-1}P\xi$. Then for any $1 \le p < \infty$ and $s, S \in (0, 1]$ there holds

$$\sup_{c \in \mathbb{T}^2} \left\langle \left| \left(\left\lceil v, \left(\cdot \right)_s \right\rceil (\partial_2 R_1 v)_s \right)_S (x) \right|^p \right\rangle^{\frac{1}{p}} \le C \left(\left(S^{\frac{1}{3}} \right)^{-\frac{1}{4}} \left(s^{\frac{1}{3}} \right)^{\frac{1}{4}} + \left(S^{\frac{1}{3}} \right)^{-\frac{3}{4}} \left(s^{\frac{1}{3}} \right)^{\frac{3}{4}} \right), \tag{5.8}$$

where the constant *C* depends only on *p*.

Proof. (5.8) is a consequence of the following two claims:

CLAIM 1 For every $s, S \in (0, 1]$ and $q \ge 1$,

$$\sup_{\mathbf{x}\in\mathbb{T}^{2}} \left\langle \left\| \frac{\partial}{\partial\xi} \left(\left\lceil v, \left(\cdot\right)_{s} \right\rceil (\partial_{2}R_{1}v)_{s} \right)_{S}(x) \right\|_{L^{2}}^{2q} \right\rangle^{\frac{1}{2q}} \lesssim \left(S^{\frac{1}{3}}\right)^{-\frac{1}{4}} \left(s^{\frac{1}{3}}\right)^{\frac{1}{4}} + \left(S^{\frac{1}{3}}\right)^{-\frac{3}{4}} \left(s^{\frac{1}{3}}\right)^{\frac{3}{4}}.$$
(5.9)

CLAIM 2 For every $s, S \in (0, 1]$ and $x_0 \in \mathbb{T}^2$,

$$\langle (\lceil v, (\cdot)_s \rceil (\partial_2 R_1 v)_s)_S (x_0) \rangle = \langle (v(\partial_2 R_1 v)_{2s})_S (x_0) \rangle - \langle (v(\partial_2 R_1 v)_s)_{s+S} (x_0) \rangle = 0.$$
(5.10)

Assuming that these claims hold, we may apply for fixed $x_0 \in \mathbb{T}^2$ the L^p -version (5.1) of the spectral gap inequality to the functional $G : \xi \mapsto ([v, (\cdot)_s](\partial_2 R_1 v)_s)_S(x_0)$, which yields (5.8) by CLAIM 1 and CLAIM 2 for $2 \le p < \infty$. For $p \in [1, 2)$ (5.8) follows by Jensen's inequality.

It is easy to see that

$$G(\xi) = \left(\left\lceil v, (\cdot)_s \right\rceil (\partial_2 R_1 v)_s \right)_S (x_0) = \iint k(z, z') \xi(z) \xi(z') \, \mathrm{d}z \, \mathrm{d}z',$$

where k(z, z') is smooth in both variables and $\iint k(z, z')^2 dz dz' < \infty$. A straightforward calculation using the Cauchy–Schwarz inequality with respect to $\langle \cdot \rangle$ and (5.4) and (5.5) we also get that $\langle |G(\xi)| \rangle \leq 1$. Hence, we may apply Lemma F.1 which implies that *G* is admissible in the spectral gap inequality (5.1).

Proof of CLAIM 1: Let $x_0 \in \mathbb{T}^2$ be fixed. We first notice that for every $s, S \in (0, 1]$, the derivative of the quadratic functional *G* is given by

$$\frac{\partial}{\partial\xi}G(\xi):\delta\xi\mapsto (\lceil\delta v,(\cdot)_s\rceil(\partial_2 R_1 v)_s)_S(x_0)+(\lceil v,(\cdot)_s\rceil(\partial_2 R_1 \delta v)_s)_S(x_0)$$

where δv is the unique solution of zero average in x_1 to $\mathcal{L}\delta v = P\delta\xi$ for $\delta\xi \in L^2(\mathbb{T}^2)$. STEP 1: We first show that

$$\sup_{\|\delta\xi\|_{L^{2}} \le 1} \left| \left(\left\lceil \delta v, (\cdot)_{s} \right\rceil (\partial_{2}R_{1}v)_{s} \right)_{S} (x_{0}) \right| \le \int_{\mathbb{R}^{2}} |y_{1}| |\psi_{s}(y)| \|\Psi_{S}(x_{0}-\cdot)(\partial_{2}R_{1}v)_{s}(\cdot-y)\|_{L^{\frac{10}{7}}} \, \mathrm{d}y \\ + \int_{\mathbb{R}^{2}} |y_{2}| |\psi_{s}(y)| \|\Psi_{S}(x_{0}-\cdot)(\partial_{2}R_{1}v)_{s}(\cdot-y)\|_{L^{\frac{10}{7}}} \, \mathrm{d}y,$$
(5.11)

 $\sup_{\|\delta\xi\|_{L^{2}} \le 1} \left| \left(\left\lceil v, (\cdot)_{s} \right\rceil (\partial_{2}R_{1}\delta v)_{s} \right)_{S} (x_{0}) \right| \lesssim \int_{\mathbb{R}^{2}} |\psi_{s}(y)| \|\Psi_{S}(x_{0}-\cdot) \left(v(\cdot-y)-v \right) \|_{L^{\frac{10}{7}}} \, \mathrm{d}y.$ (5.12)

STEP 1A (Proof of (5.11)): By Fubini and the mean value theorem we have

$$\begin{aligned} &([\delta v, (\cdot)_{s}](\partial_{2}R_{1}v)_{s})_{S}(x_{0}) \\ &= \int_{\mathbb{R}^{2}} \psi_{S}(x_{0}-x) \int_{\mathbb{R}^{2}} \psi_{s}(y) \left(\delta v(x) - \delta v(x-y)\right) \left(\partial_{2}R_{1}v\right)_{s}(x-y) \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_{\mathbb{R}^{2}} \psi_{s}(y) \int_{\mathbb{T}^{2}} \Psi_{S}(x_{0}-x) \int_{0}^{1} \left(y_{1}\partial_{1}\delta v(x-ty) + y_{2}\partial_{2}\delta v(x-ty)\right) \, \mathrm{d}t \left(\partial_{2}R_{1}v\right)_{s}(x-y) \, \mathrm{d}x \, \mathrm{d}y, \end{aligned}$$

where Ψ_S is the periodization of ψ_S . By Hölder's and Minkowski's inequalities, as well as translation invariance, it follows that

$$\begin{split} |([\delta v, (\cdot)_{s}](\partial_{2}R_{1}v)_{s})_{S}(x_{0})| \\ &\leq \int_{\mathbb{R}^{2}} |\psi_{s}(y)| \|\Psi_{S}(x_{0}-x)(\partial_{2}R_{1}v)_{s}(x-y)\|_{L_{x}^{\frac{10}{9}}} \left\| \int_{0}^{1} y_{1}\partial_{1}\delta v(x-ty) \, \mathrm{d}t \right\|_{L_{x}^{10}} \, \mathrm{d}y \\ &+ \int_{\mathbb{R}^{2}} |\psi_{s}(y)| \|\Psi_{S}(x_{0}-x)(\partial_{2}R_{1}v)_{s}(x-y)\|_{L_{x}^{\frac{10}{9}}} \left\| \int_{0}^{1} y_{2}\partial_{2}\delta v(x-ty) \, \mathrm{d}t \right\|_{L_{x}^{\frac{10}{3}}} \, \mathrm{d}y \\ &\lesssim \int_{\mathbb{R}^{2}} |y_{1}||\psi_{s}(y)| \|\Psi_{S}(x_{0}-x)(\partial_{2}R_{1}v)_{s}(x-y)\|_{L_{x}^{\frac{10}{9}}} \, \mathrm{d}y \, \|\partial_{1}\delta v\|_{L^{10}} \\ &+ \int_{\mathbb{R}^{2}} |y_{2}||\psi_{s}(y)| \|\Psi_{S}(x_{0}-x)(\partial_{2}R_{1}v)_{s}(x-y)\|_{L_{x}^{\frac{10}{9}}} \, \mathrm{d}y \, \|\partial_{2}\delta v\|_{L^{\frac{10}{3}}}. \end{split}$$

By (D.2) and (D.3) we have $\|\partial_1 \delta v\|_{L^{10}}, \|\partial_2 \delta v\|_{L^{\frac{10}{3}}} \leq \|\delta \xi\|_{L^2}$, hence (5.11) follows after taking the supremum over $\|\delta \xi\|_{L^2} \leq 1$.

STEP 1B (Proof of (5.12)): As in Step 1A, we have

$$([v, (\cdot)_s] (\partial_2 R_1 \delta v)_s)_S (x_0)$$

= $\int_{\mathbb{R}^2} \psi_s(y) \int_{\mathbb{T}^2} \Psi_S(x_0 - x) (v(x) - v(x - y)) (\partial_2 R_1 \delta v)_s (x - y) dx dy,$

where Ψ_S is the periodization of ψ_s . By Hölder's inequality, translation invariance, and the fact that R_1 is bounded on $L^{\frac{10}{3}}(\mathbb{T}^2)$, we obtain that

$$\begin{split} |\left(\left[v, (\cdot)_{s}\right](\partial_{2}R_{1}\delta v)_{s}\right)_{S}(x_{0})| \\ &\lesssim \int_{\mathbb{R}^{2}} |\psi_{s}(y)| \|\Psi_{S}(x_{0}-x)\left(v(x)-v(x-y)\right)\|_{L_{x}^{\frac{10}{7}}} \|(\partial_{2}R_{1}\delta v)_{s}(x-y)\|_{L_{x}^{\frac{10}{3}}} \, \mathrm{d}y \\ &\lesssim \int_{\mathbb{R}^{2}} |\psi_{s}(y)| \|\Psi_{S}(x_{0}-x)\left(v(x)-v(x-y)\right)\|_{L_{x}^{\frac{10}{7}}} \, \mathrm{d}y \, \|\partial_{2}\delta v\|_{L^{\frac{10}{3}}}. \end{split}$$

By (D.3), $\|\partial_2 \delta v\|_{L^{\frac{10}{3}}} \leq \|\delta \xi\|_{L^2}$, which gives (5.12) after taking the supremum over $\|\delta \xi\|_{L^2} \leq 1$. STEP 2: For any $q \geq 1$ and $x_0 \in \mathbb{T}^2$, by STEP 1 and Minkowski's inequality (since $2q \geq \max\{\frac{10}{7}, \frac{10}{9}\}$), we get that

$$\begin{split} \left\langle \left\| \frac{\partial}{\partial \xi} G(\xi) \right\|_{L^{2}}^{2q} \right\rangle^{\frac{1}{2q}} &\lesssim \int_{\mathbb{R}^{2}} |y_{1}| |\psi_{s}(y)| \|\Psi_{S}(x_{0}-x) \langle |(\partial_{2}R_{1}v)_{s}(x-y)|^{2q} \rangle^{\frac{1}{2q}} \|_{L^{\frac{10}{2}}_{x}} \, dy \\ &+ \int_{\mathbb{R}^{2}} |y_{2}| |\psi_{s}(y)| \|\Psi_{S}(x_{0}-x) \langle |(\partial_{2}R_{1}v)_{s}(x-y)|^{2q} \rangle^{\frac{1}{2q}} \|_{L^{\frac{10}{2}}_{x}} \, dy \\ &+ \int_{\mathbb{R}^{2}} |\psi_{s}(y)| \|\Psi_{S}(x_{0}-x) \langle |v(x)-v(x-y)|^{2q} \rangle^{\frac{1}{2q}} \|_{L^{\frac{10}{2}}_{x}} \, dy. \end{split}$$

By (5.4) and (5.5) this implies the bound

$$\left\langle \left\| \frac{\partial}{\partial \xi} G(\xi) \right\|_{L^2}^{2q} \right\rangle^{\frac{1}{2q}} \lesssim \left\| \Psi_S \right\|_{L^{\frac{10}{9}}} \left(s^{\frac{1}{3}} \right)^{-\frac{3}{4}} \int_{\mathbb{R}^2} |y_1| |\psi_s(y)| \, \mathrm{d}y + \left\| \Psi_S \right\|_{L^{\frac{10}{7}}} \left(s^{\frac{1}{3}} \right)^{-\frac{3}{4}} \int_{\mathbb{R}^2} |y_2| |\psi_s(y)| \, \mathrm{d}y$$

$$+ \left\|\Psi_{S}\right\|_{L^{\frac{10}{7}}} \int_{\mathbb{R}^{2}} d(0, y)^{\frac{3}{4}} |\psi_{s}(y)| \, \mathrm{d}y \\ \lesssim \left(S^{\frac{1}{3}}\right)^{-\frac{1}{4}} \left(s^{\frac{1}{3}}\right)^{1-\frac{3}{4}} + \left(S^{\frac{1}{3}}\right)^{-\frac{3}{4}} \left(s^{\frac{1}{3}}\right)^{\frac{3}{2}-\frac{3}{4}} + \left(S^{\frac{1}{3}}\right)^{-\frac{3}{4}} \left(s^{\frac{1}{3}}\right)^{\frac{3}{4}},$$

which proves CLAIM 2, since the implicit constant does not depend on x_0 .

Proof of CLAIM 2: By stationarity we know that $\langle (v(\partial_2 R_1 v)_{2s})_S(x_0) \rangle = \langle (v(\partial_2 R_1 v)_{2s})_S(0) \rangle$. If we denote by \tilde{v} the solution to (1.3) with ξ replaced by $\tilde{\xi}(x) = \xi(-x_1, x_2)$, by the symmetry of Γ (see (1.20)) and the fact that ξ and $\tilde{\xi}$ have the same law (see Assumption 1.1 (iii)), we know that $\tilde{v} = v$ in law, hence $\langle (v(\partial_2 R_1 v)_{2s})_S(0) \rangle = \langle (\tilde{v}(\partial_2 R_1 \tilde{v})_{2s})_S(0) \rangle$. By the symmetry of ψ_t for $t \in (0, 1]$ and the fact that $R_1 \tilde{v}(x) = -R_1 v(-x_1, x_2)$, we get $\langle (\tilde{v}(\partial_2 R_1 \tilde{v})_{2s})_S(0) \rangle = -\langle (v(\partial_2 R_1 v)_{2s})_S(0) \rangle$, which implies that $\langle (v(\partial_2 R_1 v)_{2s})_S(0) \rangle = 0$.

We then have the following proposition.

Proposition 5.6. Under the assumptions of Lemma 5.5, for every $\varepsilon \in (0, \frac{1}{100})$, $1 \le p < \infty$, and $0 < \tau \le t \le 1$ dyadically related there holds

$$\left\langle \left[(v\partial_2 R_1 v_{\tau})_{t-\tau} - v\partial_2 R_1 v_t \right]_{-\frac{3}{4} - \varepsilon}^p \right\rangle^{\frac{1}{p}} \le \left(t^{\frac{1}{3}} \right)^{\frac{1}{4}}.$$
(5.13)

Furthermore, the following bound holds,

$$\sup_{t\in(0,1]}\left\langle \left[v\partial_2 R_1 v_t\right]_{-\frac{3}{4}-\varepsilon}^p\right\rangle^{\frac{1}{p}} \le C.$$
(5.14)

In the above estimates, the constant *C* depends only on ε and *p*.

Proof. STEP 1: We first prove that for every $1 \le p < \infty$, $0 < \tau \le t \le 1$ (note that by assumption $\tau = \frac{t}{2^n}$ for some $n \ge 0$) and $T \in (0, 1]$ there holds,

$$\sup_{x \in \mathbb{T}^2} \left\langle |(\lceil v, (\cdot)_{t-\tau} \rceil (\partial_2 R_1 v)_{\tau})_T (x)|^p \right\rangle^{\frac{1}{p}} \le C \left(\left(T^{\frac{1}{3}} \right)^{-\frac{1}{4}} \left(t^{\frac{1}{3}} \right)^{\frac{1}{4}} + \left(T^{\frac{1}{3}} \right)^{-\frac{3}{4}} \left(t^{\frac{1}{3}} \right)^{\frac{3}{4}} \right).$$
(5.15)

Together with Lemma C.2 this implies (5.13). The telescopic sum identity

$$[v, (\cdot)_{t-\tau}](\partial_2 R_1 v)_{\tau} = \sum_{\substack{0 \le k \le n-1\\ s = \frac{t}{2^{k+1}}}} ([v, (\cdot)_s](\partial_2 R_1 v)_s)_{t-2s},$$

combined with Lemma 5.5, yields

$$\begin{split} \left\langle |(\lceil v, (\cdot)_{t-\tau} \rceil (\partial_2 R_1 v)_t)_T (x)|^p \right\rangle^{\frac{1}{p}} &\lesssim \sum_{\substack{0 \le k \le n-1 \\ s = \frac{t}{2^{k+1}}}} \left\langle |(\lceil v, (\cdot)_s \rceil (\partial_2 R_1 v)_s)_{T+t-2s} (x)|^p \right\rangle^{\frac{1}{p}} \\ &\lesssim \sum_{\substack{0 \le k \le n-1 \\ s = \frac{t}{2^{k+1}}}} \left(\left((T+t-2s)^{\frac{1}{3}} \right)^{-\frac{1}{4}} \left(s^{\frac{1}{3}} \right)^{\frac{1}{4}} + \left((T+t-2s)^{\frac{1}{3}} \right)^{-\frac{3}{4}} \left(s^{\frac{1}{3}} \right)^{\frac{3}{4}} \right) \\ &\lesssim \left(T^{\frac{1}{3}} \right)^{-\frac{1}{4}} \left(t^{\frac{1}{3}} \right)^{\frac{1}{4}} \sum_{\substack{0 \le k \le n-1 \\ 0 \le k \le n-1}} \frac{1}{2^{\frac{k+1}{12}}} + \left(T^{\frac{1}{3}} \right)^{-\frac{3}{4}} \left(t^{\frac{1}{3}} \right)^{\frac{3}{4}} \sum_{\substack{0 \le k \le n-1 \\ 0 \le k \le n-1}} \frac{1}{2^{\frac{k+1}{4}}} \\ &\lesssim \left(T^{\frac{1}{3}} \right)^{-\frac{1}{4}} \left(t^{\frac{1}{3}} \right)^{\frac{1}{4}} + \left(T^{\frac{1}{3}} \right)^{-\frac{3}{4}} \left(t^{\frac{1}{3}} \right)^{\frac{3}{4}}, \end{split}$$

which proves the desired claim.

STEP 2: We now prove that for every $\varepsilon \in (0, \frac{1}{100}), 1 \le p < \infty$ and $T \in (0, 1]$,

$$\sup_{t\in(0,1]}\sup_{x\in\mathbb{T}^2}\left\langle |(v\partial_2 R_1v_t)_T(x)|^p\right\rangle^{\frac{1}{p}} \lesssim_{\varepsilon,p} \left(T^{\frac{1}{3}}\right)^{-\frac{2}{4}-\varepsilon},\tag{5.16}$$

which together with Lemma C.2 implies (5.14).

We first assume that $t \in [\frac{T}{2}, 1]$. Then, by Definition 1.14 of negative Hölder norms and (A.14) we know that $[\partial_2 R_1 v]_{-\frac{3}{4}-2\varepsilon} \leq [R_1 v]_{\frac{3}{4}-2\varepsilon} \leq [v]_{\frac{3}{4}-\varepsilon}$. Combined with (A.1) and the fact that $||v||_{L^{\infty}} \leq [v]_{\frac{3}{4}-\varepsilon}$, we have that

$$|(v\partial_{2}R_{1}v_{t})_{T}(x)| \leq ||v\partial_{2}R_{1}v_{t}||_{L^{\infty}} \leq ||v||_{L^{\infty}} ||\partial_{2}R_{1}v_{t}||_{L^{\infty}}$$
$$\lesssim \left(t^{\frac{1}{3}}\right)^{-\frac{3}{4}-2\varepsilon} [v]_{\frac{3}{4}-\varepsilon}^{2} \lesssim \left(T^{\frac{1}{3}}\right)^{-\frac{3}{4}-2\varepsilon} [v]_{\frac{3}{4}-\varepsilon}^{2}.$$
(5.17)

By (5.7), this implies the estimate

$$\left\langle |(v\partial_{2}R_{1}v_{t})_{T}(x)|^{p}\right\rangle_{\ell}^{\frac{1}{p}} \lesssim \left(T^{\frac{1}{3}}\right)^{-\frac{3}{4}-2\varepsilon} \left\langle [v]_{\frac{3}{4}-\varepsilon}^{2p}\right\rangle_{\ell}^{\frac{1}{p}} \lesssim \left(T^{\frac{1}{3}}\right)^{-\frac{3}{4}-2\varepsilon},$$
(5.18)

for every $t \in [\frac{T}{2}, 1]$, uniformly in $x \in \mathbb{T}^2$ and $\ell \in (0, 1]$.

We now assume that $t \in (0, \frac{T}{2}]$. Then, there exists $T_* \in (\frac{T}{4}, \frac{T}{2}]$ such that $t = \frac{T_*}{2^n}$, for some $n \ge 1$. Using the semigroup property, we write

$$(v\partial_2 R_1 v_t)_T = \left((v\partial_2 R_1 v_t)_{T_* - t} - v\partial_2 R_1 v_{T_*} \right)_{T - T_* + t} + \left(v\partial_2 R_1 v_{T_*} \right)_{T - T_* + t}$$

By (5.13), the first term can be estimated as

$$\left\langle \left| \left((v\partial_2 R_1 v_t)_{T_* - t} - v\partial_2 R_1 v_{T_*} \right)_{T - T_* + t} (x) \right|^p \right\rangle^{\frac{1}{p}} \\ \lesssim \left((T - T_* + t)^{\frac{1}{3}} \right)^{-\frac{3}{4}} \left(T_*^{\frac{1}{3}} \right)^{\frac{3}{4}} + \left((T - T_* + t)^{\frac{1}{3}} \right)^{-\frac{1}{4}} \left(T_*^{\frac{1}{3}} \right)^{\frac{1}{4}} \lesssim 1,$$

where we also used that $T_* \leq \frac{T}{2} \leq T - T_* + t$. For the second term, noting that $T_* > \frac{T}{4}$ and proceeding as in (5.17) and (5.18), we obtain the bound

$$\left\langle | \left(v \partial_2 R_1 v_{T_*} \right)_{T-T_*+t} (x) |^p \right\rangle^{\frac{1}{p}} \lesssim \left(T_*^{\frac{1}{3}} \right)^{-\frac{3}{4}-2\varepsilon} \left\langle [v]_{\frac{3}{4}-\varepsilon}^{2p} \right\rangle_{\ell}^{\frac{1}{p}} \lesssim \left(T^{\frac{1}{3}} \right)^{-\frac{3}{4}-2\varepsilon}.$$

Hence, we also proved that

$$\left\langle |(v\partial_2 R_1 v_t)_T(x)|^p \right\rangle^{\frac{1}{p}} \lesssim 1 + \left(T^{\frac{1}{3}}\right)^{-\frac{3}{4} - 2\varepsilon},$$
(5.19)

for $t \in (0, \frac{T}{4}]$, uniformly in $x \in \mathbb{T}^2$.

