A NOTE ABOUT RATIONAL REPRESENTATIONS OF DIFFERENTIAL GALOIS GROUPS.

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Abstract. We give a description of the rational representations of the differential Galois group of a Picard-Vessiot extension.

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We give a description of the rational representations of the differential Galois group of a Picard-Vessiot extension (theorems 1.1 and 2.1). This gives a new description of the differential Galois correspondence. More results will be given for abelian differential extensions in a forthcoming paper [1], especially the analog of the Artin correspondence.

1. ON REPRESENTATIONS OF GALOIS GROUPS OF PICARD-VESSIOT EXTENSIONS.

In all this note we consider differential fields with algebraically closed fields of constants, denoted by $C$. The derivative will be denoted by a dash.

Let $K$ be a differential field. Let $n \geq 1$ be an integer, $M_n(K)$ and $\text{GL}_n(K)$ are the usual notations for algebra and group of $n \times n$ matrices with entries in $K$. The group $\text{GL}_n(K)$ acts on $M_n(K)$ by the following rule:

$$\text{GL}_n(K) \times M_n(K) \to M_n(K)$$

$$(U, A) \mapsto U'U^{-1} + UAU^{-1}$$

where if $U = (u_{i,j})_{1 \leq i,j \leq n}$, then $U' = (u'_{i,j})_{1 \leq i,j \leq n}$. This action can be defined in an other way, maybe more comprehensible. Consider the group

$$H_n(K) := M_n(K) \times \text{GL}_n(K),$$

Date: January 22, 2007.
the law being defined by the following formula: for all $A, B$ in $M_n(K)$ and all $F, G \in GL_n(K)$


It admits the subgroups

$$\Delta_n(K) := \{(U^{-1}U, U) / U \in GL_n(K)\},$$

$$\{0\} \times GL_n(K)$$

and $M_n(K) \times \{1\}$, this last one being normal. We set

$$Z_n(K) := \Delta_n(K) \backslash H_n(K)/\{(0) \times GL_n(K)\}.$$  

The group $GL_n(K)$ acts on $M_n(K) \times \{1\}$ by the following rule:

$$GL_n(K) \times (M_n(K) \times \{1\}) \longrightarrow (M_n(K) \times \{1\})$$

$$(U, (A, 1)) \longmapsto (U^{-1}U, (A, 1)(0, U^{-1})) = (U^{-1} + UAU^{-1}, 1)$$

With the identification $M_n(K) = (M_n(K) \times \{1\})$ and inclusion $(M_n(K) \times \{1\}) \subset H_n(K)$, it induces a canonical bijection

$$(1) \quad GL_n(K) \backslash M_n(K) \simeq \Delta_n(K) \backslash H_n(K)/\{(0) \times GL_n(K)\} = Z_n(K).$$

We will use below these two definitions of $Z_n(K)$.

Let $A$ be in $M_n(K)$ and $F$ be in $GL_n(K)$, we denote by $[(A, F)]$, resp. $[A]$, the class in $Z_n(K)$ of $(A, F) \in H_n(K)$, resp. of $A \in M_n(K)$.

Let $L/K$ be a Picard-Vessiot extension. The inclusion $H_n(K) \subseteq H_n(L)$ gives rise to a map $\alpha(L/K) : Z_n(K) \rightarrow Z_n(L)$. We set

$$Z_n(L/K) := \{a \in Z_n(K) / \alpha(L/K)(a) = [0]\}.$$ 

For any group $G$ we denote by $Rep_n(G)$ the set of equivalent classes of representations of $G$ in $GL_n(C)$, if $G = dGal(L/K)$ is the differential group of $L$ over $K$, we set $Rep_n(L/K) := Rep_n(G)$.

**Theorem 1.1.** Let $L/K$ be a Picard-Vessiot extension, then there exists a natural bijection between $Z_n(L/K)$ and $Rep_n(L/K)$.

