A construction of the fundamental solution of the Schrödinger equation with a perturbed quadratic Hamiltonian

Romain Duboscq*[a]

[a]Institut de Mathématiques de Toulouse, CNRS, UMR 5219, INSA de Toulouse, 135 avenue de Rangueil, 31077 Toulouse Cedex 4-, France

November 18, 2015

Abstract

The aim of this paper is to build a solution to the linear Schrödinger equation with a perturbed quadratic hamiltonian. The solution is given in the sense of Sussmann [30] and the construction is based on the bicharacteristics method. This is made possible under some assumptions on the hamiltonian and the regularity of the perturbative noise. Moreover, dispersive estimates and a Avron-Herbst formula are also given during the analysis of the Cauchy problem for nonlinear Schrödinger equations.

Keywords: nonlinear Schrödinger equation, stochastic partial differential equation, Hamiltonian system, bicharacteristics method, Strichartz estimates, Avron-Herbst formula

Contents

*romain.duboscq@math.univ-toulouse.fr

1 Introduction

Let us introduce the following quadratic Hamiltonian perturbed by a real-valued noise term $(\dot{w}_t)_{t\in\mathbb{R}^+}$ (considered as the time derivative of a real-valued function $(w_t)_{t\in\mathbb{R}^+}$)

$$\mathcal{H}(t,\boldsymbol{x},\boldsymbol{\xi}) = \mathcal{H}_1(\boldsymbol{x},\boldsymbol{\xi}) + \mathcal{H}_2(\boldsymbol{x},\boldsymbol{\xi})\dot{w}_t, \tag{1}$$

with

$$\begin{aligned}\mathcal{H}_1(\boldsymbol{x},\boldsymbol{\xi}) &= \frac{1}{2}\boldsymbol{\xi}\cdot\boldsymbol{\xi} + (M_{11}\boldsymbol{x})\cdot\boldsymbol{\xi} + (M_{12}\boldsymbol{x})\cdot\boldsymbol{x} + \mathbf{V}_{11}\cdot\boldsymbol{x} + \mathbf{V}_{12}\cdot\boldsymbol{\xi},\\ \mathcal{H}_2(\boldsymbol{x},\boldsymbol{\xi}) &= (M_{21}\boldsymbol{x})\cdot\boldsymbol{\xi} + (M_{22}\boldsymbol{x})\cdot\boldsymbol{x} + \mathbf{V}_{21}\cdot\boldsymbol{x} + \mathbf{V}_{22}\cdot\boldsymbol{\xi},\end{aligned} \tag{2}$$

where $M_{11}, M_{12}, M_{21}, M_{22} \in \mathcal{M}_d(\mathbb{R})$ are $d\times d$ real-valued matrices, with $d\in\mathbb{N}$. The vectors $\mathbf{V}_{11}$, $\mathbf{V}_{12}$, $\mathbf{V}_{21}$, $\mathbf{V}_{22}$ are in $\mathbb{R}^d$. The goal of this paper is to build a solution to the following linear Schrödinger equation, $\forall s\in\mathbb{R}^+$,

$$\begin{cases} i\partial_t\psi(t,\boldsymbol{x}) = \mathcal{H}_1(\boldsymbol{x},-i\nabla)\psi(t,\boldsymbol{x}) + \mathcal{H}_2(\boldsymbol{x},-i\nabla)\psi(t,\boldsymbol{x})\dot{w}_t,\ \forall t\in]s,\infty[,\ \forall\boldsymbol{x}\in\mathbb{R}^d,\\ \psi(s,\boldsymbol{x}) = \psi_s(\boldsymbol{x})\in L^2.\end{cases} \tag{3}$$

We obtain a quasi-explicit fundamental solution as well as some results concerning the existence and uniqueness of solutions to nonlinear stochastic Schrödinger equations.

Model equations

Quadratic hamiltonian are used to model systems of particles in quantum mechanics [22] or the propagation of electromagnetic plane waves in optics [1].

In [28, 29], a perturbed electrostatic field is introduced to investigate stochastic effects on two different quantum systems. In both papers, the Schrödinger equation that is used to model the systems is the following

$$\begin{cases} i\partial_t\psi(t,x) = -\frac{1}{2}\partial_x^2\psi(t,x) - x\left(F(t)+\dot{B}_t\right)\psi(t,x) + V(x)\psi(t,x),\ t\in\mathbb{R}^+, x\in\mathbb{R},\\ \psi(0,x) = \psi_0(x),\ \forall x\in\mathbb{R},\end{cases} \tag{4}$$

where F corresponds to the pulse of a laser field which is a smooth function and $(\dot{B}_t)_{t\in\mathbb{R}^+}$ is a white noise corresponding to a noise source. In [29], where the dissociation of a diatomic molecule is considered, the potential V is a Morse potential such that $V(x) = -D_e\left(1-(1-e^{-\ell x})^2\right)$, where D_e is the well depth and ℓ is the length scale. We remark that, being given a small length scale $\ell \ll 1$, the following approximation holds

$$V(x) \approx -D_e\left(1 - x^2\ell^2\right) + o(\ell^2).$$

This approximation creates a harmonic well (the constant term can be eliminated by a gauge change). In [28], the ionization of a Hydrogen atom under a laser field is studied. In this case, the potential V is a non-singular Coulomb type potential given by

$$V(x) = -\frac{1}{\sqrt{x^2+a^2}},$$

with $a\in\mathbb{R}$. Remarking that this potential is bounded, it can be treated like a perturbation to solve the Cauchy problem for equation (4).

Remark 1. *It is possible to apply an Avron-Herbst formula [5] to the solution of the equation (4) under some assumptions on the potential V. This is done in section 5 where we investigate the existence and uniqueness of global in time solutions for a class of stochastic nonlinear Schrödinger equations.*

In [17, 27], the heating effects of fluctuations in the intensity of a quadratic potential are studied. In the case of a Bose-Einstein condensate, the equation modeling this system is given by the following stochastic Gross-Pitaevskii equation

$$\begin{cases} i\partial_t \psi(t,\boldsymbol{x}) = -\frac{1}{2}\Delta\psi(t,\boldsymbol{x}) + \frac{1}{2}|\boldsymbol{x}|^2\psi(t,\boldsymbol{x})(1+\dot{B}_t) + \beta|\psi|^2\psi(t,\boldsymbol{x}),\ \forall t\in\mathbb{R}^+,\ \forall \boldsymbol{x}\in\mathbb{R}^d, \\ \psi(0,\boldsymbol{x}) = \psi_0(\boldsymbol{x}),\ \forall \boldsymbol{x}\in\mathbb{R}^d, \end{cases} \tag{5}$$

where $(\dot{B}_t)_{t\in\mathbb{R}^+}$ is a white noise and β is the intensity of the inter-atomic interaction in the condensate. This equation as been studied by de Bouard and Fukuizumi [11, 12]. Furthermore, we remark that, in the case of a rotating Bose-Einstein condensate [23], the hamiltonian associated to the linear part of the Gross-Pitaevskii equation remains quadratic and thus is still related to our original problem (3). We could also consider a perturbed electric field similar to the one in equation (4).

The bicharacteristics method

A classical construction for the solution of the Schrödinger equation (3) is to consider the following ansatz

$$\psi(t,\boldsymbol{x}) = \frac{a(t,s)}{(2\pi i(t-s))^{d/2}} \int_{\mathbb{R}^d} e^{iS(t,s,\boldsymbol{x},\boldsymbol{y})}\psi_s(\boldsymbol{y})d\boldsymbol{y}, \tag{6}$$

and to build the amplitude a and the phase S. Assuming that a and S are sufficiently smooth and then injecting the formal solution (6) into (3), we obtain the following system of equations

$$\begin{cases} \partial_t S(t,s,\boldsymbol{x},\boldsymbol{y}) = \mathcal{H}(t,\boldsymbol{x},\nabla_{\boldsymbol{x}} S),\ \text{a.e. } \boldsymbol{y}\in\mathbb{R}^d, t\in]s,\infty[, \boldsymbol{x}\in\mathbb{R}^d, \\ \partial_t a(t,s) = \frac{1}{2}\left(-\Delta_{\boldsymbol{x}} S(t,s) + \frac{d}{t-s}\right) a(t,s),\ t\in]s,\infty[. \end{cases} \tag{7}$$

Remark 2. *We can see that the first equation of (7) is independent of the amplitude a. Furthermore, the solution S of this equation is a quadratic function of $\boldsymbol{x}$ and $\boldsymbol{y}$, implying that the amplitude function $a(t,s)$ does not depend on the spatial variable. Finally, the first equation of system (7) depends on the noise term $(\dot{w}_t)_{t\in\mathbb{R}^+}$. Since we consider non-differentiable functions $(w_t)_{t\in\mathbb{R}^+}$ in this paper, we have to give a proper definition of the solution of* (7) *and* (6)*. We choose to use a definition similar to the one given by Sussmann [30] to define the function (6) as a solution of equation (3) (see definition 2).*

The first equation in (7) is a Hamilton-Jacobi equation. This kind of equation has been studied by Hamilton [19] in optics by using the characteristics method following an analogy with the classical mechanics. More recent results can be found in [3, 10, 18]. Following this approach, we prove that the solution $S(t,s,\boldsymbol{x},\boldsymbol{y})$ of the first equation in (7) is the classical action associated to the path of a particle starting at time s and position $\boldsymbol{y}$ and reaching $\boldsymbol{x}$ at time t, where the evolution of the particle is defined by the Hamilton equations associated to $\mathcal{H}$ [10]. The Hamilton equations are ordinary differential equations which give the evolution of the position and the impulsion of a particle depending on the initial position and impulsion. Therefore, finding a path connecting $\boldsymbol{x}$ and $\boldsymbol{y}$ leads to ensuring the existence and uniqueness of the solution of the Hamilton equations and then to use a change of variables between the initial impulsion and the final position.

Obtaining the amplitude a is direct since we can integrate exactly the second equation of (7) with respect to t, giving then the representation formula (6).

Such constructions were already investigated in [16, 21, 32]. However, in those previous works, the time-dependent Hamiltonian was assumed to be at least bounded almost everywhere with respect to the time variable. In [16], the following Hamiltonian is considered

$$\mathcal{H}(t,\boldsymbol{x},-i\nabla) = -\frac{1}{2}\Delta + V(t,\boldsymbol{x}), \tag{8}$$

where V is a real measurable potential function which is quadratically bounded and is $\mathcal{C}^\infty$ with respect to $\boldsymbol{x}$. In this case, the construction slightly differs from the bicharacteristics method in order to have such general assumptions on the potential. Moreover, this leads to a formulation slightly different from (6) since the phase S will not be exactly quadratic anymore (thus, the amplitude will depend on the space variables). In order to overcome this difficulty, the author considers a sequence of oscillatory integrals $\left\{(E_n(t,s))_{t\in[s,s+T]}\right\}_{n\in\mathbb{N}}$ defined as

$$E_n(t,s)\psi_s(\boldsymbol{x}) = \frac{1}{(2\pi i(t-s))^{d/2}}\int_{\mathbb{R}^d} a_n(t,s,\boldsymbol{x},\boldsymbol{y})e^{iS(t,s,\boldsymbol{x},\boldsymbol{y})}\psi_s(\boldsymbol{y})d\boldsymbol{y},$$

where the amplitudes $(a_n)_{n\in\mathbb{N}}$ are solutions of a sequence of transport equations depending on S, and proves that $(E_n(t,s))_{t\in[s,s+T]}$ converges to a certain operator $U(t,s)$ which is the exact propagator of equation (8). In [32], the considered Hamiltonian corresponds to a Schrödinger operator with magnetic fields

$$\mathcal{H}(t,\boldsymbol{x},-i\nabla) = \frac{1}{2}\left(-i\nabla - \mathbf{A}(t,\boldsymbol{x})\right)^2, \tag{9}$$

where the magnetic vector potential $\mathbf{A}(t,\boldsymbol{x})$ is assumed to be such that $\partial_{\boldsymbol{x}_j}\mathbf{A}(t,\boldsymbol{x}) \in \mathcal{C}^1(\mathbb{R}^+\times\mathbb{R}^d)$, $\forall j \in \{1,...,d\}$. The construction is similar to the one in [16]. Moreover, smoothing properties of the propagator associated to the solution of the equation are proven and allow to consider perturbative terms. In [12], a stochastic Hamiltonian is studied leading to the construction of the solution by using the bicharacteristics method. The initial stochastic Hamiltonian considered is similar to (8) with $V(t,\boldsymbol{x}) = \frac{1}{2}|\boldsymbol{x}|^2(1+\dot{w}_t)$. Prior to the construction, the noise is integrated by using the following gauge transformation

$$\psi(t,\boldsymbol{x}) \leftarrow e^{-iG(t,\boldsymbol{x})}\psi(t,\boldsymbol{x}),$$

where $G(t,\boldsymbol{x}) = \frac{1}{2}|\boldsymbol{x}|^2(t+w_t)$. This gauge change modifies the initial Hamiltonian into a Hamiltonian of the form (9), with $\mathbf{A}(t,\boldsymbol{x}) = \nabla G(t,\boldsymbol{x})$, and a construction similar to [32] is done. However, in general, such gauge transformation can not be applied to a Hamiltonian of the form (1).

Here, we have to ensure that this classical construction still holds for the stochastic Hamiltonian (1) where we do not suppose that the noise is bounded anywhere. The first step of this construction is to solve the Hamilton equations which corresponds to a linear stochastic differential equation in our case. This type of equation is well understood and does not present any difficulty. Moreover, this enables us to show that the solutions of the Hamilton equations depend on the function $(w_t)_{t\in\mathbb{R}^+}$, thus integrating the noise term and getting around the problem arising from its lack of smoothness. As stated in remark 2, we define the solution in the sense of Sussmann [30]. Therefore, we have to ensure that the solution is continuous with respect to $(w_t)_{t\in\mathbb{R}^+}$. To this end, we show that the classical action and the amplitude function verify such regularity assumptions.