Combining (5.18) and (5.19) gives (5.16) upon relabelling ε .

As a corollary we obtain,

Corollary 5.7. Let $\langle \cdot \rangle$ satisfy Assumption 1.1. There exists a unique centered and stationary random variable $\xi \mapsto F(\xi)$ such that for every $\varepsilon \in (0, \frac{1}{100})$ and $1 \le p < \infty$,

$$\lim_{t=2^{-n}\downarrow 0} \left\langle \left[F - v \partial_2 R_1 v_t \right]_{-\frac{3}{4} - \varepsilon}^p \right\rangle^{\frac{1}{p}} = 0,$$
 (5.20)

and the convergence is uniform in the class of probability measures satisfying Assumption 1.1.

Proof. We prove that for every $\varepsilon \in (0, 1]$, $1 \le p < \infty$ and $0 \le \tau \le t \le 1$ dyadic,

$$\lim_{t,\tau\downarrow 0} \left\langle \left[v\partial_2 R_1 v_\tau - v\partial_2 R_1 v_t \right]_{-\frac{3}{4}-\varepsilon}^p \right\rangle^{\frac{1}{p}} = 0.$$
(5.21)

By the triangle inequality we have that

$$\left\langle \left[v\partial_2 R_1 v_{\tau} - v\partial_2 R_1 v_t\right]_{-\frac{3}{4}-\varepsilon}^p\right\rangle^{\frac{1}{p}}$$

$$\leq \left\langle \left[v\partial_2 R_1 v_\tau - (v\partial_2 R_1 v_\tau)_{t-\tau} \right]_{-\frac{3}{4}-\varepsilon}^p \right\rangle^{\frac{1}{p}} + \left\langle \left[(v\partial_2 R_1 v_\tau)_{t-\tau} - v\partial_2 R_1 v_t \right]_{-\frac{3}{4}-\varepsilon}^p \right\rangle^{\frac{1}{p}}$$

To estimate the first term we use Propositions A.7 and (5.14) which imply that

$$\left\langle \left[v\partial_2 R_1 v_\tau - (v\partial_2 R_1 v_\tau)_{t-\tau} \right]_{-\frac{3}{4}-\varepsilon}^p \right\rangle^{\frac{1}{p}} \lesssim \left((t-\tau)^{\frac{1}{3}} \right)^{\frac{\varepsilon}{2}} \left\langle \left[v\partial_2 R_1 v_\tau \right]_{-\frac{3}{4}-\frac{\varepsilon}{2}}^p \right\rangle^{\frac{1}{p}} \lesssim \left((t-\tau)^{\frac{1}{3}} \right)^{\frac{\varepsilon}{2}}.$$

Using (5.13), the second term is estimated as

$$\left\langle \left[(v\partial_2 R_1 v_\tau)_{t-\tau} - v\partial_2 R_1 v_t \right]_{-\frac{3}{4}-\varepsilon}^p \right\rangle^{\frac{1}{p}} \lesssim \left(t^{\frac{1}{3}} \right)^{\frac{1}{4}}.$$

Hence, we have proved that

$$\left\langle \left[v\partial_2 R_1 v_\tau - v\partial_2 R_1 v_t \right]_{-\frac{3}{4}-\varepsilon}^p \right\rangle^{\frac{1}{p}} \lesssim \left((t-\tau)^{\frac{1}{3}} \right)^{\frac{\varepsilon}{2}} + \left(t^{\frac{1}{3}} \right)^{\frac{1}{4}},$$

which implies (5.21) after taking τ , $t \downarrow 0$. This implies that the sequence $\{v\partial_2 R_1 v_t\}_{t \in (0,1]}$ is Cauchy in $L^p_{\langle \cdot \rangle} C^{-\frac{3}{4}-\varepsilon}$, hence it converges to a limit F in $L^p_{\langle \cdot \rangle} C^{-\frac{3}{4}-\varepsilon}$. The fact that the convergence is uniform in the class of probability measures satisfying Assumption 1.1 follows since our estimates depend on $\langle \cdot \rangle$ only through the spectral gap inequality (1.7).

Similarly to the proof of Lemma 5.5, we can show that $\langle v\partial_2 R_1 v_t(x) \rangle = 0$ for every $x \in \mathbb{T}^2$ and $t \in (0, 1]$. Hence, as a limit in $L^p_{\langle \cdot \rangle} C^{-\frac{3}{4}-\varepsilon}$, the fact that *F* is centered and stationary follows from the corresponding properties of $v\partial_2 R_1 v_t$.

5.3. Proof of Proposition 1.9.

Proof. Let $\langle \cdot \rangle^{\text{lift}}$ be given by $\langle G(\xi, v, F) \rangle^{\text{lift}} := \langle G(\xi, v, F) \rangle$ for every bounded and continuous $G: C^{-\frac{5}{4}-\varepsilon} \times C^{\frac{3}{4}-\varepsilon} \times C^{-\frac{3}{4}-\varepsilon} \to \mathbb{R}$, with the convention that under $\langle \cdot \rangle$, $v = \mathcal{L}^{-1}P\xi$ and F is given by (5.20) (under $\langle \cdot \rangle^{\text{lift}}$ we think of (ξ, v, F) as a dummy variable in $\mathcal{T} := C^{-\frac{5}{4}-\varepsilon} \times C^{\frac{3}{4}-\varepsilon} \times C^{-\frac{3}{4}-\varepsilon}$). The fact that $\langle \cdot \rangle^{\text{lift}}$ defines a probability measure on \mathcal{T} is immediate by Corollaries 5.4 and 5.7.

Statements (i) and (ii) are immediate by the construction of $\langle \cdot \rangle^{\text{lift}}$. The first part of statement (iii), that is, (1.9), is immediate by Corollary 5.7. For the second part, note that in the case when ξ is smooth $\langle \cdot \rangle$ -almost surely, the product $v\partial_2 R_1 v$ makes sense $\langle \cdot \rangle$ -almost surely and we also have that $v\partial_2 R_1 v_t \rightarrow v\partial_2 R_1 v \langle \cdot \rangle$ -almost surely. By (5.20) we know that $v\partial_2 R_1 v_t \rightarrow F$ in $C^{-\frac{3}{4}-\varepsilon} \langle \cdot \rangle$ -almost surely along a subsequence. Hence, we should have that $F = v\partial_2 R_1 v \langle \cdot \rangle$ -almost surely.

It remains to prove the continuity statement (iv). Assume that $\{\langle \cdot \rangle_{\ell}\}_{\ell \downarrow 0}$ converges weakly to $\langle \cdot \rangle$ and consider $\{\langle \cdot \rangle_{\ell}^{\text{lift}}\}_{\ell \downarrow 0}$ and $\langle \cdot \rangle^{\text{lift}}$.

STEP 1: We prove that weak convergence of $\{\langle \cdot \rangle_{\ell}\}_{\ell \downarrow 0}$ to $\langle \cdot \rangle$ in the Schwartz topology implies weak convergence in $C^{-\frac{5}{4}-\varepsilon}$. Indeed, let $G: C^{-\frac{5}{4}-\varepsilon} \to \mathbb{R}$ be bounded and continuous. Then, we can write

$$\langle G(\xi) \rangle_{\ell} - \langle G(\xi) \rangle = (\langle G(\xi) \rangle_{\ell} - \langle G(\xi_t) \rangle_{\ell}) + (\langle G(\xi_t) \rangle_{\ell} - \langle G(\xi_t) \rangle) + (\langle G(\xi_t) \rangle - \langle G(\xi) \rangle) + \langle G(\xi_t) \rangle - \langle G(\xi) \rangle = \langle G(\xi) \rangle_{\ell} - \langle G(\xi) \rangle_{\ell} - \langle G(\xi) \rangle + \langle G(\xi) \rangle_{\ell} - \langle G(\xi) \rangle + \langle G(\xi) \rangle - \langle G(\xi) \rangle + \langle G(\xi) \rangle$$

To treat the first term, we use (A.16) and (5.6) (which holds uniform in the class of probability measures satisfying Assumption 1.1) yielding for every $1 \le p < \infty$,

$$\sup_{\ell \in (0,1]} \left\langle \left[\xi - \xi_{\ell}\right]_{-\frac{5}{4} - \varepsilon}^{p} \right\rangle_{\ell}^{\frac{1}{p}} \lesssim t^{\frac{\varepsilon}{6}} \sup_{\ell \in (0,1]} \left\langle \left[\xi\right]_{-\frac{5}{4} - \frac{\varepsilon}{2}}^{p} \right\rangle_{\ell}^{\frac{1}{p}} \lesssim t^{\frac{\varepsilon}{6}}.$$
(5.22)

For $\delta > 0$ which we fix below, we write

$$\langle G(\xi) \rangle_{\ell} - \langle G(\xi_t) \rangle_{\ell} = \langle (G(\xi) - G(\xi_t)) \mathbf{1}_{\{[\xi - \xi_t]_{-\frac{5}{4} - \varepsilon} < \delta\}} \rangle_{\ell} + \langle (G(\xi) - G(\xi_t)) \mathbf{1}_{\{[\xi - \xi_t]_{-\frac{5}{4} - \varepsilon} \ge \delta\}} \rangle_{\ell}.$$

For $\eta > 0$, by the continuity of *G* we can choose δ sufficiently small such that

$$\left| \left\langle (G(\xi) - G(\xi_t)) \mathbf{1}_{\{ [\xi - \xi_t]_{-\frac{5}{4} - \varepsilon} < \delta \}} \right\rangle_{\ell} \right| \le \frac{\eta}{4}$$

uniformly in $\ell \in (0, 1]$. By (5.22), the boundedness of G, and Chebyshev's inequality, we can choose $t \in (0, 1]$ sufficiently small such that

$$|\langle (G(\xi) - G(\xi_t)) \mathbf{1}_{\{[\xi - \xi_t]_{-\frac{5}{4} - \varepsilon} \le \delta\}} \rangle_{\ell}| \le ||G||_{L^{\infty} \left(C^{-\frac{5}{4} - \varepsilon}\right)} \sup_{\ell \in (0, 1]} \langle \mathbf{1}_{\{[\xi - \xi_t]_{-\frac{5}{4} - \varepsilon} \ge \delta\}} \rangle_{\ell} \le \frac{\eta}{4}.$$

Hence, we obtain that for *t* sufficiently small

$$\sup_{\ell \in (0,1]} |\langle G(\xi) \rangle_{\ell} - \langle G(\xi_t) \rangle_{\ell}| \le \frac{\eta}{2}.$$

Similarly, we can show that for *t* sufficiently small

$$|\langle G(\xi) \rangle - \langle G(\xi_t) \rangle| \le \frac{\eta}{2}.$$

Since $\{\langle \cdot \rangle_{\ell}\}_{\ell \downarrow 0}$ converges to $\langle \cdot \rangle$ in the Schwartz topology, for every $t \in (0, 1]$ we know that $\langle G(\xi_t) \rangle_{\ell} \to \langle G(\xi_t) \rangle$ as $\ell \downarrow 0$. In total, we have that

$$\limsup_{\ell \downarrow 0} |\langle G(\xi) \rangle_{\ell} - \langle G(\xi) \rangle| \le \eta,$$

which proves that $\langle G(\xi) \rangle_{\ell} \to \langle G(\xi) \rangle$ as $\ell \downarrow 0$ since η is arbitrary.

STEP 2: We now prove that $\{\langle \cdot \rangle_{\ell}^{\text{lift}}\}_{\ell \downarrow 0}$ converges weakly to $\langle \cdot \rangle^{\text{lift}}$ in \mathcal{T} . The argument is similar in spirit to STEP 1. Let $G : \mathcal{T} \to \mathbb{R}$ be a bounded continuous function. Then, we write

$$\begin{aligned} \langle G(\xi, v, F) \rangle_{\ell}^{\text{int}} &- \langle G(\xi, v, F) \rangle^{\text{int}} = \langle G(\xi, v, F) - G(\xi, v, v\partial_2 R_1 v_t) \rangle_{\ell} \\ &+ \langle G(\xi, v, v\partial_2 R_1 v_t) - G(\xi, v, F) \rangle \\ &+ \langle G(\xi, v, v\partial_2 R_1 v_t) \rangle_{\ell} - \langle G(\xi, v, v\partial_2 R_1 v_t) \rangle_{\ell} \end{aligned}$$

To estimate the first term, for $\delta > 0$ to be fixed below, we use the decomposition

$$\begin{aligned} \langle G(\xi, v, F) - G(\xi, v, v\partial_2 R_1 v_t) \rangle_{\ell} &= \langle (G(\xi, v, F) - G(\xi, v, v\partial_2 R_1 v_t)) \mathbf{1}_{\{[F - v\partial_2 R_1 v_t]_{-\frac{3}{4} - \varepsilon} < \delta\}} \rangle_{\ell} \\ &+ \langle (G(\xi, v, F) - G(\xi, v, v\partial_2 R_1 v_t)) \mathbf{1}_{\{[F - v\partial_2 R_1 v_t]_{-\frac{3}{4} - \varepsilon} \geq \delta\}} \rangle_{\ell}. \end{aligned}$$

For $\eta > 0$, by the continuity of *G* we can choose $\delta > 0$ sufficiently small such that

$$|\langle (G(\xi, v, F) - G(\xi, v, v\partial_2 R_1 v_t)) \mathbf{1}_{\{[F - v\partial_2 R_1 v_t]_{-\frac{3}{4} - \varepsilon} < \delta\}} \rangle_{\ell}| \le \frac{r_{\ell}}{4}$$

uniformly in $\ell \in (0, 1]$. By (5.20), the boundedness of *G*, and Chebyshev's inequality, we can choose *t* sufficiently small such that

$$\sup_{\ell \in \{0,1\}} \langle (G(\xi, v, F) - G(\xi, v, v\partial_2 R_1 v_t)) \mathbf{1}_{\{[F - v\partial_2 R_1 v_t]_{-\frac{3}{4} - \varepsilon} \ge \delta\}} \rangle_{\ell}$$

$$\leq \|G\|_{L^{\infty} \left(C^{-\frac{5}{4} - \varepsilon} \times C^{\frac{3}{4} - \varepsilon} \times C^{-\frac{3}{4} - \varepsilon}\right)} \sup_{\ell \in \{0,1\}} \langle \mathbf{1}_{\{[F - v\partial_2 R_1 v_t]_{-\frac{3}{4} - \varepsilon} \ge \delta\}} \rangle_{\ell} \leq \frac{\eta}{4}$$

Hence, we have proved that for *t* sufficiently small

$$\sup_{\ell \in (0,1]} \langle G(\xi, v, F) - G(\xi, v, v\partial_2 R_1 v_t) \rangle_{\ell} \leq \frac{\eta}{2}.$$

In a similar way we can show that for *t* sufficiently small

$$\langle G(\xi, v, F) - G(\xi, v, v\partial_2 R_1 v_t) \rangle \le \frac{\eta}{2}$$

Since $\langle \cdot \rangle_{\ell} \to \langle \cdot \rangle$ weakly as $\ell \downarrow 0$, we have that $\langle G(\xi, v, v\partial_2 R_1 v_t) \rangle_{\ell} \to \langle G(\xi, v, v\partial_2 R_1 v_t) \rangle$ as $\ell \downarrow 0$ for every $t \in (0, 1]$. Altogether, we get that

$$\lim_{\ell \downarrow 0} (\langle G(\xi, v, F) \rangle_{\ell}^{\text{lift}} - \langle G(\xi, v, F) \rangle^{\text{lift}}) \le \eta,$$

which in turn implies that $\langle G(\xi, v, F) \rangle_{\ell}^{\text{lift}} \rightarrow \langle G(\xi, v, F) \rangle^{\text{lift}}$ as $\ell \downarrow 0$ since η is arbitrary. Thus $\{\langle \cdot \rangle_{\ell}^{\text{lift}}\}_{\ell \downarrow 0}$ converges weakly to $\langle \cdot \rangle^{\text{lift}}$.

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Appendix A. Hölder spaces

The following equivalent characterization of Hölder norms relies on the "heat kernel" of the operator \mathcal{A} .

Lemma A.1 ([IO19, Lemma 10, Remark 1]). Let f be a periodic distribution on \mathbb{T}^2 .

(1) For $\beta \in (-\frac{3}{2}, 0) \setminus \{-1, -\frac{1}{2}\}$, we have

$$[f]_{\beta} \sim \sup_{T \in (0,1]} \left(T^{\frac{1}{3}}\right)^{-\beta} \|f_T\|_{L^{\infty}}.$$
 (A.1)

In the critical cases $\beta \in \{-1, -\frac{1}{2}\}$ we have

$$\sup_{T \in (0,1]} \left(T^{\frac{1}{3}} \right)^{-\beta} \| f_T \|_{L^{\infty}} \lesssim [f]_{\beta}.$$
(A.2)

(2) For $\beta \in (-\frac{3}{2}, 0) \setminus \{-1, -\frac{1}{2}\}$ and f of vanishing average, we have

$$[f]_{\beta} \sim \sup_{T \in (0,1]} \left(T^{\frac{1}{3}}\right)^{-\beta} \|T\mathcal{A}f_T\|_{L^{\infty}}.$$
(A.3)

In the critical cases $\beta \in \{-1, -\frac{1}{2}\}$ we have³⁰

$$\sup_{T \in \{0,1\}} \left(T^{\frac{1}{3}} \right)^{-\beta} \| T \mathcal{A} f_T \|_{L^{\infty}} \lesssim [f]_{\beta}.$$
(A.4)

(3) For $\alpha \in (0, \frac{3}{2}) \setminus \{1\}$ we have

$$[f]_{\alpha} \sim \sup_{T \in [0,1]} \left(T^{\frac{1}{3}}\right)^{-\alpha} \|T\mathcal{A}f_T\|_{L^{\infty}}.$$
(A.5)

In the critical case $\alpha = 1$ we have³¹

$$\sup_{T \in \{0,1\}} \left(T^{\frac{1}{3}} \right)^{-1} \| T \mathcal{A} f_T \|_{L^{\infty}} \lesssim [f]_1.$$
(A.6)

In the case of periodic distributions f of vanishing average on \mathbb{T}^2 , one can consider the supremum over all T > 0 in (A.1), (A.2), (A.3), and (A.4), while in (A.5) and (A.6) the suprema over $T \in (0, 1)$ and T > 0 are equivalent even for distributions of nonvanishing average.

We also have the following interpolation inequality.

$$\|\mathcal{A}f_{T}\|_{L^{\infty}} = \|f_{\frac{T}{2}} * \mathcal{A}\psi_{\frac{T}{2}}\|_{L^{\infty}} \le \|f_{\frac{T}{2}}\|_{L^{\infty}} \|\mathcal{A}\psi_{\frac{T}{2}}\|_{L^{1}} \le (T^{\frac{1}{3}})^{\beta} [f]_{\beta} \frac{1}{T}.$$

³¹Indeed, since $\mathcal{A}\psi_T$ has vanishing average, we write $\mathcal{A}f_T(x) = \int_{\mathbb{R}^2} \mathcal{A}\psi_T(y)(f(x-y) - f(x)) \, dy$ and deduce via Step 2 in the proof of Lemma 10 in [IO19] that

$$\|\mathcal{A}f_T\|_{L^{\infty}} \leq [f]_{\alpha} \int_{\mathbb{R}^2} |\mathcal{A}\psi_T(y)| d(y,0)^{\alpha} \, \mathrm{d}y \lesssim [f]_{\alpha} \left(T^{\frac{1}{3}}\right)^{(-3+\alpha)}$$

³⁰Indeed, using (A.2) and (1.18) we have

Lemma A.2. For every $-\frac{3}{2} < \beta < 0 < \gamma < \alpha < \frac{3}{2}$ there exists a constant C > 0 such that the following interpolation inequality holds for every $f : \mathbb{T}^2 \to \mathbb{R}$,

$$[f]_{\gamma} \le C[f]_{\beta}^{\lambda}[f]_{\alpha}^{1-\lambda}, \tag{A.7}$$

where $\lambda \in (0, 1)$ is given by $\gamma = \lambda\beta + (1 - \lambda)\alpha$. If f has vanishing average in \mathbb{T}^2 , (A.7) also holds for $\gamma = 0$ with $[f]_{\gamma}$ replaced by $||f||_{L^{\infty}}$.

