Analysis of a time-splitting scheme for a class of random nonlinear partial differential equations

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Abstract

In this paper, we consider a Lie time-splitting scheme for a nonlinear partial differential equation driven by a random time-dependent dispersion coefficient. Our main result is a uniform estimate of the error of the scheme when the time step goes to 0. Moreover, we prove that the scheme satisfies an asymptotic-preserving property. As an application, we study the order of convergence of the scheme when the dispersion coefficient approximates a (multi)fractional process.

1 Introduction

The study of partial differential equations (PDE) driven by random processes is a subject of much interest because of their numerous applications, for instance in nonlinear optics [1] or wave propagation in random media [16]. The driving random processes can model random perturbations or physical quantities whose we know only a statistical description. Besides applications, the asymptotic analysis and numerical simulations of such equations are crucial questions. This paper presents an analysis of a Lie time-splitting scheme for a class of random nonlinear partial differential equations including Schrödinger equations.

The time-splitting schemes are often used for the simulation of nonlinear evolution PDE because they are quite simple to implement. They consist of splitting the problem into two partial problems which can be solved explicitly and constructing a numerical solution by combining the solutions of the two partial problems. These methods are consistent for deterministic nonlinear Schrödinger equations [6] and can take different forms (mainly the Lie and Strang schemes). Indeed, in [6] the authors prove that the Lie scheme is of order

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1 and the Strang scheme is of order 2 (in dimension 1 or 2). The Lie scheme has also been studied for nonlinear Schrödinger equations with random white-noise dispersion [25]. In this case, it is proven that the Lie scheme is consistent and of order bounded below by $1/2$.

The asymptotic-preserving property for a given numerical scheme is of great importance for the asymptotic analysis and can be described as follows. Let $\{u^\varepsilon\}_{\varepsilon\geq 0}$ be the solutions of a family of problems such that $\lim_{\varepsilon\to 0} u^\varepsilon = u^0$, and $\{u^{\varepsilon,h}\}_{\varepsilon\geq 0}$ the family of numerical solutions approximating $\{u^\varepsilon\}_{\varepsilon\geq 0}$ for a time-step $h > 0$ and obtained from a numerical scheme. We say that the given numerical scheme is **asymptotic-preserving** if $u^{\varepsilon,h}$ approximates u^ε independently of ε and $\lim_{\varepsilon\to 0} u^{\varepsilon,h} = u^{0,h}$. There exist a lot of works dealing with Asymptotic-Preserving (AP) property in various problems (for instance [4, 10, 11, 17, 21, 23]). In particular for time-splitting schemes for Schrödinger and/or random equations, we mention for instance [2, 3, 7, 19, 25].

In this paper we consider a nonlinear PDE driven by a general random process with continuous sample paths. We analyze a Lie time-splitting scheme for this equation. We prove that the scheme converges and the uniform order of convergence is bounded below in terms of the sample paths of the driving process. The form of this lower bound and a continuity theorem are then used to establish a general asymptotic-preserving property for the time-splitting scheme. These results are then applied to nonlinear PDE driven by processes approximating fractional and multifractional processes. This generalizes results of [25] dealing with processes approximating a Brownian motion.

In section 2, we introduce the setting and study the order of convergence of the Lie time splitting scheme. Section 3 is dedicated to the proofs of the results of Section 2. In section 4, we establish the AP property and study equations driven by processes approximating (multi)fractional processes.

Notation

For a measurable space E and a normed space F, we denote by $L^2(E,F)$ the space of the square integrable functions from E to F. For the sake of clearness, we denote by L^2 the space $L^2(\mathbb{R},\mathbb{C})$. For every $p \in \mathbb{N}^*$, we introduce H^p as the Sobolev space of the square integrable functions from $\mathbb{R}$ to $\mathbb{C}$ such that the first p derivatives are square integrable. We consider $\|\cdot\|_{L^2}$, $\|\cdot\|_{H^1}$, $\|\cdot\|_{H^2}$, ..., as their associated norms. For every function $v \in L^2$, we denote the Fourier transform of v by $\mathcal{F}(v)$ or $\mathcal{F}_x(v(x))$: for every $\xi \in \mathbb{R}$,

$$\mathcal{F}(v)(\xi) = \mathcal{F}_x(v(x))(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} v(x) dx.$$

For every function $w \in L^2$, we denote the inverse Fourier transform of w by $\mathcal{F}^{-1}(w)$ or $\mathcal{F}_\xi^{-1}(w(\xi))$.

Throughout the paper, all the random variables are defined on a probability space $(\Omega, \mathcal{T}, \mathbb{P})$, the corresponding expectation being $\mathbb{E}$.

Finally, when considering a Lipschitz function g, we designate by $\|g\|_{Lip}$ its Lipschitz constant.

2 Main results

2.1 Nonlinear PDE with random dispersion

Let t_0 and T such that $0 < t_0 < T < \infty$. We consider the following nonlinear random PDE with random dispersion written in differential form:

$$\begin{aligned} u(t,x) &= u_{t_0}(x) + i\int_{t_0}^{t} P\left(i\frac{\partial}{\partial x}\right) u(\theta,x)\circ dW(\theta) \\ &\quad + \int_{t_0}^{t} g(u(\theta,x))d\theta, \qquad (t,x)\in[t_0,T]\times\mathbb{R}. \end{aligned} \tag{1}$$

The function u_{t_0} is the initial condition at t_0. The function g is the nonlinear function whose we precise the assumptions later. The notation W designates a stochastic process, which can be eventually a deterministic function; P is a polynomial with real coefficients and its degree is denoted by δ_P or δ when there is no ambiguity. The symbol $\circ$ is explained below (Remark 1).

In order to deal with existence and uniqueness of the solution of (1) we consider the corresponding linear problem

$$v(t,x) = u_{t_0}(x) + i\int_{t_0}^{t} P\left(i\frac{\partial}{\partial x}\right) v(\theta,x)\circ dW(\theta), \qquad (t,x)\in[t_0,T]\times\mathbb{R}, \tag{2}$$

whose we construct the unique solution as $v : (t,x) \to X(t_0,t)u_{t_0}(x)$ where

$$X(t_0,t)u_{t_0}(x) = \mathcal{F}^{-1}\Big(\xi \to e^{-iP(\xi)(W(t)-W(t_0))}\mathcal{F}(u_{t_0})(\xi)\Big)(x). \tag{3}$$

Remark that for every $k\in\mathbb{N}$, if $u_{t_0}\in H^k$, then

$$\|X(t_0,t)u_{t_0}\|_{H^k} = \|u_{t_0}\|_{H^k}. \tag{4}$$

Hence, Equation (1) is understood as the integral equation

$$u(t,x) = X(t_0,t)u_{t_0}(x) + \int_{t_0}^{t} X(\theta,t)g(u(\theta,x))\,d\theta, \qquad (t,x)\in[t_0,T]\times\mathbb{R}. \tag{5}$$

We have the following preliminary result.

Theorem 1. *If $u_{t_0}\in L^2$ and g is Lipschitz, then there exists a unique solution u with sample paths in $\mathcal{C}([t_0,T],L^2)$ to Equation (5). Moreover, if there exists $k\in\mathbb{N}$ such that $u_{t_0}\in H^k$, g is k times differentiable and its derivatives up to the order k are bounded, then there exists a (deterministic) constant $C_{\infty,k}>0$, independent of W, such that*

$$\max_{t\in[t_0,T]}\|u(t,\cdot)\|_{H^k} \le C_{\infty,k} < \infty \quad \textit{and} \quad \max_{t\in[t_0,T]}\|g(u(t,\cdot))\|_{H^k} \le C_{\infty,k} < \infty$$

The constant $C_{\infty,k}$ depends only on $\|u_0\|_{H^k}$.

The proof is postponed to Section 3. We define the family of operators $\{S(t_0,t)\}_{t\in[t_0,T]}$ such that $(t,x)\mapsto S(t_0,t)u_{t_0}(x)$ is the unique solution to Equation (5).

Remark 1. *If W is a Brownian motion, the solution obtained in Theorem 1 is the Stratonovich solution. This is why we use the notation $\circ$ in (1).*

2.2 Time-splitting scheme

From now on we assume that $T=1$ and we fix an initial condition $u_0\in L^2$. Throughout this section, u denotes the solution $u:(t,x)\mapsto S(0,t)u_0(x)$ to Equation (5) when $t_0=0$ and with the initial condition u_0. This section is devoted to introduce a time-splitting scheme to approximate u.