**Proof.** First of all we recall some facts that are of main importance in our proof.

1.0.1. *The representation $c_A$.* Consider the differential equation $Y' = AY$, with $A \in M_n(K)$, and let $E/K$ be a corresponding Picard-Vessiot extension, i.e. $E$ is generated over $K$ by the coefficients of a fundamental matrix $F_A$ of the equation. The rational representation $c_A$ is

$$dGal(E/K) \longrightarrow GL_n(C)$$

$$\sigma \longmapsto c_A(\sigma)$$

where $c_A(\sigma)$ is such that $\sigma(F_A) = F_A c_A(\sigma)$. Note that $c_A$ depends only on the class $[A]$ of $(A, 1)$ in $Z_n(K)$, because if $B = U^{-1}U + UAU^{-1}$ with $U \in GL_n(K)$, a fundamental matrix of the equation $Y' = BY$ is $UF_A$ and we see that $c_B = c_A$. Note also that an other fundamental matrix is of the form $F_A \gamma$, with $\gamma \in GL_n(C)$, then it gives the representation $\gamma^{-1} c_A \gamma$ equivalent to $c_A$. 
We will write equivalently $c_A$, $c_{[A]}$ or $c_E$ for this class of representations.

1.0.2. The Galois group $dGal(E/K)$. Let $A \in M_m(K)$ and

$$R = K [(X_{i,j})_{1 \leq i,j \leq m}, (\text{det})^{-1}] / q = K [(x_{i,j})_{1 \leq i,j \leq m}]$$

be a Picard-Vessiot ring over $K$ for the equation $Y' = AY$. In these formulas the $X_{i,j}$ are indeterminates, the ring $K [(X_{i,j})_{1 \leq i,j \leq m}]$ is equipped by the derivation satisfying $(X'_{i,j})_{1 \leq i,j \leq m} = A(X_{i,j})_{1 \leq i,j \leq m}$, “det” is the determinant of the matrix $(X_{i,j})_{1 \leq i,j \leq m}$, $q$ is a maximal differential ideal and $x_{i,j}$ is the image of $X_{i,j}$. Let $E = \text{Quot}(R)$ and $\mathfrak{U} = dGal(E/K)$, consider

$$K [(X_{i,j})_{1 \leq i,j \leq m}, (\text{det})^{-1}] \subseteq E [(X_{i,j})_{1 \leq i,j \leq m}, (\text{det})^{-1}]$$

$$= E [(Y_{i,j})_{1 \leq i,j \leq m}, (\text{det})^{-1}] \supseteq C [(Y_{i,j})_{1 \leq i,j \leq m}, (\text{det})^{-1}],$$

where $(Y_{i,j})_{1 \leq i,j \leq m}$ is defined by $(X_{i,j})_{1 \leq i,j \leq m} = (x_{i,j})_{1 \leq i,j \leq m}$. Let $Y'_{i,j} = 0$. We know that

$$\mathfrak{U} = dGal(E/K) = \text{SpecC} [(Y_{i,j})_{1 \leq i,j \leq m}, (\text{det})^{-1}] / J$$

where $J = qE [(Y_{i,j})_{1 \leq i,j \leq m}, (\text{det})^{-1}] \cap C [(Y_{i,j})_{1 \leq i,j \leq m}, (\text{det})^{-1}]$ ([2] proof of prop. 1.24 or the beginning of §1.5). We denote by $y_{i,j}$ the image of $Y_{i,j}$, then we have

$$\mathfrak{U} = dGal(E/K) = \text{SpecC} [(y_{i,j})_{1 \leq i,j \leq m}]$$

1.0.3. $\mathfrak{U} = dGal(E/K)$ as a torsor. We continue with the previous notations. Set $\mathfrak{T} = \text{SpecR}$, we know that $\mathfrak{T}$ is an $\mathfrak{U}$-torsor over $K$ ([2] theorem 1.30), moreover, we know that there exists a finite extension $\widetilde{K}$ of $K$ such that $\mathfrak{T} \times_K \widetilde{K} = \text{Spec} \left( R \otimes_K \widetilde{K} \right)$ is a trivial $\mathfrak{U}$-torsor over $\widetilde{K}$ ([2] cor. 1.31), this means that there exists $b \in \mathfrak{T}(\widetilde{K})$ such that the following map is an isomorphism of $\widetilde{K}$-schemes

$$\psi : \mathfrak{U} \times_C \widetilde{K} \longrightarrow \mathfrak{T} \times_K \widetilde{K}$$

$$(c_{i,j})_{1 \leq i,j \leq m} \longmapsto b(c_{i,j})_{1 \leq i,j \leq m}$$

($b$ can be seen as a matrix, on the right this is a product of matrices; see the definition of $R$ above).