Proof. By (A.1) and (A.2), we have for every $T \in (0, 1]$ and $x, y \in \mathbb{T}^2$:

$$\begin{split} |f_{2T}(x) - f_{T}(x)| &\leq \|f_{T}\|_{L^{\infty}} + \|f_{2T}\|_{L^{\infty}} \leq [f]_{\beta} \left(T^{\frac{1}{3}}\right)^{\beta}, \\ |\partial_{1}f_{2T}(x) - \partial_{1}f_{T}(x)| &\leq (\|f_{\frac{3T}{2}}\|_{L^{\infty}} + \|f_{\frac{T}{2}}\|_{L^{\infty}}) \int_{\mathbb{R}^{2}} |\partial_{1}\psi_{\frac{T}{2}}(z)| \, dz \leq [f]_{\beta} \left(T^{\frac{1}{3}}\right)^{\beta-1}, \\ |f_{T}(x) - f_{T}(y)| &\leq 2\|f_{T}\|_{L^{\infty}} \leq [f]_{\beta} \left(T^{\frac{1}{3}}\right)^{\beta}, \\ |\partial_{1}f_{T}(y)(x_{1} - y_{1})| &\leq \|\partial_{1}\psi_{\frac{T}{2}}\|_{L^{1}(\mathbb{R}^{2})} \|f_{\frac{T}{2}}\|_{L^{\infty}} d(x, y) \leq [f]_{\beta} \left(T^{\frac{1}{3}}\right)^{\beta} \frac{d(x, y)}{T^{\frac{1}{3}}}, \\ |f_{T}(x) - f_{T}(y) - \partial_{1}f_{T}(y)(x_{1} - y_{1})| \leq [f]_{\beta} \left(T^{\frac{1}{3}}\right)^{\beta} \max\left(1, \frac{d(x, y)}{T^{\frac{1}{3}}}\right), \end{split}$$
(A.8)

where we used [IO19, equation (26)]. In the case $\alpha \in (0, 1]$, we claim that for every $T \in (0, 1]$ and $x, y \in \mathbb{T}^2$:

$$|f_{2T}(x) - f_T(x)| \leq [f]_{\alpha} \left(T^{\frac{1}{3}}\right)^{\alpha}$$
 and $|f_T(x) - f_T(y)| \leq [f]_{\alpha} d^{\alpha}(x, y).$ (A.9)

Indeed, by [IO19, equation (26)], we deduce

$$\begin{split} |f_{T}(x) - f_{T}(y)| &\leq \int_{\mathbb{R}^{2}} |\psi_{T}(z)| |f(x-z) - f(y-z)| \, \mathrm{d}z \leq [f]_{\alpha} d(x,y)^{\alpha} \int_{\mathbb{R}^{2}} |\psi_{T}(z)| \, \mathrm{d}z \\ &\lesssim [f]_{\alpha} d^{\alpha}(x,y), \\ |f_{2T}(x) - f_{T}(x)| &\leq \int_{\mathbb{R}^{2}} |\psi_{T}(z)| |f_{T}(x-z) - f_{T}(x)| \, \mathrm{d}z \leq [f]_{\alpha} \int_{\mathbb{R}^{2}} |\psi_{T}(z)| d(z,0)^{\alpha} \, \mathrm{d}z \\ &\lesssim [f]_{\alpha} \left(T^{\frac{1}{3}}\right)^{\alpha}. \end{split}$$

In the case $\alpha \in (1, \frac{3}{2})$, arguing as above, we also have for every $T \in (0, 1]$ and $x, y \in \mathbb{T}^2$:

$$\begin{cases} |f_{T}(x) - f_{T}(y) - \partial_{1}f_{T}(y)(x_{1} - y_{1})| \leq [f]_{\alpha}d^{\alpha}(x, y), \\ |f_{2T}(x) - f_{T}(x)| \leq \int_{\mathbb{R}^{2}} |\psi_{T}(z)||f_{T}(x - z) - f_{T}(x) + \partial_{1}f_{T}(x)z_{1}| dz \leq [f]_{\alpha} \left(T^{\frac{1}{3}}\right)^{\alpha}, \\ |\partial_{1}f_{2T}(x) - \partial_{1}f_{T}(x)| \leq \int_{\mathbb{R}^{2}} |\partial_{1}\psi_{\frac{T}{2}}(z)||f_{\frac{3T}{2}}(x - z) - f_{\frac{T}{2}}(x - z)| dz \leq [f]_{\alpha} \left(T^{\frac{1}{3}}\right)^{\alpha-1}, \end{cases}$$
(A.10)

where we used $\int_{\mathbb{R}^2} \psi_T \, dz = 1$ and $\int_{\mathbb{R}^2} z_1 \psi_T(z) \, dz = 0$. To prove (A.7) we distinguish three different cases for γ .

CASE $\gamma \in (0, 1)$: First, assume that $\alpha \in (0, 1]$. Interpolating (A.8) and (A.9), we obtain for every $T \in (0, 1]$ and $x, y \in \mathbb{T}^2$:

$$\begin{aligned} |f_{2T}(x) - f_T(x)| &\lesssim [f]_{\beta}^{\lambda} [f]_{\alpha}^{1-\lambda} \left(T^{\frac{1}{3}}\right)^{\gamma} \\ |f_T(x) - f_T(y)| &\lesssim [f]_{\beta}^{\lambda} [f]_{\alpha}^{1-\lambda} \left(T^{\frac{1}{3}}\right)^{\lambda\beta} d(x, y)^{(1-\lambda)\alpha}. \end{aligned}$$

If $n_0 \in \mathbb{Z}$ is the largest integer such that $2^{-\frac{n_0}{3}} \ge d(x, y)$, then for every $n \ge n_0$

$$\begin{aligned} |f_{2^{-n}}(x) - f_{2^{-n}}(y)| \\ &\leq |f_{2^{-n}}(x) - f_{2^{-n_0}}(x)| + |f_{2^{-n_0}}(x) - f_{2^{-n_0}}(y)| + |f_{2^{-n}}(y) - f_{2^{-n_0}}(y)| \\ &\leq \sum_{k=n_0}^{n-1} |f_{2^{-(k+1)}}(x) - f_{2^{-k}}(x)| + |f_{2^{-n_0}}(x) - f_{2^{-n_0}}(y)| + \sum_{k=n_0}^{n-1} |f_{2^{-(k+1)}}(y) - f_{2^{-k}}(y)| \end{aligned}$$

$$\leq [f]_{\beta}^{\lambda}[f]_{\alpha}^{1-\lambda} \left(\sum_{k=n_{0}}^{n-1} \left(2^{-\frac{k}{3}} \right)^{\gamma} + \left(2^{-\frac{n_{0}}{3}} \right)^{\lambda\beta} d(x,y)^{(1-\lambda)\alpha} \right)$$

$$\leq [f]_{\beta}^{\lambda}[f]_{\alpha}^{1-\lambda} \left(\left(2^{-\frac{n_{0}}{3}} \right)^{\gamma} + \left(2^{-\frac{n_{0}}{3}} \right)^{\lambda\beta} d(x,y)^{(1-\lambda)\alpha} \right) \leq [f]_{\beta}^{\lambda}[f]_{\alpha}^{1-\lambda} d(x,y)^{\gamma},$$
(A.11)

which in turn implies (A.7) by letting $n \to \infty$.

. .

In the case $\alpha \in (1, \frac{3}{2})$, first, one needs to choose $n_0 \in \mathbb{Z}$ such that $2^{-\frac{n_0}{3}} \ge \kappa d(x, y) > 2^{-\frac{n_0+1}{3}}$ with a constant $\kappa > 0$, depending only on γ , which we fix below. Second, one needs to estimate the intermediate term $|f_{2^{-n_0}}(x) - f_{2^{-n_0}}(y)|$ differently. For this, we need to use that for every $T \in (0, 1]$, by (A.1), (A.2), Definition 1.14, and since $\gamma < 1$,

$$\|\partial_1 f_T\|_{L^{\infty}} \lesssim [\partial_1 f]_{\gamma-1} (T^{\frac{1}{3}})^{\gamma-1} \le c_0 [f]_{\gamma} (T^{\frac{1}{3}})^{\gamma-1}$$

where $c_0 > 0$ depends only on γ . By interpolating between (A.8) and (A.10), we estimate the intermediate term for a constant C > 0 (depending on α , β , γ) by,

$$\begin{split} |f_{2^{-n_{0}}}(x) - f_{2^{-n_{0}}}(y)| \\ &\leq |f_{2^{-n_{0}}}(x) - f_{2^{-n_{0}}}(y) - \partial_{1}f_{2^{-n_{0}}}(y)(x_{1} - y_{1})| + |\partial_{1}f_{2^{-n_{0}}}(y)(x_{1} - y_{1})| \\ &\leq C[f]_{\beta}^{\lambda}[f]_{\alpha}^{1-\lambda} \left(2^{-\frac{n_{0}}{3}}\right)^{\lambda\beta} d(x,y)^{(1-\lambda)\alpha} \max\left(1, \frac{d(x,y)}{2^{-\frac{n_{0}}{3}}}\right)^{\lambda} + c_{0}[f]_{\gamma}(2^{-\frac{n_{0}}{3}})^{\gamma-1}d(x,y) \\ &\leq C_{\kappa}[f]_{\beta}^{\lambda}[f]_{\alpha}^{1-\lambda}d(x,y)^{\gamma} + c_{0}[f]_{\gamma}\kappa^{\gamma-1}d(x,y)^{\gamma}. \end{split}$$

Choosing $\kappa > 0$ such that $c_0 \kappa^{\gamma-1} = \frac{1}{2}$ and proceeding as in (A.11) (where now the change of n_0 affects the implicit constant by a factor depending on κ), after passing to the limit $n \to \infty$ we obtain

$$|f(x) - f(y)| \le \left(C[f]_{\beta}^{\lambda}[f]_{\alpha}^{1-\lambda} + \frac{1}{2}[f]_{\gamma} \right) d(x, y)^{\gamma}.$$

Dividing by $d(x, y)^{\gamma}$ and taking the supremum over $x \neq y$ we finally obtain (A.7).

CASE $\gamma \in (1, \frac{3}{2})$: Since $\alpha > \gamma$ we also have $\alpha \in (1, \frac{3}{2})$. Interpolating (A.8) and (A.10), we obtain for every $T \in (0, 1]$ and $x, y \in \mathbb{T}^2$,

$$\begin{split} |f_{2T}(x) - f_T(x)| &\lesssim [f]_{\beta}^{\lambda} [f]_{\alpha}^{1-\lambda} \left(T^{\frac{1}{3}}\right)^{\gamma}, \\ |\partial_1 f_{2T}(x) - \partial_1 f_T(x)| &\lesssim [f]_{\beta}^{\lambda} [f]_{\alpha}^{1-\lambda} \left(T^{\frac{1}{3}}\right)^{\gamma-1}, \\ |f_T(x) - f_T(y) - \partial_1 f_T(y)(x_1 - y_1)| &\lesssim [f]_{\beta}^{\lambda} [f]_{\alpha}^{1-\lambda} \left(T^{\frac{1}{3}}\right)^{\lambda\beta} d(x, y)^{(1-\lambda)\alpha} \max\left(1, \frac{d(x, y)}{T^{\frac{1}{3}}}\right)^{\lambda}. \end{split}$$

If $n_0 \in \mathbb{Z}$ is the largest integer such that $2^{-\frac{n_0}{3}} \ge d(x, y)$, the same argument as in the previous case yields for $n \ge n_0$,

$$\begin{split} |f_{2^{-n}}(x) - f_{2^{-n}}(y) - \partial_{1}f_{2^{-n}}(y)(x_{1} - y_{1})| \\ &\leq |f_{2^{-n_{0}}}(x) - f_{2^{-n_{0}}}(y) - \partial_{1}f_{2^{-n_{0}}}(y)(x_{1} - y_{1})| + |f_{2^{-n}}(x) - f_{2^{-n_{0}}}(x)| \\ &+ |f_{2^{-n}}(y) - f_{2^{-n_{0}}}(y)| + |\partial_{1}f_{2^{-n}}(y) - \partial_{1}f_{2^{-n_{0}}}(y)||x_{1} - y_{1}| \\ &\leq [f]_{\beta}^{\lambda}[f]_{\alpha}^{1-\lambda} \left(\left(2^{-\frac{n_{0}}{3}}\right)^{\lambda\beta} d(x, y)^{(1-\lambda)\alpha} + \sum_{k=n_{0}}^{n-1} \left(2^{-\frac{k}{3}}\right)^{\gamma} + d(x, y) \sum_{k=n_{0}}^{n-1} \left(2^{-\frac{k}{3}}\right)^{\gamma-1} \right) \\ &\leq [f]_{\beta}^{\lambda}[f]_{\alpha}^{1-\lambda} d(x, y)^{\gamma}, \end{split}$$

which yields (A.7) by letting $n \to \infty$.

CASE $\gamma = 1$: Take a sequence $(\gamma_n) \subset (1, \frac{3}{2})$ such that $\gamma_n \searrow 1$, and consider the corresponding exponents $\lambda_n \to \lambda$. Then by [IO19, Remark 2] and the previous case, we have

$$[f]_1 \leq [f]_{\gamma_n} \leq [f]_{\beta}^{\lambda_n} [f]_{\alpha}^{1-\lambda_n},$$

with implicit constants independent of *n*. We can therefore perform the limit $n \to \infty$ to conclude the estimate for $\gamma = 1$.

CASE $\gamma = 0$ and f has vanishing average: If $[f]_{\alpha} = 0$, then $f \equiv 0,^{3^2}$ so (A.7) holds trivially. Assume that $[f]_{\alpha} \neq 0$. If $\alpha \in (0, 1]$, we have for every T > 0,

$$\begin{split} |f(x)| &\leq |f(x) - f_T(x)| + |f_T(x)| \leq \int_{\mathbb{R}^2} |\psi_T(z)| |f(x-z) - f(x)| \, \mathrm{d}z + \left(T^{\frac{1}{3}}\right)^{\beta} [f]_{\beta} \\ &\leq \left(T^{\frac{1}{3}}\right)^{\alpha} [f]_{\alpha} + \left(T^{\frac{1}{3}}\right)^{\beta} [f]_{\beta}, \end{split}$$

while if $\alpha \in (1, \frac{3}{2})$,

$$\begin{split} |f(x)| &\leq |f(x) - f_T(x)| + |f_T(x)| \leq \int_{\mathbb{R}^2} |\psi_T(z)| |f(x-z) - f(x) + \partial_1 f(x) z_1| \, \mathrm{d}z + \left(T^{\frac{1}{3}}\right)^{\beta} [f]_{\beta} \\ &\leq \left(T^{\frac{1}{3}}\right)^{\alpha} [f]_{\alpha} + \left(T^{\frac{1}{3}}\right)^{\beta} [f]_{\beta}, \end{split}$$

where we used $\int_{\mathbb{R}^2} z_1 \psi_T(z) \, dz = 0$ and Lemma A.1 in the case of distributions with vanishing average, as *T* can be larger than 1. Choosing $T^{\frac{1}{3}} = \left(\frac{[f]_{\beta}}{[f]_{\alpha}}\right)^{\frac{1}{\alpha-\beta}}$ leads to the conclusion.

Remark A.3. One can also prove that for $-\frac{1}{2} < \beta < 1 < \alpha < \frac{3}{2}$ the interpolation estimate

$$\partial_1 f \|_{L^{\infty}} \lesssim [f]^{\lambda}_{\beta} [f]^{1-\lambda}_{\alpha} \tag{A.12}$$

holds for $\lambda \in (0, 1)$ given by $1 = \lambda \beta + (1 - \lambda)\alpha$. Indeed, by Lemma A.2 (the case $\gamma = 0$) we know that

$$\|\partial_1 f\|_{L^{\infty}} \lesssim [\partial_1 f]^{\lambda}_{\beta-1} [\partial_1 f]^{1-\lambda}_{\alpha-1}.$$

By Definition 1.14, we have $[\partial_1 f]_{\beta-1} \leq [f]_{\beta}$ and by [IO19, Lemma 12], we know $[\partial_1 f]_{\alpha-1} \leq [f]_{\alpha}$, so the desired estimate follows.

We also need the following lemma for the Hilbert transform acting on Hölder spaces.

Lemma A.4.

(1) For $\beta \in (-\frac{3}{2}, 0)$ and $\varepsilon > 0$ such that $\beta - \varepsilon \in (-\frac{3}{2}, 0)$, there exists a constant C > 0 such that for every periodic distribution f,

$$[R_1 f]_{\beta - \varepsilon} \le C[f]_{\beta}. \tag{A.13}$$

(2) For $\alpha \in (0, \frac{3}{2})$ and $\varepsilon > 0$ such that $\alpha - \varepsilon \in (0, \frac{3}{2})$ there exists a constant C > 0 such that for every $f : \mathbb{T}^2 \to \mathbb{R}$,

$$[R_1 f]_{\alpha - \varepsilon} \le C[f]_{\alpha}. \tag{A.14}$$

Proof. To prove (A.13), we claim that for every $\varepsilon \in (0, \frac{3}{2})$ and $T \in (0, 1]$, we have that

$$\|R_1 f_T\|_{L^{\infty}} \lesssim \left(T^{\frac{1}{3}}\right)^{-\epsilon} \|f_{\frac{T}{2}}\|_{L^{\infty}}$$

Indeed, if we write Ψ_T for the periodization of ψ_T , by the semigroup property and Young's inequality for convolution we have for $p = \frac{5}{2\varepsilon} > 1$,

$$\|R_1 f_T\|_{L^{\infty}} \lesssim \|f_{\frac{T}{2}}\|_{L^p} \|R_1 \Psi_{\frac{T}{2}}\|_{L^{\frac{p}{p-1}}} \lesssim \|f_{\frac{T}{2}}\|_{L^p} \|\Psi_{\frac{T}{2}}\|_{L^{\frac{p}{p-1}}} \lesssim \|f_{\frac{T}{2}}\|_{L^{\infty}} \left(T^{\frac{1}{3}}\right)^{-\frac{5}{2p}}$$

³²This is clear for $\alpha \in (0, 1)$ since f has vanishing average. For $\alpha \in (1, \frac{3}{2})$ notice that by [IO19, Lemma 12], $\|\partial_1 f\|_{L^{\infty}} = 0$, hence by Definition 1.13 f is constant, and this constant is 0 since f has vanishing average.

where we used that the Hilbert transform is bounded³³ on $L^{\frac{p}{p-1}}(\mathbb{T}^2)$ for $\frac{p}{p-1} > 1$ and the bound $\|\Psi_{\frac{T}{2}}\|_{L^{\frac{p}{p-1}}} \leq (T^{\frac{1}{3}})^{-\frac{5}{2p}}$, which follows from Remark 1.12. Then (A.13) follows via the characterization of negative Hölder norms (A.1) and (A.2) if $\beta - \epsilon \neq -1, -\frac{1}{2}$. In those cases, consider $\gamma \in (\beta - \epsilon, \beta)$ and use the previous case together with [IO19, Remark 2].

Equation (A.14) is essentially [IO19, Lemma 7], noting that we can assume that f is of vanishing average, as R_1 annihilates constants and $[f - \int_{\mathbb{T}^2} f \, dx]_{\alpha} \leq [f]_{\alpha}$.

Lemma A.5. Let $\alpha \in (-\frac{3}{2}, \frac{3}{2}) \setminus \{0\}$ and s > 0 such that $\alpha - s \in (-\frac{3}{2}, \frac{3}{2}) \setminus \{-1, -\frac{1}{2}, 0, 1\}$. There exists a constant C > 0 such that for every periodic $f \in C^{\alpha}$,

$$[|\partial_1|^s f]_{\alpha-s} \le C[f]_{\alpha}.$$

Proof. Without loss of generality, we may assume that f is of vanishing average because $[f - \int_{\mathbb{T}^2} f \, dx]_{\alpha} \leq [f]_{\alpha}$ and $[|\partial_1|^s f]_{\alpha-s}$ is invariant by adding a constant to f. Then by the semigroup property, $|\partial_1|^s f_T = (\frac{T}{2})^{-\frac{s}{3}} f_{\frac{T}{2}} * (|\partial_1|^s \psi)_{\frac{T}{2}}$ for every $T \in (0, 1]$ and since $|\partial_1|^s \psi \in L^1(\mathbb{R}^2)$, we deduce $||T\mathcal{A}|\partial_1|^s f_T||_{L^{\infty}} \leq (T^{\frac{1}{3}})^{-s} ||T\mathcal{A}f_{\frac{T}{2}}||_{L^{\infty}}$. Hence, we obtain that

$$\left(T^{\frac{1}{3}}\right)^{-(\alpha-s)} \|T\mathcal{A}|\partial_1|^s f_T\|_{L^{\infty}} \lesssim \left(T^{\frac{1}{3}}\right)^{-\alpha} \|T\mathcal{A}f_{\frac{T}{2}}\|_{L^{\infty}},$$

and the conclusion follows by Lemma A.1.

Lemma A.6. Let $\alpha \in (-\frac{3}{2}, \frac{3}{2}) \setminus \{0\}$. For every sequence $\{f_n\}_{n \ge 1} \subset C^{\alpha}$ with $\sup_{n \ge 1} [f_n]_{\alpha} < \infty$, there exists $f \in C^{\alpha}$ with

$$[f]_{\alpha} \leq \liminf_{n \to \infty} [f_n]_{\alpha},$$

such that $f_n \to f$ in $C^{\alpha-\varepsilon}$ for every $\varepsilon > 0$ along a subsequence. In particular, the embedding $C^{\alpha} \hookrightarrow C^{\alpha-\varepsilon}$ is compact.

Proof. First, assume that $\alpha - \varepsilon \neq -1, -\frac{1}{2}, 0, 1$. Let $\{f_n\}_{n \ge 1} \subset C^{\alpha}$ such that $K := \sup_{n \ge 1} [f_n]_{\alpha} < \infty$,

and, if $\alpha \in (0, \frac{3}{2})$, we assume in addition that $\{\|f_n\|_{L^{\infty}}\}_{n \ge 1}$ is uniformly bounded (hence, up to subtraction, we may assume that f_n has zero average for any $n \in \mathbb{N}$).

By [IO19, Lemma 13] there exists $f \in C^{\alpha}$ such that $f_n \to f$ (along a subsequence) in the sense of distributions and

$$[f]_{\alpha} \le \liminf_{n \to \infty} [f_n]_{\alpha} \le K$$

For T, t > 0 we have that

$$\begin{aligned} \|(\mathcal{A}f - \mathcal{A}f_n)_T\|_{L^{\infty}} &\leq \|(\mathcal{A}f - \mathcal{A}f_t)_T\|_{L^{\infty}} + \|(\mathcal{A}f - \mathcal{A}f_n)_{t+T}\|_{L^{\infty}} \\ &+ \|((\mathcal{A}f_n)_t - \mathcal{A}f_n)_T\|_{L^{\infty}}. \end{aligned}$$
(A.15)

By (A.3), (A.5), and (A.16) below the following estimates hold

$$\begin{aligned} \|(\mathcal{A}f - \mathcal{A}f_t)_T\|_{L^{\infty}} &\lesssim \left(T^{\frac{1}{3}}\right)^{\alpha - \varepsilon - 3} [f - f_t]_{\alpha - \varepsilon} \lesssim K\left(T^{\frac{1}{3}}\right)^{\alpha - \varepsilon - 3} \left(t^{\frac{1}{3}}\right)^{\varepsilon}, \\ \|((\mathcal{A}f_n)_t - \mathcal{A}f_n)_T\|_{L^{\infty}} &\lesssim \left(T^{\frac{1}{3}}\right)^{\alpha - \varepsilon - 3} [(f_n)_t - f_n]_{\alpha - \varepsilon} \lesssim K\left(T^{\frac{1}{3}}\right)^{\alpha - \varepsilon - 3} \left(t^{\frac{1}{3}}\right)^{\varepsilon}. \end{aligned}$$

By Young's inequality for convolution and (1.18) we further have that

$$\|(\mathcal{A}f - \mathcal{A}f_n)_{t+T}\|_{L^{\infty}} \lesssim \left(T^{\frac{1}{3}}\right)^{\alpha-\varepsilon-3} \|(\mathcal{A}f - \mathcal{A}f_n)_t\|_{L^{\frac{5}{2(3-\alpha+\varepsilon)}}}$$

³³This follows from the fact that the Hilbert transform R_1 is bounded on $L^q(\mathbb{R}^2)$ for any $q \in (1, \infty)$ and the transference of multipliers method [Gra14, Theorem 4.3.7].