For $t_0\in[0,1]$ and $u_{t_0}\in L^2$, we introduce the problem

$$w(t,x)=u_{t_0}(x)+\int_{t_0}^{t} g(w(\theta,x))\,d\theta, \qquad (t,x)\in[t_0,1]\times\mathbb{R}. \tag{6}$$

If g is a Lipschitz function, the unique solution w of (6) is given by Theorem 1 (with $W\equiv 0$). We then define the family of operators $Y=\{Y(t)\}_{t\geq 0}$ such that for every $(t,x)\in[t_0,1]\times\mathbb{R}$, $w(t,x)=Y(t-t_0)u_{t_0}$. We define the (Lie) splitting operator by

$$Z(t_0,t):=Y(t-t_0)X(t_0,t). \tag{7}$$

For every $k\in\mathbb{N}$ and $h\in(0,1]$, we set $S^{k,h}:=S((k-1)h,kh)$ $Z^{k,h}:=Z((k-1)h,kh)$. For every $n\in\mathbb{N}$, we set

$$u^{n,h}:=Z^{n,h}\cdots Z^{1,h}u_0.$$

We aim to prove that $\{u^{n,h}\}_{n\in\{1,\dots,N\}}$ approximates $\{u(nh,\cdot)\}_{n\in\{1,\dots,N\}}$ in some sense for $h\to 0$.

For every $0\leq t_0<t\leq 1$ we define

$$\mathcal{I}_W(t_0,t):=\int_{t_0}^{t}\left(|W(t)-W(\theta)|+\int_{t_0}^{\theta}|W(\theta)-W(\sigma)|d\sigma\right)d\theta. \tag{8}$$

Our main result is the following.

Theorem 2. *We assume that $u_0\in H^\delta$, W admits finite first-order moments, g is δ times differentiable, its derivatives up to the order $\delta+2$ are bounded. There exists a constant C which depends only on g and $||u_0||_{H^\delta}$, such that for every $h\in(0,1]$,*

$$\mathbb{E}\left[\max_{n\in\{1,\dots,N\}}||u^{n,h}-u(nh,\cdot)||_{L^2}\right]\leq C\sum_{n=1}^{N}\mathbb{E}\left[\mathcal{I}_W((n-1)h,nh)\right]. \tag{9}$$

Remark 2 (Fundamental remark)**.** *Notice that the constant C appearing in Theorem 2 is independent of the process W. This is a key point of the result.*

We can easily deduce the following corollary about processes with stationary increments.

Corollary 1. *Under the assumptions of Theorem 2, if W has stationary increments, then (9) can be written as*

$$\mathbb{E}\left[\max_{n\in\{1,\ldots,N\}} ||u^{n,h} - u(nh,\cdot)||_{L^2}\right] \leq CN \int_0^h \mathbb{E}\left[|W(\theta)|\right] d\theta. \tag{10}$$

We refer the reader to Section 4 for applications of Corollary 1 to equations driven by Brownian motions and (multi-)fractional processes.

3 Proofs

3.1 Proof of Theorem 1

Throughout this section, we consider a continuous sample path W of a given stochastic process. **Notice that all the constants appearing in this proof are independent of** W, even though the other quantities do depend on W. For the sake of simplicity, we assume that $t_0 = 0$ and $T = 1$.

Let Γ be the application from $\mathcal{C}([0,1], L^2)$ to itself such that for every $U \in \mathcal{C}([0,1], L^2)$,

$$\Gamma(U)(t,x) = X(0,t)u_0(x) + \int_0^t X(s,t)g(U(s,\cdot))(x)ds.$$

We define the sequence $\{U_j\}_{j\in\mathbb{N}} \in \left(\mathcal{C}([0,1], L^2)\right)^{\mathbb{N}}$ by $U_0 := u_0$ and $U_{j+1} := \Gamma(U_j)$ for all $j > 0$. Using a classical fixed-point procedure, we get the following result.

Lemma 1. *There exists a unique solution $u \in \mathcal{C}([0,1], L^2)$ to (5).*

Proof. Obviously, for every U and V in $\mathcal{C}([0,1], L^2)$ and every $t \in [0,1]$,

$$\sup_{t'\in[0,t]} \|\Gamma(U(t')) - \Gamma(V(t'))\|_{L^2} \leq \|g\|_{Lip} \int_0^t \sup_{t'\in[0,\theta]} \|U(t') - V(t')\|_{L^2} d\theta \tag{11}$$

Moreover, we have

$$\sup_{t'\in[0,t]} \|\Gamma(U(t'))\|_{L^2} \leq \|u_0\|_{L^2} + \|g\|_{Lip} \int_0^t \sup_{t'\in[0,\theta]} \|U(t')\|_{L^2} d\theta$$

and thus

$$\sup_{t\in[0,1]} \|\Gamma(U(t))\|_{L^2} \leq \|u_0\|_{L^2} \exp(\|g\|_{Lip}). \tag{12}$$

We deduce from (11) and (12) that the sequence $\{U_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0,1], L^2)$. Then there exists a solution u to (5). The uniqueness is a direct consequence of (11). □

We now prove the estimates on the H^k-norm of the solution u. Let us first give the following useful lemma.

Lemma 2. *Let $n \geq 1$, $\psi \in \mathcal{C}^n(\mathbb{R}, \mathbb{R}^2)$ and $\phi \in \mathcal{C}^n(\mathbb{R}^2, \mathbb{R})$. There exists a constant $C > 0$ such that for every $x \in \mathbb{R}$,*

$$\left|(\phi \circ \psi)^{(n)}(x)\right| \leq C \sum_{k=1}^{n} \|\nabla^k \phi\|_\infty \sum_{\substack{1 \leq l_1 \leq .. \leq l_k \\ l_1 + \ldots + l_k = n}} \prod_{r=1}^{k} \left\|\psi^{(l_r)}(x)\right\| \tag{13}$$

Remark 3. *In the previous result, remark that*

$$\sum_{k=1}^{n} \|\nabla^k \phi\|_\infty \sum_{\substack{1 \leq l_1 \leq .. \leq l_k \\ l_1 + \ldots + l_k = n}} \prod_{r=1}^{k} \left\|\psi^{(l_r)}(x)\right\| = \|\nabla \phi\|_\infty \left\|\psi^{(n)}(x)\right\| + \sum_{k=2}^{n} \|\nabla^k \phi\|_\infty \sum_{\substack{1 \leq l_1 \leq .. \leq l_k < n \\ l_1 + \ldots + l_k = n}} \prod_{r=1}^{k} \left\|\psi^{(l_r)}(x)\right\|$$

The proof of Lemma 2 is a direct consequence of Lemma 13 stated in the appendix. We now establish the following lemma.

Lemma 3. *Let $u_0 \in H^n$, $n \geq 1$. Then there exists a deterministic constant $C_n = C(g, \|u_0\|_{H^n})$ depending only on $\|u_0\|_{H^n}$ and g such that the solution u of (5) satisfies*

$$\sup_{t \in [0,1]} \|u(t)\|_{H^n} \leq C_n. \tag{14}$$

The proof of Lemma 3 is a direct consequence of Lemma 4 stated just below.