1.0.4. Galois actions. Let $\sigma$ be an element of $\mathfrak{U} = dGal(E/K)$, the action of $\sigma$ on $R$ is given by the images of the $x_{i,j}$, which are defined by the matrix formula $(\sigma(x_{i,j})) = (x_{i,j})c_E(\sigma)$. We denote by $\sigma^\flat$ the morphism induces by $\sigma$ on $\mathfrak{T}$ or on $\mathfrak{T} \times_K \widetilde{K}$, this is the action of $\mathfrak{U}$ which defines the torsor structure. An element of $\mathfrak{T}(\widetilde{K})$ can be represented by a matrix $a = (a_{i,j})_{1 \leq i,j \leq m}$ with $a_{i,j}$ in $\widetilde{K}$, its image is $\sigma^\flat(a) = ac_E(\sigma)$. For any $\sigma$ in $\mathfrak{U}$ denote by $\lambda_\sigma$ the right translation on $\mathfrak{U}$ by $\sigma$, i.e.

$$\lambda_\sigma : \mathfrak{U} \rightarrow \mathfrak{U}$$

$$\tau \longmapsto \tau \sigma$$
Write again \( \lambda \sigma \) for \( \lambda \sigma \times \text{Id}_{\overline{K}} : \mathfrak{U} \times_{C} \overline{K} \to \mathfrak{U} \times_{C} \overline{K} \), then the morphism \( \psi \) of (1.0.3) is equivariant, this means that for any \( \sigma \in \mathfrak{U} \) the following diagram is commutative

\[
\begin{array}{ccc}
\mathfrak{U} \times_{C} \overline{K} & \xrightarrow{\psi} & \mathfrak{T} \times_{K} \overline{K} \\
\lambda \sigma \downarrow & & \downarrow \sigma^b \\
\mathfrak{U} \times_{C} \overline{K} & \xrightarrow{\psi} & \mathfrak{T} \times_{K} \overline{K}
\end{array}
\]

Proof of the theorem, the map \( Z_n(L/K) \to R_n(L/K) \).

Let \( A \in M_n(K) \) such that \([A] \in Z_n(L/K)\), then there exists \( U \in \text{GL}_n(L) \) such that

\[
A = U' U^{-1},
\]

this means that \( U \) is a fundamental matrix of the equation \( Y' = AY \), as it is with entries in \( L \), it exists a differential subextension \( E \) of \( L \) which is a Picard-Vessiot extension for the equation \( Y' = AY \). Denote by \( \rho_A \) the representation

\[
(2) \quad \rho_A : d\text{Gal}(L/K) \xrightarrow{\text{restriction}} d\text{Gal}(E/K) \xrightarrow{c_A} \text{GL}_n(C).
\]

Now we prove that this representation \( \rho_A \) does not depend on the class of \( A \) in \( Z_n(K) \) and of the choice of \( U \in \text{GL}_n(L) \) such that \( A = U' U^{-1} \).

Let \( B \in M_n(K) \) such that \([B] = [A] \) in \( Z_n(K) \), then there exists \( W, T \in \text{GL}_n(K) \) such that

\[
(B, 1) = (W' W^{-1}, W)(A, 1)(0, T),
\]

it follows that \( B = W' W^{-1} + W A W^{-1} \), this means that \( W U \) is a fundamental matrix of the equation \( Y' = BY \). We see that \( \rho_A = \rho_B \), and we denote this representation by \( \rho_{[A]} \).

Let \( V \in \text{GL}_n(L) \) such that \( A = U' U^{-1} = V' V^{-1} \), then we see that \((V^{-1} U')^t = 0\), this means that there exists \( \gamma \in \text{GL}_n() \) such that \( U = V \gamma \) and the two representations define as before in (2) are conjugate.

Then to each element \([A]\) of \( Z_n(L/K) \) we have associated the element \( \rho_{[A]} \) of \( \text{Rep}_n(L/K) \).

Proof of the theorem, the map \( R_n(L/K) \to Z_n(L/K) \).

Let \( \rho : d\text{Gal}(L/K) \to \text{GL}_n(C) \) be a rational representation. Let \( E \) be the fixed field of \( \ker \rho \), we set \( \mathfrak{U} = d\text{Gal}(E/K) \) and we denote again by \( \rho \) the representation \( \mathfrak{U} \hookrightarrow \text{GL}_n(C) \) coming from the given one. The field \( E \) is a Picard-Vessiot extension corresponding to an equation \( Y' = AY \), with \( A \in M_n(K) \). Our aim is to prove that one can chose \( A \) in \( M_n(K) \), i.e. \( m = n \), and that this gives the inverse map of \([A] \mapsto \rho_{[A]} \).