Since $f_n \to f$ in the sense of distributions, $(\mathcal{A}f - \mathcal{A}f_n)_t \to 0$ as $n \to \infty$ pointwise for every $t \in (0, 1]$, and by (A.3) and (A.5) we know that

$$\|(\mathcal{A}f - \mathcal{A}f_n)_t\|_{L^{\infty}} \lesssim \left(t^{\frac{1}{3}}\right)^{\alpha-3} [f - f_n]_{\alpha} \lesssim K\left(t^{\frac{1}{3}}\right)^{\alpha-3}$$

Hence, by dominated convergence theorem, $\|(\mathcal{A}f - \mathcal{A}f_n)_t\|_{L^{\frac{5}{2(3-\alpha+\varepsilon)}}} \to 0$ as $n \to \infty$ for every $t \in (0, 1]$. Taking $n \to \infty$ in (A.15) and using again (A.3) and (A.5) we obtain that

$$\limsup_{n\to\infty} [f-f_n]_{\alpha-\varepsilon} \lesssim K\left(t^{\frac{1}{3}}\right)^{\varepsilon},$$

which completes the proof if we let $t \rightarrow 0$.

If $\alpha - \varepsilon \in \{-1, -\frac{1}{2}, 0, 1\}$, consider $\gamma \in (\alpha - \varepsilon, \alpha)$; in view of [IO19, Remark 2] and the above result we then have $[f - f_n]_{\alpha - \varepsilon} \leq [f - f_n]_{\gamma} \to 0$ as $n \to \infty$.

Proposition A.7. For every $\alpha \in (-\frac{3}{2}, \frac{3}{2}) \setminus \{0\}$ and $\varepsilon > 0$ such that $\alpha - \varepsilon \in (-\frac{3}{2}, \frac{3}{2}) \setminus \{-1, -\frac{1}{2}, 0, 1\}$ the following estimate holds:

$$[f - f_t]_{\alpha - \varepsilon} \lesssim \left(t^{\frac{1}{3}}\right)^{\varepsilon} [f]_{\alpha}. \tag{A.16}$$

Proof. To prove (A.16) we use the definition of ψ_t and the semigroup property to estimate for $t, T \in (0, 1]$

$$\begin{aligned} \|\mathcal{A} \left(f - f_t\right)_T \|_{L^{\infty}} &\leq \int_T^{t+T} \|\mathcal{A}\partial_s f_s\|_{L^{\infty}} \, \mathrm{d}s = \int_T^{t+T} \|\mathcal{A}^2 f_s\|_{L^{\infty}} \, \mathrm{d}s \\ &= \int_T^{t+T} \|\mathcal{A}\psi_{\frac{s}{2}} * \mathcal{A}f_{\frac{s}{2}}\|_{L^{\infty}} \, \mathrm{d}s \leq \int_T^{t+T} \|\mathcal{A}\psi_{\frac{s}{2}}\|_{L^1} \|\mathcal{A}f_{\frac{s}{2}}\|_{L^{\infty}} \, \mathrm{d}s. \end{aligned}$$

Since $\|\mathcal{A}\psi_{\frac{s}{2}}\|_{L^1} \leq s^{-1}$, (A.3) and (A.5) imply that

$$\begin{aligned} \|\mathcal{A}(f-f_t)_T\|_{L^{\infty}} &\lesssim [f]_{\alpha} \int_T^{t+T} (s^{\frac{1}{3}})^{-3+\alpha} \frac{\mathrm{d}s}{s} \lesssim [f]_{\alpha} \left(T^{\frac{1}{3}}\right)^{-3+\alpha-\varepsilon} \int_T^{t+T} (s^{\frac{1}{3}})^{\varepsilon} \frac{\mathrm{d}s}{s} \\ &\lesssim [f]_{\alpha} \left(T^{\frac{1}{3}}\right)^{-3+\alpha-\varepsilon} \left(t^{\frac{1}{3}}\right)^{\varepsilon}, \end{aligned}$$

so that (A.16) follows from (A.3) and (A.5).

Appendix B. Besov spaces

In the next lemma we summarize some useful properties of the Besov seminorms that we often use in this article.

Lemma B.1. Let $0 < s < s' \le 1$, $1 \le p \le q < \infty$, and $j \in \{1, 2\}$. The following estimates hold

$$\|f\|_{\dot{\mathcal{B}}^{s'}_{\sigma:1}} \le [f]_{s'}, \tag{B.1}$$

$$\|f\|_{\dot{\mathcal{B}}^{s}_{p;2}} \leq [f]_{\frac{3}{2}s}, \qquad (B.2)$$

$$\|f\|_{\dot{\mathcal{B}}^{s}_{p;j}} \leq \|f\|_{\dot{\mathcal{B}}^{s'}_{p;j}} \quad and \quad \|f\|_{\dot{\mathcal{B}}^{s'}_{p;j}} \leq \|f\|_{\dot{\mathcal{B}}^{s'}_{q;j}}, \tag{B.3}$$

$$\||\partial_{j}|^{s}f\|_{L^{p}} \leq C(s,s')\|f\|_{\dot{\mathcal{B}}^{s'}_{p;j}},\tag{B.4}$$

for every function $f : \mathbb{T}^2 \to \mathbb{R}$, where the constant C(s, s') > 0 depends only on s and s'.

Proof. Estimates (B.1) and (B.2) are immediate from Definitions 1.13 and 2.1. Both estimates in (B.3) follow from Definition 2.1 and Jensen's inequality. To prove (B.4) we first notice that by a simple calculation of the Fourier coefficients we have the identity

$$|\partial_j|^s f(x) = k_s \int_{\mathbb{R}} \frac{\partial_j^h f(x)}{|h|^{s+1}} \,\mathrm{d}h, \quad x \in \mathbb{T}^2,$$

for some constant $k_s > 0$ which depends only on *s*, where we interpret the integral as a principle value. Then by Minkowski's inequality we get

$$\begin{split} \||\partial_{j}|^{s}f\|_{L^{p}} &\leq k_{s} \int_{\mathbb{R}} \left(\int_{\mathbb{T}^{2}} \frac{|\partial_{j}^{h}f(x)|^{p}}{|h|^{(s+1)p}} \, \mathrm{d}x \right)^{\frac{1}{p}} \, \mathrm{d}h \\ &\leq k_{s} \int_{|h| \leq 1} \left(\int_{\mathbb{T}^{2}} \frac{|\partial_{j}^{h}f(x)|^{p}}{|h|^{(s+1)p}} \, \mathrm{d}x \right)^{\frac{1}{p}} \, \mathrm{d}h + k_{s} \int_{|h| \geq 1} \left(\int_{\mathbb{T}^{2}} \frac{|\partial_{j}^{h}f(x)|^{p}}{|h|^{(s+1)p}} \, \mathrm{d}x \right)^{\frac{1}{p}} \, \mathrm{d}h. \end{split}$$

The first term in the last inequality is estimated by

$$\int_{|h| \le 1} \left(\int_{\mathbb{T}^2} \frac{|\partial_j^h f(x)|^p}{|h|^{(s+1)p}} \, \mathrm{d}x \right)^{\frac{1}{p}} \, \mathrm{d}h = 2 \int_0^1 \left(\int_{\mathbb{T}^2} \frac{|\partial_j^h f(x)|^p}{h^{(s+1)p}} \, \mathrm{d}x \right)^{\frac{1}{p}} \, \mathrm{d}h$$
$$\le 2 \int_0^1 \frac{h^{s'-s}}{h} \, \mathrm{d}h \, \|f\|_{\dot{\mathcal{B}}^{s'}_{p;j}} = \frac{2}{s'-s} \|f\|_{\dot{\mathcal{B}}^{s'}_{p;j}},$$

where we also used translation invariance of the torus and that s' > s. The second term is estimated by Minkowski's inequality

$$\int_{|h|\ge 1} \left(\int_{\mathbb{T}^2} \frac{|\partial_j^h f(x)|^p}{|h|^{(s+1)p}} \, \mathrm{d}x \right)^{\frac{1}{p}} \, \mathrm{d}h \le 4 \int_1^\infty \frac{1}{h^{s+1}} \, \mathrm{d}h \, \|f\|_{L^p} = \frac{4}{s} \|f\|_{L^p},$$

where we used again translation invariance of the torus. If f has vanishing average in x_j , then $\|f\|_{L^p} \leq \|f\|_{\dot{\mathcal{B}}^{s'}_{p;j}}$. Otherwise $\|f - \int_0^1 f \, \mathrm{d} x_j\|_{L^p} \leq \|f\|_{\dot{\mathcal{B}}^{s'}_{p;j}}$ and we can replace f by $f - \int_0^1 f \, \mathrm{d} x_j$. This completes the proof.

Lemma B.2.

(i) For every $s \in (0, 1)$, $p \ge 1$ and $f, g : \mathbb{T}^2 \to \mathbb{R}$ we have $\|fg\|_{\dot{B}^s_{p;1}} \le \|f\|_{\dot{B}^s_{p;1}} \|g\|_{L^{\infty}} + \|f\|_{L^p} [g]_s,$ $\|fg\|_{\dot{B}^s_{p;2}} \le \|f\|_{\dot{B}^s_{p;2}} \|g\|_{L^{\infty}} + \|f\|_{L^p} [g]_{\frac{3}{2}s}.$

(ii) For every $s \in (0, 1)$, there exists a constant C(s) > 0 depending only on s such that for every $\varepsilon \in (0, 1 - s)$ and $f, g : \mathbb{T}^2 \to \mathbb{R}$ we have

$$\begin{aligned} \||\partial_{1}|^{s}(fg)\|_{L^{2}} &\leq 2\||\partial_{1}|^{s}f\|_{L^{2}}\|g\|_{L^{\infty}} + \frac{C(s)}{\sqrt{\varepsilon}}\|f\|_{L^{2}}[g]_{s+\varepsilon}, \\ \||\partial_{2}|^{s}(fg)\|_{L^{2}} &\leq 2\||\partial_{2}|^{s}f\|_{L^{2}}\|g\|_{L^{\infty}} + \frac{C(s)}{\sqrt{\varepsilon}}\|f\|_{L^{2}}[g]_{\frac{3}{2}(s+\varepsilon)} \end{aligned}$$

Proof. (i) For $j \in \{1, 2\}$ we have that

$$\partial_j^h(fg)(x) = (\partial_j^h f)(x)g(x+he_j) + f(x)(\partial_j^h g)(x)$$

hence by Minkowski's inequality, $\|\partial_{j}^{h}(fg)\|_{L^{p}} \leq \|(\partial_{j}^{h}f)g(\cdot + he_{j})\|_{L^{p}} + \|f(\partial_{j}^{h}g)\|_{L^{p}}$. It follows that $\|fg\|_{\dot{B}^{s}_{p,j}} \leq \|f\|_{\dot{B}^{s}_{p,j}} \|g\|_{L^{\infty}} + \|f\|_{L^{p}}[g]_{s_{j}}$ with $s_{1} = s$ and $s_{2} = \frac{3}{2}s$.

(ii) By (2.2) we know that

$$\||\partial_{j}|^{2}(fg)\|_{L^{2}}^{2} = c_{s} \int_{\mathbb{R}} \frac{1}{|h|^{2s}} \int_{\mathbb{T}^{2}} |\partial_{j}^{h}(fg)(x)|^{2} dx \frac{dh}{|h|}$$

Similarly to (i) we can prove that

$$c_{s} \int_{\mathbb{R}} \frac{1}{|h|^{2s}} \int_{\mathbb{T}^{2}} |\partial_{j}^{h}(fg)(x)|^{2} dx \frac{dh}{|h|}$$

$$\leq 2c_{s} \int_{\mathbb{R}} \frac{1}{|h|^{2s}} \int_{\mathbb{T}^{2}} |\partial_{j}^{h}f(x)|^{2} dx \frac{dh}{|h|} ||g||_{L^{\infty}}^{2} + 2c_{s} ||f||_{L^{2}}^{2} \int_{\mathbb{R}} \frac{1}{|h|^{2s}} \sup_{x \in \mathbb{T}^{2}} |\partial_{j}^{h}g(x)|^{2} \frac{dh}{|h|}$$

$$= 2 |||\partial_j|^s f||_{L^2}^2 ||g||_{L^{\infty}}^2 + 2c_s ||f||_{L^2}^2 \int_{\mathbb{R}} \frac{1}{|h|^{2s}} \sup_{x \in \mathbb{T}^2} |\partial_j^h g(x)|^2 \frac{dh}{|h|},$$

where in the last step we used again (2.2). Using the periodicity of g and the fact that for h > 1 we can write $h = h_{\text{fr}} + h_{\text{int}}$ with $h_{\text{fr}} \in (0, 1]$ and $h_{\text{int}} \in \mathbb{Z}$, we notice that $|\partial_j^h g(x)| = |\partial_j^{h_{\text{fr}}} g(x)| \leq [g]_{s_j} h_{\text{fr}}^{s_j} \leq [g]_{s_j}$ with $s_j \in \{s + \varepsilon, \frac{3}{2}(s + \varepsilon)\}$. Then the result follows from

$$\left(\int_0^1 + \int_1^\infty\right) \frac{1}{h^{2s}} \sup_{x \in \mathbb{T}^2} |\partial_j^h g(x)|^2 \frac{\mathrm{d}h}{h} \lesssim \left(\frac{1}{\varepsilon} + \int_1^\infty \frac{\mathrm{d}h}{h^{1+2s}}\right) [g]_{s_j}^2.$$

Lemma B.3. For every $s \in (0, 1)$ and $\gamma \in (s, 1]$, there exists a constant $C(s, \gamma) > 0$ depending only on s and γ such that the following duality estimate holds for every $f, g : \mathbb{T}^2 \to \mathbb{R}$,

$$\left| \int_{\mathbb{T}^2} fg \, \mathrm{d}x \right| \le C(s,\gamma) \left(\||\partial_1|^{\gamma} f\|_{L^1} + \||\partial_2|^{\frac{2}{3}\gamma} f\|_{L^1} + \|f\|_{L^1} \right) [g]_{-s}.$$

Proof. By the mean value theorem and the definition of the kernel ψ_T we have that

$$\int_{\mathbb{T}^2} f(g-g_1) \, \mathrm{d}x = \int_0^{\frac{1}{2}} \int_{\mathbb{T}^2} f \, \partial_T g_{2T} \, \mathrm{d}x \, \mathrm{d}T = \int_0^{\frac{1}{2}} \int_{\mathbb{T}^2} f \, \left(|\partial_1|^3 - \partial_2^2 \right) g_{2T} \, \mathrm{d}x \, \mathrm{d}T.$$

Let $\psi^{(1)} = |\partial_1|^{3-\gamma}\psi$ and $\psi^{(2)} = |\partial_2|^{2-\frac{2}{3}\gamma}\psi$. Recalling that $|\partial_1|^{\alpha}\psi$, $|\partial_2|^{\alpha}\psi \in L^1(\mathbb{R}^2)$ for every $\alpha \ge 0,^{34}$ integrating by parts and using the semigroup property we obtain by (A.1) and Remark 1.12,

$$\begin{aligned} \left| \int_{0}^{\frac{1}{2}} \int_{\mathbb{T}^{2}} f\left(|\partial_{1}|^{3} - \partial_{2}^{2} \right) g_{2T} \, \mathrm{d}x \, \mathrm{d}T \right| &= \left| \int_{0}^{\frac{1}{2}} \int_{\mathbb{T}^{2}} \left(|\partial_{1}|^{3} - \partial_{2}^{2} \right) f_{T} \, g_{T} \, \mathrm{d}x \, \mathrm{d}T \right| \\ &\lesssim_{s} \int_{0}^{\frac{1}{2}} \int_{\mathbb{T}^{2}} T^{-\frac{3-\gamma}{3}} \left| |\partial_{1}|^{\gamma} f * \psi_{T}^{(1)} \right| \, [g]_{-s} T^{-\frac{s}{3}} \, \mathrm{d}x \, \mathrm{d}T \\ &+ \int_{0}^{\frac{1}{2}} \int_{\mathbb{T}^{2}} T^{-\frac{3-\gamma}{3}} \left| |\partial_{2}|^{\frac{2}{3}\gamma} f * \psi_{T}^{(2)} \right| \, [g]_{-s} T^{-\frac{s}{3}} \, \mathrm{d}x \, \mathrm{d}T \\ &\lesssim_{s} \left(|||\partial_{1}|^{\gamma} f||_{L^{1}} + |||\partial_{2}|^{\frac{2}{3}\gamma} f||_{L^{1}} \right) \, [g]_{-s} \int_{0}^{\frac{1}{2}} T^{-\frac{3-\gamma+s}{3}} \, \mathrm{d}T. \end{aligned}$$

The proof is complete since the last integral is finite for $\gamma > s$ and by (A.1) we also have the estimate

$$\left|\int_{\mathbb{T}^2} fg_1 dx\right| \leq \|f\|_{L^1} \|g_1\|_{L^{\infty}} \lesssim_s \|f\|_{L^1} [g]_{-s}.$$

The following Lemma establishes an optimal Sobolev embedding with respect to our anisotropic metric. Recall that in our context the (scaling) dimension of the space is dim = $\frac{5}{2}$, and ∂_2 costs as much as $\frac{3}{2}$ of ∂_1 . Therefore, the critical exponent of the embedding $H^1_{\text{anisotropic}} \subset L^{2^*}$ is given by $2^* = \frac{2 \dim}{\dim -2} = 10$.

³⁴As explained in [IO19, Proof of Lemma 10], the kernel ψ factorizes into a Gaussian in x_2 and a kernel $\varphi(x_1)$ that is smooth and $|x_1|^3 \partial_1^k \varphi \in L^{\infty}(\mathbb{R})$ for every $k \ge 0$ because in Fourier space $\xi_1^k e^{-|\xi_1|^3}$ has integrable derivatives up to order 3.

Lemma B.4. There exists a constant C > 0 such that for any function $f : \mathbb{T}^2 \to \mathbb{R}$ of vanishing average in x_1 ,

$$\|f\|_{L^{10}} \le C\left(\|\partial_1 f\|_{L^2} + \|\partial_2\|^{\frac{2}{3}} f\|_{L^2}\right),\tag{B.5}$$

$$\|f\|_{L^{\frac{10}{3}}} \le C\left(\||\partial_1|^{\frac{1}{2}}f\|_{L^2} + \||\partial_2|^{\frac{1}{3}}f\|_{L^2}\right),\tag{B.6}$$

$$\|f\|_{L^{\frac{5}{1-2\epsilon}}} \le C\left(\||\partial_{1}|^{\frac{3}{4}+\epsilon}f\|_{L^{2}} + \||\partial_{2}|^{\frac{2}{3}(\frac{3}{4}+\epsilon)}f\|_{L^{2}}\right), \quad \epsilon \in [0, \frac{1}{4}).$$
(B.7)

Proof. We first prove (B.5). For that, by the one-dimensional Gagliardo-Nirenberg-Sobolev inequalities, using that $f(\cdot, x_2)$ has vanishing average, we have that for every $x_1, x_2 \in [0, 1)$,

$$\begin{split} \|f(\cdot, x_2)\|_{L^{\infty}_{x_1}} &\lesssim \|\partial_1 f(\cdot, x_2)\|_{L^{2}_{x_1}}^{\frac{1}{6}} \|f(\cdot, x_2)\|_{L^{10}_{x_1}}^{\frac{2}{6}}, \\ \|f(x_1, \cdot)\|_{L^{\infty}_{x_2}} &\lesssim \||\partial_2|^{\frac{2}{3}} f(x_1, \cdot)\|_{L^{2}_{x_2}}^{\frac{3}{8}} \|f(x_1, \cdot)\|_{L^{10}_{x_2}}^{\frac{5}{8}} + \|f(x_1, \cdot)\|_{L^{1}_{x_2}}^{\frac{1}{8}}. \end{split}$$

Hence, Hölder's inequality implies

$$\begin{split} \left\| \|f\|_{L^{\infty}_{x_{1}}} \right\|_{L^{6}_{x_{2}}} &\lesssim \|\partial_{1}f\|_{L^{2}}^{\frac{1}{6}} \|f\|_{L^{10}}^{\frac{5}{6}}, \\ \left\| \|f\|_{L^{\infty}_{x_{2}}} \right\|_{L^{4}_{x_{1}}} &\lesssim \|\partial_{2}|^{\frac{2}{3}}f\|_{L^{2}}^{\frac{3}{8}} \|f\|_{L^{10}}^{\frac{5}{8}} + \|f\|_{L^{4}} &\lesssim \|\partial_{2}|^{\frac{2}{3}}f\|_{L^{2}}^{\frac{3}{8}} \|f\|_{L^{10}}^{\frac{5}{8}} + \|f\|_{L^{10}} \end{split}$$

Therefore, we obtain

$$\|f\|_{L^{10}} \le \left\|\|f\|_{L^{\infty}_{x_{1}}}\right\|_{L^{6}_{x_{2}}}^{\frac{3}{5}} \left\|\|f\|_{L^{\infty}_{x_{2}}}\right\|_{L^{4}_{x_{1}}}^{\frac{2}{5}} \le \|\partial_{1}f\|_{L^{2}}^{\frac{1}{10}} \|\partial_{2}|^{\frac{2}{3}}f\|_{L^{2}}^{\frac{3}{20}} \|f\|_{L^{10}}^{\frac{3}{4}} + \|\partial_{1}f\|_{L^{2}}^{\frac{1}{10}} \|f\|_{L^{10}}^{\frac{9}{10}}$$

It follows by Young's inequality that for any $\varepsilon \in (0, 1)$ there exists a constant $C(\varepsilon) > 0$ such that

$$\|f\|_{L^{10}} \leq \varepsilon \|f\|_{L^{10}} + C(\varepsilon) \left(\|\partial_1 f\|_{L^2} + \|\partial_2\|^{\frac{2}{3}} f\|_{L^2} \right),$$

from which (B.5) follows.