Lemma 4. *Let $u_0 \in H^n$, $n \geq 1$. Then there exists a deterministic constant $C_n = C(g, \|u_0\|_{H^n})$ depending only on $\|u_0\|_{H^n}$ and g such that for every $j \geq 1$, U_j satisfies*

$$\sup_{t \in [0,1]} \|U_j(t)\|_{H^n} \leq C_n. \tag{15}$$

Proof. We prove by induction that for all $n \in \mathbb{N}$ and $j \in \mathbb{N}$,

$$\|U_{j+1}(t)\|_{H^n} \leq \|u_0\|_{H^n} + c_n \int_0^t \|U_j(s)\|_{H^n} ds \tag{16}$$

where c_n depends on g and $\|u_0\|_{H^n}$ so that the proof can be concluded by Gronwall's lemma. Inequality (16) is obvious for all $j \in \mathbb{N}$ and $n = 1$. We suppose that (16) is true up to the rank $n \geq 2$ and we consider $u_0 \in H^{n+1}$. Because of Lemma 2 and Remark 3 there exists $C > 0$ such that for all $j \in \mathbb{N}$,

$$\begin{aligned} \left|\partial_x^{n+1} g(U_j)\right| &\leq C \sum_{k=1}^{n+1} \|\nabla^k g\|_\infty \sum_{\substack{1 \leq l_1 \leq .. \leq l_k < n \\ l_1 + \ldots + l_k = n+1}} \prod_{r=1}^{k} \left|\partial_x^{l_r} U_j\right| \\ &\leq C \|\nabla g\|_\infty \left|\partial_x^{n+1} U_j\right| + C \sum_{k=2}^{n+1} \|\nabla^k g\|_\infty \sum_{\substack{1 \leq l_1 \leq .. \leq l_k < n \\ l_1 + \ldots + l_k = n+1}} \prod_{r=1}^{k} \left|\partial_x^{l_r} U_j\right| \end{aligned}$$

Using the Sobolev inequality $\|\phi\|_{L^\infty} \le C\|\phi\|_{H^1}$, we get

$$
\begin{aligned}
&\left\|\partial_x^{n+1} g(U_j)\right\|_{L^2}\\
&\le C\|\nabla g\|_\infty \left\|\partial_x^{n+1} U_j\right\|_{L^2} + C\sum_{k=2}^{n+1}\|\nabla^k g\|_\infty \sum_{\substack{1\le l_1\le ..\le l_k\le n\\ l_1+...+l_k=n+1}} \left\|\prod_{r=1}^{k}\partial_x^{l_r} U_j\right\|_{L^2}\\
&\le C\|\nabla g\|_\infty \left\|\partial_x^{n+1} U_j\right\|_{L^2} + C\sum_{k=2}^{n+1}\|\nabla^k g\|_\infty \sum_{\substack{1\le l_1\le ..\le l_k\le n\\ l_1+...+l_k=n+1}} \left\|\partial_x^{l_1} U_j\right\|_{L^2}\prod_{r=2}^{k}\left\|\partial_x^{l_r} U_j\right\|_{L^\infty}\\
&\le C\|\nabla g\|_\infty \left\|\partial_x^{n+1} U_j\right\|_{L^2} + C\sum_{k=2}^{n+1}\|\nabla^k g\|_\infty \sum_{\substack{1\le l_1\le ..\le l_k\le n\\ l_1+...+l_k=n+1}} \|U_j\|_{H^{l_1}}\prod_{r=2}^{k}\|U_j\|_{H^{l_r+1}}
\end{aligned}
$$

In the last sum on the set $\{1 \le l_1 \le .. \le l_k \le n; l_1 + ... + l_k = n+1\}$, at most one index from $\{l_1, l_2+1, ..., l_k+1\}$ can be equal to $n+1$. Moreover, by assumption, we have $\max_{\theta\in[0,1]} \|U_j(\theta)\|_{H^l} \le C_l$ for every $l \in \{0, ..., n\}$. We then get that for all t,

$$
\left\|\partial_x^{n+1} U_{j+1}(t)\right\|_{L^2} \le \left\|\partial_x^{n+1} u_0\right\|_{L^2} + K\int_0^t \|U_j(\theta)\|_{H^{n+1}} d\theta,
$$

where K depends on g and $\|u_0\|_{H^{n+1}}$. Using (16), this prove by induction that

$$
\|U_{j+1}(t)\|_{H^{n+1}} \le \|u_0\|_{H^{n+1}} + c_{n+1}\int_0^t \|U_j(\theta)\|_{H^{n+1}} d\theta,
$$

where c_{n+1} depends on g and $\|u_0\|_{H^{n+1}}$. This concludes the proof. □

3.2 Preliminary results

We now state some lemmas for the proof of Theorem 2. The first lemma deals with the boundedness of the splitting operators $\{Z(t_0,t)\}_{t\in[t_0,1]}$ in H^k for $t_0 \in [0,1[$ under some assumptions on g.

Lemma 5. *Let $v_0 \in L^2$. If there exists $k \in \mathbb{N}$ such that $v_0 \in H^k$, g is k times differentiable and its derivatives up to the order k are bounded, then there exists a (deterministic) constant $C_{\infty,k} > 0$ such that for every $(t_0,t) \in [0,1]^2$ satisfying $t_0 < t$ we have*

$$
\max_{\theta\in[t_0,t]} \|Z(t_0,\theta)v_0\|_{H^k} \le C_{\infty,k} \text{ and } \max_{\theta\in[t_0,t]} \|g(Z(t_0,\theta)v_0)\|_{H^k} \le C_{\infty,k}
$$

The constant $C_{\infty,k}$ depends only on $\|u_0\|_{H^k}$.

Proof. We follow the same lines as the proof of Lemma 3 (and Lemma 4). □

The two following Lemmas provide useful estimates on the operators X and Z. Lemma 6 essentially states the continuity of $\{X(t_0,t)\}_{t\in[t_0,1]}$ for $t_0 \in [0,1[$.

Lemma 6. *Let $v_0 \in H^\delta$. There exists a constant C_P only depending on P such that for every $(t_0, t) \in [0,1]^2$ satisfying $t_0 < t$ we have*

$$||X(t_0,t)v_0 - v_0||_{L^2} \leq C_P|W(t) - W(t_0)|\|v_0\|_{H^\delta}.$$

Proof. Using the estimate $|\exp(ix) - 1|^2 \leq x^2$, we get

$$\begin{aligned}
&||X(t_0,t)v_0 - v_0||_{L^2}^2 \\
&\quad = \int_{-\infty}^{+\infty} \Big|\mathcal{F}(v_0)(\theta)\exp\Big(-i(W(t)-W(t_0))P(\theta)\Big) - \mathcal{F}(v_0)(\theta)\Big|^2 d\theta \\
&\quad \leq (W(t)-W(t_0))^2 \int_{-\infty}^{+\infty} |P(\theta)\mathcal{F}(v_0)(\theta)|^2 \, d\theta \\
&\quad \leq C_P^2 (W(t)-W(t_0))^2 ||v_0||_{H^\delta}^2
\end{aligned}$$

where, for instance, $C_P^2 = \delta \sum_{j=0}^{\delta} p_j^2$. □

Finally, the Lemma stated below proves the Lipschitz property of the operators $Z(t_0,t)$ (for $t_0 < t$) from L^2 to itself.

Lemma 7. *Let $v_0, v_1 \in L^2$. If there exists $k \in \mathbb{N}$ such that v_0 and $v_1 \in H^k$, g is k times differentiable and its derivatives up to the order k are bounded, then there exists a (deterministic) constant $C_g > 0$ such that for every $(t_0,t) \in [0,1]^2$ satisfying $t_0 < t$ we have*

$$\|Z(t_0,t)v_0 - Z(t_0,t)v_1\|_{L^2} \leq \|v_0 - v_1\|_{L^2} \exp(C_g(t-t_0)).$$

The constant C_g depends only on g.

Proof. Because of (4), it is enough to prove that there exists a (deterministic) constant $C_g > 0$ such that for every $(t_0,t) \in [0,1]^2$ satisfying $t_0 < t$ we have

$$\|Y(t-t_0)v_0 - Y(t-t_0)v_1\|_{L^2} \leq \|v_0 - v_1\|_{L^2} \exp(C_g(t-t_0)).$$

Since (6) and because g is Lipschitz, we have

$$\begin{aligned}
&\|Y(t-t_0)v_0 - Y(t-t_0)v_1\|_{L^2} \\
&\qquad \leq \|v_0 - v_1\|_{L^2} + \int_{t_0}^{t} \|g(Y(\theta-t_0)v_0) - g(Y(\theta-t_0)v_1)\|_{L^2} d\theta \\
&\qquad \leq \|v_0 - v_1\|_{L^2} + \|g\|_{Lip} \int_{t_0}^{t} \|Y(\theta-t_0)v_0 - Y(\theta-t_0)v_1\|_{L^2} d\theta.
\end{aligned}$$

We conclude by Gronwall's lemma. □

3.3 Proof of Theorem 2

To prove Theorem 2 we deal with the local error of the scheme (Lemma 8) and then we prove the estimate of the global error.