Main results

As stated in remark 2, we follow the idea of Sussmann [30] to define the solution of the stochastic Schrödinger equation (3). This choice is motivated by the fact that we construct a semi-explicit solution. Our definition uses an extra regularity of the function $(w_t)_{t\in\mathbb{R}^+}$ which is assumed to be in the γ-Hölder space, with $\gamma \in]0,1]$.

Definition 1. *(γ-Hölder function) Let $I = [a,b]$, with $a < b$ and $\gamma \in]0,1]$. We define the space of γ-Hölder function, denoted $\mathcal{C}^\gamma(I,\mathbb{R})$, as the set of all continuous functions $f \in \mathcal{C}^0(I,\mathbb{R})$ such that*

$$\|f\|_{\dot{\mathcal{C}}^\gamma} := \sup_{\substack{t,s\in I\\ t\neq s}} \frac{|f(t)-f(s)|}{|t-s|^\gamma} < \infty.$$

It is endowed with the norm

$$\|f\|_{\mathcal{C}^\gamma_\alpha} := \|f\|_{\mathcal{C}^0} + \|f\|_{\dot{\mathcal{C}}^\gamma}.$$

Furthermore, we extend the norm $\|\cdot\|_{\dot{\mathcal{C}}^\gamma}$ to the case $\gamma = 0$ by setting

$$\|f\|_{\dot{\mathcal{C}}^0} := \sup_{\substack{t,s\in I\\ t\neq s}} |f(t)-f(s)|.$$

We now give the definition of the solution of the stochastic problem (3).

Definition 2. *Let $T > 0$, $s \in \mathbb{R}^+$, $\alpha \in \mathbb{R}$ and $(w_t)_{t\in[s,s+T]} \in \mathcal{C}^\gamma([s,s+T],\mathbb{R})$, $\gamma \in]0,1]$. The function ψ_w is a weak solution to the problem (3) if there exists a neighborhood $\mathcal{Q}$ of $(w_t)_{t\in[s,s+T]}$ in $\mathcal{C}^\gamma([s,s+T],\mathbb{R})$ such that*

- *For every $(\tilde{w}_t)_{t\in[s,s+T]} \in \mathcal{Q} \cap \mathcal{C}^1([s,s+T],\mathbb{R})$, $\psi_{\tilde{w}}$ is a weak solution of (3).*
- *The Itô map $\mathcal{I} : \mathcal{C}^\gamma([s,s+T],\mathbb{R}) \to \mathcal{C}^0([s,s+T],L^2(\mathbb{R}^d))$ defined by*

$$\mathcal{I}(w) = \psi_w,$$

  is continuous.

To obtain a solution of the form (6), we have to make the following assumptions on the Hamiltonian $\mathcal{H}$ and the noise.

Assumption 1. *We assume that $(w_t)_{t\in\mathbb{R}^+}$ is a γ-Hölder function, with $\gamma \in]0,1]$, if not stated otherwise. Moreover, concerning the Hamiltonian $\mathcal{H}$ defined by (1), we assume that one of the two assumptions below is satisfied*

- *$\gamma \in]1/2,1]$,*
- *$M_{21} = 0$ and $\mathbf{V}_{22} = 0$.*

Remark 3. *As we can see in the previous assumptions, the type of perturbed Hamiltonian that can be taken into account in this construction depends on the irregularity of the noise. This fact can be seen in the proof of proposition 8. Moreover, these assumptions includes the hamiltonian operators that are considered in [12].*

Remark 4. *An example of stochastic process satisfying the previous assumptions is the fractional brownian motion B^H with Hurst index $H \in]0,1[$ [26]. Indeed, a trajectory $(B^H_t)_{t\in\mathbb{R}^+}$ is a $(H-\varepsilon)$-Hölder function for $\varepsilon > 0$. Furthermore, we remark that, in the case where $H = 1/2$, this process corresponds to the brownian motion whose time derivative is the white noise $(\dot{B}_t)_{t\in\mathbb{R}^+}$. Thus, we can consider equations (4) and (5) by using our approach.*

Let us introduce now the propagator $(U_w(t,s))_{t\in[s,s+T]}$, $\forall \psi_s \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,

$$U_w(t,s)\psi_s(\boldsymbol{x}) = \frac{a_w(t,s)}{(2\pi i(t-s))^{d/2}} \int_{\mathbb{R}^d} e^{iS_w(t,s,\boldsymbol{x},\boldsymbol{y})}\psi_s(\boldsymbol{y})d\boldsymbol{y}, \tag{10}$$

where the amplitude function a_w and the classical action S_w are constructed by the previous strategy. We introduce the Σ^n functional spaces which arise naturally in the context of a quadratic Hamiltonian [9, 32].

Definition 3. *We define the Σ^n functional space, $\forall n \in \mathbb{N}$, as*

$$\Sigma^n = \left\{ \varphi \in L^2; \sum_{|\alpha+\beta|\le n} \|\boldsymbol{x}^\alpha \partial_{\boldsymbol{x}}^\beta \varphi\|_{L^2} := \|\varphi\|_{\Sigma^n} < \infty \right\},$$

with $\Sigma^0 = L^2$.

Under these notations and assumptions 1, we can prove the following result.

Theorem 1. *Let $n \in \mathbb{N}$. Then, there exists $T > 0$ such that, for any $\psi_s \in \Sigma^n$, the function ψ_w defined by*

$$\psi_w(t,\boldsymbol{x}) = U_w(t,s)\psi_s(\boldsymbol{x}),\ \forall t \in [s,s+T],\ \forall \boldsymbol{x} \in \mathbb{R}^d, \tag{11}$$

where $(U_w(t,s))_{t\in[s,s+T]}$ is given by (10), is a solution in the sense of definition 2 to the problem (3). Moreover, ψ_w belongs to the functional space $\mathcal{C}^0([s,s+T],\Sigma^n)$.

Remark 5. *In theorem 1, the propagator $(U_w(t,s))_{t\in[s,s+T]}$ can be extended to $t \in [s, s+T_0]$, $\forall T_0 > 0$, by setting*

$$U_w(t,s) = U_w(s_n,s_{n-1})U_w(s_n,s_{n-1})...U_w(s_1,s_0),$$

where $\{s_j\}_{j\in\{0,\dots,n\}}$ is a subdivision of such that $s_n = t$, $s_0 = s$ and $|s_j - s_{j-1}| < T$, $\forall j \in \{1,...,n\}$.

The particular form of the solution (6) allows us to prove dispersive estimates on the propagator $(U_w(t,s))_{t\in[s,s+T]}$. This property has been used to prove the existence and uniqueness of a solution for nonlinear Schrödinger equations with a potential at most quadratic [6, 7, 25], a time-dependent potential at most quadratic [8], an electromagnetic potential [24], a stochastic potential at most quadratic [12] or a quadratic potential and a rotation term [2]. Furthermore, we can prove the existence and uniqueness of a solution for the following nonlinear *mild* equation

$$\psi(t,\boldsymbol{x}) = U_w(t,s)\psi_s(\boldsymbol{x}) - i\beta \int_s^t U_w(t,\tau)|\psi(\tau,\boldsymbol{x})|^{2\sigma}\psi(\tau,\boldsymbol{x})d\tau,\ \forall t \in [s,\infty[,\ \forall \boldsymbol{x} \in \mathbb{R}^d, \tag{12}$$

with $\beta \in \mathbb{R}$ and $\sigma > 0$. We obtain the following result.

Theorem 2. *Suppose that the hamiltonian operators $\mathcal{H}_1(\boldsymbol{x}, -i\nabla)$ and $\mathcal{H}_2(\boldsymbol{x}, -i\nabla)$, given by (2), are symmetric operators on the Schwartz space $\mathcal{S}$ and that the Hamiltonian $\mathcal{H}$ satisfies assumptions 1.*

- *Let $\psi_s \in L^2$ and $0 < \sigma < \frac{2}{d}$. Then, there exists a unique solution $\psi \in \mathcal{C}^0([s,\infty[, L^2) \cap L^r([s,\infty[, L^{2\sigma+2})$ to the problem (12), where $r = \frac{4(\sigma+1)}{d\sigma}$. In addition, if $\psi_s \in \Sigma^1$, then the solution ψ belongs to $\mathcal{C}^0([s,\infty[,\Sigma^1)$.*

- *Let $\psi_s \in \Sigma^1$ and $0 < \sigma < \infty$ if $d = 1, 2$ or $0 < \sigma < \frac{2}{d-2}$ if $d \geq 3$. Then the problem (12) admits a unique solution ψ such that, $\forall j \in \{1, .., d\}$,*

$$\psi,\ \boldsymbol{x}_j\psi,\ \partial_{\boldsymbol{x}_j}\psi \in \mathcal{C}^0([s, s+T[, L^2) \cap L^r([s, s+T[, L^{2\sigma+2}),$$

where $T > 0$ depends on $\|\psi_s\|_{\Sigma^1}$ and $r = \frac{4(\sigma+1)}{d\sigma}$.

Remark 6. *In the previous theorem, we remark that the upper bound for the value of σ corresponds to the L^2 (resp. H^1) critical index for the standard nonlinear Schrödinger equation when ψ_s belongs to L^2 (resp. H^1).*

Since the hamiltonian operator $\mathcal{H}$ in the previous statement is time-dependent, this implies a loss of *a priori* estimates in Σ^1 for the solution of the nonlinear equation (12) by using quantities such as the natural candidate for the energy

$$\mathcal{E}(t, \phi) = \int_{\mathbb{R}^d} \Re\left[\frac{1}{2}\phi^*(\boldsymbol{x})\mathcal{H}(t, \boldsymbol{x}, -i\nabla)\phi(\boldsymbol{x}) + \frac{\beta}{2\sigma+2}|\phi(\boldsymbol{x})|^{2\sigma+2}\right] d\boldsymbol{x}, \tag{13}$$

which is conserved when the hamiltonian $\mathcal{H}$ is time-independent (*i.e.* $\dot{w} \equiv 0$) and $\mathcal{H}_1 = \frac{1}{2}|\xi|^2 + (M_{12}\boldsymbol{x} + \mathbf{V}_{11}) \cdot \boldsymbol{x}$ (see for instance [9]). Therefore, without further assumptions on the noise or the hamiltonian operator, the question of the existence and uniqueness of global in time solutions in Σ^1 for H^1-subcritical nonlinearities is quite delicate. However, this problem can be partially addressed with the help of an Avron-Herbst formula [5] under the following additional assumptions.

Assumption 2. *The Hamiltonian $\mathcal{H}$ defined by (1) is such that*

- *$M_{21} = 0$ and $M_{22} = 0$,*
- *M_{11} is a skew-adjoint matrix.*

Moreover, the coefficient β in equation (12) is positive.

This leads us to our last result.

Theorem 3. *Let $\psi_s \in \Sigma^1$ and $0 < \sigma < \infty$ with $\sigma < \frac{2}{d-2}$ for $d \geq 3$. In addition, suppose that the assumptions 1 and 2 are verified, that the hamiltonian operators $\mathcal{H}_1(\boldsymbol{x}, -i\nabla)$ and $\mathcal{H}_2(\boldsymbol{x}, -i\nabla)$, given by Hamiltonians, are symmetric operators on the Schwartz space $\mathcal{S}$ and that the Hamiltonian $\mathcal{H}$. Then, there exists a unique solution $\psi \in \mathcal{C}^0([s, \infty[, \Sigma^1)$ to the problem (12).*

Structure of the paper

The paper is organized as follows. In section 2, we study the solutions of the Hamilton equations, also called classical orbits, associated to the problem (3) for a continuous trajectory. Moreover, we prove the change of variables between the initial impulsion and the final position. These solutions and the change of variables lead to construct the classical action S, which is done in section 3. The classical action is shown to be a solution of the Hamilton-Jacobi equation (33) associated to the hamiltonian $\mathcal{H}$ given by (1) for a $\mathcal{C}^1$ trajectory. Furthermore, we show that we can extend the classical action in the case of a γ-Hölder trajectory thanks to its continuity with respect to the trajectories. This leads, in section 4, to the construction of the propagator $(U_w(t,s))_{t\in[s,s+T]}$ formulated in (10). By using definition 2, we construct the solution of the linear problem (3). Finally, in section 5, we prove theorem 2 with the help of dispersive estimates and show that theorem 3 follows from an Avron-Herbst formula.

2 Classical orbits and changes of variables

We start by solving the Hamilton equations associated to the Hamiltonian $\mathcal{H}$ and prove some properties on the classical orbits $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{\xi}}$.