For the second inequality observe that $\frac{3}{10} = \frac{1}{2} \cdot \frac{1}{10} + \frac{1}{2} \cdot \frac{1}{2}$, so that (B.6) follows from complex interpolation (with change of measure). Indeed, by inequality (B.5) in the form

$$\|f\|_{L^{10}}^2 \lesssim \sum_{k \in (2\pi\mathbb{Z})^2} (|k_1|^2 + |k_2|^{\frac{4}{3}}) |\widehat{f}(k)|^2,$$

it follows that $\mathrm{Id} : L^2((2\pi\mathbb{Z})^2, w \,\mathrm{d}\chi) \to L^{10}(\mathbb{T}^2, \mathrm{d}x)$ is bounded, where $w(k) := |k_1|^2 + |k_2|^{\frac{4}{3}}$ and χ is the counting measure, while Parseval's identity $||f||_{L^2}^2 = \sum_{k \in (2\pi\mathbb{Z})^2} |\widehat{f}(k)|^2$ shows that $\mathrm{Id} : L^2((2\pi\mathbb{Z})^2, \mathrm{d}\chi) \to L^2(\mathbb{T}^2, \mathrm{d}x)$ is bounded. Hence, by interpolation (see [BL76, Corollary 5.5.4]), it follows that $\mathrm{Id} : L^2((2\pi\mathbb{Z})^2, w^{\frac{1}{2}} \,\mathrm{d}\chi) \to L^{\frac{10}{3}}(\mathbb{T}^2, \mathrm{d}x)$ is bounded, which implies (B.6). The bound (B.7) follows similarly by interpolation.

Lemma B.5. There exists a constant C > 0 such that for every $s_1 \in [0, 1]$ and $s_2 \in [0, \frac{2}{3}]$ with $s_1 + \frac{3}{2}s_2 \leq 1$ and every periodic function $f : \mathbb{T}^2 \to \mathbb{R}$ of vanishing average in x_1 the following holds

$$\||\partial_1|^{s_1}|\partial_2|^{s_2}f\|_{L^2}^2 \le C\mathcal{H}(f).$$

In particular, any sublevel set of \mathcal{E} (respectively \mathcal{H}) over \mathcal{W} is relatively compact in L^2 .

Proof. The desired estimate is immediate by Hölder's inequality in Fourier space and (2.15). To prove that the sublevel sets of \mathcal{E} (respectively \mathcal{H}) over \mathcal{W} are relatively compact, we notice that $\mathcal{H}(f)$ controls $\||\partial_1|^{\frac{1}{2}}f\|_{L^2}$ and $\||\partial_2|^{\frac{1}{2}}f\|_{L^2}$, hence also $\|f\|_{L^2}$ since f has vanishing average. By the compact embedding $H^{\frac{1}{2}} \subset L^2$ and (2.14), we deduce that any sublevel set of \mathcal{E} (respectively \mathcal{H}) over \mathcal{W} is relatively compact in L^2 .

Below we use the following notation for s > 0 and a periodic function $f : \mathbb{T}^2 \to \mathbb{R}$ with vanishing average in x_1 ,

$$|||f|||_{s}^{2} := \int_{\mathbb{T}^{2}} \left(|\partial_{1}|^{s} f \right)^{2} \, \mathrm{d}x + \int_{\mathbb{T}^{2}} \left(|\partial_{1}|^{-\frac{s}{2}} |\partial_{2}|^{s} f \right)^{2} \, \mathrm{d}x.$$
(B.8)

Lemma B.6. Let s > 0. There exists a constant C > 0 such that for any periodic function $f : \mathbb{T}^2 \to \mathbb{R}$ with vanishing average in x_1 there holds

$$\||\partial_1|^s f\|_{L^2} + \||\partial_2|^{\frac{2}{3}s} f\|_{L^2} \le C \|\|f\||_s.$$
(B.9)

Proof. Note that by the definition of $|||f|||_s$ we only have to bound $|||\partial_2|^{\frac{2}{3}s}f||_{L^2} \leq |||f|||_s$. This follows easily by an application of Hölder's inequality in Fourier space. Indeed, by (2.2) we have that³⁵

$$\begin{split} \int_{\mathbb{T}^2} ||\partial_2|^{\frac{2}{3}s} f|^2 \mathrm{d}x &= \sum_{k \in (2\pi\mathbb{Z})^2} |k_2|^{\frac{4}{3}s} |\widehat{f}(k)|^2 = \sum_{k \in (2\pi\mathbb{Z})^2} |k_1|^{-\frac{2}{3}s} |k_2|^{\frac{4}{3}s} |\widehat{f}(k)|^{\frac{4}{3}s} |k_1|^{\frac{2}{3}s} |\widehat{f}(k)|^{\frac{2}{3}s} \\ &\leq \left(\sum_{k \in (2\pi\mathbb{Z})^2} |k_1|^{-s} |k_2|^{2s} |\widehat{f}(k)|^2 \right)^{\frac{2}{3}} \left(\sum_{k \in (2\pi\mathbb{Z})^2} |k_1|^{2s} |\widehat{f}(k)|^2 \right)^{\frac{1}{3}s} \\ &= \left(\int_{\mathbb{T}^2} ||\partial_1|^{-\frac{s}{2}} |\partial_2|^s f|^2 \mathrm{d}x \right)^{\frac{2}{3}s} \left(\int_{\mathbb{T}^2} |\partial_1 f|^s \mathrm{d}x \right)^{\frac{1}{3}s} \\ &\leq \frac{2}{3} \int_{\mathbb{T}^2} ||\partial_1|^{-\frac{s}{2}} |\partial_2|^s f|^2 \mathrm{d}x + \frac{1}{3} \int_{\mathbb{T}^2} ||\partial_1|^s f|^2 \mathrm{d}x \leq \frac{2}{3} |||f|||_s^2. \end{split}$$

Lemma B.7. Let $\epsilon \in (0, \frac{1}{4}]$. There exists a constant C > 0 such that for any $f, g : \mathbb{T}^2 \to \mathbb{R}$ with vanishing average in x_1 there holds

$$\left\| |\partial_1|^{\frac{1}{2}} (fg) \right\|_{L^2} \le C \| |f||_1 \| |g||_{\frac{3}{4} + \epsilon}.$$

In particular, we have that $\||\partial_1|^{\frac{1}{2}}(fg)\|_{L^2} \leq C\mathcal{H}(f)^{\frac{1}{2}}\mathcal{H}(g)^{\frac{1}{2}}.$

Proof. By (2.2), writing $g^h := g(\cdot + h)$, we have

$$\begin{aligned} \left\| |\partial_{1}|^{\frac{1}{2}}(fg) \right\|_{L^{2}}^{2} &= \int_{\mathbb{R}} \int_{\mathbb{T}^{2}} |\partial_{1}^{h}(fg)|^{2} \, \mathrm{d}x \frac{\mathrm{d}h}{|h|^{2}} \\ &\lesssim \int_{\mathbb{R}} \int_{\mathbb{T}^{2}} |\partial_{1}^{h}f|^{2} |g^{h}|^{2} \, \mathrm{d}x \frac{\mathrm{d}h}{|h|^{2}} + \int_{\mathbb{R}} \int_{\mathbb{T}^{2}} |\partial_{1}^{h}g|^{2} |f|^{2} \, \mathrm{d}x \frac{\mathrm{d}h}{|h|^{2}}. \end{aligned}$$
(B.10)

To estimate the first term on the right-hand side of (B.10) we use Lemma B.4. We fix $\epsilon \in (0, \frac{1}{4}]$ and let $p'_{\epsilon} = \frac{5+\kappa_{\epsilon}}{2}$ with $\kappa_{\epsilon} = \frac{10\epsilon}{1-2\epsilon}$, and $p_{\epsilon} = \frac{5+\kappa_{\epsilon}}{3+\kappa_{\epsilon}} > 1$. Then $\frac{1}{p_{\epsilon}} + \frac{1}{p'_{\epsilon}} = 1$, so that by Hölder's inequality,

$$\begin{split} \int_{\mathbb{R}} \int_{\mathbb{T}^2} |\partial_1^h f|^2 |g^h|^2 \, \mathrm{d}x \frac{\mathrm{d}h}{|h|^2} &\leq \int_{\mathbb{R}} \left(\int_{\mathbb{T}^2} |\partial_1^h f|^{2p_\epsilon} \, \mathrm{d}x \right)^{\frac{1}{p_\epsilon}} \left(\int_{\mathbb{T}^2} |g^h|^{5+\kappa_\epsilon} \, \mathrm{d}x \right)^{\frac{2}{5+\kappa_\epsilon}} \frac{\mathrm{d}h}{|h|^2} \\ &= \|g\|_{L^{5+\kappa_\epsilon}}^2 \int_{\mathbb{R}} \left(\int_{\mathbb{T}^2} |\partial_1^h f|^{2p_\epsilon} \, \mathrm{d}x \right)^{\frac{1}{p_\epsilon}} \frac{\mathrm{d}h}{|h|^2}, \end{split}$$

³⁵Recall that f has vanishing average in x_1 , in particular $\hat{f}(0, k_2) = 0$ for all $k_2 \in 2\pi\mathbb{Z}$.

where in the last step we also used translation invariance. Note that the exponent $2p_{\epsilon} \in (2, 10)$, so that we may interpolate

$$\begin{split} \int_{\mathbb{R}} \left(\int_{\mathbb{T}^2} |\partial_1^h f|^{2p_{\epsilon}} \, \mathrm{d}x \right)^{\frac{1}{p_{\epsilon}}} \frac{\mathrm{d}h}{|h|^2} &= \int_{\mathbb{R}} \left(\int_{\mathbb{T}^2} |\partial_1^h f|^2 \, \mathrm{d}x \right)^{\theta_{\epsilon}} \left(\int_{\mathbb{T}^2} |\partial_1^h f|^{10} \, \mathrm{d}x \right)^{\frac{1-\theta_{\epsilon}}{5}} \frac{\mathrm{d}h}{|h|^2} \\ &\lesssim \|f\|_{L^{10}}^{2(1-\theta_{\epsilon})} \int_{\mathbb{R}} \left(\int_{\mathbb{T}^2} |\partial_1^h f|^2 \, \mathrm{d}x \right)^{\theta_{\epsilon}} \frac{\mathrm{d}h}{|h|^2}, \end{split}$$

with $\theta_{\epsilon} = \frac{1}{2} + \epsilon$. By Jensen's inequality, the mean-value theorem, and Poincaré's inequality for functions with zero average in x_1 , it then follows that

$$\int_{\mathbb{R}} \left(\int_{\mathbb{T}^2} |\partial_1^h f|^2 \, \mathrm{d}x \right)^{\theta_{\epsilon}} \frac{\mathrm{d}h}{|h|^2} \lesssim \|f\|_{L^2}^{1+2\epsilon} \left(\int_{|h| \ge 1} \frac{\mathrm{d}h}{|h|^2} \right)^{\theta_{\epsilon}} + \|\partial_1 f\|_{L^2}^{1+2\epsilon} \left(\int_{|h| < 1} \frac{\mathrm{d}h}{|h|^{1-2\epsilon}} \right)^{\theta_{\epsilon}} \lesssim \|\partial_1 f\|_{L^2}^{1+2\epsilon}.$$

By Lemmata B.4 and B.6 we can estimate $||f||_{L^{10}} \leq |||f|||_1$ and $||g||_{L^{5+\kappa_{\epsilon}}} \leq |||g|||_{\frac{3}{4}+\epsilon}$, hence

$$\int_{\mathbb{R}} \int_{\mathbb{T}^2} |\partial_1^h f|^2 |g^h|^2 \, \mathrm{d}x \frac{\mathrm{d}h}{|h|^2} \lesssim |||f|||_1^2 ||g|||_{\frac{3}{4}+\epsilon}^2$$

It remains to bound the second term on the right-hand side of (B.10). Similar to the fist term, we get by Hölder's inequality and interpolation that

$$\begin{split} \int_{\mathbb{R}} \int_{\mathbb{T}^2} |\partial_1^h g|^2 |f|^2 \, \mathrm{d}x \frac{\mathrm{d}h}{|h|^2} &\leq \int_{\mathbb{R}} \left(\int_{\mathbb{T}^2} |\partial_1^h g|^{\frac{5}{2}} \, \mathrm{d}x \right)^{\frac{3}{5}} \left(\int_{\mathbb{T}^2} |f|^{10} \, \mathrm{d}x \right)^{\frac{1}{5}} \frac{\mathrm{d}h}{|h|^2} \\ &\leq \||f\|_{L^{10}}^2 \int_{\mathbb{R}} \left(\int_{\mathbb{T}^2} |\partial_1^h g|^2 \, \mathrm{d}x \right)^{\frac{2}{3}} \left(\int_{\mathbb{T}^2} |\partial_1^h g|^5 \, \mathrm{d}x \right)^{\frac{2}{15}} \frac{\mathrm{d}h}{|h|^2} \\ &\lesssim \||f\|_{L^{10}}^2 \|g\|_{L^5}^{\frac{2}{3}} \int_{\mathbb{R}} \left(\int_{\mathbb{T}^2} |\partial_1^h g|^2 \, \mathrm{d}x \right)^{\frac{2}{3}} \frac{\mathrm{d}h}{|h|^2}. \end{split}$$

If $\varepsilon < \frac{1}{4}$, splitting the integral in *h*, it follows with Jensen's inequality that

$$\begin{split} \int_{\mathbb{R}} \left(\int_{\mathbb{T}^2} |\partial_1^h g|^2 \, \mathrm{d}x \right)^{\frac{2}{3}} \frac{\mathrm{d}h}{|h|^2} &= \int_{|h| \ge 1} \left(\int_{\mathbb{T}^2} |\partial_1^h g|^2 \, \mathrm{d}x \right)^{\frac{2}{3}} \frac{\mathrm{d}h}{|h|^2} + \int_{|h| < 1} \left(\int_{\mathbb{T}^2} |\partial_1^h g|^2 \, \mathrm{d}x \right)^{\frac{2}{3}} \frac{\mathrm{d}h}{|h|^2} \\ &\lesssim \|g\|_{L^2}^{\frac{4}{3}} \left(\int_{|h| \ge 1} \frac{\mathrm{d}h}{|h|^2} \right)^{\frac{2}{3}} + \int_{|h| < 1} \left(\int_{\mathbb{T}^2} \frac{|\partial_1^h g|^2}{|h|^{\frac{3}{2} + 6\epsilon}} \, \mathrm{d}x \right)^{\frac{2}{3}} \frac{\mathrm{d}h}{|h|^{1 - 4\epsilon}} \\ &\lesssim \|g\|_{L^2}^{\frac{4}{3}} + \left(\int_{|h| < 1} \int_{\mathbb{T}^2} \frac{|\partial_1^h g|^2}{|h|^{\frac{3}{2} + 2\epsilon}} \, \mathrm{d}x \frac{\mathrm{d}h}{|h|} \right)^{\frac{2}{3}} \lesssim \|g\|_{L^2}^{\frac{4}{3}} + \||\partial_1|^{\frac{3}{4} + \epsilon}g\|_{L^2}^{\frac{4}{3}}. \end{split}$$

Using again Lemmata B.4 and B.6, we may bound $||f||_{L^{10}} \leq |||f|||_1$ and $||g||_{L^5} \leq |||g|||_{\frac{3}{4}+\epsilon}$, the conclusion follows with $||g||_{L^2} \leq |||g|||_{\frac{3}{4}+\epsilon}$. If $\epsilon = \frac{1}{4}$, we use instead the estimate

$$\int_{|h|<1} \left(\int_{\mathbb{T}^2} |\partial_1^h g|^2 \, \mathrm{d}x \right)^{\frac{2}{3}} \frac{\mathrm{d}h}{|h|^2} \le \|\partial_1 g\|_{L^2}^{\frac{4}{3}} \int_{|h|<1} \frac{\mathrm{d}h}{|h|^{\frac{2}{3}}} \le \|\|g\|\|_1^{\frac{4}{3}}.$$

As for the Sobolev embedding in Lemma B.4, we next prove the embedding $H_{\text{anistropic}}^1 \subset C^{\alpha}$, which is optimal and the critical exponent is given by $\alpha = 1 - \frac{\dim}{2} = -\frac{1}{4}$.

Lemma B.8 (Besov embedding into Hölder spaces). There exists a constant C > 0 such that for any periodic distribution $f : \mathbb{T}^2 \to \mathbb{R}$ of vanishing average, and $s \in (0, \frac{11}{4}) \setminus \{\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{9}{4}\}$, there holds

$$[f]_{-\frac{5}{4}+s} \le C(||\partial_1|^s f||_{L^2} + ||\partial_2|^{\frac{4}{3}s} f||_{L^2}), \tag{B.11}$$

In particular, $[w]_{-\frac{1}{4}} \leq C\mathcal{H}(w)^{\frac{1}{2}}$ for every $w \in \mathcal{W}$.

Proof. Since *f* is of vanishing average, by (A.3) and (A.5) we know that for $s \in (0, \frac{11}{4}) \setminus \{\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{9}{4}\}$,

$$[f]_{-\frac{5}{4}+s} \sim \sup_{T \in (0,1]} \left(T^{\frac{1}{3}}\right)^{\frac{5}{4}-s} \|T\mathcal{A}f_T\|_{L^{\infty}}$$

Writing $|\partial_1|^3 f_T = |\partial_1|^s f * |\partial_1|^{3-s} \psi_T$ and $\partial_2^2 f_T = |\partial_2|^{\frac{2}{3}s} f * |\partial_2|^{2-\frac{2}{3}s} \psi_T$, and using that $|\partial_j|^{\alpha} \psi \in L^2(\mathbb{R}^2)$ for every $\alpha \ge 0$ and j = 1, 2, we deduce with Young's inequality for convolution of functions (as in Remark 1.12) that

$$\begin{split} \|\mathcal{A}f_{T}\|_{L^{\infty}} &\leq \||\partial_{1}|^{s}f\|_{L^{2}}\||\partial_{1}|^{3-s}\psi_{T}\|_{L^{2}} + \||\partial_{2}|^{\frac{2}{3}s}f\|_{L^{2}}\||\partial_{2}|^{2-\frac{s}{3}s}\psi_{T}\|_{L^{2}} \\ &\lesssim \left(T^{\frac{1}{3}}\right)^{-3+s-\frac{5}{4}} \left(\||\partial_{1}|^{s}f\|_{L^{2}} + \||\partial_{2}|^{\frac{2}{3}s}f\|_{L^{2}}\right). \end{split}$$

This implies that

$$[f]_{-\frac{5}{4}+s} \sim \sup_{T \in \{0,1\}} \left(T^{\frac{1}{3}}\right)^{\frac{5}{4}-s} \|T\mathcal{A}f_T\|_{L^{\infty}} \lesssim \||\partial_1|^s f\|_{L^2} + \||\partial_2|^{\frac{2}{3}s} f\|_{L^2}.$$

Combining (B.11) for s = 1 and (2.15), we conclude that $[w]_{-\frac{1}{4}} \leq \mathcal{H}(w)^{\frac{1}{2}}$ for every $w \in \mathcal{W}$.

The next proposition is the classical 1-dimensional embedding of Besov spaces into L^p spaces in the periodic setting. 36

Proposition B.9. For every $p \in (1, \infty]$ and $q \in [1, p]$ with $(p, q) \neq (\infty, 1)$, there exists a constant C(p, q) > 0 such that for every periodic $f : [0, 1) \rightarrow \mathbb{R}$ with vanishing average

$$\left(\int_{0}^{1} |f(z)|^{p} \, \mathrm{d}z\right)^{\frac{1}{p}} \leq C(p,q) \int_{0}^{1} \frac{1}{h^{\frac{1}{q}-\frac{1}{p}}} \left(\int_{0}^{1} |\partial_{1}^{h}f(z)|^{q} \, \mathrm{d}z\right)^{\frac{1}{q}} \frac{\mathrm{d}h}{h},\tag{B.12}$$

with the usual interpretation for $p = \infty$ or $q = \infty$.

Proof. We prove the statement for $p \in (1, \infty)$ (the case $p = \infty$ follows similarly). For $T \in (0, 1]$ let $\varphi_T(h) = \frac{1}{\sqrt{4\pi T}} e^{-\frac{h^2}{4T}}$, $h \in \mathbb{R}$, be the heat semigroup, and denote its periodization by

$$\Phi_T(h) = \frac{1}{\sqrt{4\pi T}} \sum_{k \in \mathbb{Z}} e^{-\frac{(h-k)^2}{4T}} = \sum_{k \in \mathbb{Z}} e^{-4\pi^2 k^2 T} e^{2\pi i k h}, \quad h \in [0,1).$$

Note that Φ_T is smooth, $\Phi_T \ge 0$, $\|\Phi_T\|_{L^1} = \int_0^1 \Phi_T \, dh = 1$, and for every $T \in (0, 1]$

$$\|\Phi_T\|_{L^{\infty}} \leq 1 + \sum_{k \in \mathbb{Z} \setminus \{0\}} e^{-4\pi^2 k^2 T} \lesssim 1 + \int_{\mathbb{R}} \widehat{\varphi}_T(\xi) \, \mathrm{d}\xi \lesssim 1 + \varphi_T(0) \lesssim \frac{1}{\sqrt{T}}.$$

Therefore, by interpolation, for every $r \in [1, \infty]$ and $T \in (0, 1]$ we have

$$\|\Phi_T\|_{L^r} \lesssim \sqrt{T}^{-(1-\frac{1}{r})}.$$
 (B.13)

We also claim that for every $T \in (0, 1]$

$$\sup_{|h| \le \frac{1}{2}} \frac{h^2}{\sqrt{T}} \Phi_T(h) \lesssim 1.$$
(B.14)

³⁶The nonperiodic version of the statement is essentially a combination of [BCD11, Proposition 2.20 and Theorem 2.36].