We use the previous notations and descriptions of \( E, R, \mathfrak{U}, \mathfrak{T} \) etc. We set \( \text{GL}_n(C) = \text{Spec} C[\{(T_{r,s})_{1 \leq r,s \leq n}, (\det)^{-1}\}] \), let

\[
\rho^\#: C \left[ \{(T_{r,s})_{1 \leq r,s \leq n}, (\det)^{-1}\} \right] \to C \left[ \{(Y_{i,j})_{1 \leq i,j \leq m}, (\det)^{-1}\} \right] / J
\]
be the comorphism of $\rho : \mathfrak{U} \hookrightarrow \text{GL}_n(C)$; $\rho^\sharp$ is onto. Set $I = \ker(\rho^\sharp)$, then we have an isomorphism induces by $\rho^\sharp$.

$$\bar{\rho} : C \left[ \left( T_{r,s} \right)_{1 \leq r,s \leq n} , (\det)^{-1} \right] / I \simeq C \left[ \left( Y_{i,j} \right)_{1 \leq i,j \leq n} , (\det)^{-1} \right] / J.$$  

Let $t_{r,s}$ be the image of $T_{r,s}$ in the quotient on the left, and recall that $y_{i,j}$ are that of $Y_{i,j}$ in the quotient on the right, then the preceding formula can be written

$$\bar{\rho} : C \left[ \left( t_{r,s} \right)_{1 \leq r,s \leq n} \right] \simeq C \left[ \left( y_{i,j} \right)_{1 \leq i,j \leq n} \right] .$$

Set $\mathfrak{W} = \text{Spec} \left( C \left[ \left( t_{r,s} \right)_{1 \leq r,s \leq n} \right] \right)$, this is an algebraic subgroup of $\text{GL}_n(C)$, it is isomorphic to $\mathfrak{U}$ via the morphism induces by $\bar{\rho}$, denoted by abuse of language $\rho : \mathfrak{U} \simeq \mathfrak{W}$.

The composed morphism (see (1.0.3))

$$(3) \quad \varphi : \mathfrak{T} \otimes_K \widetilde{K} \xrightarrow{\psi^{-1}} \mathfrak{U} \times_C \widetilde{K} \xrightarrow{\rho \times \text{Id}_K} \mathfrak{W} \times_C \widetilde{K}$$

is an isomorphism of $\widetilde{K}$-schemes, equivariant for the actions of $\mathfrak{U}$ and $\mathfrak{W}$, this means that for any $\sigma$ in $\mathfrak{U}$ we have $\varphi \circ \sigma^\sharp = \lambda_{\rho(\sigma)} \circ \varphi$, where, as before, $\lambda_{\rho(\sigma)}$ is the endomorphism of $\mathfrak{W} \times_C \widetilde{K}$ coming from the right translation by $\rho(\sigma)$ on $\mathfrak{W}$ (1.0.4).

**Lemma 1.2.** Let $\varphi^\sharp$ be the comorphism of $\varphi$ (see (2)) and for any $r,s = 1, \ldots, n$ set $z_{r,s} = \varphi^\sharp(t_{r,s})$ (recall that $\mathfrak{W} = \text{Spec} C\left[ \left( t_{r,s} \right)_{1 \leq r,s \leq n} \right]$).

Then, for all $\sigma \in \mathfrak{U}$, there exists a matrix $a(\sigma) \in \text{GL}_n(C)$ such that we have the equality of matrices: $(\sigma(z_{r,s}))_{1 \leq r,s \leq n} = (z_{r,s})_{1 \leq r,s \leq n} a(\sigma)$.

**Proof.** Denote by $\lambda_{\rho(\sigma)}^\sharp$ the comorphism of the right translation by $\rho(\sigma)$ on $\mathfrak{W} \times_C \widetilde{K}$, we have the equalities of matrices

$$(\sigma(z_{r,s}))_{1 \leq r,s \leq n} = (\sigma(\varphi^\sharp(t_{r,s})))_{1 \leq r,s \leq n} = (\varphi^\sharp \left( \lambda_{\rho(\sigma)}^\sharp(t_{r,s}) \right))_{1 \leq r,s \leq n} ,$$

because $\varphi$ is equivariant, and

$$\left( \lambda_{\rho(\sigma)}^\sharp(t_{r,s}) \right)_{1 \leq r,s \leq n} = (t_{r,s})_{1 \leq r,s \leq n} a(\rho(\sigma))$$

where for any $\tau \in \mathfrak{W}$ the matrix $a(\tau)$ is in $\text{GL}_n(C)$ and is such that the formula $(\tau(t_{r,s}))_{1 \leq r,s \leq n} = (t_{r,s})_{1 \leq r,s \leq n} a(\tau)$ defines the images of the $t_{r,s}$ by the comorphism $\lambda_{\tau}^\sharp$ of the right translation on $\mathfrak{W}$ by $\tau$. We have find

$$(\sigma(z_{r,s}))_{1 \leq r,s \leq n} = (z_{r,s})_{1 \leq r,s \leq n} a(\rho(\sigma))$$

with $a(\rho(\sigma))$ in $\text{GL}_n(C)$.