First, we assume that the trajectories $(w_t)_{t\in\mathbb{R}^+}$ are $\mathcal{C}^1$. The Hamilton equations are given by, $\forall s \in \mathbb{R}^+$,

$$\left\{ \begin{array}{l} \partial_t \bar{\boldsymbol{x}}_w(t,s) = \bar{\boldsymbol{\xi}}_w(t,s) + M_{11}\bar{\boldsymbol{x}}_w(t,s) + \mathbf{V}_{12} + M_{21}\bar{\boldsymbol{x}}_w(t,s)\dot{w}_t + \mathbf{V}_{22}\dot{w}_t,\ \forall t \in [s,\infty[, \\ \partial_t \bar{\boldsymbol{\xi}}_w(t,s) = -(M_{12}+M_{12}^*)\bar{\boldsymbol{x}}_w(t,s) - M_{11}^*\bar{\boldsymbol{\xi}}_w(t,s) - \mathbf{V}_{11} - (M_{22}+M_{22}^*)\bar{\boldsymbol{x}}_w(t,s)\dot{w}_t \\ \qquad\qquad -M_{21}^*\bar{\boldsymbol{\xi}}_w(t,s)\dot{w}_t - \mathbf{V}_{21}\dot{w}_t,\ \forall t\in[s,\infty[, \end{array} \right. \tag{14}$$

with the initial conditions

$$\bar{\boldsymbol{x}}_w(s,s) = \boldsymbol{y} \in \mathbb{R}^d \quad \text{and} \quad \bar{\boldsymbol{\xi}}_w(s,s) = \boldsymbol{\eta} \in \mathbb{R}^d.$$

Denoting by $\bar{\boldsymbol{x}}_w(t,s,\boldsymbol{y},\boldsymbol{\eta})$ and $\bar{\boldsymbol{\xi}}_w(t,s,\boldsymbol{y},\boldsymbol{\eta})$ these trajectories with the initial conditions $\boldsymbol{y}$ and $\boldsymbol{\eta}$, this system can also be rewritten as

$$\partial_t \boldsymbol{\chi}_w(t,s,\boldsymbol{y},\boldsymbol{\eta}) = M_1\boldsymbol{\chi}_w(t,s,\boldsymbol{y},\boldsymbol{\eta}) + M_2\boldsymbol{\chi}_w(t,s,\boldsymbol{y},\boldsymbol{\eta})\dot{w}_t + \mathbf{V}_1 + \mathbf{V}_2\dot{w}_t, \tag{15}$$

with

$$\boldsymbol{\chi}_w(t,s,\boldsymbol{y},\boldsymbol{\eta}) = \begin{pmatrix} \bar{\boldsymbol{x}}_w(t,s,\boldsymbol{y},\boldsymbol{\eta}) \\ \bar{\boldsymbol{\xi}}_w(t,s,\boldsymbol{y},\boldsymbol{\eta}) \end{pmatrix}, \quad M_1 = \begin{pmatrix} M_{11} & 1 \\ -M_{12}-M_{12}^* & -M_{11}^* \end{pmatrix},$$

$$M_2 = \begin{pmatrix} M_{21} & 0 \\ -M_{22}-M_{22}^* & -M_{21}^* \end{pmatrix}, \quad \mathbf{V}_1 = \begin{pmatrix} \mathbf{V}_{12} \\ -\mathbf{V}_{11} \end{pmatrix} \text{ and } \mathbf{V}_2 = \begin{pmatrix} \mathbf{V}_{22} \\ -\mathbf{V}_{21} \end{pmatrix}.$$

Remark 7. *From equation (15), we can see that classical flow $\boldsymbol{\chi}_w$ is an affine transformation in the phase space $\mathbb{R}^{2d}$. Thus, the solution $\boldsymbol{\chi}_w$ can be expressed thanks to the resolvent matrix $\mathfrak{e}(t,s)$ associated to the problem when $\mathbf{V}_1 = \mathbf{V}_2 = 0$. This resolvent matrix is given by*

$$\mathfrak{e}(t,s) = \sum_{k=0}^{\infty} \sum_{I\in\{1,2\}^k} M_I \int_s^t \dot{\mathbf{w}}_\tau^{n,I} d\tau,$$

where we set, $\forall I \in \mathbb{Z}^k$, $\forall k \in \mathbb{N}$,

$$M_I = M_{I_1}...M_{I_k} \quad \text{and} \quad \int_s^t \dot{\mathbf{w}}_\tau^{n,I} d\tau = \int_s^t \int_s^{\tau_1} ... \int_s^{\tau_{k-1}} \dot{w}_{\tau_1}^{n,I_1} ... \dot{w}_{\tau_k}^{n,I_1} d\tau_1 ... d\tau_k,$$

with $w_t^{n,1} = t$ and $w_t^{n,2} = w_t^n$, $\forall t \in [s,s+T]$. Since $(w_t)_{t\in\mathbb{R}^+}$ is assume to be $\mathcal{C}^1$, we can easily bound $\mathfrak{e}$ and, hence, obtain directly the existence and uniqueness of a solution of (15)

It follows from Duhamel's formula that $\boldsymbol{\chi}_w$ can also be formulated as the solution of the following integral equation, $\forall s \in \mathbb{R}^+$, $\forall \boldsymbol{y},\boldsymbol{\eta} \in \mathbb{R}^d$,

$$\forall t \in [s,s+T], \quad \boldsymbol{\chi}_w(t,s,\boldsymbol{y},\boldsymbol{\eta}) = e^{M_2(w_t-w_s)}\boldsymbol{\chi}_w(s,s,\boldsymbol{y},\boldsymbol{\eta}) + \int_s^t e^{M_2(w_t-w_\tau)} M_1 \boldsymbol{\chi}_w(\tau,s,\boldsymbol{y},\boldsymbol{\eta}) d\tau + \int_s^t e^{M_2(w_t-w_\tau)} (\mathbf{V}_1 + \mathbf{V}_2\dot{w}_\tau)\, d\tau. \tag{16}$$

The last term of the right hand side of equation (16) can be expressed as

$$\int_s^t e^{M_2(w_t-w_\tau)}\mathbf{V}_2\dot{w}_\tau d\tau = \sum_{j=0}^{\infty}\frac{M_2^k(w_t-w_s)^{k+1}}{(k+1)!}\mathbf{V}_2. \tag{17}$$

Hence, the existence and uniqueness of a solution $\boldsymbol{\chi}_w = (\bar{\boldsymbol{x}}_w, \bar{\boldsymbol{\xi}}_w)$ of equation (15) follows from a standard fixed point argument in $\mathcal{C}^0([s,s+T],\mathbb{R}^{2d})$, with $T>0$ sufficiently small, and thanks to a Gronwall inequality. Furthermore, by using (16) and (17), it is quite straightforward to prove that the existence and uniqueness of a solution $\boldsymbol{\chi}_w$ of (15) extends to the case of a continuous trajectory $(w_t)_{t\in\mathbb{R}^+}$ thanks to the following definition [13, 30].

Definition 4. *Let $T>0$ and $(w_t)_{t\in[s,s+T]}\in\mathcal{C}^0$. We define $\boldsymbol{\chi}_w$ as the solution of (15) if there exist a neighborhood $\mathcal{Q}$ of $(w_t)_{t\in\mathbb{R}^+}$ in $\mathcal{C}^0([s,s+T],\mathbb{R})$ and an application $\mathcal{J}:\ \mathcal{Q}\to\mathcal{C}^0([s,s+T],\mathbb{R}^{2d})$ such that*

- *$\mathcal{J}$ is continuous with respect to the norm*

$$\|\mathcal{J}(w)\|_{\mathcal{C}^0([s,s+T],\mathbb{R}^{2d})} := \sup_{t\in[s,s+T]}|\mathcal{J}(w,t)|.$$

- *For all $(\tilde{w}_t)_{t\in[s,s+T]}\in\mathcal{Q}\cap\mathcal{C}^1$, $\boldsymbol{\chi}_{\tilde{w}}$ is a solution to the ordinary differential equation (15).*
- *We have $\mathcal{J}(w)=\boldsymbol{\chi}_w$.*

To obtain the change of variable between the initial impulsion $\boldsymbol{\eta}$ and the final position $\bar{\boldsymbol{x}}_w(t,s,\boldsymbol{y},\boldsymbol{\eta})$, we need to develop the classical orbits with respect to the time variable while working with trajectories in $\mathcal{C}^\gamma$, $\gamma\in]0,1]$. To this end, we state a new formulation of $\boldsymbol{\chi}_w$ which is easier to manipulate. To prove the continuity of the classical orbits with respect to the trajectories, we assume that the trajectories belong to

$$B_W(\mathcal{C}^0) := \left\{ v\in\mathcal{C}^0([s,s+T],\mathbb{R});\ \sup_{t\in[s,s+T]}|v(t)|\le W \right\},$$

for a certain $W>0$. This does not restrict the class of possible trajectories since they are always considered as bounded.

Proposition 1. *Let $s\ge 0$ and $(w_t)_{t\in[s,s+T]}\in\mathcal{C}^0$. Then, there exists $T>0$ sufficiently small and three unique continuous mappings $\Xi_{0,w}(t,s)$, $\Xi_{1,w}(t,s)$, $\Xi_{2,w}(t,s)\in\mathcal{C}^0([s,s+T],\mathcal{L}(\mathbb{R}^{2d},\mathbb{R}^{2d}))$ such that, $\forall t\in[s,s+T]$, $\forall\boldsymbol{x},\boldsymbol{\eta}\in\mathbb{R}^d$,*

$$\boldsymbol{\chi}_w(t,s,\boldsymbol{y},\boldsymbol{\eta}) = \Xi_{0,w}(t,s)\begin{pmatrix}\boldsymbol{y}\\ \boldsymbol{\eta}\end{pmatrix} + \Xi_{1,w}(t,s)\mathbf{V}_1 + \Xi_{2,w}(t,s)\mathbf{V}_2. \tag{18}$$

Moreover, let $W>0$ and $j\in\{0,1,2\}$, for all $(w_t)_{t\in[s,s+T]},(\tilde{w}_t)_{t\in[s,s+T]}\in B_W(\mathcal{C}^0)$, we have the following estimate

$$\sup_{t\in[s,s+T]}\|\Xi_{j,w}(t,s)-\Xi_{j,\tilde{w}}(t,s)\|_{\mathcal{L}(\mathbb{R}^d,\mathbb{R}^d)} \le C_{T,s,W}\|w-\tilde{w}\|_{\dot{\mathcal{C}}^0([s,s+T])}. \tag{19}$$

Proof. Let $\{(w_t^n)_{t\in\mathbb{R}^+}\}_{n\in\mathbb{N}}\subset\mathcal{C}^1$ be a sequence which converges to $(w_t)_{t\in\mathbb{R}^+}$ in $\mathcal{C}^0$. We consider $(w_t^n)_{t\in\mathbb{R}^+}$, $\forall n\in\mathbb{N}$, and the Picard sequence $(\boldsymbol{\chi}_{k,w^n})_{k\in\mathbb{N}}$ given by

$$\begin{cases} \boldsymbol{\chi}_{k+1,w^n}(t,s,\boldsymbol{y},\boldsymbol{\eta}) = \boldsymbol{\chi}_{k,w^n}(t,s,\boldsymbol{y},\boldsymbol{\eta}) + M_1\displaystyle\int_s^t \boldsymbol{\chi}_{k,w^n}(\tau,s,\boldsymbol{y},\boldsymbol{\eta})d\tau + M_2\int_s^t \boldsymbol{\chi}_{k,w^n}(\tau,s,\boldsymbol{y},\boldsymbol{\eta})\dot{w}_\tau^n d\tau \\ \qquad\qquad + \mathbf{V}_1\displaystyle\int_s^t d\tau + \mathbf{V}_2\int_s^t \dot{w}_\tau^n d\tau,\ \forall t\in[s,s+T], \\ \boldsymbol{\chi}_{0,w^n}(t,s,\boldsymbol{y},\boldsymbol{\eta}) = \begin{pmatrix}\boldsymbol{y}\\ \boldsymbol{\eta}\end{pmatrix}. \end{cases}$$

Since, for $T > 0$ sufficiently small, the Picard sequence $(\boldsymbol{\chi}_{k,w^n})_{k\in\mathbb{N}}$ converge to the solution $\boldsymbol{\chi}_{w^n}$ of the equation (15) in $\mathcal{C}^0([s,s+T],\mathbb{R}^{2d})$, we deduce the following formulation

$$\boldsymbol{\chi}_{w^n}(t,s,\boldsymbol{y},\boldsymbol{\eta}) = \sum_{k=0}^{\infty}\sum_{I\in\{1,2\}^k} M_I \int_s^t d\mathbf{w}_\tau^{n,I}\begin{pmatrix}\boldsymbol{y}\\ \boldsymbol{\eta}\end{pmatrix} + \sum_{k=0}^{\infty}\sum_{J\in\{1,2\}^k\times\{-1\}} M_J \int_s^t d\mathbf{w}_\tau^{n,J}\mathbf{V}_1$$
$$+ \sum_{k=0}^{\infty}\sum_{K\in\{1,2\}^k\times\{-2\}} M_K \int_s^t d\mathbf{w}_\tau^{n,K}\mathbf{V}_2,$$

where we set, $\forall I \in \mathbb{Z}^k$, $\forall k \in \mathbb{N}$,

$$M_I = M_{I_1}...M_{I_k} \quad \text{and} \quad \int_s^t d\mathbf{w}_\tau^{n,I} = \int_s^t dw_{\tau_1}^{n,I_1}\int_s^{\tau_1} dw_{\tau_2}^{n,I_1}...\int_s^{\tau_{k-1}} dw_{\tau_k}^{n,I_k},$$

with $w_t^{n,1} = w_t^{n,-1} = t$, $w_t^{n,2} = w_t^{n,-2} = w_t^n$, $\forall t \in [s,s+T]$, and $M_{-1} = M_{-2} = \mathrm{Id}$, the identity operator. Introducing

$$\Xi_{0,w^n}(t,s) = \sum_{k=0}^{\infty}\sum_{I\in\{1,2\}^k} M_I \int_s^t d\mathbf{w}_\tau^{n,I}, \quad \Xi_{1,w^n}(t,s) = \sum_{k=0}^{\infty}\sum_{J\in\{1,2\}^k\times\{-1\}} M_J \int_s^t d\mathbf{w}_\tau^{n,J},$$
$$\text{and } \Xi_{2,w^n}(t,s) = \sum_{k=0}^{\infty}\sum_{K\in\{1,2\}^k\times\{-2\}} M_K \int_s^t d\mathbf{w}_\tau^{n,K},$$

and thanks to the formulation (16) and a Gronwall inequality, we obtain that, $\forall j \in \{0,1,2\}$, $(\Xi_{j,w^n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}^0([s,s+T],\mathcal{L}(\mathbb{R}^{2d},\mathbb{R}^{2d}))$. Therefore, there exist three continuous mapping $\Xi_{0,w}$, $\Xi_{1,w}$ and $\Xi_{2,w}$ such that

$$\boldsymbol{\chi}_{w^n}(t,s,\boldsymbol{y},\boldsymbol{\eta}) \underset{n\to\infty}{\longrightarrow} \Xi_{0,w}(t,s)\begin{pmatrix}\boldsymbol{y}\\ \boldsymbol{\eta}\end{pmatrix} + \Xi_{1,w}(t,s)\mathbf{V}_1 + \Xi_{2,w}(t,s)\mathbf{V}_2,$$

in $\mathcal{C}^0([s,s+T],\mathbb{R}^{2d})$, which leads to formula (18). We now prove that, $\forall j \in \{0,1,2\}$, $\Xi_{j,w}$ is continuous with respect to $(w_t)_{t\in[s,s+T]}$ *i.e.* we prove (19). By using the integral formulation (16), (17) and a Grownall inequality, we obtain that, for all $(w_t)_{t\in[s,s+T]}$, $(\tilde{w}_t)_{t\in[s,s+T]} \in B_W(\mathcal{C}^0)$,

$$\sup_{t\in[s,s+T]} \|\Xi_w(t,s) - \Xi_{\tilde{w}}(t,s)\|_{\mathcal{L}(\mathbb{R}^d,\mathbb{R}^d)} \le C_{T,s,W}\sup_{t\in[s,s+T]} |w_t - \tilde{w}_t + w_s - \tilde{w}_s| \le C_{T,s,W}\|w-\tilde{w}\|_{\dot{\mathcal{C}}^0},$$

ending hence the proof. □

The following result provides an expansion of the applications $\Xi_{0,w}$, $\Xi_{1,w}$ and $\Xi_{2,w}$ with respect to the time variable. Moreover, we directly deduce an expansion of the classical orbits.