Indeed, using that for every $k \in \mathbb{Z} \setminus \{0\}$, $|h| \leq \frac{1}{2}$, and $T \in (0, 1]$, $\frac{(h-k)^2}{4T} \geq \frac{(|k|-\frac{1}{2})^2}{4T} \geq \frac{1}{16}$, we obtain³⁷ $e^{-\frac{(h-k)^2}{4T}} \leq \frac{4T}{(h-k)^2} \leq \frac{T}{k^2}$. It then follows that

$$\frac{\sqrt{4\pi}h^2}{\sqrt{T}}\Phi_T(h) = \frac{h^2}{T}e^{-\frac{h^2}{4T}} + \sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{h^2}{T}e^{-\frac{(h-k)^2}{4T}} \lesssim \sup_{z\in\mathbb{R}}z^2e^{-\frac{z^2}{4}} + h^2\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{1}{k^2} \lesssim 1.$$

By the semigroup property and the periodicity of f we know that for $z \in (-\frac{1}{2}, \frac{1}{2})$,

$$f * \Phi_{2T}(z) - f * \Phi_T(z) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\partial_1^{-h} f * \Phi_T(z) \right) \Phi_T(h) \, \mathrm{d}h.$$

Using Minkowski's inequality and Young's inequality for convolution with exponents $1 + \frac{1}{p} = \frac{1}{r} + \frac{1}{q}$ with $r \in [1, \infty)$ we deduce by (B.13),

$$\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |f * \Phi_{2T}(z) - f * \Phi_{T}(z)|^{p} dz \right)^{\frac{1}{p}} \lesssim \int_{-\frac{1}{2}}^{\frac{1}{2}} \Phi_{T}(h) \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |\partial_{1}^{-h} f * \Phi_{T}(z)|^{p} dz \right)^{\frac{1}{p}} dh$$

$$\lesssim \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |\Phi_{T}(z)|^{r} dz \right)^{\frac{1}{r}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Phi_{T}(h) \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |\partial_{1}^{h} f(z)|^{q} dz \right)^{\frac{1}{q}} dh$$

$$\lesssim \frac{1}{\sqrt{T}^{\left(\frac{1}{q} - \frac{1}{p}\right)}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Phi_{T}(h) \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |\partial_{1}^{h} f(z)|^{q} dz \right)^{\frac{1}{q}} dh,$$

$$(B.15)$$

where we also used a change of variables and periodicity to replace -h by h.

To prove (B.12) we now write

$$f(z) = \sum_{k=1}^{\infty} \left(f * \Phi_{2 \cdot 2^{-k}}(z) - f * \Phi_{2^{-k}}(z) \right) - f * \Phi_1(z),$$

and obtain by (B.15) that

$$\left(\int_{0}^{1} |f(z)|^{p} dz\right)^{\frac{1}{p}} = \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |f(z)|^{p} dz\right)^{\frac{1}{p}} \lesssim \Sigma^{(1)} + \Sigma^{(2)} + \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |f * \Phi_{1}(z)|^{p} dz\right)^{\frac{1}{p}},$$
(B.16)

where

$$\Sigma^{(1)} = \sum_{k=1}^{\infty} \sqrt{2}^{k\left(\frac{1}{q} - \frac{1}{p}\right)} \int_{\{|h| \le \frac{1}{2}: |h|\sqrt{2}^{k} \le 1\}} \Phi_{2^{-k}}(h) \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |\partial_{1}^{h}f(z)|^{q} dz\right)^{\frac{1}{q}} dh,$$

$$\Sigma^{(2)} = \sum_{k=1}^{\infty} \sqrt{2}^{k\left(\frac{1}{q} - \frac{1}{p}\right)} \int_{\{|h| \le \frac{1}{2}: |h|\sqrt{2}^{k} > 1\}} \Phi_{2^{-k}}(h) \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |\partial_{1}^{h}f(z)|^{q} dz\right)^{\frac{1}{q}} dh.$$

Using that by (B.13), $\sup_{|h| \le \frac{1}{2}} |\Phi_{2^{-k}}(h)| \le \sqrt{2}^k$, we can estimate $\Sigma^{(1)}$ as follows:

$$\begin{split} \Sigma^{(1)} &\lesssim \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{\{k \ge 1: \, |h| \sqrt{2}^k \le 1\}} \sqrt{2}^{k\left(\frac{1}{q} - \frac{1}{p} + 1\right)} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |\partial_1^h f(z)|^q \, \mathrm{d}z \right)^{\frac{1}{q}} \, \mathrm{d}h \\ &\lesssim \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{|h|^{\frac{1}{q} - \frac{1}{p}}} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |\partial_1^h f(z)|^q \, \mathrm{d}z \right)^{\frac{1}{q}} \frac{\mathrm{d}h}{|h|}. \end{split}$$

³⁷Recall that $e^{-x} \leq \frac{1}{x}$ for $x \geq \frac{1}{16}$.

For $\Sigma^{(2)}$, by (B.14) and the fact that $\frac{1}{q} - \frac{1}{p} - 1 < 0$ (since $(p, q) \neq (\infty, 1)$), we have

$$\begin{split} \Sigma^{(2)} &\lesssim \sum_{k=1}^{\infty} \sqrt{2}^{k \left(\frac{1}{q} - \frac{1}{p} - 1\right)} \int_{\{|h| \leq \frac{1}{2}: |h| \sqrt{2}^{k} > 1\}} \frac{1}{|h|} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |\partial_{1}^{h} f(z)|^{q} \, \mathrm{d}z \right)^{\frac{1}{q}} \frac{\mathrm{d}h}{|h|} \\ &\lesssim_{\frac{1}{q} - \frac{1}{p}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{|h|^{\frac{1}{q} - \frac{1}{p}}} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |\partial_{1}^{h} f(z)|^{q} \, \mathrm{d}z \right)^{\frac{1}{q}} \frac{\mathrm{d}h}{|h|}. \end{split}$$

For the last term in (B.16) we use that f has vanishing average and periodicity, which implies that $\int_{-\frac{1}{2}}^{\frac{1}{2}} f * \Phi_1(z+h) dh = 0$ for every $z \in (-\frac{1}{2}, \frac{1}{2})$, to obtain the bound

$$\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |f * \Phi_{1}(z)|^{p} dz \right)^{\frac{1}{p}} = \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| f * \Phi_{1}(z) - \int_{-\frac{1}{2}}^{\frac{1}{2}} f * \Phi_{1}(z+h) dh \right|^{p} dz \right)^{\frac{1}{p}}$$

$$\leq \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |\partial_{1}^{h} f * \Phi_{1}(z)|^{p} dz \right)^{\frac{1}{p}} dh \lesssim \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |\partial_{1}^{h} f(z)|^{q} dz \right)^{\frac{1}{q}} dh$$

$$\leq \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{|h|^{\frac{1}{q} - \frac{1}{p}}} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |\partial_{1}^{h} f(z)|^{q} dz \right)^{\frac{1}{q}} \frac{dh}{|h|},$$

$$(B.17)$$

where we also used Minkowski's inequality, Young's inequality for convolution and the fact that $\frac{1}{q} - \frac{1}{p} + 1 > 0$. The right hand side of (B.17) is estimated by twice the right hand side of (B.12), so the conclusion follows.

The next lemma allows us to connect the estimate (2.4) with regularity in Besov spaces.

Lemma B.10. There exists a constant C > 0 such that for every $s \in (0, 1)$, $p \in [1, \infty)$ and every periodic function $f : [0, 1) \rightarrow \mathbb{R}$ the following estimate holds:

$$\sup_{h \in \{0,1\}} \frac{1}{h^s} \left(\int_0^1 |\partial_1^h f(z)|^p \, \mathrm{d}z \right)^{\frac{1}{p}} \le C \sup_{h \in \{0,1\}} \frac{1}{h^s} \left(\frac{1}{h} \int_0^h \int_0^1 |\partial_1^{h'} f(z)|^p \, \mathrm{d}z \, \mathrm{d}h' \right)^{\frac{1}{p}}. \tag{B.18}$$

Proof. Let $h \in (0, 1]$. Then for $h' \in (0, h]$ we have that

$$\int_0^1 |\partial_1^h f(z)|^p \, \mathrm{d}z \le 2^{p-1} \int_0^1 |\partial_1^{h'} f(z)|^p \, \mathrm{d}z + 2^{p-1} \int_0^1 |\partial_1^{h-h'} f(z+h')|^p \, \mathrm{d}z$$

Integrating over $h' \in [\frac{h}{2}, h]$ we obtain that

$$h\int_{0}^{1} |\partial_{1}^{h}f(z)|^{p} dz \leq 2^{p}\int_{\frac{h}{2}}^{h}\int_{0}^{1} |\partial_{1}^{h'}f(z)|^{p} dz dh' + 2^{p}\int_{\frac{h}{2}}^{h}\int_{0}^{1} |\partial_{1}^{h-h'}f(z+h')|^{p} dz dh'.$$

By the change of variables h'' = h - h' and z' = z + h - h'', upon relabelling, we see that

$$\int_{\frac{h}{2}}^{h} \int_{0}^{1} |\partial_{1}^{h-h'} f(z+h')|^{p} \, \mathrm{d}z \, \mathrm{d}h' = \int_{0}^{\frac{h}{2}} \int_{0}^{1} |\partial_{1}^{h'} f(z)|^{p} \, \mathrm{d}z \, \mathrm{d}h',$$

where we also used periodicity in z. Hence we have proved that

$$\int_0^1 |\partial_1^h f(z)|^p \, \mathrm{d}z \le \frac{2^p}{h} \int_0^h \int_0^1 |\partial_1^{h'} f(z)|^p \, \mathrm{d}z \, \mathrm{d}h',$$

which in turn implies (B.18).

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Appendix C. Stochastic estimates

We show that solutions of the linearized equation

$$\mathcal{L}v = P\xi \tag{C.1}$$

almost surely have (negative) infinite total energy (see (1.1) for the definition) under the law of white noise.

Proposition C.1. Assume that $\langle \cdot \rangle$ is the law of white noise. If v is the solution of vanishing average in x_1 -direction to $\mathcal{L}v = P\xi$, then $E_{tot}(v) = -\infty \langle \cdot \rangle$ -almost surely.

Proof. Recall that in Fourier space we have an explicit representation of the solution to (C.1) as

$$\widehat{v}(k) = \frac{\xi(k)}{k_1^2 + |k_1|^{-1}k_2^2}$$
 for $k_1 \neq 0$ and $\widehat{v}(0, k_2) = 0$ for all $k_2 \in 2\pi\mathbb{Z}$.

A short calculation shows that the harmonic part of the energy is

$$\mathcal{H}(v) = \int_{\mathbb{T}^2} \xi v \, \mathrm{d} x$$

so that

$$E_{tot}(v) = -\mathcal{H}(v) + \int_{\mathbb{T}^2} \left(|\partial_1|^{-\frac{1}{2}} \partial_1 \frac{v^2}{2} \right)^2 \, \mathrm{d}x - 2 \int_{\mathbb{T}^2} \left(|\partial_1|^{-\frac{1}{2}} \partial_1 \frac{v^2}{2} \right) \left(|\partial_1|^{-\frac{1}{2}} \partial_2 v \right) \, \mathrm{d}x.$$

By Young's inequality, we have

$$E_{tot}(v) \leq -\frac{1}{2}\mathcal{H}(v) + 3\int_{\mathbb{T}^2} \left(|\partial_1|^{-\frac{1}{2}} \partial_1 \frac{v^2}{2} \right)^2 \,\mathrm{d}x. \tag{C.2}$$

By (B.1), (B.4) and [IO19, Lemma 12], we may estimate

$$\int_{\mathbb{T}^2} \left(|\partial_1|^{-\frac{1}{2}} \partial_1 \frac{v^2}{2} \right)^2 \, \mathrm{d}x = \int_{\mathbb{T}^2} \left(R_1 |\partial_1|^{\frac{1}{2}} \frac{v^2}{2} \right)^2 \, \mathrm{d}x = \int_{\mathbb{T}^2} \left(|\partial_1|^{\frac{1}{2}} \frac{v^2}{2} \right)^2 \, \mathrm{d}x \lesssim \left[v^2 \right]_{\frac{1}{2} + \varepsilon}^2 \lesssim \left[v \right]_{\frac{3}{4} - \varepsilon}^4,$$

where $[v]_{\frac{3}{4}-\varepsilon}$ is finite $\langle \cdot \rangle$ -almost surely by (5.7)

We next show that

k

$$\mathcal{H}(v) = \sum_{k_1 \neq 0} \frac{|\xi(k)|^2}{k_1^2 + |k_1|^{-1}k_2^2}$$

diverges $\langle \cdot \rangle$ -almost surely. Since $\hat{\xi}(-k) = -\overline{\hat{\xi}(k)}$, we have that

$$\mathcal{H}(v) = 2 \sum_{k \in 2\pi\mathbb{Z}^2 \setminus \{k_1 \le 0\}} \frac{|\xi(k)|^2}{k_1^2 + |k_1|^{-1}k_2^2} =: 2 \sum_{k \in 2\pi\mathbb{Z}^2 \setminus \{k_1 \le 0\}} a_k |\widehat{\xi}(k)|^2.$$

By the independence of $\{|\hat{\xi}(k)|\}_{k \in 2\pi\mathbb{Z}^2 \setminus \{k_1 \leq 0\}}$ and Kolmogorov's 0-1 law, we know that the probability of the event $\{\mathcal{H}(v) = +\infty\}$ is either 1 or 0. Hence, it is enough to show that $\langle \{\mathcal{H}(v) = +\infty\} \rangle > 0$. We first notice that

$$\sum_{\epsilon \ge \pi \mathbb{Z}^2 \setminus \{k_1 \le 0\}} a_k |\widehat{\xi}(k)|^2 \ge \sum_{k \in 2\pi \mathbb{Z}^2 \setminus \{k_1 \le 0\}} a_k \mathbf{1}_{\{|\widehat{\xi}(k)|^2 \ge 1\}} =: \sum_{k \in 2\pi \mathbb{Z}^2 \setminus \{k_1 \le 0\}} a_k X_k$$

and since the random variables $\xi(k)$ are identically distributed, there exists $p \in (0, 1)$ such that $\langle X_k \rangle = p$ for every $k \in 2\pi \mathbb{Z}^2 \setminus \{k_1 \leq 0\}$. Given $M \geq 1$, there exists a finite subset $J \subset 2\pi \mathbb{Z}^2 \setminus \{k_1 \leq 0\}$ such that $\frac{p}{2} \sum_{k \in J} a_k \geq M$ and the following estimate holds

$$p\sum_{k\in J}a_k = \left\langle\sum_{k\in J}a_kX_k\right\rangle$$

1

$$\leq \frac{p}{2} \sum_{k \in J} a_k \left(1 - \left(\left\{ \sum_{k \in J} a_k X_k \ge M \right\} \right) \right) + \sum_{k \in J} a_k \left(\left\{ \sum_{k \in J} a_k X_k \ge M \right\} \right)$$

Then, it is easy to see that

$$\left\langle \left\{ \sum_{k \in J} a_k X_k \ge M \right\} \right\rangle \ge \frac{p}{2-p} > 0,$$

which in turn implies

$$\left\langle \left\{ \sum_{k \in 2\pi \mathbb{Z}^2 \setminus \{k_1 \le 0\}} a_k X_k = +\infty \right\} \right\rangle = \lim_{M \uparrow +\infty} \left\langle \left\{ \sum_{k \in 2\pi \mathbb{Z}^2 \setminus \{k_1 \le 0\}} a_k X_k \ge M \right\} \right\rangle \ge \frac{p}{2-p} > 0.$$

Thus, we obtain that

$$\langle \{\mathcal{H}(v) = +\infty\} \rangle \ge \left\langle \left\{ \sum_{k \in 2\pi \mathbb{Z}^2 \setminus \{k_1 \le 0\}} a_k X_k = +\infty \right\} \right\rangle > 0,$$

which proves the desired claim.

The next lemma is a Kolmogorov-type criterion for periodic random fields.

Lemma C.2. Let $\{g(x)\}_{x \in \mathbb{T}^2}$ be a random field and assume that for some $\alpha \in (0, \frac{3}{2})$ and every $1 \le p < \infty$

$$\sup_{T\in(0,1]} \left(T^{\frac{1}{3}}\right)^{\alpha} \sup_{x\in\mathbb{T}^2} \langle |g_T(x)|^p \rangle^{\frac{1}{p}} < \infty.$$

Then, for every $\varepsilon \in (0, \frac{3}{2} - \alpha)$, $g \in C^{-\alpha-\varepsilon}$ almost surely and for every $1 \le p < \infty$ there exists $C(\varepsilon, p) > 0$ such that

$$\left\langle \left[g\right]_{-\alpha-\varepsilon}^{p}\right\rangle^{\frac{1}{p}} \leq C(\varepsilon,p).$$

Proof. First assume that $\alpha + \varepsilon \neq 1, \frac{1}{2}$. Let $p_{\varepsilon} > \frac{5}{2\varepsilon}$. We first claim that

$$[g]_{-\alpha-\varepsilon} \lesssim \sum_{n\geq 1} \left(2^{\frac{1}{3}}\right)^{-n\left(\varepsilon-\frac{5}{2p_{\varepsilon}}\right)} \left(2^{\frac{1}{3}}\right)^{-n\alpha} \|g_{2^{-n}}\|_{L^{p_{\varepsilon}}}.$$

Indeed, for every $T \in (0, 1]$ we can find $n \ge 1$ such that $2^{-n} < T \le 2^{-n+1}$ and by the semigroup property and Remark 1.12, we obtain

$$\left(T^{\frac{1}{3}}\right)^{\alpha+\varepsilon} \|g_{T}\|_{L^{\infty}} \lesssim \left(2^{\frac{1}{3}}\right)^{-n(\alpha+\varepsilon)} \|(g_{2^{-n}})_{T-2^{-n}}\|_{L^{\infty}} \lesssim \left(2^{\frac{1}{3}}\right)^{-n(\alpha+\varepsilon)} \|g_{2^{-n}}\|_{L^{\infty}}$$
$$\lesssim \left(2^{\frac{1}{3}}\right)^{-n(\alpha+\varepsilon)} \left(2^{\frac{1}{3}}\right)^{(n+1)\frac{5}{2p_{\varepsilon}}} \|g_{2^{-n-1}}\|_{L^{p_{\varepsilon}}}$$
$$\lesssim \sum_{n\geq 1} \left(2^{\frac{1}{3}}\right)^{-n\left(\varepsilon-\frac{5}{2p_{\varepsilon}}\right)} \left(2^{\frac{1}{3}}\right)^{-n\alpha} \|g_{2^{-n}}\|_{L^{p_{\varepsilon}}}$$

and, taking the supremum over all $T \in (0, 1]$, we conclude the above claim via (A.1).

Then, for every $p \ge p_{\varepsilon}$, Minkowski's and Jensen's inequality imply

$$\begin{split} \left\langle [g]_{-\alpha-\varepsilon}^{p} \right\rangle^{\frac{1}{p}} &\lesssim \left\langle \left(\left(\sum_{n\geq 1} \left(2^{\frac{1}{3}} \right)^{-n\left(\varepsilon - \frac{5}{2p_{\varepsilon}}\right)} \left(2^{\frac{1}{3}} \right)^{-n\alpha} \|g_{2^{-n}}\|_{L^{p_{\varepsilon}}} \right)^{p} \right\rangle^{\frac{1}{p}} \\ &\leq \sum_{n\geq 1} \left(2^{\frac{1}{3}} \right)^{-n\left(\varepsilon - \frac{5}{2p_{\varepsilon}}\right)} \left(2^{\frac{1}{3}} \right)^{-n\alpha} \left\langle \|g_{2^{-n}}\|_{L^{p_{\varepsilon}}}^{p} \right\rangle^{\frac{1}{p}} \\ &\leq \sum_{n\geq 1} \left(2^{\frac{1}{3}} \right)^{-n\left(\varepsilon - \frac{5}{2p_{\varepsilon}}\right)} \left(2^{\frac{1}{3}} \right)^{-n\alpha} \left\langle \|g_{2^{-n}}\|_{L^{p}}^{p} \right\rangle^{\frac{1}{p}} \end{split}$$

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$$\lesssim \sum_{n\geq 1} \left(2^{\frac{1}{3}}\right)^{-n\left(\varepsilon-\frac{5}{2p_{\varepsilon}}\right)} \sup_{T\in(0,1]} \left(T^{\frac{1}{3}}\right)^{\alpha} \sup_{x\in\mathbb{T}^2} \langle |g_T(x)|^p\rangle^{\frac{1}{p}},$$

where in the final step we used the estimate $\langle ||g_T||_{L^p}^p \rangle^{\frac{1}{p}} \leq \sup_{x \in \mathbb{T}^2} \langle |g_T(x)|^p \rangle^{\frac{1}{p}}$. As $\varepsilon > \frac{5}{2p_{\varepsilon}}$, our hypothesis yields the conclusion. For $p \in [1, p_{\varepsilon})$, one concludes via Jensen's inequality.

In the critical case $\alpha + \varepsilon = 1, \frac{1}{2}$, one considers $\gamma < \alpha + \varepsilon$ and applies the above to conclude $g \in C^{-\gamma}$, which together with [IO19, Remark 2] gives $g \in C^{-\alpha-\varepsilon}$.