Lemma 8. *Let $v_0 \in H^\delta$. There exists a (deterministic) constant $C = C(g, \|v_0\|_{H^\delta}) > 0$ depending only on g and $\|v_0\|_{H^\delta}$ such that for every $(t_0, t) \in [0,1]^2$ satisfying $t_0 < t$ we have*

$$\|S(t_0,t)v_0 - Y(t-t_0)X(t_0,t)v_0\|_{L^2} \leq C\mathcal{I}_W(t_0,t). \tag{17}$$

Proof. Throughout the proof, the letter C stands for a deterministic constant, can vary from line to line and depends only on P, g and $\|v_0\|_{H^\delta}$. First, we remark that

$$Z(t_0,t)v_0 = Y(t-t_0)X(t_0,t)v_0 = X(t_0,t)v_0 + \int_{t_0}^t g(Y(\theta-t_0)X(t_0,t)v_0)d\theta. \tag{18}$$

Thus, we have, using (5) and (18),

$$S(t_0,t)v_0 - Z(t_0,t)v_0 = R_1(t_0,t) + R_2(t_0,t) + R_3(t_0,t) \tag{19}$$

where

$$\begin{aligned}
R_1(t_0,t) &= \int_{t_0}^t X(\theta,t)\{g(S(t_0,\theta)v_0) - g(Z(t_0,\theta)v_0)\}d\theta, \\
R_2(t_0,t) &= \int_{t_0}^t X(\theta,t)g(Z(t_0,\theta)v_0) - g(Z(t_0,\theta)v_0)d\theta, \\
R_3(t_0,t) &= \int_{t_0}^t \left(g(Z(t_0,\theta)v_0) - g(Y(\theta-t_0)X(t_0,t)v_0)\right)d\theta \\
&= \int_{t_0}^t \left(g(Y(\theta-t_0)X(t_0,\theta)v_0) - g(Y(\theta-t_0)X(t_0,t)v_0)\right)d\theta
\end{aligned}$$

Because of (4), we have

$$\|R_1(t_0,t)\|_{L^2} \quad \leq \quad \|g\|_{Lip}\int_{t_0}^t ||S(t_0,\theta)v_0 - Z(t_0,\theta)v_0||_{L^2}\,d\theta. \tag{20}$$

Since Lemmas 6 and 5 we have

$$\begin{aligned}
\|R_2(t_0,t)\|_{L^2} &\leq C\int_{t_0}^t |W(t)-W(\theta)|\|g(Z(z_0,z')v_0)\|_{L^2}d\theta \\
&\leq C\left(\int_{t_0}^t |W(t)-W(\theta)|d\theta\right)\max_{s\in[t_0,t]}\|g(Z(t_0,s)v_0)\|_{L^2}. \qquad (21)
\end{aligned}$$

From Lemmas 6 and 7 we obtain that

$$\begin{aligned}
||R_3(t_0,t)||_{L^2} &\leq \int_{t_0}^t ||g(Z(t_0,\theta)v_0) - g(Y(\theta-t_0)X(t_0,t)v_0)||_{L^2}\,d\theta \\
&\leq C\int_{z_0}^z ||X(t_0,\theta)v_0 - X(t_0,t)v_0||_{L^2}\,d\theta \\
&\leq C\left(\int_{t_0}^t |W(t)-W(\theta)|d\theta\right). \qquad (22)
\end{aligned}$$

Then, by (19), (20), (21) and (22), for every $(t_0, t) \in [0,1]^2$ satisfying $t_0 < t$ we get

$$\begin{aligned} \|S(t_0,t)v_0 - Z(t_0,t)v_0\|_{L^2} &\leq C\int_{t_0}^{t} |W(t) - W(\theta)|d\theta \\ &\quad + \|g\|_{Lip}\int_{t_0}^{t} \|S(t_0,\theta)v_0 - Z(t_0,\theta)v_0\|_{L^2} d\theta. \end{aligned}$$

We complete the proof by using the modified Gronwall lemma recalled below (Lemma 9). □

Lemma 9 (Modified Gronwall lemma)**.** *Let ϕ and f be two nonnegative functions defined on an interval $[a, b]$. We assume that there exists a constant $c > 0$ such that for every $t \in [a, b]$,*

$$\phi(t) \leq f(t) + c\int_a^t \phi(\theta)d\theta. \tag{23}$$

Then, for every $t \in [a, b]$,

$$\phi(t) \leq f(t) + ce^{ct}\int_a^t e^{-c\theta} f(\theta)d\theta. \tag{24}$$

Remark 4. *Remark that we **do not** assume f to be increasing.*

Now we prove Theorem 2.

Proof. (Theorem 2) We write

$$u_n^h - u(nh,\cdot) = \sum_{j=1}^{n} \left(Z^{n,h}\cdots Z^{j,h}S^{j-1,h}\cdots S^{1,h}u_0 - Z^{n,h}\cdots Z^{j+1,h}S^{j,h}\cdots S_1^h u_0\right).$$

From Lemma 7, there exists a (deterministic) constant $C_g > 0$ depending only on g such that

$$||u^{n,h} - u(nh,\cdot)||_{L^2} \leq \sum_{j=1}^{n} e^{C_g(n-j)}||(Z^{j,h} - S^{j,h})S^{j-1,h}\cdots S^{1,h}u_0||_{L^2}.$$

By Lemmas 8 and 1, there exists a (deterministic) constant $C(g, \|u_0\|_{H^\delta}) > 0$ depending only on g and $\|u_0\|_{H^\delta}$ such that

$$\max_{n\in\{1,\dots,N\}} ||u^{n,h} - u(nh,\cdot)||_{L^2} \leq C(g, \|u_0\|_{H^\delta})\sum_{j=1}^{N} \mathcal{I}_W((j-1)h, jh).$$

This concludes the proof. □

4 An application: asymptotic-preserving property

4.1 General setting and notation

This section is devoted to establish the so-called asymptotic-preserving property of the Lie scheme. From now on, the process W driving Equation (1) may vary. We then introduce new notations to take account of the dependence of all quantities with respect to the process. For a given stochastic process W whose sample paths are continuous on $[0,1]$ and $(t_0,t) \in [0,1]^2$ satisfying $t_0 < t$, we define the operators $S_W(t_0,t)$, $X_W(t_0,t)$ and $Y_W(t_0,t)$ such that for all $u_{t_0} \in L^2$ the functions $S_W(t_0,\cdot)u_{t_0}$, $X_W(t_0,\cdot)u_{t_0}$ and $Y_W(t_0,\cdot)u_{t_0}$ are respectively solutions of (1), (2) and (6). We let $Z_W(t_0,t) = Y_W(t_0,t)X_W(t_0,t)$. For all $N \in \mathbb{N}^*$ and $k \in \{1,...,N\}$, denoting $h := 1/N$, we define $S_W^{k,h} := S_W((k-1)h, kh)$ and $Z_W^{k,h} := Z_W((k-1)h, kh)$. For every $n \in \{1,...,N\}$ and an initial condition $u_0 \in H^\delta$, we set $u_W^{n,h} := Z_W^{n,h} \cdots Z_W^{1,h} u_0$. We denote by u_W the solution of Equation (1) with $t_0 = 0$ and driven by W and we set $u_W^{\cdot,h} := \{u_W^{n,h}\}_{n\in\{1,\dots,N\}}$.

We consider a family of continuous processes $\{W^\varepsilon\}_{\varepsilon>0}$ and another continuous process W^0 such that the solution u_{W^ε} converges to u_{W^0} as $\varepsilon \to 0$. We established and proved in the previous sections that the schemes $u_{W^\varepsilon}^{\cdot,h}$ and $u_{W^0}^{\cdot,h}$ converge respectively to W^ε and W^0 as the time step h goes to 0. Under suitable assumptions on the sequence of processes $\{W^\varepsilon\}_{\varepsilon>0}$ we prove in this section that

- the scheme $u_{W^\varepsilon}^{\cdot,h}$ converges (in some sense) to u_{W^ε} **uniformly** with respect to $\varepsilon > 0$ when $h \to 0$,
- and the scheme $u_{W^\varepsilon}^{\cdot,h}$ converges (in some sense) to $u_{W^0}^{\cdot,h}$ as $\varepsilon \to 0$ for every $h > 0$.