Proposition 2. *Let $s \ge 0$. Then, there exists $T > 0$ sufficiently small such that the following expansions hold, $\forall t \in [s,s+T]$,*

$$\Xi_{0,w}(t,s) = Id + M_1(t-s) + M_2(w_t - w_s) + (t-s)^{\varrho_0}R_{0,w}(t,s),$$
$$\Xi_{1,w}(t,s) = Id(t-s) + (t-s)^{\varrho_1}R_{1,w}(t,s),$$
$$\Xi_{2,w}(t,s) = Id(w_t - w_s) + (t-s)^{\varrho_2}R_{2,w}(t,s),$$

where $\varrho_j > 1$, $\forall j \in \{0,1,2\}$, and, $\forall j \in \{0,1,2\}$, $R_{j,w}$ is an operator uniformly bounded with respect to $t \in]s, s+T]$. Let $W > 0$. Then, we have, for all $(w_t)_{t\in[s,s+T]}, (\tilde{w}_t)_{t\in[s,s+T]} \in \mathcal{C}^\gamma \cap B_W(\mathcal{C}^0)$, $\forall j \in \{0,1,2\}$,

$$\sup_{t\in[s,s+T]}(t-s)^{\varrho_j}\|R_{j,w}(t,s) - R_{j,,\tilde{w}}(t,s)\|_{\mathcal{L}(\mathbb{R}^{2d},\mathbb{R}^{2d})} \leq C_{T,s,W}\|w-\tilde{w}\|_{\dot{\mathcal{C}}^0([s,s+T])}.$$

Moreover, we have

$$\begin{aligned}
&\int_s^t \Xi_{0,w}(\tau,s)\dot{w}_\tau d\tau = Id(w_t - w_s) + (t-s)^{\tilde{\varrho}_0}\tilde{R}_{0,w}(t,s),\\
&\int_s^t \Xi_{1,w}(\tau,s)\dot{w}_\tau d\tau = (t-s)^{\tilde{\varrho}_1}\tilde{R}_{1,w}(t,s),\\
&\int_s^t \Xi_{2,w}(\tau,s)\dot{w}_\tau d\tau = Id\frac{(w_t-w_s)^2}{2} + (t-s)^{\tilde{\varrho}_2}\tilde{R}_{2,w}(t,s),
\end{aligned} \tag{20}$$

where $\tilde{\varrho}_j > 1$, $\forall j \in \{0,1,2\}$, and, $\forall j \in \{0,1,2\}$, $\tilde{R}_{j,w}$ is an operator uniformly bounded with respect to $t \in]s, s+T]$. Let $W > 0$. Then, we have, for all $(w_t)_{t\in[s,s+T]}, (\tilde{w}_t)_{t\in[s,s+T]} \in \mathcal{C}^\gamma \cap B_W(\mathcal{C}^0)$, $\forall j \in \{0,1,2\}$,

$$\sup_{t\in[s,s+T]}(t-s)^{\tilde{\varrho}_j}\|\tilde{R}_{j,w}(t,s) - \tilde{R}_{j,,\tilde{w}}(t,s)\|_{\mathcal{L}(\mathbb{R}^{2d},\mathbb{R}^{2d})} \leq C_{T,s,W}\|w-\tilde{w}\|_{\dot{\mathcal{C}}^0([s,s+T])}.$$

Proof. The integral formulation (16) leads to, $\forall t \in [s, s+T]$,

$$\begin{aligned}
\Xi_{0,w}(t,s) = e^{M_2(w_t-w_s)} &+ \int_s^t e^{M_2(w_t-w_\tau)}M_1 e^{M_2(w_\tau-w_s)}d\tau\\
&+ \int_s^t e^{M_2(w_t-w_\tau)}M_1\int_s^\tau e^{M_2(w_\tau-w_\iota)}\Xi_{0,w}(\iota,s)d\iota d\tau.
\end{aligned} \tag{21}$$

By using the expansion

$$e^{M_2(w_t-w_s)} = \mathrm{Id} + M_2(w_t-w_s) + \sum_{j=2}^\infty \frac{M_2^j(w_t-w_s)^j}{j!}, \tag{22}$$

and if $\gamma \in]1/2, 1[$, $(w_t)_{t\in\mathbb{R}^+}$ being γ-Hölder, we have, on one hand,

$$\left\|\sum_{j=2}^\infty \frac{M_2^j(w_t-w_s)^j}{j!}\right\|_{\mathcal{L}(\mathbb{R}^{2d},\mathbb{R}^{2d})} \leq (t-s)^{2\gamma}\sum_{j=2}^\infty \frac{\|M_2\|^j_{\mathcal{L}(\mathbb{R}^{2d},\mathbb{R}^{2d})}\|w\|^j_{\mathcal{C}^{0,\gamma}}(t-s)^{\gamma(j-2)}}{j!},$$

and, if $M_{21} = 0$, we obtain, on the other hand,

$$\sum_{j=2}^\infty \frac{M_2^j(w_t-w_s)^j}{j!} = 0.$$

Therefore, in (22), the sum appearing in the right hand side is such that

$$\left\|\sum_{j=2}^\infty \frac{M_2^j(w_t-w_s)^j}{j!}\right\|_{\mathcal{L}(\mathbb{R}^{2d},\mathbb{R}^{2d})} = \underset{t\to s}{o}(t-s)^{\varrho},$$

with $\varrho > 1$. Concerning the second term from the expansion (21), we have

$$\int_s^t e^{M_2(w_t-w_\tau)} M_1 e^{M_2(w_\tau-w_s)} d\tau = M_1(t-s) + \int_s^t \sum_{\substack{j,k\geq 0\\ j+k\geq 1}} \frac{M_2^j(w_t-w_\tau)^j}{j!} M_1 \frac{M_2^k(w_\tau-w_s)^k}{k!} d\tau. \tag{23}$$

By using (21), (22) and finally (23), we obtain the first result for the mapping $\Xi_{0,w}$. The expansions of the mappings $\Xi_{1,w}$ and $\Xi_{2,w}$ are obtained by using (16), (17) and similar expansions of the exponential matrices.

Concerning the second result, it is obtained similarly through integrations by parts. □

From Proposition 2, we have the following corollary.

Corollary 1. *Let $s \geq 0$. Then, there exists $T > 0$ such that the classical orbits can be expanded as, $\forall t \in [s, s+T]$,*

$$\begin{aligned}
\bar{\boldsymbol{x}}_w(t,s,\boldsymbol{y},\boldsymbol{\eta}) &= \boldsymbol{y} + (t-s)\mathbf{V}_{12} + (w_t-w_s)\mathbf{V}_{22} + (t-s)\boldsymbol{\eta} + (M_{11}(t-s) + M_{21}(w_t-w_s))\,\boldsymbol{y}\\
&\quad + (t-s)^\varrho r_{0,1}(t,s)\boldsymbol{y} + (t-s)^\varrho r_{0,2}(t,s)\boldsymbol{\eta} + (t-s)^\varrho r_{1,1}(t,s)\mathbf{V}_{12} + (t-s)^\varrho r_{1,2}(t,s)\mathbf{V}_{22},\\
\bar{\boldsymbol{\xi}}_w(t,s,\boldsymbol{y},\boldsymbol{\eta}) &= \boldsymbol{\eta} - (t-s)\mathbf{V}_{11} - (w_t-w_s)\mathbf{V}_{21} - ((M_{12}+M_{12}^*)(t-s) + (M_{22}+M_{22}^*)(w_t-w_s))\,\boldsymbol{y}\\
&\quad - M_{21}\boldsymbol{\eta}(w_t-w_s) - M_{11}^*\boldsymbol{\eta}(t-s) + (t-s)^\varrho r_{0,3}(t,s)\boldsymbol{y} + (t-s)^\varrho r_{0,4}(t,s)\boldsymbol{\eta}\\
&\quad + (t-s)^\varrho r_{1,3}(t,s)\mathbf{V}_{11} + (t-s)^\varrho r_{1,4}(t,s)\mathbf{V}_{21},
\end{aligned}$$

where $r_{0,j}, r_{1,j} \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ are uniformly bounded with respect to $t \in [s, s+T]$, $\forall j \in \{1,2,3,4\}$, and $\varrho > 1$.

Our aim is now to obtain the following variables changes

$$(\boldsymbol{y},\boldsymbol{\eta}) \leftrightarrow (\bar{\boldsymbol{x}}_w(t,s,\boldsymbol{y},\boldsymbol{\eta}),\boldsymbol{y}) \quad \text{and} \quad (\boldsymbol{y},\boldsymbol{\eta}) \leftrightarrow (\bar{\boldsymbol{x}}_w(t,s,\boldsymbol{y},\boldsymbol{\eta}),\boldsymbol{\eta}).$$

To this end, we prove that the applications

$$\Theta_{1,t,s,w}: \ (\boldsymbol{y},\boldsymbol{\eta}) \to (\bar{\boldsymbol{x}}_w(t,s,\boldsymbol{y},\boldsymbol{\eta}),\boldsymbol{y}),$$

and

$$\Theta_{2,t,s,w}: \ (\boldsymbol{y},\boldsymbol{\eta}) \to (\bar{\boldsymbol{x}}_w(t,s,\boldsymbol{y},\boldsymbol{\eta}),\boldsymbol{\eta}),$$

are diffeomorphisms from $\mathbb{R}^d$ to itself.

We have a first result on the dependence of $\bar{\boldsymbol{x}}$ with respect to $\boldsymbol{y}$ and the dependence of $\bar{\boldsymbol{\xi}}$ with respect to $\boldsymbol{\eta}$. The proof is a consequence of proposition 1 and corollary 1.

Lemma 1. *Let $s \geq 0$. Then there exists $T > 0$ such that we have, for all $t \in [s, s+T]$,*

$$\begin{aligned}
&\frac{\partial \bar{\boldsymbol{x}}}{\partial \boldsymbol{y}}(t,s) = 1 + (t-s)^\gamma A_{1,w}(t,s), \quad \frac{\partial \bar{\boldsymbol{\xi}}}{\partial \boldsymbol{\eta}}(t,s) = 1 + (t-s)^\gamma A_{2,w}(t,s),\\
&\frac{\partial \bar{\boldsymbol{x}}}{\partial \mathbf{V}_{12}}(t,s) = (t-s) + (t-s)^\varrho A_{3,w}(t,s), \quad \frac{\partial \bar{\boldsymbol{x}}}{\partial \mathbf{V}_{22}}(t,s) = (w_t-w_s) + (t-s)^\varrho A_{4,w}(t,s),
\end{aligned}$$

where $\{A_{j,w}\}_{j\in\{1,2,3,4\}} \subset \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ are operators uniformly bounded with respect to $t \in [s, s+T]$. Let $W > 0$. Then, we have, for all $(w_t)_{t\in[s,s+T]}, (\tilde{w}_t)_{t\in[s,s+T]} \in \mathcal{C}^\gamma \cap B_W(\mathcal{C}^0)$, $\forall j \in \{1,2,3,4\}$,

$$\sup\nolimits_{t\in[s,s+T]} (t-s)^\gamma \|A_{j,w}(t,s) - A_{j,,\tilde{w}}(t,s)\|_{\mathcal{L}(\mathbb{R}^d,\mathbb{R}^d)} \leq C_{T,s,W} \|w - \tilde{w}\|_{\dot{\mathcal{C}}^0([s,s+T])}.$$

To obtain a similar result between the variables $\bar{\boldsymbol{x}}$ and $\boldsymbol{\eta}$, we introduce a new variable $\boldsymbol{\zeta} := (t-s)\boldsymbol{\eta}$. The application of Corollary 1 leads to the following result.