The fact that $\varphi$ is an isomorphism implies that $R \otimes_K \widetilde{K}$ is generated over $\widetilde{K}$ by the $z_{r,s}$, $1 \leq r,s \leq n$, indeed $R \otimes_K \widetilde{K}$ is generated over $\widetilde{K}$ by the $C$-space $V := \sum_{1 \leq r,s \leq n} Cz_{r,s}$ and the lemma shows that this space $V$ is (globally) invariant under the action of the Galois group $\mathfrak{U}$. The (ordinary) Galois group $\text{Gal}(\widetilde{K}/K)$ acts as usual on the right
hand factor of $R \otimes_K \bar{K}$ and trivially on the left one, then we see that $R$ is generated over $K$ by the $z_{r,s}$, $1 \leq r, s \leq n$.

Another consequence of the previous lemma is that the matrix

$$D \overset{\text{def}}{=} (z'_{r,s})_{1 \leq r,s \leq n},$$

is in $M_n(K)$, then, because $\varphi^z$ is an isomorphism, the ring $R$ is generated by the entries of a fundamental matrix of the equation $Y' = DY$, we know also that $R$ is a simple differential ring. It follows that $R$, resp. $E$, is the Picard-Vessiot ring, resp. field, over $K$ of this equation.

To a rational representation $\rho : dGal(L/K) \to GL_n(C)$ we have associated an element $[D]$ of $Z_n(L/K)$, this is clearly the inverse map of $[A] \mapsto \rho_{[A]}$. \hfill \Box

2. A CORRESPONDANCE.

Let $K^{\text{diff}}$ be a universal Picard-Vessiot extension of $K$ and set $C^{\text{diff}} = dGal(K^{\text{diff}}/K)$. We choose once of all an identification $GL_n(C) = GL(C^n)$.

Let $\text{Rep}_n(C^{\text{diff}})$ be the category of representations of $C^{\text{diff}}$ in $GL_n(C)$: the objects are morphisms $\rho : C^{\text{diff}} \to GL_n(C)$, an arrow $f : \rho_1 \to \rho_2$ is a $C$-linear map from $C^n$ into itself such that, for any $g \in C^{\text{diff}}$, the following diagram is commutative

$$
\begin{array}{ccc}
C^n & \overset{\rho_1(g)}{\longrightarrow} & C^n \\
\downarrow f & & \downarrow f \\
C^n & \overset{\rho_2(g)}{\longrightarrow} & C^n
\end{array}
$$

To define the category $Z_n(K)$ we need the following remarks. Let $M$ and $N$ be two elements of $M_n(K)$, we say that they are equivalent if there exists $U$ and $V$ in $GL_n(K)$ such that $N = VMU$. We denote by $\overline{M}$ the equivalent class of $M$. Let $A_i \in M_n(K)$, $i = 1, 2$ and let $M \in M_n(K)$ such that

\begin{equation}
M' = A_2M - MA_1.
\end{equation}

Let $B_i \in [A_i]$, let $U_i \in GL_n(K)$ such that

$$A_i = U_i'U_i^{-1} + U_iB_iU_i^{-1},$$

then an easy calculation shows that

$$(U_2^{-1}MU_1)' = B_2(U_2^{-1}MU_1) - (U_2^{-1}MU_1)B_2.$$ 

Suppose that $M \in GL_n(K)$ and satisfies (4), then

$$(M^{-1})' = A_1M^{-1} - M^{-1}A_2.$$ 

Now we can define the category $Z_n(K)$. Its objects are elements of $Z_n(K)$ (see (1)), an arrow $[A_1] \to [A_2]$, where $A_1$ and $A_2$ are elements of $M_n(K)$, is an equivalence class $\overline{M}$ in $M_n(K)$ such that there exists $M \in \overline{M}$ satisfying (4). The two preceding formulas show that this definition does not depend on the choice of $A_i$ in $[A_i]$, $i = 1, 2$, and
that invertible arrows in $\mathbb{Z}_n(K)$ correspond to equivalence classes of invertible matrices. We explain the composition of arrows. Let $\bar{M} : [A_1] \to [A_2]$ and $\bar{N} : [A_2] \to [A_3]$ two arrows of $\mathbb{Z}_n(K)$, choose $M \in \bar{M}$, $N \in \bar{N}$ such that

$$M' = A_2 M - MA_1 \quad \text{and} \quad N' = A_3 N - NA_2,$$

then we see that

$$(NM)' = A_3 NM - NMA_1.$$  

The composed arrow is $\bar{N} \circ \bar{M} = \bar{NM}$, for a good choice of representing elements of the different classes of matrices.