This implies that the limit $\varepsilon \to 0$ does not affect the convergence of the time-splitting scheme. This is the so-called **Asymptotic-Preserving (AP) property**, which is usually represented by the diagram

$$\begin{array}{ccc} u_{W^\varepsilon}^{\cdot,h} & \stackrel{\varepsilon\to 0}{\longrightarrow} & u_{W^0}^{\cdot,h} \\ \downarrow h\to 0 & & \downarrow h\to 0 \\ u_{W^\varepsilon} & \stackrel{\varepsilon\to 0}{\longrightarrow} & u_{W^0} \end{array} \tag{25}$$

Notice that AP property has been studied in various problems (for instance [4, 10, 11, 17, 21, 23]) and in particular for time-splitting schemes for Schrödinger and/or random equations (for instance [2, 3, 7, 19, 25]).

4.2 Main results

We establish and prove the main results of this section. The first one concerns the convergence of u_{W^ε} when $\varepsilon \to 0$.

Theorem 3. *Let $u_0 \in H^\delta$. The mapping*

$$\begin{array}{rcl} \mathcal{C}([0,1],\mathbb{R}) & \to & \mathcal{C}([0,1],L^2) \\ w & \longmapsto & u_w \end{array}$$

is Lipschitz. As a consequence, when $\varepsilon \to 0$, if W^ε converges in distribution to W^0 in $\mathcal{C}([0,1],\mathbb{R})$, then u_{W^ε} converges in distribution to u_{W^0} in $\mathcal{C}([0,1],L^2)$.

Proof. Let w_1 and w_2 in $\mathcal{C}([0,1],\mathbb{R})$. We have

$$\begin{aligned} u_{w_1}(t,x) - u_{w_2}(t,x) &= X_{w_1}(0,t)u_0(x) - X_{w_2}(0,t)u_0(x) \\ &\quad + \int_0^t (X_{w_1}(\theta,t) - X_{w_2}(\theta,t))g(u_{w_1}(\theta,x))\,d\theta \\ &\quad + \int_0^t X_{w_2}(\theta,t)(g(u_{w_1}(\theta,x)) - g(u_{w_2}(\theta,x)))\,d\theta. \end{aligned}$$

We then deduce that there exists a constant $C > 0$ which only depends on g and $||u_0||_{H^\delta}$ such that

$$||u_{w_1}(t) - u_{w_2}(t)||_{L^2} \leq C||w_1 - w_2||_\infty + \|\nabla g\|_\infty \int_0^t ||u_{w_1}(\theta) - u_{w_2}(\theta)||_{L^2} d\theta.$$

By Gronwall's lemma,

$$\sup_{t\in[0,1]} ||u_{w_1}(t) - u_{w_2}(t)||_{L^2} \leq C\exp(\|\nabla g\|_\infty)||w_1 - w_2||_\infty,$$

which concludes the proof. □

The second main result of this section deals with the convergence of $u^{\cdot,h}_{W^\varepsilon}$ as $\varepsilon \to 0$.

Theorem 4. *Let $u_0 \in H^\delta$, $N \in \mathbb{N}^*$ and $h = 1/N$. The mapping*

$$\begin{array}{rcl} \mathcal{C}([0,1],\mathbb{R}) & \to & (L^2)^{N+1} \\ w & \longmapsto & \{u^{j,h}_w\}_{j=0,\dots,N} \end{array}$$

is Lipschitz. As a consequence, when $\varepsilon \to 0$, if W^ε converges in distribution to W^0 in $\mathcal{C}([0,1],\mathbb{R})$, then $\{u^{j,h}_{W^\varepsilon}\}_{j=0,\dots,N}$ converges in distribution to $\{u^{j,h}_{W^0}\}_{j=0,\dots,N}$ in $(L^2)^{N+1}$ as $\varepsilon \to 0$.

Proof. By induction, it is enough to show that there exists $C > 0$ such for all $j = 1,...,N$ and (w_1,w_2) in $\mathcal{C}([0,1],\mathbb{R})^2$,

$$\|u^{j,h}_{w_1} - u^{j,h}_{w_2}\|_{L^2} \leq C\|u^{j-1,h}_{w_1} - u^{j-1,h}_{w_2}\|_{L^2} + C\|w_1 - w_2\|_\infty. \tag{26}$$

Throughout this proof, C stands for a positive constant which depends on $\|u_0\|_{H^\delta}$, g and N and can vary from line to line. We have, using (18),

$$\begin{aligned} u^{j,h}_{w_1} - u^{j,h}_{w_2} &= X^{j,h}_{w_1}u^{j-1,h}_{w_1} - X^{j,h}_{w_2}u^{j-1,h}_{w_2} \\ &\quad + \int_{(j-1)h}^{jh} \Big(g\big(Y(\theta-(j-1)h)X^{j,h}_{w_1}u^{j-1,h}_{w_1}\big) \\ &\qquad - g\big(Y(\theta-(j-1)h)X^{j,h}_{w_2}u^{j-1,h}_{w_2}\big)\Big)\,d\theta. \end{aligned}$$

Taking the L^2-norm and because g is Lipschitz, we get

$$\begin{aligned}\|u^{j,h}_{w_1}-u^{j,h}_{w_2}\|_{L^2} &\leq \|X^{j,h}_{w_1}u^{j-1,h}_{w_1}-X^{j,h}_{w_2}u^{j-1,h}_{w_2}\|_{L^2}\\ &\quad+\int_{(j-1)h}^{jh}\Big\|g\big(Y(\theta-(j-1)h)X^{j,h}_{w_1}u^{j-1,h}_{w_1}\big)\\ &\qquad\qquad -g\big(Y(\theta-(j-1)h)X^{j,h}_{w_2}u^{j-1,h}_{w_2}\big)\Big\|_{L^2}d\theta.\\ &\leq C\|X^{j,h}_{w_1}u^{j-1,h}_{w_1}-X^{j,h}_{w_2}u^{j-1,h}_{w_2}\|_{L^2}.\end{aligned}$$

Because of Lemmas 3 and 6, we obtain

$$\begin{aligned}\|u^{j,h}_{w_1}-u^{j,h}_{w_2}\|_{L^2} &\leq C\|X^{j,h}_{w_1}(u^{j-1,h}_{w_1}-u^{j-1,h}_{w_2})\|_{L^2}+C\|X^{j,h}_{w_1}u^{j-1,h}_{w_2}-X^{j,h}_{w_2}u^{j-1,h}_{w_2}\|_{L^2}\\ &\leq C\|u^{j-1,h}_{w_1}-u^{j-1,h}_{w_2}\|_{L^2}+C\|w_1-w_2\|_\infty.\end{aligned}$$

This concludes the proof. □

From now on, we consider that $\{W^\varepsilon\}_{\varepsilon>0}$ and W^0 satisfy the following assumptions.

- **Assumption (A_1).** As $\varepsilon\to 0$, W^ε converges in distribution to W^0 in the space $\mathcal{C}([0,1],\mathbb{R})$.
- **Assumption (A_2).** There exist three constants $K>0$, $\gamma\geq 1$ and $\beta\geq 1$ such that for all t_1 and $t_2\in[0,1]$ and $\varepsilon>0$,

$$\mathbb{E}\left[(W^\varepsilon(t_1)-W^\varepsilon(t_2))^\gamma\right]\leq K|t_1-t_2|^\beta. \tag{27}$$

We state the last main result of this section.

Theorem 5. *We assume that $u_0\in H^\delta$, g is δ times differentiable, its derivatives up to the order δ are bounded. Then there exists a constant C which depends only on g and $||u_0||_{H^\delta}$, such that for every $h\in(0,1]$ and every $\varepsilon\geq 0$,*

$$\mathbb{E}\left[\max_{n\in\{1,\dots,N\}}\|u^{n,h}_{W^\varepsilon}-u_{W^\varepsilon}(nh,\cdot)\|_{L^2}\right]\leq Ch^{\beta/\gamma}. \tag{28}$$

This implies (25) and then the AP property.