Lemma 2. *Let $s \geq 0$. Then there exists $T > 0$ such that we have, for all $t \in]s, s+T]$,*

$$\frac{\partial \bar{\boldsymbol{x}}}{\partial \boldsymbol{\zeta}}(t,s) = 1 + (t-s)^{\varrho-1} B_w(t,s),$$

where $\varrho - 1 > 0$, and $B_w \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ is an operator uniformly bounded with respect to $t \in [s, s+T]$.

Moreover, let $W > 0$. Then, for all $(w_t)_{t\in[s,s+T]}$, $(\tilde{w}_t)_{t\in[s,s+T]} \in \mathcal{C}^\gamma \cap B_W(\mathcal{C}^0)$, we have the following inequality

$$\sup_{t\in[s,s+T]} (t-s)^{\varrho-1} \|B_w(t,s) - B_{\tilde{w}}(t,s)\|_{\mathcal{L}(\mathbb{R}^d,\mathbb{R}^d)} \leq C_{T,s,W} \|w - \tilde{w}\|_{\dot{\mathcal{C}}^0([s,s+T])}.$$

We finally obtain the main result of this section.

Proposition 3. *There exists $T > 0$ such that the families of $\mathcal{C}^\infty(\mathbb{R}^{2d}, \mathbb{R}^{2d})$ mappings*

$$\Theta_{1,t,s,w} : \ (\boldsymbol{y}, \boldsymbol{\zeta}(\boldsymbol{\eta})) \to (\bar{\boldsymbol{x}}_w(t,s,\boldsymbol{y},\boldsymbol{\eta}), \boldsymbol{y}),$$

and

$$\Theta_{2,t,s,w} : \ (\boldsymbol{y}, \boldsymbol{\zeta}(\boldsymbol{\eta})) \to (\bar{\boldsymbol{x}}_w(t,s,\boldsymbol{y},\boldsymbol{\eta}), \boldsymbol{\zeta}(\boldsymbol{\eta})),$$

are diffeomorphisms, $\forall t \in]s, s+T]$.

Proof. Let us first remark that the applications are one-to-one since we have the uniqueness of the classical orbits with respect to their initial conditions. For $\varepsilon > 0$, we choose $T > 0$ such that, $\forall t \in [s, s+T]$,

$$\sum_{j,k=1}^{d} (t-s)^\gamma |A_{1,w,j,k}(t,s)| + (t-s)^\gamma |A_{2,w,j,k}(t,s)| + (t-s)^{\varrho-1} |B_{w,j,k}(t,s)| < \varepsilon. \tag{24}$$

We now focus our attention on the map $\Theta_{1,t,s,w}$ since the proof is similar for $\Theta_{2,t,s,w}$.

The jacobian matrix of the application $\Theta_{1,t,s,w}$ has the following expression, $\forall t \in]s, s+T]$,

$$\begin{aligned}
\partial_{\boldsymbol{y},\boldsymbol{\zeta}} \Theta_{1,t} &= \begin{pmatrix} 1 + (t-s)^\gamma A_{1,w}(t,s) & 1 + (t-s)^{\varrho-1} B_w(t,s) \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} (t-s)^\gamma A_{1,w}(t,s) & (t-s)^{\varrho-1} B_w(t,s) \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Since the previous matrix is a perturbation of an invertible matrix, we can use Neumann's series to conclude that $\partial_{\boldsymbol{y},\boldsymbol{\zeta}} \Theta_{1,t}$ is invertible, *i.e.*

$$(\partial_{\boldsymbol{y},\boldsymbol{\zeta}} \Theta_{1,t})^{-1} = \lim_{N\to\infty} \sum_{j=0}^{N} (-1)^j \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-j} \begin{pmatrix} (t-s)^\gamma A_{1,w}(t,s) & (t-s)^{\varrho-1} B_w(t,s) \\ 0 & 0 \end{pmatrix}^j \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1}.$$

In view of (24), the series clearly converges. Hence, we deduce that the application $\Theta_{1,t,s,w}$ is a diffeomorphism from $\mathbb{R}^{2d}$. ☐

Thanks to the diffeomorphism $\Theta_{1,t,s,w}$, the variables $\boldsymbol{\zeta}$ and $\boldsymbol{\eta}$ can be defined as

$$\forall t \in [s, s+T],\ \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d, \quad \boldsymbol{\zeta} = \bar{\boldsymbol{\zeta}}_w(t, s, \boldsymbol{x}, \boldsymbol{y}),$$

$$\forall t \in]s, s+T],\ \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d, \quad \boldsymbol{\eta} = \bar{\boldsymbol{\eta}}_w(t, s, \boldsymbol{x}, \boldsymbol{y}) = \frac{1}{t-s}\bar{\boldsymbol{\zeta}}_w(t, s, \boldsymbol{x}, \boldsymbol{y}).$$

Furthermore, we remark that the function $\bar{\boldsymbol{\eta}}_w$ verifies

$$\bar{\boldsymbol{x}}_w(t, s, \boldsymbol{y}, \bar{\boldsymbol{\eta}}_w(t, s, \boldsymbol{x}, \boldsymbol{y})) = \boldsymbol{x}. \tag{25}$$

We now state some properties of $\bar{\boldsymbol{\eta}}$.

Proposition 4. *The function $\bar{\boldsymbol{\eta}}$ is linear with respect to the space variables $\boldsymbol{x}$ and $\boldsymbol{y}$. Furthermore, it satisfies the following inequality, $\forall(\alpha_1, \alpha_2) \in \mathbb{N}^d \times \mathbb{N}^d$ such that $|\alpha_1 + \alpha_2| \leq 1$, $\forall t \in]s, s+T]$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$,*

$$|\partial_{\boldsymbol{x}}^{\alpha_1} \partial_{\boldsymbol{y}}^{\alpha_2} \bar{\boldsymbol{\eta}}_w(t, s, \boldsymbol{x}, \boldsymbol{y})| \leq \frac{C_{T,s,w}}{t-s}\left(|\mathbf{V}_{12}| + |\mathbf{V}_{22}| + |\boldsymbol{x}|^{1-|\alpha_1+\alpha_2|} + |\boldsymbol{y}|^{1-|\alpha_1+\alpha_2|}\right). \tag{26}$$

We also have the following expansion: $\forall t \in]s, s+T]$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$,

$$\begin{aligned}\bar{\boldsymbol{\eta}}_w(t, s, \boldsymbol{x}, \boldsymbol{y}) = &\frac{\boldsymbol{x}-\boldsymbol{y}}{t-s} - \frac{w_t - w_s}{t-s}\left(M_{21}\boldsymbol{y} + \mathbf{V}_{22}\right) - \left(M_{11}\boldsymbol{y} + \mathbf{V}_{12}\right)\\ &+\mathbf{r}(t, s, \boldsymbol{x}, \boldsymbol{y}),\end{aligned} \tag{27}$$

where $\mathbf{r}$ is a linear function with respect to $\boldsymbol{x}$ and $\boldsymbol{y}$ and such that, $\forall(\alpha_1, \alpha_2) \in \mathbb{N}^d \times \mathbb{N}^d$ with $|\alpha_1 + \alpha_2| \leq 1$,

$$|\partial_{\boldsymbol{x}}^{\alpha_1} \partial_{\boldsymbol{y}}^{\alpha_2} \mathbf{r}(t, s, \boldsymbol{x}, \boldsymbol{y})| \underset{t \to s}{\longrightarrow} 0.$$

Proof. From corollary 1 and equation (25), we obtain, $\forall t \in]s, s+T]$,

$$\begin{aligned}\boldsymbol{x} = &\boldsymbol{y} + (t-s)\mathbf{V}_{12} + (w_t - w_s)\mathbf{V}_{22} + (t-s)\bar{\boldsymbol{\eta}}_w(t, s, \boldsymbol{x}, \boldsymbol{y}) + \left(M_{11}(t-s) + M_{21}(w_t - w_s)\right)\boldsymbol{y}\\ &+(t-s)^\varrho r_{0,1}(t, s)\boldsymbol{y} + (t-s)^\varrho r_{0,2}(t, s)\bar{\boldsymbol{\eta}}_w(t, s, \boldsymbol{x}, \boldsymbol{y}) + (t-s)^\varrho r_{1,1}(t, s)\mathbf{V}_{12} + (t-s)^\varrho r_{1,2}(t, s)\mathbf{V}_{22}.\end{aligned} \tag{28}$$

Therefore, for $T > 0$ sufficiently small, we have

$$\begin{aligned}\bar{\boldsymbol{\eta}}_w(t, s, \boldsymbol{x}, \boldsymbol{y}) =&(1 - (t-s)^{\varrho-1} r_{0,2}(t, s))^{-1}\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{t-s} - \frac{w_t - w_s}{t-s}M_{21}\boldsymbol{y} - M_{11}\boldsymbol{y}\right)\\ &- (1 - (t-s)^{\varrho-1} r_{0,2}(t, s))^{-1}\left(\mathbf{V}_{12} + \frac{w_t - w_s}{t-s}\mathbf{V}_{22}\right)\\ &- (1 - (t-s)^{\varrho-1} r_{0,2}(t, s))^{-1}(t-s)^{\varrho-1}\left[r_{0,1}(t, s)\boldsymbol{y} + r_{1,1}(t, s)\mathbf{V}_{12} + r_{1,2}(t, s)\mathbf{V}_{22}\right],\end{aligned}$$

which leads to (26). Concerning the estimate (27), we make use again (28) and then (26). □

Combining Lemma 2 and proposition 3, we deduce the following result.

Proposition 5. *For all $t \in [s, s+T]$, we have*

$$\begin{aligned}&\frac{\partial \bar{\boldsymbol{\zeta}}}{\partial \boldsymbol{x}}(t, s) = 1 + (t-s)^{\varrho-1} D_{1,w}(t, s), \quad \frac{\partial \bar{\boldsymbol{\zeta}}}{\partial \boldsymbol{y}}(t, s) = -1 + (t-s)^{\varrho-1} D_{2,w}(t, s),\\ &\frac{\partial \bar{\boldsymbol{\zeta}}}{\partial \mathbf{V}_{12}}(t, s) = -(t-s) + (t-s)^\varrho D_{3,w}(t, s), \quad \frac{\partial \bar{\boldsymbol{\zeta}}}{\partial \mathbf{V}_{22}}(t, s) = -(w_t - w_s) + (t-s)^{\varrho-1} D_{4,w}(t, s),\\ &(t-s)\frac{\partial \bar{\boldsymbol{\xi}}}{\partial \boldsymbol{x}}(t, s) = 1 + (t-s)^{\varrho-1} D_{5,w}(t, s),\end{aligned} \tag{29}$$

where $\{D_{j,w}\}_{j\in\{1,...,5\}} \subset \mathcal{L}(\mathbb{R}^d,\mathbb{R}^d)$ *are operators uniformly bounded with respect to* $t \in [s, s+T]$.

Moreover, let $W > 0$. *Then, we have the following estimates: for all* $(w_t)_{t\in[s,s+T]}, (\tilde{w}_t)_{t\in[s,s+T]} \in \mathcal{C}^\gamma \cap B_W(\mathcal{C}^0)$, $\forall j \in \{1, ..., 5\}$,

$$\sup_{t\in[s,s+T]}(t-s)^{\varrho-1}\|D_{j,w}(t,s) - D_{j,\tilde{w}}(t,s)\|_{\mathcal{L}(\mathbb{R}^d,\mathbb{R}^d)} \leq C_{T,s,W}\|w-\tilde{w}\|_{\dot{\mathcal{C}}^0([s,s+T])}.$$

Proof. Thanks to (25), we have

$$\frac{\partial \bar{\boldsymbol{x}}(t,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w(t,s,\boldsymbol{x},\boldsymbol{y}))}{\partial \boldsymbol{x}} = 1$$

and

$$\frac{\partial \bar{\boldsymbol{x}}_w(t,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w(t,s,\boldsymbol{x},\boldsymbol{y}))}{\partial \boldsymbol{x}} = \frac{\partial \bar{\boldsymbol{x}}_w}{\partial \boldsymbol{\zeta}}(t,s,\boldsymbol{x},\boldsymbol{y})\frac{\partial \bar{\boldsymbol{\zeta}}_w}{\partial \boldsymbol{x}}(t,s,\boldsymbol{x},\boldsymbol{y}) = 1.$$

The application of lemma 2 shows that

$$(1+(t-s)^{\varrho-1}B_w(t,s))\frac{\partial \bar{\boldsymbol{\zeta}}_w}{\partial \boldsymbol{x}}(t,s,\boldsymbol{x},\boldsymbol{y}) = 1.$$

We can invert $(1+(t-s)^{\varrho-1}B_w(t,s))$ by proposition 3 and deduce (by using a Neumann's series) the first equality of (29). Thanks to the equality (25), we obtain

$$\frac{\partial \bar{\boldsymbol{x}}_w}{\partial \boldsymbol{y}}(t,s,\boldsymbol{x},\boldsymbol{y}) + \frac{\partial \bar{\boldsymbol{x}}_w}{\partial \boldsymbol{\zeta}}(t,s,\boldsymbol{x},\boldsymbol{y})\frac{\partial \bar{\boldsymbol{\zeta}}_w}{\partial \boldsymbol{y}}(t,s,\boldsymbol{x},\boldsymbol{y}) = 0.$$

Applying lemmas 1 and 2 lead to

$$1+(t-s)^\gamma A_{1,w}(t,s) + (1+(t-s)^{\varrho-1}B_w(t,s))\frac{\partial \bar{\boldsymbol{\zeta}}_w}{\partial \boldsymbol{y}}(t,s,\boldsymbol{x},\boldsymbol{y}) = 0,$$

and to the second equality of (29) (by computing the Neumann's series of $(1+(t-s)^{\varrho-1}B_w(t,s))^{-1}$). Concerning the third and fourth equalities of (29), the proof is similar to the second one. We differentiate (25) with respect to $\mathbf{V}_{12}$ and $\mathbf{V}_{22}$ and then use the expansion of $(1+(t-s)^\varrho B_w(t,s))^{-1}$. Finally, the last equality of (29) is proved by remarking that

$$(t-s)\frac{\partial \bar{\boldsymbol{\xi}}_w}{\partial \boldsymbol{x}}(t,s,\boldsymbol{x},\boldsymbol{y}) = \frac{\partial \bar{\boldsymbol{\xi}}_w}{\partial \boldsymbol{\eta}}(t,s,\boldsymbol{x},\boldsymbol{y})\frac{\partial \bar{\boldsymbol{\zeta}}_w}{\partial \boldsymbol{x}}(t,s,\boldsymbol{x},\boldsymbol{y}),$$

which concludes the first part of this proposition.