Appendix D. Some estimates for the linear equation

Lemma D.1. There exists C > 0 such that for every $\varepsilon \in (0, \frac{1}{8})$ and every $\xi \in C^{-\frac{5}{4}-\varepsilon}$, the solution v of vanishing average in x_1 to the equation $\mathcal{L}v = P\xi$ satisfies $|\partial_1|^{-1}\partial_2 v \in C^{\frac{1}{4}-\varepsilon}$ with

$$\left[\left|\partial_{1}\right|^{-1}\partial_{2}v\right]_{\frac{1}{4}-\varepsilon} \leq C[\xi]_{-\frac{5}{4}-\varepsilon}.$$

Proof. Recalling our notation $\mathcal{A} = |\partial_1|\mathcal{L}$, we have that

$$P\xi_T = \mathcal{L}v_T = |\partial_1|^{-1}\mathcal{A}v_T.$$

For $g = |\partial_1|^{-1} \partial_2 v$, this yields

$$\mathcal{A}g_T = \mathcal{A}|\partial_1|^{-1}\partial_2 v_T = \partial_2 P\xi_T.$$

Hence, for $T \in (0, 1]$ we have that

$$\begin{aligned} \|T\mathcal{A}g_{T}\|_{L^{\infty}} &= T \|\partial_{2}P\xi_{T}\|_{L^{\infty}} = T \|P\xi_{\frac{T}{2}} * \partial_{2}\psi_{\frac{T}{2}}\|_{L^{\infty}} = 2^{\frac{1}{2}} \left(T^{\frac{1}{3}}\right)^{3-\frac{1}{2}} \|P\xi_{\frac{T}{2}} * (\partial_{2}\psi)_{\frac{T}{2}}\|_{L^{\infty}} \\ &\lesssim \left(T^{\frac{1}{3}}\right)^{\frac{3}{2}} \|P\xi_{\frac{T}{2}}\|_{L^{\infty}} \lesssim \left(T^{\frac{1}{3}}\right)^{\frac{1}{4}-\varepsilon} [P\xi]_{-\frac{5}{4}-\varepsilon}, \end{aligned}$$

where we used the characterisation of negative Hölder spaces from Lemma A.1. Note that the implicit constant is universal for ε small, in particular for $\varepsilon \in (0, \frac{1}{8})$. Hence, we obtain that

$$\left[\left|\partial_{1}\right|^{-1}\partial_{2}v\right]_{\frac{1}{4}-\varepsilon} = \left[g\right]_{\frac{1}{4}-\varepsilon} \lesssim \sup_{T \in (0,1]} \left(T^{\frac{1}{3}}\right)^{-\frac{1}{4}+\varepsilon} \|T\mathcal{A}g_{T}\|_{L^{\infty}} \lesssim \left[\xi\right]_{-\frac{5}{4}-\varepsilon}.$$

Proposition D.2. There exists a constant C > 0 such that for every $\delta \xi \in L^2(\mathbb{T}^2)$, the solution δv of vanishing average in x_1 to $\mathcal{L}\delta v = P\delta \xi$ satisfies the following estimates:

$$[\delta v]_{\frac{3}{2}} \le C \|\delta \xi\|_{L^2},\tag{D.1}$$

$$\|\partial_1 \delta v\|_{L^{10}} \le C \|\delta \xi\|_{L^2},\tag{D.2}$$

$$\|\partial_2 \delta v\|_{L^{\frac{10}{3}}} \le C \|\delta \xi\|_{L^2}. \tag{D.3}$$

Proof. Writing $\partial_1^2 \delta v$ and $|\partial_2|^{\frac{4}{3}} \delta v$ as Fourier series and using Young's inequality in the form $|k_2|^{\frac{8}{3}} \leq k_1^4 + \frac{k_2^4}{k_1^2}$ for every $k_1, k_2 \in 2\pi\mathbb{Z}$, we obtain that

$$\|\partial_{1}^{2}\delta v\|_{L^{2}}^{2}+\||\partial_{2}|^{\frac{4}{3}}\delta v\|_{L^{2}}^{2} \lesssim \|\mathcal{L}\delta v\|_{L^{2}}^{2} \lesssim \|P\delta\xi\|_{L^{2}}^{2} \lesssim \|\delta\xi\|_{L^{2}}^{2},$$

which leads to (D.1) via (B.11) for s = 2. Using the same argument, we also obtain that

$$\|\partial_1^2 \delta v\|_{L^2}^2 + \||\partial_2|^{\frac{4}{3}} \partial_1 \delta v\|_{L^2}^2 \lesssim \|\delta \xi\|_{L^2}^2, \tag{D.4}$$

$$\|\partial_1 |\partial_2|^{\frac{2}{3}} \delta v\|_{L^2}^2 + \|\partial_2\|^{\frac{4}{3}} \delta v\|_{L^2}^2 \lesssim \|\delta\xi\|_{L^2}^2, \tag{D.5}$$

$$\||\partial_1|^{\frac{1}{2}} \partial_2 \delta v\|_{L^2}^2 + \||\partial_2|^{\frac{1}{3}} \partial_2 \delta v\|_{L^2}^2 \lesssim \|\delta\xi\|_{L^2}^2.$$
(D.6)

By the embedding result in Lemma B.4 we know that

$$\|\partial_1 \delta v\|_{L^{10}} \lesssim \|\partial_1^2 \delta v\|_{L^2} + \||\partial_2|^{\frac{2}{3}} \partial_1 \delta v\|_{L^2}, \tag{D.7}$$

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$$\|\partial_{2}^{h}\delta v\|_{L^{10}} \lesssim \|\partial_{1}\partial_{2}^{h}\delta v\|_{L^{2}} + \||\partial_{2}|^{\frac{2}{3}}\partial_{2}^{h}\delta v\|_{L^{2}}, \tag{D.8}$$

$$\|\partial_{2}\delta v\|_{L^{\frac{10}{3}}} \lesssim \||\partial_{1}|^{\frac{1}{2}} \partial_{2}\delta v\|_{L^{2}} + \||\partial_{2}|^{\frac{1}{3}} \partial_{2}\delta v\|_{L^{2}}.$$
 (D.9)

Combining (D.4) with (D.7) and (D.6) with (D.9) we obtain (D.2) and (D.3).

Lemma D.3. Let ξ be smooth. Then the solution v of $\mathcal{L}v = P\xi$ with vanishing average in x_1 is smooth.

Proof. If ξ is smooth, then $\partial_1^m \partial_2^n \xi \in L^2(\mathbb{T}^2)$ for all $m, n \in \mathbb{N}_0$. It follows by a simple calculation in Fourier space that

$$\begin{aligned} \|\partial_1^m \partial_2^n v\|_{L^2}^2 &= \sum_k |k_1|^{2m} |k_2|^{2n} |\widehat{v}(k)|^2 = \sum_{k_1 \neq 0} |k_1|^{2m} |k_2|^{2n} \frac{|k_1|^2 |\xi(k)|^2}{|k_1|^3 + |k_2|^2} \\ &\leq \sum_{k_1 \neq 0} |k_1|^{2m+2} |k_2|^{2n} |\widehat{\xi}(k)|^2 = \|\partial_1^{m+1} \partial_2^n \xi\|_{L^2}^2 < \infty \end{aligned}$$

for all $m, n \in \mathbb{N}_0$, in particular, v is smooth by Sobolev embedding.

Appendix E. Regularity of finite-energy solutions for smooth data

In this section we develop an L^2 -based regularity theory for weak solutions u with finite energy $\mathcal{H}(u) < \infty$ of the Euler–Lagrange equation

$$\mathcal{L}u = -P\left(uR_1\partial_2 u - \frac{1}{2}uR_1\partial_1 u^2\right) - \frac{1}{2}R_1\partial_2 u^2 + P\xi.$$
(E.1)

For $h \in (0, 1]$, define the difference quotients

$$D_i^h u = |h|^{-\alpha_i} \partial_i^h u, \quad \alpha_1 = 1, \alpha_2 = \frac{2}{3}.$$
 (E.2)

Proposition E.1 (H^{2-} estimate). Assume that $\xi \in L^2$. There exists a constant C > 0 such that for any solution u of the Euler–Lagrange equation (E.1) with $\mathcal{H}(u) < \infty$ we have

$$\sup_{h \in (0,1]} \mathcal{H}(D_i^h u) \le C \left(1 + \|\xi\|_{L^2}^2 + \mathcal{H}(u)^{12} \right).$$

In particular, for any $s \in [0,2)$ we have that $\||\partial_1|^s u\|_{L^2} + \||\partial_2|^{\frac{2}{3}s} u\|_{L^2} < \infty$.

Proposition E.2 (H^{3-} estimate). Assume that ξ satisfies $\mathcal{H}(\xi) < \infty$. There exists a constant C > 0 such that for any solution u of the Euler–Lagrange equation (E.1) with $\mathcal{H}(u) < \infty$ we have

$$\sup_{h,h'\in\{0,1]} \mathcal{H}(D_i^h D_i^{h'} u) \le C \left(1 + \mathcal{H}(\xi)^{22} + \mathcal{H}(u)^{132}\right)$$

In particular, for any $s \in [0,3)$ we have that $\||\partial_1|^s u\|_{L^2} + \||\partial_2|^{\frac{2}{3}s} u\|_{L^2} < \infty$.

For the proof of Propositions E.1 and E.2 it is convenient to work with the scale of norms

$$|||f|||_{s}^{2} = \int_{\mathbb{T}^{2}} \left(|\partial_{1}|^{s} f \right)^{2} \, \mathrm{d}x + \int_{\mathbb{T}^{2}} \left(|\partial_{1}|^{-\frac{s}{2}} |\partial_{2}|^{s} f \right)^{2} \, \mathrm{d}x,$$

for s > 0 and periodic functions $f : \mathbb{T}^2 \to \mathbb{R}$ with vanishing average in x_1 , as defined in (B.8). These norms are adapted to the harmonic energy \mathcal{H} , in particular, we have that $|||f|||_1 = \mathcal{H}(f)^{\frac{1}{2}}$, and one may think of the norms $||| \cdot |||_s$ defining a scale of (anisotropic) Sobolev spaces. Indeed, as shown in Lemma B.6 the norms $||| \cdot |||_s$ control an anisotropic fractional gradient in L^2 .

Lemma E.3. There exists a constant C > 0 such that for any $u \in W$ there holds

$$\sup_{h \in (0,1]} \|D_i^h u\|_{L^2} \le C \|\|u\|\|_1, \quad i = 1, 2.$$

Proof. We treat the two directions i = 1, 2 separately. For i = 1, the claim follows easily by the mean-value theorem, which implies that

$$\sup_{h \in (0,1]} \|D_1^h u\|_{L^2} \le \|\partial_1 u\|_{L^2}.$$

For i = 2 we appeal to Lemma B.10 (or rather its analogue for functions on \mathbb{T}^2) to estimate

$$\sup_{h \in \{0,1\}} \|D_{2}^{h}u\|_{L^{2}}^{2} = \sup_{h \in \{0,1\}} \frac{1}{h^{\frac{4}{3}}} \int_{\mathbb{T}^{2}} |\partial_{2}^{h}u|^{2} dx \lesssim \sup_{h \in \{0,1\}} \frac{1}{h^{\frac{4}{3}}} \frac{1}{h} \int_{0}^{h} \int_{\mathbb{T}^{2}} |\partial_{2}^{h'}u|^{2} dx dh'$$

$$\lesssim \sup_{h \in \{0,1\}} \int_{0}^{h} \int_{\mathbb{T}^{2}} \frac{|\partial_{2}^{h'}u|^{2}}{h'^{\frac{4}{3}}} dx \frac{dh'}{h'} \lesssim \int_{\mathbb{R}} \int_{\mathbb{T}^{2}} \frac{|\partial_{2}^{h}u|^{2}}{|h|^{\frac{4}{3}}} dx \frac{dh}{|h|} \overset{(2.2)}{\lesssim} \||\partial_{2}|^{\frac{2}{3}}u\|_{L^{2}}^{2}.$$

The conclusion then follows from Lemma B.6.

The proof of Propositions E.1 and E.2 mainly relies on the following two lemmata:

Lemma E.4. Let $\epsilon \in (0, \frac{1}{4})$. There exists a constant $C_{\epsilon} > 0$ such that for all periodic functions $f, g, \varphi : \mathbb{T}^2 \to \mathbb{R}$ of vanishing average in x_1 we have

$$\left| \int_{\mathbb{T}^2} f(\partial_2 g) \varphi \, \mathrm{d}x \right| \le C_\epsilon |||f|||_1 |||g|||_1 |||\varphi|||_{\frac{3}{4} + \epsilon}$$

Proof. This follows easily from the definition of $\|| \cdot \||_1$ and Lemma B.7. Indeed, we have

$$\left| \int_{\mathbb{T}^2} f(\partial_2 g) \varphi \, \mathrm{d}x \right| = \left| \int_{\mathbb{T}^2} |\partial_1|^{-\frac{1}{2}} \partial_2 g \, |\partial_1|^{\frac{1}{2}} (f\varphi) \, \mathrm{d}x \right| \lesssim |||g|||_1 ||\partial_1|^{\frac{1}{2}} (f\varphi)||_{L^2} \lesssim |||g|||_1 |||f|||_1 |||\varphi|||_{\frac{3}{4} + \epsilon}.$$

Lemma E.5. There exists a constant C > 0 such that for all periodic functions $f, g, h, \varphi : \mathbb{T}^2 \to \mathbb{R}$ of vanishing average in x_1 we have

$$\left| \int_{\mathbb{T}^2} fR_1(g\partial_1 h) \varphi \, \mathrm{d}x \right| \le C |||f|||_1 |||g|||_1 |||h|||_1 |||\varphi|||_{\frac{1}{2}}$$

Proof. We first use Cauchy-Schwarz to bound

$$\left| \int_{\mathbb{T}^2} fR_1(g\partial_1 h)\varphi \, \mathrm{d}x \right| = \left| \int_{\mathbb{T}^2} (\partial_1 h)gR_1(f\varphi) \, \mathrm{d}x \right| \le \|\partial_1 h\|_{L^2} \|gR_1(f\varphi)\|_{L^2} \le \|\|h\|\|_1 \|gR_1(f\varphi)\|_{L^2}.$$

By Hölder's inequality and the boundedness of R_1 on $L^{\frac{5}{2}}(\mathbb{T}^2)$ (A.4) , we may further estimate

$$\|gR_{1}(f\varphi)\|_{L^{2}} \leq \|g\|_{L^{10}} \|R_{1}(f\varphi)\|_{L^{\frac{5}{2}}} \leq \|g\|_{L^{10}} \|f\varphi\|_{L^{\frac{5}{2}}} \leq \|g\|_{L^{10}} \|f\|_{L^{10}} \|\varphi\|_{L^{\frac{10}{3}}}.$$

from which the claim follows with the Sobolev-type embeddings

$$\|g\|_{L^{10}} \stackrel{(B.5)}{\lesssim} \|\partial_1 g\|_{L^2} + \|\partial_2|^{\frac{2}{3}} g\|_{L^2} \stackrel{(B.9)}{\lesssim} \|g\|_{1},$$

$$\|\varphi\|_{L^{\frac{10}{3}}} \stackrel{(B.6)}{\lesssim} \|\partial_1|^{\frac{1}{2}} \varphi\|_{L^2} + \|\partial_2|^{\frac{1}{3}} \varphi\|_{L^2} \stackrel{(B.9)}{\lesssim} \|\varphi\|_{\frac{1}{2}}.$$

Proof of Proposition E.1. Since $|||u|||_1^2 = \mathcal{H}(u) < \infty$, we can test the Euler–Lagrange equation (E.1) with $\psi = D_i^{-h} D_i^h u$, where D_i^h denotes the difference quotient introduced in (E.2) for $h \in (0, 1]$. Then the left-hand side of (E.1) turns into

$$\int_{\mathbb{T}^2} \mathcal{L} u D_i^{-h} D_i^h u \, \mathrm{d} x = \mathcal{H}(D_i^h u) = |||D_i^h u|||_1^2$$

We continue with estimating the terms on the right-hand side of the Euler-Lagrange equation:

TERM 1 $(uR_1\partial_2 u)$. By a discrete integration by parts, we can write

$$\int_{\mathbb{T}^2} uR_1 \partial_2 u D_i^{-h} D_i^h u \, \mathrm{d}x = \int_{\mathbb{T}^2} D_i^h u (\partial_2 R_1 u) D_i^h u \, \mathrm{d}x + \int_{\mathbb{T}^2} u^h \partial_2 (D_i^h R_1 u) D_i^h u \, \mathrm{d}x,$$

where $u^h = u(\cdot + h)$. Applying Lemma E.4 to each of these two terms, together with

$$|||R_1u|||_1 = |||u|||_1$$
 and $|||u^h|||_1 = |||u|||_1$, (E.3)

then yields

$$\left| \int_{\mathbb{T}^2} D_i^h u(\partial_2 R_1 u) D_i^h u \, dx \right| \leq |||D_i^h u|||_1 |||R_1 u|||_1 |||D_i^h u|||_{\frac{4}{5}} \leq |||D_i^h u|||_1 |||u|||_1 |||D_i^h u|||_{\frac{4}{5}},$$
$$\left| \int_{\mathbb{T}^2} u^h \partial_2 (D_i^h R_1 u) D_i^h u \, dx \right| \leq |||u^h|||_1 |||R_1 D_i^h u|||_1 |||D_i^h u|||_{\frac{4}{5}} \leq |||u|||_1 |||D_i^h u|||_1 |||D_i^h u|||_{\frac{4}{5}}.$$

Note that by interpolation (which is easily seen in Fourier space) and Lemma E.3, there holds

$$|||D_{i}^{h}u|||_{\frac{4}{5}} \leq ||D_{i}^{h}u||_{L^{2}}^{\frac{1}{5}}|||D_{i}^{h}u|||_{1}^{\frac{4}{5}} \leq |||u|||_{1}^{\frac{1}{5}}|||D_{i}^{h}u|||_{1}^{\frac{4}{5}},$$
(E.4)

hence

$$\int_{\mathbb{T}^2} uR_1 \partial_2 u D_i^{-h} D_i^h u \, \mathrm{d}x \bigg| \lesssim |||u|||_1^{\frac{6}{5}} |||D_i^h u|||_1^{\frac{9}{5}}.$$

TERM 2 $(R_1 \partial_2 u^2)$. As for the first term, after a discrete integration by parts

$$\int_{\mathbb{T}^2} R_1 \partial_2 u^2 D_i^{-h} D_i^h u \, \mathrm{d}x = \int_{\mathbb{T}^2} D_i^h (u \partial_2 u) R_1 D_i^h u \, \mathrm{d}x$$
$$= \int_{\mathbb{T}^2} (D_i^h u) \partial_2 u R_1 D_i^h u \, \mathrm{d}x + \int_{\mathbb{T}^2} u^h D_i^h \partial_2 u R_1 D_i^h u \, \mathrm{d}x,$$

an application of Lemma E.4, the interpolation (E.4), and Lemma E.3 implies

$$\left| \int_{\mathbb{T}^2} R_1 \partial_2 u^2 D_i^{-h} D_i^h u \, dx \right| \lesssim |||D_i^h u|||_1 |||u|||_1 |||D_i^h u|||_{\frac{4}{5}} \lesssim |||u|||_1^{\frac{6}{5}} |||D_i^h u|||_1^{\frac{9}{5}}.$$

TERM 3 $(u\partial_1 R_1 u^2)$. For the cubic term in *u* we have

$$\int_{\mathbb{T}^2} u\partial_1 R_1 u^2 D_i^{-h} D_i^h u \, \mathrm{d}x = \int_{\mathbb{T}^2} (D_i^h u) R_1 (u\partial_1 u) D_i^h u \, \mathrm{d}x + \int_{\mathbb{T}^2} u^h R_1 (D_i^h u\partial_1 u) D_i^h u \, \mathrm{d}x + \int_{\mathbb{T}^2} u^h R_1 (u^h \partial_1 D_i^h u) D_i^h u \, \mathrm{d}x.$$

By Lemma E.5 we may estimate the first term by

$$\int_{\mathbb{T}^2} (D_i^h u) R_1(u\partial_1 u) D_i^h u \, \mathrm{d}x \lesssim |||D_i^h u|||_1 |||u|||_1 |||u|||_1 |||D_i^h u|||_{\frac{1}{2}},$$

and similarly for the other two terms, so that, together with (E.3),

$$\left| \int_{\mathbb{T}^2} u \partial_1 R_1 u^2 D_i^{-h} D_i^h u \, dx \right| \lesssim |||u|||_1^2 |||D_i^h u|||_1 |||D_i^h u|||_{\frac{1}{2}}.$$

Interpolation and Lemma E.3 then yield

$$|||D_{i}^{h}u|||_{\frac{1}{2}} \leq ||D_{i}^{h}u||_{L^{2}}^{\frac{1}{2}}|||D_{i}^{h}u|||_{1}^{\frac{1}{2}} \leq |||u|||_{1}^{\frac{1}{2}}|||D_{i}^{h}u|||_{1}^{\frac{1}{2}},$$

hence

$$\left| \int_{\mathbb{T}^2} u \partial_1 R_1 u^2 D_i^{-h} D_i^h u \, \mathrm{d}x \right| \lesssim |||u|||_1^{\frac{5}{2}} |||D_i^h u|||_1^{\frac{3}{2}}.$$

TERM 4 ($P\xi$). Note that by assumption $\xi \in L^2$, so that Cauchy–Schwarz and Lemma E.3 imply

$$\left| \int_{\mathbb{T}^2} P\xi D_i^{-h} D_i^h u \, dx \right| \le \|P\xi\|_{L^2} \|D_i^{-h} D_i^h u\|_{L^2} \le \|P\xi\|_{L^2} \|D_i^h u\|_{1}$$

With the above estimates on the three superlinear terms on the right-hand side of the Euler-Lagrange equation, we have therefore shown that

$$|||D_{i}^{h}u|||_{1}^{2} \leq |||u|||_{1}^{\frac{6}{5}} |||D_{i}^{h}u|||_{1}^{\frac{9}{5}} + |||u|||_{1}^{\frac{5}{2}} |||D_{i}^{h}u|||_{1}^{\frac{3}{2}} + ||\xi||_{L^{2}} |||D_{i}^{h}u|||_{1},$$

which by Young's inequality implies that

$$|||D_{i}^{h}u|||_{1}^{2} \leq |||u|||_{1}^{12} + |||u|||_{1}^{10} + ||P\xi||_{L^{2}}^{2} \leq 1 + |||u|||_{1}^{12} + ||P\xi||_{L^{2}}^{2},$$

with an implicit constant that does not depend on *h*. Finally, we notice that a bound on the quantity $\sup_{h \in \{0,1\}} |||D_i^h u|||_1$ implies a bound on $|||\partial_1|^s u||_{L^2} + |||\partial_2|^{\frac{2}{3}s} u||_{L^2}$ for $s \in [0,1)$ by Lemma B.6 and (2.2).

Proof of Proposition E.2. The proof is very similar to the proof of Proposition E.1. Under the assumptions of Proposition E.2, Proposition E.1 implies that

$$\sup_{h \in \{0,1\}} \||D_i^h u||_1 \leq 1 + \||u|\|_1^6 + \|P\xi\|_{L^2} \leq 1 + \||u\|\|_1^6 + \||\xi\|\|_1,$$
(E.5)

so that we may test the Euler-Lagrange equation (E.1) with $\psi = D_i^{-h'} D_i^{h'} D_i^{-h} D_i^h u$. Then the left-hand side of (E.1) turns into

$$\int_{\mathbb{T}^2} \mathcal{L} u \, D_i^{-h'} D_i^{h'} D_i^{-h} D_i^h u \, \mathrm{d} x = \|| D_i^{h'} D_i^h u \||_1^2.$$

We proceed by estimating each term on the right-hand side of (E.1).