Proof. By Theorem 2 and Remark 2, there exists a constant C which depends only on g and $||u_0||_{H^\delta}$, such that for every $h\in(0,1]$ and every $\varepsilon\geq 0$,

$$\mathbb{E}\left[\max_{n\in\{1,\dots,N\}}\|u^{n,h}_{W^\varepsilon}-u_{W^\varepsilon}(nh,\cdot)\|_{L^2}\right]\leq C\sum_{n=1}^{N}\mathbb{E}\left[\mathcal{I}_{W^\varepsilon}((n-1)h,nh)\right]. \tag{29}$$

By Hölder's inequality and (27), we have

$$\begin{aligned}\mathcal{I}_{W^\varepsilon}((n-1)h, nh) &\leq \int_{(n-1)h}^{nh} \mathbb{E}[|W^\varepsilon(nh) - W^\varepsilon(\theta)|^\gamma]^{1/\gamma} d\theta \\ &\quad + \int_{(n-1)h}^{nh} \left(\int_{(n-1)h}^{\theta} \mathbb{E}[|W^\varepsilon(\theta) - W^\varepsilon(\sigma)|^\gamma]^{1/\gamma} d\sigma \right) d\theta \\ &\leq 2h^{1+\beta/\gamma}.\end{aligned}$$

Combining the last inequality with (29) we conclude the proof. □

The remaining part of the section is devoted to apply Theorem 5 to different frameworks. **From now on, we assume that $u_0 \in H^\delta$, g is δ times differentiable, its derivatives up to the order δ are bounded**.

4.3 Diffusion approximation

In this subsection we improve results from [25]. For all $\varepsilon > 0$, consider the solution $u^\varepsilon : [0,1] \times \mathbb{R} \to \mathbb{C}$ of the equation

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(t,x) = \dfrac{i}{\varepsilon} m\left(\dfrac{t}{\varepsilon^2}\right) P\left(i\dfrac{\partial}{\partial x}\right) u^\varepsilon(t,x) + g(u^\varepsilon(t,x)), \\ u^\varepsilon(t=0,x) = u_0(x), \qquad (t,x) \in [0,1] \times \mathbb{R}, \end{cases} \tag{30}$$

where m is a continuous, centered and mixing process [16]. Let B be a Brownian motion and c_0 be a positive constant defined by

$$c_0^2 = 2\int_0^\infty \mathbb{E}[m(0)m(\theta)]d\theta.$$

For every $t \geq 0$ we set

$$\mathcal{S}^\varepsilon(t) = \frac{1}{\varepsilon}\int_0^t m\left(\frac{\theta}{\varepsilon^2}\right) d\theta.$$

It well-known that $\{\mathcal{S}^\varepsilon(t)\}_{t\in[0,1]}$ converges in distribution to $c_0 B$ in $\mathcal{C}([0,1])$ as $\varepsilon \to 0$ (the functional Donsker theorem, see [16] for instance). As a consequence, by Theorem 3, u^ε converges in distribution to u in the space $\mathcal{C}([0,1], L^2)$ as $\varepsilon \to 0$, where u is the solution of

$$\begin{aligned} u(t,x) &= u_0(x) + ic_0 \int_0^t P\left(i\frac{\partial}{\partial x}\right) u(\theta,x) \circ dB(\theta) \\ &\quad + \int_0^t g(u(\theta,x))d\theta, \qquad (t,x) \in [0,1] \times \mathbb{R}. \end{aligned} \tag{31}$$

For every $N \in \mathbb{N}^*$ (with $h = 1/N$), let $\{u^{n,h,\varepsilon}\}_{0\leq n\leq N}$ be the Lie scheme associated to u^ε and $\{u^{n,h}\}_{0\leq n\leq N}$ be the Lie scheme associated to u. We now establish the AP property in this framework.

Theorem 6. *For all $N \in \mathbb{N}^*$, $\{u^{n,h,\varepsilon}\}_{0\le n\le N}$ converges in distribution to $\{u^{n,h}\}_{0\le n\le N}$ in $(L^2)^{N+1}$ as $\varepsilon \to 0$. Moreover, there exists a constant C which depends only on g and $||u_0||_{H^\delta}$, such that for every $h \in (0,1]$ and every $\varepsilon > 0$,*

$$\mathbb{E}\left[\max_{n\in\{1,\dots,N\}} \|u^{n,h,\varepsilon} - u^\varepsilon(nh,\cdot)\|_{L^2}\right] \le Ch^{1/2} \tag{32}$$

and

$$\mathbb{E}\left[\max_{n\in\{1,\dots,N\}} \|u^{n,h} - u(nh,\cdot)\|_{L^2}\right] \le Ch^{1/2}. \tag{33}$$

A weaker form of this result has been proven in [25]. More precisely, it has been shown that, for $P(\xi) = \xi^2$, there exists a constant C which depends only on g and $||u_0||_{H^2}$ (because $\delta = 2$ in this case), such that for every $h \in (0,1]$ and every $\varepsilon > 0$,

$$\max_{n\in\{1,\dots,N\}} \mathbb{E}\left[\|u^{n,h,\varepsilon} - u^\varepsilon(nh,\cdot)\|_{L^2}\right] \le C(h^{1/2} + \varepsilon). \tag{34}$$

The improvement comes from the general formulation of the error estimate in Theorem 2 and in particular from a more subtle use of Gronwall's lemmas in its proof.

Proof. The convergence of $\{u^{n,h,\varepsilon}\}_{0\le n\le N}$ to $\{u^{n,h}\}_{0\le n\le N}$ is a direct consequence of Theorem 4 and the functional Donsker theorem. To prove (32) and (33), we show Assumptions (A1) and (A2). For all $s < t \in [0,1]$,

$$\begin{aligned}\mathbb{E}\left[\left(\mathcal{S}^\varepsilon(t) - \mathcal{S}^\varepsilon(s)\right)^2\right] &\le \frac{1}{\varepsilon^2}\int_s^t d\theta \int_s^t d\sigma \left|\mathbb{E}\left[m\left(\frac{\theta}{\varepsilon^2}\right) m\left(\frac{\sigma}{\varepsilon^2}\right)\right]\right| \\ &\le (t-s)\int_0^\infty d\sigma \left|\mathbb{E}\left[m(\sigma)\, m(0)\right]\right|.\end{aligned}$$

Hence, Assumptions (A1) and (A2) are satisfied ending hence the proof. □

4.4 Approximation by a fractional Brownian motion

A fractional Brownian motion $B_H = \{B_H(t)\}_{t\ge 0}$ (see [29]) with Hurst index $H \in (0,1)$ is a Gaussian process with mean 0 and satisfying for all t and $s \ge 0$,

$$\mathbb{E}[B_H(t)B_H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Notice that a fractional Brownian motion with $H = 1/2$ is a Brownian motion.

The class of fractional Brownian motions is important in applications of stochastic processes because it satisfies the invariance principle stated below.

Let $H \in (0,1)$ and m be a stationary Gaussian process with mean 0. For every $\varepsilon \in (0,1)$ we define $\mathcal{S}^\varepsilon = \{\mathcal{S}^\varepsilon(t)\}_{t\ge 0}$ such that for every $t \ge 0$,

$$\mathcal{S}^\varepsilon(t) = \varepsilon^{2H}\int_0^{t/\varepsilon^2} m(s)ds.$$

We assume that one of the three following properties holds

- If $H \in (1/2, 1)$, there exists $\sigma_H > 0$ such that $\mathbb{E}[m(0)m(t)] \sim \sigma_H t^{2H-2}$ as $t \to \infty$.
- If $H \in (0, 1/2)$, there exists $\sigma_H < 0$ such that $\mathbb{E}[m(0)m(t)] \sim \sigma_H t^{2H-2}$ as $t \to \infty$ and $\int_0^\infty \mathbb{E}[m(0)m(t)]dt = 0$.
- If $H = 1/2$, $\int_0^\infty |\mathbb{E}[m(0)m(t)]|dt < \infty$ and $\int_0^\infty \mathbb{E}[m(0)m(t)]dt > 0$.

We have the following result (invariance principle, see [29]).