Finally, the estimates of the operators $D_{j,w}$, $j \in \{1, ..., 5\}$, with respect to the trajectory $(w_t)_{t\in[s,s+T]}$ are proved by using lemmas 1 and 2. □

We now prove the continuity of the function $\bar{\boldsymbol{\eta}}$ with respect to $(w_t)_{t\in[s,s+T]}$ thanks to the previous result.

Corollary 2. *Let* $W > 0$. *We have, for all* $(w_t)_{t\in[s,s+T]}, (\tilde{w}_t)_{t\in[s,s+T]} \in \mathcal{C}^\gamma \cap B_W(\mathcal{C}^0)$, $\forall t \in]s, s+T]$,

$$|\bar{\boldsymbol{\eta}}_w(t,s,\boldsymbol{x},\boldsymbol{y}) - \bar{\boldsymbol{\eta}}_{\tilde{w}}(t,s,\boldsymbol{x},\boldsymbol{y})| \leq \frac{C_{T,s,W}}{t-s}\|w-\tilde{w}\|_{\dot{\mathcal{C}}^0([s,s+T])}(|\mathbf{V}_{12}| + |\mathbf{V}_{22}| + |\boldsymbol{x}| + |\boldsymbol{y}|).$$

Proof. We remark that, since the function $\bar{\boldsymbol{\eta}}$ is linear with respect to the space variables $\boldsymbol{x}$, $\boldsymbol{y}$, $\mathbf{V}_{12}$ and $\mathbf{V}_{22}$, $\forall t \in [s, s+T]$, we have

$$(t-s)\bar{\boldsymbol{\eta}}_w(t,s,\boldsymbol{x},\boldsymbol{y}) = (t-s)\frac{\partial \bar{\boldsymbol{\eta}}_w}{\partial \mathbf{V}_{12}}(t,s)\mathbf{V}_{12} + (t-s)\frac{\partial \bar{\boldsymbol{\eta}}_w}{\partial \mathbf{V}_{22}}(t,s)\mathbf{V}_{22} + (t-s)\frac{\partial \bar{\boldsymbol{\eta}}_w}{\partial \boldsymbol{x}}(t,s)\boldsymbol{x} + (t-s)\frac{\partial \bar{\boldsymbol{\eta}}_w}{\partial \boldsymbol{y}}(t,s)\boldsymbol{y}.$$

Thanks to proposition 5, this leads to: $\forall t \in [s, s+T]$,

$$\begin{aligned}(t-s)\bar{\boldsymbol{\eta}}_w(t,s,\boldsymbol{x},\boldsymbol{y}) &= \boldsymbol{x} - \boldsymbol{y} - (t-s)\mathbf{V}_{12} - (w_t - w_s)\mathbf{V}_{22} \\ &\quad + (t-s)^{\varrho-1}\left(D_{1,w}(t,s)\boldsymbol{x} + D_{2,w}(t,s)\boldsymbol{y} + (t-s)D_{3,w}\mathbf{V}_{12} + D_{4,w}(t,s)\mathbf{V}_{22}\right),\end{aligned}$$

showing finally the continuity of the operators $\{D_{j,w}\}_{j\in\{1,\ldots,4\}}$ in proposition 5. □

3 The classical action

Let us assume that $(w_t)_{t\in[s,s+T]} \in \mathcal{C}^1$. Then, we define the classical action S as: $\forall s \in \mathbb{R}^+$, $\forall t \in [s, s+T]$, $\forall \boldsymbol{y}, \boldsymbol{\eta} \in \mathbb{R}^d$,

$$S(t,s,\boldsymbol{\chi}_w(\cdot,s,\boldsymbol{y},\boldsymbol{\eta})) := \int_s^t \partial_t \bar{\boldsymbol{x}}_w(\tau,s,\boldsymbol{y},\boldsymbol{\eta}) \cdot \bar{\boldsymbol{\xi}}_w(\tau,s,\boldsymbol{y},\boldsymbol{\eta}) - \int_s^t \mathcal{H}(\tau, \bar{\boldsymbol{x}}_w(\tau,s,\boldsymbol{y},\boldsymbol{\eta}), \bar{\boldsymbol{\xi}}_w(\tau,s,\boldsymbol{y},\boldsymbol{\eta}))d\tau. \quad (30)$$

Here, we denote by $\boldsymbol{\chi}_w(\cdot,s,\boldsymbol{y},\boldsymbol{\eta})$ the classical orbits as a function of the time variable $t \in [s, s+T]$. In this section, our aim is to prove that the classical action is a solution of the Hamilton-Jacobi equation in (7) with the associated hamiltonian $\mathcal{H}$ (see proposition 6). Then, in proposition 7, we obtain a formulation of the action where the singular terms are explicit when $t \to s$. In addition, we prove that the action is continuous with respect to the trajectory $(w_t)_{t\in[s,s+T]}$. These results are used in the next section to prove that the propagator given by (10) is continuous with respect to the time variable t and the trajectory.

First, we have the following result concerning the differentiation of the classical action with respect to the classical orbits.

Lemma 3. *Let us assume that* $(w_t)_{t\in[s,s+T]} \in \mathcal{C}^1$. *Then, we have,* $\forall \boldsymbol{\chi}'(t) = (\bar{\boldsymbol{x}}'(t), \bar{\boldsymbol{\xi}}'(t)) \in \mathcal{C}^1([s,s+T], \mathbb{R}^{2d})$,

$$D_{\boldsymbol{\chi}} S(t,s,\boldsymbol{\chi}_w(\cdot,s,\boldsymbol{y},\boldsymbol{\eta}))(\boldsymbol{\chi}') = \bar{\boldsymbol{\xi}}_w(t,s,\boldsymbol{y},\boldsymbol{\eta}) \cdot \bar{\boldsymbol{x}}'(t) - \boldsymbol{\eta} \cdot \bar{\boldsymbol{x}}'(s).$$

Proof. The proof is straightforward by using an integration by part and the Hamilton equations (14). □

Combining the previous result and proposition 3, we can prove that, by considering the function $\boldsymbol{\eta} = \bar{\boldsymbol{\eta}}_w(t,s,\boldsymbol{x},\boldsymbol{y})$ as an initial data, the classical action becomes a solution of the Hamilton-Jacobi equation from (7). Furthermore, it is a generating function of the diffeomorphism $\Theta_{1,t,s,w}$.

Proposition 6. *Let us assume that* $(w_t)_{t\in[s,s+T]} \in \mathcal{C}^1$ *and let us consider the classical action as the integral of the lagrangian* $\mathcal{L}$*, associated to the hamiltonian* $\mathcal{H}$*, all along the unique path* $\bar{\boldsymbol{x}}$ *starting at* $\boldsymbol{y}$ *at time* s *and reaching* $\boldsymbol{x}$ *at time* t *given by the Hamilton equations (14), i.e.* $\forall t \in]s, s+T]$, $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$,

$$S_w(t,s,\boldsymbol{x},\boldsymbol{y}) := \int_s^t \partial_t \bar{\boldsymbol{x}}_w(\tau,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w) \cdot \bar{\boldsymbol{\xi}}_w(\tau,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w)d\tau - \int_s^t \mathcal{H}(\tau, \bar{\boldsymbol{x}}_w(\tau,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w), \bar{\boldsymbol{\xi}}_w(\tau,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w))d\tau. \quad (31)$$

Then, S_w *is a generating function of the diffeomorphism* $\Theta_{1,t,s,w}$*, i.e.* $\forall t \in]s, s+T]$, $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$,

$$\frac{\partial S_w}{\partial \boldsymbol{x}}(t,s,\boldsymbol{x},\boldsymbol{y}) = \bar{\boldsymbol{\xi}}_w(t,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w) \quad \textit{and} \quad \frac{\partial S_w}{\partial \boldsymbol{y}}(t,s,\boldsymbol{x},\boldsymbol{y}) = -\bar{\boldsymbol{\eta}}_w(t,s,\boldsymbol{x},\boldsymbol{y}). \quad (32)$$

Furthermore, S_w also satisfies the Hamilton-Jacobi equation: $\forall t\in]s,s+T]$, $\forall \boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^d$,

$$\frac{\partial S_w}{\partial t}(t,s,\boldsymbol{x},\boldsymbol{y})+\mathcal{H}\left(t,\boldsymbol{x},\frac{\partial S_w}{\partial \boldsymbol{x}}(t,s,\boldsymbol{x},\boldsymbol{y})\right)=0. \tag{33}$$

Proof. We differentiate the classical action S_w with respect to $\boldsymbol{x}$ (respectively $\boldsymbol{y}$) by using the lemma 3. With the help of equation (25) and the fact that the initial position is $\bar{\boldsymbol{x}}_w(s,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w(t,s,\boldsymbol{x},\boldsymbol{y}))=\boldsymbol{y}$, we obtain the first (respectively the second) equation of (32).

Let us now prove that the classical action is solution to the Hamilton-Jacobi equation. We have, $\forall t\in]s,s+T]$, $\forall \boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^d$,

$$\partial_t S_w(t,s,\boldsymbol{x},\boldsymbol{y})=\frac{\partial S_w}{\partial t}(t,s,\boldsymbol{\chi}_w(\cdot,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w))+D_{\boldsymbol{\chi}}S(t,s,\boldsymbol{\chi}_w(\cdot,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w))\left(\frac{\partial \boldsymbol{\chi}_w(\cdot,s,\boldsymbol{y},\boldsymbol{\eta})}{\partial \boldsymbol{\eta}}\partial_t\bar{\boldsymbol{\eta}}_w\right).$$

By using (25), we obtain, $\forall t\in]s,s+T]$, $\forall \boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^d$,

$$\begin{aligned}\partial_t S_w(t,s,\boldsymbol{x},\boldsymbol{y})&=\partial_t\bar{\boldsymbol{x}}_w(t,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w)\cdot\bar{\boldsymbol{\xi}}_w(t,s,\boldsymbol{y},\bar{\boldsymbol{\eta}})-\mathcal{H}(t,\bar{\boldsymbol{x}}_w(t,\boldsymbol{y},\bar{\boldsymbol{\eta}}),\bar{\boldsymbol{\xi}}_w(t,\boldsymbol{y},\bar{\boldsymbol{\eta}}))\\&\quad+\frac{\partial\bar{\boldsymbol{x}}_w}{\partial\boldsymbol{\eta}}(t,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w)\partial_t\bar{\boldsymbol{\eta}}_w(t,s,\boldsymbol{x},\boldsymbol{y})\cdot\bar{\boldsymbol{\xi}}_w(t,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w)\\&=-\mathcal{H}(t,\bar{\boldsymbol{x}}_w(t,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}),\bar{\boldsymbol{\xi}}_w(t,s,\boldsymbol{y},\bar{\boldsymbol{\eta}})).\end{aligned}$$

Finally, replacing $\bar{\boldsymbol{\xi}}_w(t,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w(t,s,\boldsymbol{x},\boldsymbol{y}))=\frac{\partial S_w}{\partial\boldsymbol{x}}(t,s,\boldsymbol{x},\boldsymbol{y})$ in the previous equation, we deduce (33). ☐

Before extending the classical action to the case where $(w_t)_{t\in[s,s+T]}\in\mathcal{C}^\gamma$, we state the following lemma regarding the dependence of the action with respect to $\mathbf{V}_{11}$, $\mathbf{V}_{12}$, $\mathbf{V}_{21}$ and $\mathbf{V}_{22}$. The proof follows directly from lemma 3.

Lemma 4. *Let us assume that $(w_t)_{t\in[s,s+T]}\in\mathcal{C}^1$. Then, we have, $\forall t\in]s,s+T]$, $\forall \boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^d$,*

$$\begin{aligned}&\frac{\partial S_w}{\partial\mathbf{V}_{12}}(t,s,\boldsymbol{x},\boldsymbol{y})=-\int_s^t\bar{\boldsymbol{\xi}}_w(\tau,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w)d\tau,\quad \frac{\partial S_w}{\partial\mathbf{V}_{22}}(t,s,\boldsymbol{x},\boldsymbol{y})=-\int_s^t\bar{\boldsymbol{\xi}}_w(\tau,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w)\dot{w}_\tau d\tau,\\&\frac{\partial S_w}{\partial\mathbf{V}_{11}}(t,s,\boldsymbol{x},\boldsymbol{y})=-\int_s^t\bar{\boldsymbol{x}}_w(\tau,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w)d\tau\quad\text{and}\quad\frac{\partial S_w}{\partial\mathbf{V}_{21}}(t,s,\boldsymbol{x},\boldsymbol{y})=-\int_s^t\bar{\boldsymbol{x}}_w(\tau,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w)\dot{w}_\tau d\tau.\end{aligned}\tag{34}$$

We can now show that the classical action can be extended to the case where the trajectories are γ-Hölder. The following result is proved by using the continuity of $\boldsymbol{\chi}_w$ and $\bar{\boldsymbol{\eta}}_w$ with respect to the trajectories $(w_t)_{t\in[s,s+T]}$. Moreover, we show a semi-explicit formulation of the action.