Then $\mathbb{Z}_n(K)$ is a category, indeed it is easily to see that it is an additive category.

**Theorem 2.1.** The two categories $\mathbb{Z}_n(K)$ and $\text{Rep}_n(G_{\text{diff}})$ are equivalent. On objects, this equivalence is $[A] \mapsto c_{[A]}$ (see (1.0.1)).

**Proof.** Note that here to write $c_{[A]}$ is an abuse of notation, if $L/K$ is the Picard-Vessiot extension (contained in $K_{\text{diff}}$) associated to the equation $Y' = AY$, we denote always $c_{[A]}$ the representation

$$G_{\text{diff}}\text{ restriction} \xrightarrow{\text{dGal}(L/K)} c_{[A]} \xrightarrow{\text{GL}_n(C)}.$$

The map $[A] \mapsto c_{[A]}$ on objects of the categories has been constructed in the previous theorem, it is one to one. Let $[A_1]$ and $[A_2]$ be two objects of $\mathbb{Z}_n(K)$ and $\bar{M} : [A_1] \to [A_1]$ be an arrow, select $M \in \bar{M}$ such that $M' = A_2 M - MA_1$. Let $F_1, F_2 \in \text{GL}_n(K_{\text{diff}})$ be fundamental matrices for respectively the equations $Y' = A_1 Y$ and $Y' = A_2 Y$. Then $F'_i = A_i F_i, i = 1, 2$. Let $f = F_2^{-1} M F_1$, a priori $f$ is in $\text{GL}_n(K)$, but

$$f' = (F_2^{-1})' M F_1 + F_2^{-1} M' F_1 + F_2^{-1} M F_1 = (-F_2^{-1} A_2) M F_1 + F_2^{-1} (A_2 M - MA_1) F_1 + F_2^{-1} M A_1 F_1 = 0.$$

Then $f = F_2^{-1} M F_1$ is in $\text{GL}_n(C)$. Now we prove that $f$ is a morphism from $c_{[A_1]}$ to $c_{[A_2]}$. Let $g$ be an element of $G_{\text{diff}}$. Applying $g$ to the relation $f = F_2^{-1} M F_1$ we find

$$f = g(F_2^{-1}) M g(F_1) = g(F_2^{-1}) F_2 f F_1^{-1} g(F_1) = c_{[A_2]}(g) f c_{[A_1]}(g),$$

(see (1.0.1)) for all $g$. This means that $f : c_{[A_1]} \to c_{[A_2]}$ is a map in $\text{Rep}_n(G_{\text{diff}})$.

Conversely let $f : \rho_1 \to \rho_2$ be an arrow of $\text{Rep}_n(G_{\text{diff}})$, then we can see $f$ as a matrix with coefficient in $C$. We know that there exists $A_i$ in $M_n(K)$ such that $\rho_i = c_{[A_i]}$, $i = 1, 2$. Let as before $F_i$ be a fundamental matrix for the equation $Y' = A_i Y$. Set $M = F_2 f F_1^{-1}$.

- We prove that $M$ is in $M_n(K)$. The fact that $f$ is a morphism of representations means that for all $g$ in $G_{\text{diff}}$ we have

$$f c_{[A_1]}(g) = c_{[A_1]}(g) f,$$
which is equivalent to
\[ fF_1^{-1}g(F_1) = F_2^{-1}g(F_2)f, \]
then
\[ F_2fF_1^{-1} = g(F_2)f g(F_1^{-1}) = g(F_2fF_1^{-1}). \]
This proves that the entries of $M$ are in $K$.

- We prove the formula $M' = A_2M - MA_1$. We have
\[
M' = F'_2fF_1^{-1} + F_2f(F_1^{-1})' = A_2F_2fF_1^{-1} + F_2f(-F_1^{-1}A_1)
\]
which is the expected formula. □

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