Lemma 10. *As $\varepsilon \to 0$, $\mathcal{S}^\varepsilon$ converges in distribution in $\mathcal{C}([0,\infty))$ to $c_H B_H$ where B_H is a fractional Brownian motion with Hurst index H and c_H is a positive constant defined by $c_H^2 = 2\int_0^\infty \mathbb{E}[m(0)m(t)]dt$ if $H = 1/2$ and by $c_H^2 = \sigma_H/H(2H-1)$ if $H \neq 1/2$.*

For all $\varepsilon > 0$, consider the solution $u^\varepsilon : [0,1] \times \mathbb{R} \to \mathbb{C}$ of the equation

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(t,x) = \dfrac{i}{\varepsilon^{2-2H}} m\left(\dfrac{t}{\varepsilon^2}\right) P\left(i\dfrac{\partial}{\partial x}\right) u^\varepsilon(t,x) + g(u^\varepsilon(t,x)), \\ \\ u^\varepsilon(t=0,x) = u_0(x), \qquad (t,x) \in [0,1] \times \mathbb{R} \end{cases} \tag{35}$$

where m is defined just above. Thanks to Lemma 10 and Lemma 3, u^ε converges in distribution to $u : [0,1] \times \mathbb{R} \to \mathbb{C}$ in the space $\mathcal{C}([0,1], L^2)$ as $\varepsilon \to 0$, where u is the solution of

$$\begin{aligned} u(t,x) \quad = \quad & u_0(x) + ic_H \int_0^t P\left(i\frac{\partial}{\partial x}\right) u(\theta,x) \circ dB_H(\theta) \\ & + \int_0^t g(u(\theta,x))d\theta, \qquad (t,x) \in [0,1] \times \mathbb{R}. \end{aligned} \tag{36}$$

In the equation above, c_H is the constant defined in Lemma 10, B_H is a fractional Brownian motion with Hurst index H. For every $N \in \mathbb{N}$ (with $h = 1/N$), let $\{u^{n,h,\varepsilon}\}_{1\leq n\leq N}$ be the Lie scheme associated to u^ε and $\{u^{n,h}\}_{1\leq n\leq N}$ be the Lie scheme associated to u. We establish the AP property.

Theorem 7. *For every $N \in \mathbb{N}^*$, $\{u^{n,h,\varepsilon}\}_{1\leq n\leq N}$ converges in distribution to $\{u^{n,h}\}_{1\leq n\leq N}$ as $\varepsilon \to 0$. Moreover, there exists a constant C which depends only on g and $||u_0||_{H^\delta}$, such that for every $h \in (0,1]$ and every $\varepsilon > 0$,*

$$\mathbb{E}\left[\max_{n\in\{1,\ldots,N\}} \|u^{n,h,\varepsilon} - u^\varepsilon(nh,\cdot)\|_{L^2}\right] \leq Ch^H,$$

and

$$\mathbb{E}\left[\max_{n\in\{1,\ldots,N\}} \|u^{n,h} - u(nh,\cdot)\|_{L^2}\right] \leq Ch^H.$$

Proof. By Lemma 10 and Theorem 4, we get the convergence of $\{u^{n,h,\varepsilon}\}_{1\leq n\leq N}$ as $\varepsilon \to 0$. By Lemma 10 again, $\mathcal{S}^\varepsilon$ satisfies Assumptions (A1). Moreover, there exists $C > 0$ such that for all t and s,

$$\mathbb{E}[(\mathcal{S}^\varepsilon(t) - \mathcal{S}^\varepsilon(s))^2] \leq C|t-s|^{2H}.$$

Then, $\mathcal{S}^\varepsilon$ satisfies assumptions (A1) and (A2), which concludes the proof by Theorem 5. □

Notice that, if $H = 1/2$, then Theorem 7 is Theorem 6 in the Gaussian case.

4.5 Approximation in a long-range random medium

Recently, long-range random media have attracted a lot of attention in applications to wave propagation ([18, 27] for instance). A fractional Brownian motion with Hurst index $H > 1/2$ is a basic model for long-range dependence. Nevertheless, this model is Gaussian and we also need non-Gaussian models. A natural generalization of fractional Brownian motions is the class of Hermite processes. Let $K \in \mathbb{N}^*$ and $H = (2-\gamma K)/2 \in (1/2, 1)$. We define the K-th Hermite process of index H for every $t > 0$ by

$$B_{H,K}(t) = \int_{\mathbb{R}^K} \mathcal{G}_{H,K}(t, x_1, ..., x_K) \prod_{k=1}^{K} \tilde{B}(dx_k)$$

with

$$\mathcal{G}_{H,K}(t, x_1, ..., x_K) = \frac{\left(e^{-it\sum_{j=1}^K x_j} - 1\right)}{C(H)\sum_{j=1}^K x_j} \prod_{k=1}^{K} \frac{x_k}{|x_k|^{(H-1)/K+3/2}}$$

where $C(H)$ a normalizing constant, $\tilde{B}(dx)$ is the Fourier transform of a Brownian measure and the multiple stochastic integral is in the sense of [12].

Notice that for $K = 1$, $B_{H,K} = B_{H,1}$ is a fractional Brownian motion with Hurst index $H > 1/2$. More generally, for every K, $B_{H,K}$ is centered and admits the same covariance as a fractional Brownian motion, that is, for all t and $s \geq 0$,

$$\mathbb{E}[B_{H,K}(t)B_{H,K}(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Moreover, $B_{H,K}$ is Gaussian if and only if $K = 1$.

As the class of fractional Brownian motions, Hermite processes are important in applications of stochastic processes because they satisfy the invariance principle [13, 31, 32]. Let m be a continuous Gaussian process, centered, stationary, satisfying $\mathbb{E}\left[m(0)^2\right] = 1$ and such that

$$\mathbb{E}\left[m(0)m(t)\right] \sim c_m t^{-\gamma}$$

as $t \to \infty$, where $1 < \gamma < 1/K$ and $c_m > 0$. For every $\varepsilon \in (0, 1)$ we define $\mathcal{S}^\varepsilon = \{\mathcal{S}^\varepsilon(t)\}_{t\geq 0}$ for every $t \geq 0$ by

$$\mathcal{S}^\varepsilon(t) = \varepsilon^{-\gamma K} \int_0^t \Phi\left(m\left(\frac{s}{\varepsilon^2}\right)\right) ds$$

where Φ is a continuous function in $L^2(e^{-x^2/2}dx)$ with Hermite index equal to $K \in \mathbb{N}^*$. This means that if we denote the k-th Hermite coefficient of Φ by

$$\Phi_k = \int_{-\infty}^{\infty} P_k(x)\Phi(x)\frac{e^{-x^2/2}}{k!\sqrt{2\pi}}dx$$

where P_k is the k-th Hermite polynomial, then we have

$$\Phi = \sum_{k=K}^{\infty} \Phi_k P_k$$

with $\Phi_K \neq 0$. The invariance principle for Hermite processes can be stated as below.

Lemma 11. *As $\varepsilon \to 0$, $\mathcal{S}^\varepsilon$ converges in distribution to $c_{H,K}B_{H,K}$ in $\mathcal{C}([0,\infty))$, where $c_{H,K}^2 = c_m^K \Phi_K^2/(K!)^2$.*

For every $\varepsilon > 0$, we consider the solution u^ε of the equation

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(t,x) = \dfrac{i}{\varepsilon^{\gamma K}} m\left(\dfrac{t}{\varepsilon^2}\right) P\left(i\dfrac{\partial}{\partial x}\right) u^\varepsilon(t,x) + g(u^\varepsilon(t,x)), \\ \\ u^\varepsilon(t=0,x) = u_0(x), \quad (t,x) \in [0,1] \times \mathbb{R}, \end{cases} \tag{37}$$

where m is defined just above. Thanks to Lemma 11 and Lemma 3, u^ε converges in distribution to $u : [0,1] \times \mathbb{R} \to \mathbb{C}$ in the space $\mathcal{C}([0,1], L^2)$ as $\varepsilon \to 0$, where u is the solution of

$$\begin{aligned} u(t,x) \quad &= \quad u_0(x) + ic_{H,K}\int_0^t P\left(i\frac{\partial}{\partial x}\right) u(\theta,x) \circ dB_{H,K}(\theta) \\ &\quad + \int_0^t g(u(\theta,x))d\theta, \qquad (t,x) \in [0,1] \times \mathbb{R}, \end{aligned} \tag{38}$$

with the constant $c_{H,K}$ defined as in Lemma 11, $B_{H,K}$ is a K-th Hermite process of order K. For every $N \in \mathbb{N}^*$ (with $h = 1/N$), let $\{u^{n,h,\varepsilon}\}_{1\le n\le N}$ be the Lie scheme associated to u^ε and $\{u^{n,h}\}_{1\le n\le N}$ be the Lie scheme associated to u. We now establish the AP property.