Proposition 7. *Let $W>0$. For all $(\alpha_1,\alpha_2)\in\mathbb{N}^d\times\mathbb{N}^d$, with $|\alpha_1+\alpha_2|\le 2$, and for all $(w_t)_{t\in[s,s+T]}$, $(\tilde{w}_t)_{t\in[s,s+T]}\in\mathcal{C}^\gamma\cap B_W(\mathcal{C}^0)$, we have the following estimate, $\forall t\in]s,s+T]$,*

$$\begin{aligned}&|\partial_{\boldsymbol{x}}^{\alpha_1}\partial_{\boldsymbol{y}}^{\alpha_2}S_w(t,s,\boldsymbol{x},\boldsymbol{y})-\partial_{\boldsymbol{x}}^{\alpha_1}\partial_{\boldsymbol{y}}^{\alpha_2}S_{\tilde{w}}(t,s,\boldsymbol{x},\boldsymbol{y})|\\&\qquad\le\frac{C_{T,s,W}}{t-s}\|w-\tilde{w}\|_{\dot{\mathcal{C}}^0([s,s+T])}(1+|\boldsymbol{x}|^{2-|\alpha_1+\alpha_2|}+|\boldsymbol{y}|^{2-|\alpha_1+\alpha_2|}).\end{aligned}\tag{35}$$

Moreover, for any $(w_t)_{t\in[s,s+T]}\in\mathcal{C}^\gamma$, the following formulation of the action S_w holds

$$\begin{aligned}S_w(t,s,\boldsymbol{x},\boldsymbol{y})=&\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{2(t-s)}-\frac{w_t-w_s}{2(t-s)}(\mathbf{V}_{22}+M_{21}^*\boldsymbol{y}+M_{21}\boldsymbol{x})\cdot(\boldsymbol{x}-\boldsymbol{y})-\frac{1}{2}(\mathbf{V}_{12}+M_{11}^*\boldsymbol{y}+M_{11}\boldsymbol{x})\cdot(\boldsymbol{x}-\boldsymbol{y})\\&+r_w(t,s,\boldsymbol{x},\boldsymbol{y}),\end{aligned}\tag{36}$$

where r_w is a quadratic function with respect to $\boldsymbol{x}$ and $\boldsymbol{y}$ and such that, $\forall(\alpha_1,\alpha_2)\in\mathbb{N}^d\times\mathbb{N}^d$ verifying $|\alpha_1+\alpha_2|\leq 2$,

$$\left|\partial_{\boldsymbol{x}}^{\alpha_1}\partial_{\boldsymbol{y}}^{\alpha_2} r_w(t,s,\boldsymbol{x},\boldsymbol{y})\right| \to_{t\to s} 0,$$

and, for all $(w_t)_{t\in[s,s+T]}$, $(\tilde{w}_t)_{t\in[s,s+T]}\in\mathcal{C}^\gamma\cap B_W(\mathcal{C}^0)$,

$$\begin{aligned}&\left|\partial_{\boldsymbol{x}}^{\alpha_1}\partial_{\boldsymbol{y}}^{\alpha_2}(r_w(t,s,\boldsymbol{x},\boldsymbol{y})-r_w(t,s,\boldsymbol{x},\boldsymbol{y}))\right|\\ &\qquad\leq C_{T,s,W}\|w-\tilde{w}\|_{\dot{\mathcal{C}}^0([s,s+T])}(1+|\boldsymbol{x}|^{2-|\alpha_1+\alpha_2|}+|\boldsymbol{y}|^{2-|\alpha_1+\alpha_2|}).\end{aligned}$$

Proof. Let us assume that the trajectory $(w_t)_{t\in[s,s+T]}$ is in $\mathcal{C}^1$. First, we remark that if $\boldsymbol{x}=\boldsymbol{y}=\mathbf{V}_{jk}=0$, $\forall j,k\in\{1,2\}$, then $S_w(t,s,0,0)=0$. Since $\bar{\boldsymbol{\xi}}$ and $\bar{\boldsymbol{\eta}}$ are linear with respect to $\boldsymbol{x}$ and $\boldsymbol{y}$, we obtain an expansion of the action with respect to these variables: $\forall t\in]s,s+T]$, $\forall\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^d$,

$$S_w(t,s,\boldsymbol{x},\boldsymbol{y})=S_w(t,s,0,0)+\int_0^1\partial_{\boldsymbol{y}}S_w(t,s,0,\iota\boldsymbol{y})\cdot\boldsymbol{y}d\iota+\int_0^1\partial_{\boldsymbol{x}}S_w(t,s,\iota\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{x}d\iota. \tag{37}$$

By using proposition 6, we have: for all $(w_t)_{t\in[s,s+T]}$ and $(\tilde{w}_t)_{t\in[s,s+T]}\in\mathcal{C}^1$,

$$\begin{aligned}|S_w(t,s,\boldsymbol{x},\boldsymbol{y})-S_{\tilde{w}}(t,s,\boldsymbol{x},\boldsymbol{y})|&\leq|S_w(t,s,0,0)-S_{\tilde{w}}(t,s,0,0)|+\int_0^1|\bar{\boldsymbol{\eta}}_w(t,s,0,\iota\boldsymbol{y})-\bar{\boldsymbol{\eta}}_{\tilde{w}}(t,s,0,\iota\boldsymbol{y})|d\iota|\boldsymbol{y}|\\ &\quad+\int_0^1|\bar{\boldsymbol{\xi}}_w(t,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w(t,s,\iota\boldsymbol{x},\boldsymbol{y}))-\bar{\boldsymbol{\xi}}_{\tilde{w}}(t,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w(t,s,\iota\boldsymbol{x},\boldsymbol{y}))||\boldsymbol{x}|d\iota\\ &\quad+\int_0^1|\bar{\boldsymbol{\xi}}_{\tilde{w}}(t,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w(t,s,\iota\boldsymbol{x},\boldsymbol{y})-\bar{\boldsymbol{\eta}}_{\tilde{w}}(t,s,\iota\boldsymbol{x},\boldsymbol{y}))||\boldsymbol{x}|d\iota.\end{aligned}$$

Thanks to proposition 1 and corollary 2, we deduce the estimate, $\forall t\in]s,s+T]$, $\forall\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^d$,

$$\begin{aligned}|S_w(t,s,\boldsymbol{x},\boldsymbol{y})-S_{\tilde{w}}(t,s,\boldsymbol{x},\boldsymbol{y})|&\leq|S_w(t,s,0,0)-S_{\tilde{w}}(t,s,0,0)|\\ &\quad+\frac{C_{T,s,W}}{t-s}\|w-\tilde{w}\|_{\dot{\mathcal{C}}^0([s,s+T])}(1+|\boldsymbol{x}|^2+|\boldsymbol{y}|^2).\end{aligned}$$

Concerning the first term appearing in the right hand side of the previous estimate, we use an expansion similar to (37) with respect to the variables $\mathbf{V}_{jk}$, $j,k\in\{1,2\}$, that is

$$S_w(t,s,0,0)=\sum_{j,k=1}^{2}\int_0^1\partial_{\mathbf{V}_{jk}}S_{jk,w}(t,s,\iota\mathbf{V}_{jk})\cdot\mathbf{V}_{jk}d\iota, \tag{38}$$

where $S_{jk,w}$ is the classical action with $\boldsymbol{x}=\boldsymbol{y}=\mathbf{V}_{\ell m}=0$ if $m+2(\ell-1)<k+2(j-1)$. Then lemma 4 enables us to deduce that

$$\begin{aligned}S_w(t,s,0,0)=&-\int_0^1\int_s^t\bar{\boldsymbol{x}}_{11,w}(\tau,s,\iota\mathbf{V}_{11})\cdot\mathbf{V}_{11}d\tau d\iota-\int_0^1\int_s^t\bar{\boldsymbol{\xi}}_{12,w}(\tau,s,\iota\mathbf{V}_{12})\cdot\mathbf{V}_{12}d\tau d\iota\\ &-\int_0^1\int_s^t\bar{\boldsymbol{x}}_{21,w}(\tau,s,\iota\mathbf{V}_{21})\cdot\mathbf{V}_{21}\dot{w}_\tau d\tau d\iota-\int_0^1\int_s^t\bar{\boldsymbol{\xi}}_{22,w}(\tau,s,\iota\mathbf{V}_{22})\cdot\mathbf{V}_{22}\dot{w}_\tau d\tau d\iota,\end{aligned}$$

where $\bar{\boldsymbol{x}}_{jk,w}$ (resp. $\bar{\boldsymbol{\xi}}_{jk,w}$) is the classical position (resp. momentum) with $\boldsymbol{x} = \boldsymbol{y} = \mathbf{V}_{\ell m} = 0$ if $m+2(\ell-1) < k+2(j-1)$. Furthermore, with the help of proposition 2, we have the following development of the integrals involving $\bar{\boldsymbol{x}}$

$$\begin{aligned}\int_0^1 \int_s^t \bar{\boldsymbol{x}}_{11,w}(\tau, s, \iota \mathbf{V}_{11}) d\tau d\iota = & \int_s^t \Xi_{0,w}(\tau, s) d\tau \begin{pmatrix} 0 \\ \bar{\boldsymbol{\eta}}_w(t,s,0,0) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ & + \int_s^t \Xi_{1,w}(\tau, s) d\tau \begin{pmatrix} \mathbf{V}_{12} \\ -\frac{1}{2}\mathbf{V}_{11} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ & + \int_s^t \Xi_{2,w}(\tau, s) d\tau \begin{pmatrix} \mathbf{V}_{22} \\ -\mathbf{V}_{21} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{aligned} \tag{39}$$

and

$$\begin{aligned}\int_0^1 \int_s^t \bar{\boldsymbol{x}}_{21,w}(\tau, s, \iota \mathbf{V}_{21}) \dot{w}_\tau d\tau d\iota = & \int_s^t \Xi_{0,w}(\tau, s) \dot{w}_\tau d\tau \begin{pmatrix} 0 \\ \bar{\boldsymbol{\eta}}_w(t,s,0,0) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ & + \int_s^t \Xi_{2,w}(\tau, s) \dot{w}_\tau d\tau \begin{pmatrix} \mathbf{V}_{22} \\ -\frac{1}{2}\mathbf{V}_{21} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ = & \frac{(w_t - w_s)^2}{2} \mathbf{V}_{22} + (t-s) \tilde{R}_{3,w}(t,s), \end{aligned} \tag{40}$$

where $\tilde{R}_{3,w}$ is an operator uniformly bounded with respect to $t \in]s, s+T]$ and continuous with respect to w (as in proposition 2). We can proceed in the same way for of the integrals involving $\bar{\boldsymbol{\xi}}$ and obtain

$$\begin{aligned}\int_0^1 \int_s^t \bar{\boldsymbol{\xi}}_{12,w}(\tau, s, \iota \mathbf{V}_{12}) d\tau d\iota = & \int_s^t \Xi_{0,w}(\tau, s) d\tau \begin{pmatrix} 0 \\ \bar{\boldsymbol{\eta}}_w(t,s,0,0) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ & + \int_s^t \Xi_{1,w}(\tau, s) d\tau \begin{pmatrix} \frac{1}{2}\mathbf{V}_{12} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ & + \int_s^t \Xi_{2,w}(\tau, s) d\tau \begin{pmatrix} \mathbf{V}_{22} \\ -\mathbf{V}_{21} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned} \tag{41}$$

and

$$\begin{aligned}\int_0^1 \int_s^t \bar{\boldsymbol{\xi}}_{22,w}(\tau, s, \iota \mathbf{V}_{22}) \dot{w}_\tau d\tau d\iota = & \int_s^t \Xi_{0,w}(\tau, s) \dot{w}_\tau d\tau \begin{pmatrix} 0 \\ \bar{\boldsymbol{\eta}}_w(t,s,0,0) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ & + \int_s^t \Xi_{2,w}(\tau, s) \dot{w}_\tau d\tau \begin{pmatrix} \frac{1}{2}\mathbf{V}_{22} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ = & (w_t - w_s) \bar{\boldsymbol{\eta}}_w(t,s,0,0) + (t-s) \tilde{R}_{4,w}(t,s), \end{aligned} \tag{42}$$

where $\tilde{R}_{4,w}$ is an operator uniformly bounded with respect to $t \in]s, s+T]$ and continuous with respect to w (as in proposition 2). Thus, with corollary 2, this yield the estimate

$$|S_w(t,s,0,0) - S_{\tilde{w}}(t,s,0,0)| \leq \frac{C_{T,s,W}}{t-s} \|w - \tilde{w}\|_{\dot{\mathcal{C}}^0([s,s+T])}.$$

Therefore, we obtain the estimate (35) when $\alpha_1 = \alpha_2 = 0$. By considering a sequence of trajectories $((w_t^n)_{t\in[s,s+T]})_{n\in\mathbb{N}}$ converging towards $(w_t)_{t\in[s,s+T]}$, we extend the classical action to the case of a γ-Hölder trajectory. The estimate (35) for $|\alpha_1+\alpha_2|\leq 2$ is deduced from proposition 1, corollary 2, proposition 5 and finally proposition 6.