TERM 1 ($uR_1\partial_2 u$). Integrating by parts twice (with respect to $D_i^{-h'}$ and D_i^{-h} , we obtain four terms, all of which can be estimated by either

$$|||u|||_{1}|||D_{i}^{h'}D_{i}^{h}u|||_{1}|||D_{i}^{h'}D_{i}^{h}u|||_{\frac{4}{5}}, \quad \text{or} \quad |||D_{i}^{h'}u|||_{1}|||D_{i}^{h}u|||_{1}|||D_{i}^{h'}D_{i}^{h}u|||_{\frac{4}{5}}.$$
 (E.6)

By interpolation, see (E.4), and Lemma E.3, there holds

$$|||D_{i}^{h'}D_{i}^{h}u|||_{\frac{4}{5}} \leq ||D_{i}^{h'}D_{i}^{h}u||_{L^{2}}^{\frac{1}{5}}|||D_{i}^{h'}D_{i}^{h}u|||_{1}^{\frac{4}{5}} \leq |||D_{i}^{h}u|||_{1}^{\frac{1}{5}}|||D_{i}^{h'}D_{i}^{h}u|||_{1}^{\frac{4}{5}}$$

TERM 2 $(R_1 \partial_2 u^2)$. This term is treated like the previous one, with the same bounds (E.6). TERM 3 $(u\partial_1 R_1 u^2)$. For this term we again integrate by parts twice to obtain nine terms, all of which can be bounded by one of the expressions

$$||u||_{1}||D_{i}^{h'}u||_{1}||D_{i}^{h}u||_{1}||D_{i}^{h'}D_{i}^{h}u|||_{\frac{1}{2}}, \text{ or } ||u||_{1}^{2}||D_{i}^{h'}D_{i}^{h}u|||_{1}||D_{i}^{h'}D_{i}^{h}u||_{\frac{1}{2}},$$

where by interpolation and Lemma E.3

$$|||D_{i}^{h'}D_{i}^{h}u|||_{\frac{1}{2}} \leq ||D_{i}^{h'}D_{i}^{h}u||_{L^{2}}^{\frac{1}{2}}|||D_{i}^{h'}D_{i}^{h}u|||_{1}^{\frac{1}{2}} \leq |||D_{i}^{h}u|||_{1}^{\frac{1}{2}}|||D_{i}^{h'}D_{i}^{h}u|||_{1}^{\frac{1}{2}}.$$

TERM 4 (*P* ξ). Under the regularity assumption $|||\xi|||_1 < \infty$ on ξ we can also estimate with Cauchy– Schwarz and Lemma E.3

$$\left| \int_{\mathbb{T}^2} P\xi D_i^{-h'} D_i^{h'} D_i^{-h} D_i^h u \, \mathrm{d}x \right| = \left| \int_{\mathbb{T}^2} D_i^{h'} P\xi D_i^{h'} D_i^{-h} D_i^h u \, \mathrm{d}x \right| \le \|D_i^{h'} \xi\|_{L^2} \|D_i^{-h} D_i^{h'} D_i^h u\|_{L^2} \le \|\|\xi\|\|_1 \|\|D_i^{h'} D_i^h u\|\|_1.$$

In total, we can therefore bound

$$\begin{split} \||D_{i}^{h'}D_{i}^{h}u\||_{1}^{2} &\leq \||u\||_{1}\||D_{i}^{h}u\||_{1}^{\frac{1}{5}}\||D_{i}^{h'}D_{i}^{h}u\||_{1}^{\frac{9}{5}} + \||D_{i}^{h'}u\||_{1}\||D_{i}^{h}u\||_{1}^{\frac{6}{5}}\||D_{i}^{h'}D_{i}^{h}u\||_{1}^{\frac{9}{5}} \\ &+ \||u\||_{1}\||D_{i}^{h'}u\||_{1}\||D_{i}^{h}u\||_{1}^{\frac{3}{2}}\||D_{i}^{h'}D_{i}^{h}u\||_{1}^{\frac{1}{2}} + \||u\||_{1}^{2}\||D_{i}^{h}u\||_{1}^{\frac{1}{2}}\||D_{i}^{h'}D_{i}^{h}u\||_{1}^{\frac{3}{2}} \\ &+ \||\xi\||_{1}\||D_{i}^{h'}D_{i}^{h}u\||_{1}, \end{split}$$

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from which it follows by Young's inequality and (E.5) that

 $|||D_i^{h'}D_i^hu|||_1^2 \lesssim 1 + |||u|||_1^{132} + |||\xi|||_1^{22},$

with an implicit constant that does not depend on h, h'. As in the proof of Proposition E.1, we notice that a bound on $\sup_{h,h'\in(0,1]} |||D_i^h D_i^{h'} u|||_1^2$ implies a bound on $|||\partial_1|^s u||_{L^2} + |||\partial_2|^{\frac{2}{3}s} u||_{L^2}$ for $s \in [2,3)$ by Lemma B.6 and (2.2).

Appendix F. Approximation of quadratic functionals of the noise by cylinder functionals

In this section we show the following approximation result, which seems classical but since we could not find a proof in the literature we include it here:

Lemma F.1. Let K be a linear operator on the space of Schwartz distributions $S'(\mathbb{T}^2)$ such that $K(S'(\mathbb{T}^2)) \subseteq C^{\infty}(\mathbb{T}^2)$ and $K : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$ is Hilbert–Schmidt. Assume further that the probability measure $\langle \cdot \rangle$ satisfies Assumption 1.1 (iv). Consider the quadratic functional $G : S'(\mathbb{T}^2) \to \mathbb{R}$ given by

$$G(\xi) := \xi(K\xi).$$

Then under the assumption that $\langle |G(\xi)| \rangle < \infty$, G is well-approximated by cylindrical functionals in with respect to the norm $\langle |G(\xi)|^{2p} \rangle^{\frac{1}{2p}} + \langle \| \frac{\partial G}{\partial F}(\xi) \|_{L^2}^{2p} \rangle^{\frac{1}{2p}}$ for every $1 \le p < \infty$.

Proof. Without loss of generality, we may assume that *K* is symmetric. Since *K* is Hilbert–Schmidt, there exists an orthonormal system $\{\varphi_n\}_{n \in \mathbb{N}}$ of $L^2(\mathbb{T}^2)$ and a sequence $\{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ such that

$$K = \sum_{n \in \mathbb{N}} \lambda_n(\varphi_n, \cdot)\varphi_n, \quad \text{and} \quad \|K\|_{HS}^2 = \sum_n \lambda_n^2 < \infty.$$
(F.1)

Note that by the assumption $K(\mathcal{S}'(\mathbb{T}^2)) \subseteq C^{\infty}(\mathbb{T}^2)$ there holds $\{\varphi_n\}_{n \in \mathbb{N}} \subset C^{\infty}(\mathbb{T}^2)$, in particular $\xi(\varphi_n)$ is well-defined for any $n \in \mathbb{N}$, and given $N \in \mathbb{N}$ we may define $K_N \xi := \sum_{n \leq N} \lambda_n \xi(\varphi_n) \varphi_n$. STEP 1 We first show that for any $\xi \in \mathcal{S}'(\mathbb{T}^2)$, $K_N \xi \to K \xi$ in $\mathcal{S}'(\mathbb{T}^2)$ as $N \to \infty$.

Indeed, with $\xi_t := \xi * \psi_t \in C^{\infty}(\mathbb{T}^2)$, for any $\phi \in C^{\infty}(\mathbb{T}^2)$ we have

$$|(K_N\xi - K\xi, \phi)| \le |(K_N\xi - K_N\xi_t, \phi)| + |(K_N\xi_t - K\xi_t, \phi)| + |(K\xi_t - K\xi, \phi)|.$$

Note that $\xi_t \to \xi$ in S' as $t \downarrow 0$, hence for any $\epsilon > 0$ we may choose t > 0 small enough so that by symmetry of K_N and K,

$$|(K_N\xi - K_N\xi_t, \phi)| = |(\xi - \xi_t)(K_N\phi)| < \frac{\epsilon}{3},$$
$$|(K\xi_t - K\xi, \phi)| = |(\xi - \xi_t)(K\phi)| < \frac{\epsilon}{3},$$

for any $N \in \mathbb{N}$. The remaining term can be bounded using Cauchy-Schwarz and Bessel's inequality for orthonormal systems to obtain

$$\begin{split} |(K_N\xi_t - K\xi_t, \phi)| &= \left| \left(\sum_{n > N} \lambda_n(\varphi_n, \xi_t)(\varphi_n, \phi) \right) \right| \le \left(\sum_{n > N} \lambda_n^2 |(\varphi_n, \xi_t)|^2 \right)^{\frac{1}{2}} \left(\sum_{n \in \mathbb{N}} |(\varphi_n, \phi)|^2 \right)^{\frac{1}{2}} \\ &\le \|\xi_t\|_{L^2} \left(\sum_{n > N} \lambda_n^2 \right)^{\frac{1}{2}} \|\phi\|_{L^2} < \frac{\epsilon}{3} \end{split}$$

if *N* is chosen suitably large, since $\{\lambda_n\}_n$ is square-summable.

STEP 2 We next claim that $G_N(\xi) := \xi(K_N\xi)$ satisfies $G_N(\xi) \to G(\xi)$ as $N \to \infty$ for all $\xi \in \mathcal{S}'(\mathbb{T}^2)$. For the proof of this claim we again appeal to the convergence $\xi_t \to \xi$ in \mathcal{S}' by splitting

$$\begin{aligned} |G_N(\xi) - G(\xi)| &= |\xi(K_N\xi) - \xi(K\xi)| \\ &\leq |\xi(K_N\xi) - \xi_t(K_N\xi)| + |\xi_t(K_N\xi) - \xi_t(K\xi)| + |\xi_t(K\xi) - \xi(K\xi)|. \end{aligned}$$

As in STEP 1, there holds

$$|\xi(K_N\xi) - \xi_t(K_N\xi)| \to 0, \quad |\xi_t(K\xi) - \xi(K\xi)| \to 0$$

as $t \downarrow 0$ for any $N \in \mathbb{N}$, since $K_N \xi, K \xi \in C^{\infty}(\mathbb{T}^2)$. Moreover, by STEP 1, we have

$$|\xi_t(K_N\xi) - \xi_t(K\xi)| = |(K_N\xi - K\xi, \xi_t)| \to 0$$

as $N \to \infty$.

STEP 3 We show that $\frac{\partial G_N(\xi)}{\partial \xi} \to \frac{\partial G(\xi)}{\partial \xi}$ as $N \to \infty$ in $L^p_{\langle \cdot \rangle} L^2_x$ for any $1 \le p < \infty$. Indeed, by symmetry of *K* and *K*_N we have that

$$\frac{\partial G_N(\xi)}{\partial \xi} = 2K_N\xi = 2\sum_{n \le N} \lambda_n \xi(\varphi_n)\varphi_n, \quad \text{and} \quad \frac{\partial G(\xi)}{\partial \xi} = 2K\xi = 2\sum_{n \in \mathbb{N}} \lambda_n \xi(\varphi_n)\varphi_n.$$

With this we obtain by orthonormality of the $\{\varphi_n\}_n$ that

$$\left\|\frac{\partial(G-G_N)(\xi)}{\partial\xi}\right\|_{L^2}^2 = 4\sum_{n>N}\lambda_n^2\xi(\varphi_n)^2,$$

hence, applying (5.1) in the form of

$$\left\langle \xi(\varphi_n)^{2p} \right\rangle^{\frac{1}{2p}} \lesssim_p \|\varphi_n\|_{L^2} \lesssim_p 1,$$

for all $n \in \mathbb{N}$, it follows that

$$\begin{split} \left\langle \left\| \frac{\partial G(\xi)}{\partial \xi} - \frac{\partial G_N(\xi)}{\partial \xi} \right\|_{L^2}^{2p} \right\rangle^{\frac{1}{2p}} &= \left\langle \left(4 \sum_{n > N} \lambda_n^2 \xi(\varphi_n)^2 \right)^p \right\rangle^{\frac{1}{2p}} \le 2 \left(\sum_{n > N} \lambda_n^2 \left\langle \xi(\varphi_n)^{2p} \right\rangle^{\frac{1}{p}} \right)^{\frac{1}{2}} \\ &\lesssim_p \left(\sum_{n > N} \lambda_n^2 \right)^{\frac{1}{2}} \xrightarrow{N \to \infty} 0, \end{split}$$

by the finiteness of $||K||_{HS}$.

STEP 4 We show that the sequence $\{G_N(\xi) - \langle G_N(\xi) \rangle\}_{N \in \mathbb{N}}$ is Cauchy in $L^{2p}_{\langle \cdot \rangle}$ for any $1 \leq p < \infty$, in particular, there exists a centred random variable $G_*(\xi)$ such that $G_N(\xi) - \langle G_N(\xi) \rangle \rightarrow G_*(\xi)$ for $\langle \cdot \rangle$ -almost every ξ as $N \rightarrow \infty$ along a subsequence.

Indeed, since $G_N(\xi)$ is a cylinder functional and $\langle \cdot \rangle$ satisfies Assumption 1.1 (iv), in particular Proposition 5.1, we can apply (5.1) to obtain the bound

$$\begin{split} \left\langle |G_N(\xi) - \left\langle G_N(\xi) \right\rangle - \left(G_M(\xi) - \left\langle G_M(\xi) \right\rangle \right)|^{2p} \right\rangle^{\frac{1}{2p}} &= \left\langle |G_N(\xi) - G_M(\xi) - \left\langle G_N(\xi) - G_M(\xi) \right\rangle|^{2p} \right\rangle^{\frac{1}{2p}} \\ &\lesssim_p \left\langle \left\| \frac{\partial (G_N - G_M)(\xi)}{\partial \xi} \right\|_{L^2}^{2p} \right\rangle^{\frac{1}{2p}}, \end{split}$$

for $N \ge M$. Hence, by Step 3,

$$\langle |G_N(\xi) - \langle G_N(\xi) \rangle - (G_M(\xi) - \langle G_M(\xi) \rangle)|^{2p} \rangle \xrightarrow{\frac{1}{2p}} \xrightarrow{M,N \to \infty} 0$$

for any $1 \le p < \infty$.

STEP 5 We claim that $G_N(\xi) \to G(\xi)$ in $L^{2p}_{\langle \cdot \rangle}$ as $N \to \infty$.

Indeed, by STEP 2 we know that $G_N(\xi) \to G(\xi)$ almost surely, so that with the result from STEP 4 we may conclude that

$$\langle G_N(\xi) \rangle = G_N(\xi) - (G_N(\xi) - \langle G_N(\xi) \rangle) \rightarrow G(\xi) - G_*(\xi)$$

almost surely along a subsequence as $N \to \infty$. But since $\langle G_N(\xi) \rangle$ is constant in ξ (recall that ξ denotes the dummy variable over which $\langle \cdot \rangle$ integrates), the random variable $G(\xi) - G_*(\xi)$ must be almost surely constant, i.e.

$$G(\xi) - G_*(\xi) = \langle G(\xi) - G_*(\xi) \rangle = \langle G(\xi) \rangle,$$

since $\langle G \rangle < \infty$ by assumption and G_* is centred. Hence $\langle G_N(\xi) \rangle \rightarrow \langle G(\xi) \rangle$ as $N \rightarrow \infty$ along a subsequence. Since the above argument can be repeated for any subsequence, we obtain

that $\langle G_N(\xi) \rangle \to \langle G(\xi) \rangle$ as $N \to \infty$. Since $G_N(\xi) \to G(\xi)$ almost surely, we conclude that $G_*(\xi) = G(\xi) - \langle G(\xi) \rangle$ almost surely, which together with STEP 4 implies that $G_N(\xi) \to G(\xi)$ in $L^{2p}_{\ell,\cdot}$ as $N \to \infty$.

By STEP 3–5, we conclude that $G_N \to G$ with respect to the norm $\langle |G(\xi)|^{2p} \rangle^{\frac{1}{2p}} + \langle || \frac{\partial G}{\partial \xi}(\xi) ||_{L^2}^{2p} \rangle^{\frac{1}{2p}}$.

References

- [BCD11] H. BAHOURI, J.-Y. CHEMIN, and R. DANCHIN, Fourier Analysis and Nonlinear Partial Differential Equations. Grundlehren der mathematischen Wissenschaften 343. Springer, Berlin, Heidelberg, 2011. ISBN: 978-3-642-16829-1. MR 2768550. Zbl 1227.35004.
- [BL76] J. BERGH, and J. LÖFSTRÖM, Interpolation Spaces. An Introduction. Grundlehren der mathematischen Wissenschaften 223. Springer, Berlin, Heidelberg, 1976. ISBN: 978-3-642-66453-3. MR 0482275. Zbl 0344.46071.
- [BBH94] F. BETHUEL, H. BREZIS, and F. HÉLEIN, Ginzburg-Landau vortices. Reprint of the 1994 edition. Modern Birkhäuser Classics. Birkhäuser/Springer, Cham, 2017. ISBN: 978-3-319-66672-3. MR 3618899. Zbl 1372.35002.
- [CH16] A. CHANDRA, and M. HAIRER, An analytic BPHZ theorem for regularity structures. Preprint (2016). arXiv:1612.08138.
- [CMW19] A. CHANDRA, A. MOINAT, and H. WEBER, A priori bounds for the Φ^4 equation in the full sub-critical regime. *Preprint* (2019). arXiv:1910.13854.
- [Dal93] G. DAL MASO, *An Introduction to* Γ*-convergence*. Progress in Nonlinear Differential Equations and their Applications **8**. Birkhäuser, Boston, MA, 1993. ISBN: 978-1-4612-6709-6. MR 1201152. Zbl 0816.49001.
- [DD03] G. DA PRATO, and A. DEBUSSCHE, Strong solutions to the stochastic quantization equations. *The Annals of Probability* 31 (2003), no. 4, 1900–1916. MR 2016604. Zbl 1071.81070.
- [DG20] M. DUERINCKX, and A. GLORIA, Multiscale functional inequalities in probability: Constructive approach. Annales Henri Lebesgue 3 (2020), 825–872. MR 4149827.
- [FH14] P.K. FRIZ, and M. HAIRER, A Course on Rough Paths. With an Introduction to Regularity Structures. 2nd edition. Universitext. Springer, Cham, 2020. ISBN: 978-3-030-41555-6. MR 3289027. Zbl 07214193.
- [GJO15] M. GOLDMAN, M. JOSIEN, and F. OTTO, New bounds for the inhomogenous Burgers and the Kuramoto– Sivashinsky equations. *Communications in Partial Differential Equations* 40 (2015), no. 12, 2237–2265. MR 3421759. Zbl 06539046.
- [Gra14] L. GRAFAKOS, *Classical Fourier Analysis*, 3rd edition. Graduate Texts in Mathematics **249**. Springer, New York, 2014. ISBN: 978-1-4939-1193-6. MR 3243734. Zbl 1304.42001.
- [GH19] M. GUBINELLI, and M. HOFMANOVÁ, Global solutions to elliptic and parabolic Φ⁴ models in Euclidean space. *Communications in Mathematical Physics* **368** (2019), no. 3, 1201–1266. MR 3951704. Zbl 1420.35481.
- [Hai14] M. HAIRER, A theory of regularity structures. *Inventiones Mathematicae* 198 (2014), no. 2, 269–504.
 MR 3274562. Zbl 1332.60093.
- [HS17] M. HAIRER, and H. SHEN, A central limit theorem for the KPZ equation. The Annals of Probability 45 (2017), no. 6B, 4167–4221. MR 3737909. Zbl 1388.60111.
- [Har68] K.J. HARTE, Theory of Magnetization Ripple in Ferromagnetic Films. Journal of Applied Physics 39 (1968), 1503–1524.
- [Hel98] B. HELFFER, Remarks on the decay of correlations and Witten Laplacians, Brascamp–Lieb inequalities and semiclassical limit. *Journal of Functional Analysis* **155** (1998), no. 2, 571–586. MR 1624506. Zbl 0921.35141.
- [Hof68] H. HOFFMANN, Theory of magnetization ripple. *IEEE Transactions on Magnetics* 4 (1968), no. 1, 32–38.
- [IJ19] R. IGNAT, and R.L. JERRARD, Renormalized energy between vortices in some Ginzburg–Landau models on 2-dimensional Riemannian manifolds. Archive for Rational Mechanics and Analysis 239 (2021), no. 3, 1577–1666. MR 4215198. Zbl 1462.35375.
- [IM16] R. IGNAT, and R. MOSER, Interaction energy of domain walls in a nonlocal Ginzburg-Landau type model from micromagnetics. Archive for Rational Mechanics and Analysis 221 (2016), no. 1, 419–485. MR 3483899. Zbl 1342.35362.
- [IO19] R. IGNAT, and F. OTTO, The magnetization ripple: A nonlocal stochastic PDE perspective. Journal de Mathématiques Pures et Appliquées 130 (2019), 157–199. MR 4001632. Zbl 07107304.
- [JO20] M. JOSIEN, and F. OTTO, The annealed Calderón–Zygmund estimate as convenient tool in quantitative stochastic homogenization. *Journal of Functional Analysis* 283 (2022), no. 7, Paper No. 109594, 74 pp. MR 4444733. Zbl 07557253.
- [Kuro6] M. KURZKE, Boundary vortices in thin magnetic films. Calculus of Variations and Partial Differential Equations 26 (2006), no. 1, 1–28. MR 2214879. Zbl 1151.35006.
- [MW17a] J.-C. MOURRAT, and H. WEBER, Global well-posedness of the dynamic Φ^4 model in the plane. *The Annals of Probability* **45** (2017), no. 4, 2398–2476. MR 3693966. Zbl 1381.60098.

- [MW17b] J.-C. MOURRAT, and H. WEBER, The dynamic Φ_3^4 comes down from infinity. *Communications in Mathematical Physics* **356** (2017), no. 3, 673–753. MR 3719541. Zbl 1384.81068.
- [OS10] F. OTTO, and J. STEINER, The concertina pattern: from micromagnetics to domain theory. *Calculus of Variations and Partial Differential Equations* **39** (2010), no. 1–2, 139–181. MR 2659683. Zbl 1237.78036.

[OW19] F. OTTO, and H. WEBER, Quasilinear SPDEs via rough paths. *Archive for Rational Mechanics and Analysis* 232 (2019), no. 2, 873–950. MR 3925533. Zbl 1426.60090.

- [SS07] E. SANDIER, and S. SERFATY, Vortices in the magnetic Ginzburg–Landau model. Progress in Nonlinear Differential Equations and their Applications 70, Birkhäuser Boston, Boston, MA, 2007. ISBN: 978-0-8176-4316-4. MR 2279839. Zbl 1112.35002.
- [SS15] E. SANDIER, and S. SERFATY, 2D Coulomb gases and the renormalized energy. The Annals of Probability 43 (2015), no. 4, 2026–2083. MR 3353821. Zbl 1328.82006.
- [SSWMO12] J. STEINER, R. SCHÄFER, H. WIECZORECK, J. MCCORD, and F. OTTO, Formation and coarsening of the concertina magnetization pattern in elongated thin-film elements. *Physical Review B* 85 (2012), 104407.
- [TW18] P. TSATSOULIS, and H. WEBER, Spectral gap for the stochastic quantization equation on the 2-dimensional torus. Annales de l'Institut Henri Poincaré Probabilités et Statistiques 54 (2018), no. 3, 1204–1249. MR 3825880. Zbl 1403.81030.

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