Theorem 8. *For every $N \in \mathbb{N}^*$, $\{u^{n,h,\varepsilon}\}_{1\le n\le N}$ converges in distribution to $\{u^{n,h}\}_{1\le n\le N}$ as $\varepsilon \to 0$. Moreover, there exists a constant C which depends only on g and $||u_0||_{H^\delta}$, such that for every $h \in (0,1]$ and every $\varepsilon > 0$,*

$$\mathbb{E}\left[\max_{n\in\{1,\dots,N\}} \|u^{n,h,\varepsilon} - u^\varepsilon(nh,\cdot)\|_{L^2}\right] \le Ch^H,$$

and

$$\mathbb{E}\left[\max_{n\in\{1,\dots,N\}} \|u^{n,h} - u(nh,\cdot)\|_{L^2}\right] \le Ch^H.$$

Proof. The proof is similar to the proof of Theorem 7. It is a consequence of Lemma 11 and Theorems 4 and 5. $\square$

4.6 Generalization to multifractional media

Fractional Brownian motions with Hurst index $H > 1/2$ and Hermite processes fit very well for modeling long-range media. Nevertheless, their range properties are governed by the constant Hurst index, which implies a strong homogeneity. To deal with less homogeneous media, multifractional processes have been introduced [5, 28]. The main interest of multifractional processes lies in the fact that they have a Hurst index varying along their trajectories. This implies more flexibility in the choice of the models. Applications of multifractional models to waves in random media have been studied recently [27]. In this subsection we deal with the AP property of the Lie time-splitting scheme in the case of convergence to multifractional processes. We restrict our study to a simple Gaussian framework, but it can be easily generalized to non-Gaussian settings as discussed at the end of this subsection.

Let a Gaussian field $m = \{m(t,H)\}_{(t,H)\in\mathbb{R}\times(1/2,1)}$. We assume that m is centered and satisfies for every compact set $K \subset (1/2,1)$,

$$\lim_{|t_1-t_2|\to\infty} \sup_{(H_1,H_2)\in K^2} \Big| R(H_1,H_2) \\ -|t_1-t_2|^{2-H_1-H_2}\mathbb{E}[m(t_1,H_1)m(t_2,H_2)]\Big| = 0, \quad (39)$$

where $R : (1/2,1)^2 \to (0,\infty)$ is a continuous function. This is long-range assumption in a multifractional setting. Let $\mathcal{H} : [0,\infty) \to [a,b] \subset (1/2,1)$ be a continuous function. We define $\mathcal{S}^\varepsilon = \{\mathcal{S}^\varepsilon(t)\}_{t\geq 0}$ such that for every $t \geq 0$,

$$\mathcal{S}^\varepsilon(t) = \int_0^{t/\varepsilon^2} \varepsilon^{2\mathcal{H}(\varepsilon^2 s)} m(s,\mathcal{H}(\varepsilon^2 s))ds = \int_0^t \varepsilon^{2\mathcal{H}(s)-2} m(s/\varepsilon^2,\mathcal{H}(s))ds.$$

The following result establish an invariance principle for Gaussian multifractional processes ([9], see [27, 26] for generalizations and applications).

Lemma 12. *As $\varepsilon \to 0$, $\mathcal{S}^\varepsilon$ converges in distribution to a process $\mathcal{S}_\mathcal{H}$ in $\mathcal{C}([0,\infty))$, where $\mathcal{S}_\mathcal{H}$ is Gaussian, centered and satisfies for all t and $s \geq 0$,*

$$\mathbb{E}\left[\mathcal{S}_\mathcal{H}(t)\mathcal{S}_\mathcal{H}(s)\right] = \int_0^t d\theta \int_0^s d\sigma R(\mathcal{H}(\theta),\mathcal{H}(\sigma))|\theta-\sigma|^{\mathcal{H}(\theta)+\mathcal{H}(\sigma)-2}.$$

A detailed study of $\mathcal{S}_\mathcal{H}$ can be found in [9]. In particular, it is proven that $S_\mathcal{H}$ satisfies the main properties of a multifractional process.

For every $\varepsilon > 0$, we consider the solution $u^\varepsilon : [0,1]\times\mathbb{R} \to \mathbb{C}$ of the equation

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(t,x) = \dfrac{i}{\varepsilon^{2-2\mathcal{H}(s)}} m\left(\dfrac{s}{\varepsilon^2},\mathcal{H}(s)\right) P\left(i\dfrac{\partial}{\partial x}\right) u^\varepsilon(t,x) + g(u^\varepsilon(t,x)), \\ u^\varepsilon(t=0,x) = u_0(x), \quad (t,x)\in[0,1]\times\mathbb{R}. \end{cases} \quad (40)$$

where m is defined just above. Because of Lemma 12 and Lemma 3, u^ε converges in distribution to $u : [0,1]\times\mathbb{R} \to \mathbb{C}$ in the space $\mathcal{C}([0,1],L^2)$ as $\varepsilon \to 0$, where u

is the solution of

$$u(t,x) = u_0(x) + i\int_0^t P\left(i\frac{\partial}{\partial x}\right)u(\theta,x)\circ d\mathcal{S}_{\mathcal{H}}(\theta) + \int_0^t g(u(\theta,x))d\theta, \qquad (t,x)\in[0,1]\times\mathbb{R}. \tag{41}$$

In the equation above, $\mathcal{S}_{\mathcal{H}}$ is the multifractional process defined in Lemma 12. For every $N\in\mathbb{N}^*$ (with $h=1/N$), let $\{u^{n,h,\varepsilon}\}_{1\le n\le N}$ be the Lie scheme associated to u^ε and $\{u^{n,h}\}_{1\le n\le N}$ be the Lie scheme associated to u. We have the following result.

Theorem 9. *For every $N\in\mathbb{N}^*$, $\{u^{n,h,\varepsilon}\}_{1\le n\le N}$ converges in distribution to $\{u^{n,h}\}_{1\le n\le N}$ as $\varepsilon\to 0$. Moreover, there exists a constant C which depends only on g and $||u_0||_{H^\delta}$, such that for every $h\in(0,1]$ and every $\varepsilon>0$,*

$$\mathbb{E}\left[\max_{n\in\{1,\dots,N\}}\|u^{n,h,\varepsilon}-u^\varepsilon(nh,\cdot)\|_{L^2}\right]\le Ch^{\min\mathcal{H}}$$

and

$$\mathbb{E}\left[\max_{n\in\{1,\dots,N\}}\|u^{n,h}-u(nh,\cdot)\|_{L^2}\right]\le Ch^{\min\mathcal{H}}.$$

Proof. From the covariance of $S_{\mathcal{H}}$, we can easily deduce that there exists $C>0$ such that for all t and s,

$$\mathbb{E}[(\mathcal{S}^\varepsilon(t)-\mathcal{S}^\varepsilon(s))^2]\le C|t-s|^{2\min\mathcal{H}}.$$

The remaining part of the proof is similar to the proof of Theorem 7. □

To conclude this section, notice that we can also pursue the same study for non-Gaussian multifractional models by using Hermite processes and replacing Lemma 12 for suitable limit theorems (see [27, 26] for invariance principle for non-Gaussian multifractional processes).

A Technical lemma

In this section we establish a technical lemma we use in the proof of Lemma 3. It can be proven by induction.

Lemma 13. *Let $n\ge 1$, $\psi=(\psi_1,\psi_2)\in\mathcal{C}^n(\mathbb{R},\mathbb{R}^2)$ and $\phi\in\mathcal{C}^n(\mathbb{R}^2,\mathbb{R})$. Then*

$$(\phi\circ\psi)^{(n)} = \sum_{\substack{j,k\ge 0\\ 1\le j+k\le n}}\ \sum_{\substack{1\le l_1\le\ldots\le l_k\\ 1\le m_1\le\ldots\le m_j\\ l_1+\ldots+m_1+\ldots=n}} w_{l_1,\ldots,m_1,\ldots}\left(\phi^{(k,j)}\circ\psi\right)\left(\prod_{r=1}^k\psi_1^{(l_r)}\right)\left(\prod_{r=1}^j\psi_2^{(m_r)}\right) \tag{42}$$

where all the coefficients $w_{l_1,\ldots,m_1,\ldots}$ are integer.

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