Applying (37), we can now develop the classical action. We restrict our expansion to the singular or time-independent terms and include the remaining terms in a function r_w. The singularities arise when $t\to s$ and are related to the function $\bar{\boldsymbol{\eta}}$. It follows from proposition 4 that $\bar{\boldsymbol{\eta}}_w(t,s,0,0)$ is of order $O((t-s)^{-\gamma})$ when $\mathbf{V}_{22}\neq 0$ and order $O(1)$ when $\mathbf{V}_{22}=0$. By using the relation (38) and its development given by the integrals (39), (40), (41) and (42), we first remark that $S_w(t,s,0,0)$ is not singular since, thanks to proposition 2, each integrals of the form $\int_s^t \Xi_{j,w}(\tau,s)d\tau$, for $j\in\{0,1,2\}$, is of order $O(t-s)$. Hence, $S_w(t,s,0,0)$ can be immediately included in r_w. For the two remaining terms we have, on one hand, thanks to corollary 1 and proposition 4,

$$\begin{aligned}\int_0^1 \bar{\boldsymbol{\xi}}_w(t,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w(t,s,\iota\boldsymbol{x},\boldsymbol{y}))\cdot\boldsymbol{x}d\iota &= \int_0^1 \bar{\boldsymbol{\eta}}_w(t,s,\iota\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{x}-(w_t-w_s)M_{21}\bar{\boldsymbol{\eta}}_w(t,s,\iota\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{x}d\iota\\ &\quad -\int_0^1 (t-s)M_{11}^*\bar{\boldsymbol{\eta}}_w(t,s,\iota\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{x}d\iota + r_w(t,s,\boldsymbol{x},\boldsymbol{y})\\ &= \frac{|\boldsymbol{x}|^2-2\boldsymbol{x}\cdot\boldsymbol{y}}{2(t-s)}-\frac{w_t-w_s}{2(t-s)}(M_{21}\boldsymbol{x}+\mathbf{V}_{22})\cdot\boldsymbol{x}-\frac{1}{2}(M_{11}\boldsymbol{x}+\mathbf{V}_{12})\cdot\boldsymbol{x}\\ &\quad +r_w(t,s,\boldsymbol{x},\boldsymbol{y}),\end{aligned}$$

and, on the other hand, by using proposition 4,

$$-\int_0^1 \bar{\boldsymbol{\eta}}_w(t,s,0,\iota\boldsymbol{y})\cdot\boldsymbol{y}d\iota = \frac{|\boldsymbol{y}|^2}{2(t-s)}+\frac{w_t-w_s}{2(t-s)}(M_{21}\boldsymbol{y}+\mathbf{V}_{22})\cdot\boldsymbol{y}+\frac{1}{2}(M_{11}\boldsymbol{y}+\mathbf{V}_{12})\cdot\boldsymbol{y}+r_w(t,s,\boldsymbol{x},\boldsymbol{y}).$$

This therefore ends the proof. □

4 Construction of the propagator

Let us now consider the operator U_w defined by: $\forall\psi_0\in\mathcal{C}_0^\infty(\mathbb{R}^d)$, $\forall t\in]s,s+T]$,

$$U_w(t,s)\psi_0(\boldsymbol{x}) := \frac{a_w(t,s)}{(2\pi i(t-s))^{d/2}}\int_{\mathbb{R}^d} e^{iS_w(t,s,\boldsymbol{x},\boldsymbol{y})}\psi_0(\boldsymbol{y})d\boldsymbol{y}. \tag{43}$$

We first prove an explicit formula for the amplitude function a_w by solving the equation

$$\begin{cases} \partial_t a_w(t,s) = \dfrac{1}{2}\left(-\Delta_{\boldsymbol{x}} S_w(t,s)+\dfrac{d}{t-s}\right)a(t,s),\ \forall t\in]s,s+T],\\ a(s,s)=1.\end{cases} \tag{44}$$

In addition, we prove that the family of operators $(U_w(t,s))_{t\in[s,s+T]}$ is an isometric propagator which is strongly continuous in $L^2(\mathbb{R}^d)$ and that the operator $U(t,s)$ is continuous with respect to the trajectory $(w_t)_{t\in[s,s+T]}$. Finally, the propagator $(U_w(t,s))_{t\in[s,s+T]}$ is also proven to be linear in the functional spaces Σ^n, $n\in\mathbb{N}$.

We first solve equation (44). The following lemma gives a partial formulation of the amplitude function a_w.

Lemma 5. *We have, for all $t \in [s, s+T]$,*

$$a_w(t,s) = \exp\left(\frac{1}{2}\int_s^t \left[\frac{w_\tau - w_s}{\tau - s}\, Tr(M_{21}) + Tr(M_{11}) - \Delta_{\boldsymbol{x}} r_w(\tau, s)\right] d\tau\right).$$

Furthermore, $a_w(\cdot, s)$ is a continuous function.

Proof. Using (36), we obtain, $\forall t \in]s, s+T]$,

$$-\Delta_{\boldsymbol{x}} S_w(t,s) + \frac{d}{t-s} = \frac{w_t - w_s}{t-s}\mathrm{Tr}(M_{21}) + \mathrm{Tr}(M_{11}) - \Delta_{\boldsymbol{x}} r_w(t,s).$$

Therefore, integrating equation (44) with respect to the time variable, we deduce the expected result. □

Let us now prove that the family of operators $(U_w(t,s))_{t\in]s,s+T]}$ is continuous from L^2 to itself. To this end, we use the following theorem [4, 15].

Theorem 4. *Consider the following oscillatory integral*

$$\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d),\ \forall \upsilon > 0, \quad \mathcal{I}_\upsilon \psi_0(\boldsymbol{x}) = \int_{\mathbb{R}^d} e^{\upsilon i f(\boldsymbol{x},\boldsymbol{y})}\varphi(\boldsymbol{y}) d\boldsymbol{y},$$

where f is a real-valued smooth function in $\mathbb{R}^d \times \mathbb{R}^d$. We also suppose that there exist two constants $C_1, C_2 > 0$ such that, $\forall(\alpha_1, \alpha_2) \in \mathbb{N}^d \times \mathbb{N}^d$ verifying $|\alpha_1| + |\alpha_2| \geq 2$,

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d, \quad \left|\det\left(\frac{\partial^2 f(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{x}_j \partial \boldsymbol{y}_k}\right)\right| \geq C_1 \quad \textit{and} \quad \left|\partial_{\boldsymbol{x}}^{\alpha_1}\partial_{\boldsymbol{y}}^{\alpha_2} f(\boldsymbol{x},\boldsymbol{y})\right| \leq C_2. \tag{45}$$

Then, there exists a constant $C(C_1, C_2) > 0$, which is independent of f, such that

$$\forall \upsilon > 0, \quad \|\mathcal{I}_\upsilon \varphi(\boldsymbol{x})\|_{L^2_{\boldsymbol{x}}} \leq C\upsilon^{-d/2}\|\varphi\|_{L^2_{\boldsymbol{x}}},$$

To apply theorem 4 to the integral (43), we set

$$f(\boldsymbol{x},\boldsymbol{y}) = (t-s)S_w(t,s,\boldsymbol{x},\boldsymbol{y})$$

and

$$\upsilon = \frac{1}{t-s}.$$

By using proposition 7, one gets

$$(t-s)\frac{\partial^2 S_w}{\partial \boldsymbol{x}\partial \boldsymbol{y}}(t,s) = (t-s)\frac{\partial \bar{\boldsymbol{\eta}}}{\partial \boldsymbol{x}} = -1 - (t-s)E_w(t,s).$$

This proves that, for $T > 0$ small enough, there exists a constant $C_1 > 0$ such that, for all $t \in]s, s+T]$,

$$\left|\det\left((t-s)\frac{\partial^2 S_w}{\partial \boldsymbol{x}_j \partial \boldsymbol{y}_k}(t,s)\right)\right| \geq C_1.$$

Hence, the first assumption of (45) is fulfilled. Concerning the second hypothesis, we simply remark that: $\forall(\alpha_1, \alpha_2) \in \mathbb{N}^d \times \mathbb{N}^d$ such that $|\alpha_1| + |\alpha_2| \geq 2$,

$$(t-s)\partial_{\boldsymbol{x}}^{\alpha_1}\partial_{\boldsymbol{y}}^{\alpha_2} S_w(t,s) = 0.$$

By applying theorem 4 to (43), we obtain that the propagator $(U(t,s))_{t\in[s,s+T]}$ is a bounded linear operator from L^2 to itself and that $U_w(\cdot, s) \in L^\infty([s, s+T], L^2)$.

Let us now show the following useful result which follows a similar proof as in [12, 16, 32].

Lemma 6. *Consider the operator L_w defined by:* $\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$

$$L_w\varphi(t,s,\boldsymbol{x},\boldsymbol{y}) = \frac{\partial_{\boldsymbol{y}} S_w(t,s,\boldsymbol{x},\boldsymbol{y})}{(t-s)|\partial_{\boldsymbol{y}} S_w(t,s,\boldsymbol{x},\boldsymbol{y})|^2}.\partial_{\boldsymbol{y}}\varphi(\boldsymbol{y}).$$

Its adjoint operator in L^2 is then given by

$$L_w^*\varphi(t,s,\boldsymbol{x},\boldsymbol{y}) = -\partial_{\boldsymbol{y}}.\left(\frac{\partial_{\boldsymbol{y}} S_w(t,s,\boldsymbol{x},\boldsymbol{y})}{(t-s)|\partial_{\boldsymbol{y}} S_w(t,s,\boldsymbol{x},\boldsymbol{y})|^2}\varphi(\boldsymbol{y})\right).$$

Moreover, we remark that

$$L_w e^{iS_w(t,s,\boldsymbol{x},\boldsymbol{y})} = \frac{i}{(t-s)} e^{iS_w(t,s,\boldsymbol{x},\boldsymbol{y})}. \tag{46}$$

Let $W > 0$. Then, for all $\alpha \in \mathbb{N}^d$, $(w_t)_{t\in[s,s+T]}$, $(\tilde{w}_t)_{t\in[s,s+T]} \in \mathcal{C}^\gamma \cap B_W(\mathcal{C}^0))$ and for all $\tilde{R} > 0$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ such that $|\boldsymbol{y}| \le \tilde{R}$ and $|\boldsymbol{x}| \ge R$, with $R > 0$ large enough, we have the estimates

$$\left|\partial_{\boldsymbol{y}}^\alpha \left(L_w^* - L_{\tilde{w}}^*\right)\varphi(t,s,\boldsymbol{x},\boldsymbol{y})\right| \le C_{T,s,W,R,\tilde{R}}|\boldsymbol{x}|^{-1}\left(|\partial_{\boldsymbol{y}}^\alpha\varphi(\boldsymbol{y})| + |\partial_{\boldsymbol{y}}\varphi(\boldsymbol{y})|\right)\|w-\tilde{w}\|_{\dot{\mathcal{C}}^0([s,s+T])}, \tag{47}$$

and, $\forall m \in \mathbb{N}$,

$$|\partial_{\boldsymbol{y}}^\alpha (L_w^*)^m\varphi(t,s,\boldsymbol{x},\boldsymbol{y})| \le C_{T,s,W,R,\tilde{R}}|\boldsymbol{x}|^{-m}\textstyle\sum_{|\beta|\le m+|\alpha|}|\partial_{\boldsymbol{y}}^\beta\varphi(\boldsymbol{y})|. \tag{48}$$

Proof. The formulation of the adjoint operator L_w^* and equation (46) are directly obtained.

Let us now consider (48). We have: $\forall t \in [s,s+T]$, $\forall \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^d$,

$$(t-s)\partial_{\boldsymbol{y}_j} S_w(t,s,\boldsymbol{x},\boldsymbol{y}) = (t-s)\partial_{\boldsymbol{y}_j} S_w(t,s,0,\boldsymbol{y}) + \sum_{k=1}^d (t-s)\partial_{\boldsymbol{x}_k}\partial_{\boldsymbol{y}_j} S_w(t,s)\boldsymbol{x}_k.$$

By using proposition 7, we prove that,

$$\min_{t\in[s,s+T]} \|(t-s)\partial_{\boldsymbol{x}_k}\partial_{\boldsymbol{y}_j} S(t,s)\|_{\mathcal{L}(\mathbb{R}^d,\mathbb{R}^d)} = 1,$$

and, $\forall \boldsymbol{y} \in \mathbb{R}^d$,

$$\sup_{t\in[s,s+T]} \|(t-s)\partial_{\boldsymbol{y}_j} S_w(t,s,0,\boldsymbol{y})\| \le C_{T,s,W}(1+|\boldsymbol{y}|).$$

This leads to the following inequality

$$|(t-s)\partial_{\boldsymbol{y}} S_w(t,s,\boldsymbol{x},\boldsymbol{y})| \ge |\boldsymbol{x}| - C_{T,s,W}(1+|\boldsymbol{y}|).$$

For all $\tilde{R} > 0$, by taking $|\boldsymbol{y}| \le \tilde{R}$ and $|\boldsymbol{x}| \ge R := 2C_{T,s,W}(1+\tilde{R})$, we obtain

$$|(t-s)\partial_{\boldsymbol{y}} S_w(t,s,\boldsymbol{x},\boldsymbol{y})| \ge \frac{|\boldsymbol{x}|}{2}. \tag{49}$$

Moreover, we have, $\forall \alpha \in \mathbb{N}^d$,

$$\partial_{\boldsymbol{y}}^\alpha\left(\frac{\partial_{\boldsymbol{y}} S_w}{(t-s)|\partial_{\boldsymbol{y}} S_w|^2}\right) = \frac{1}{(t-s)}\sum_{\beta\le\alpha} C_{\alpha,\beta}\partial_{\boldsymbol{y}}^{\alpha-\beta}\left(\partial_{\boldsymbol{y}} S_w\right)\partial_{\boldsymbol{y}}^\beta\left(|\partial_{\boldsymbol{y}} S_w|^{-2}\right),$$

and, for all $\beta \in \mathbb{N}^d$ such that $\beta \leq \alpha$,

$$\partial_{\boldsymbol{y}}^{\beta} \left(|\partial_{\boldsymbol{y}} S_w|^{-2} \right) = \sum_{\gamma \leq \beta, \ |\gamma| \geq 1} (-1)^{|\gamma|} |\gamma|! |\partial_{\boldsymbol{y}} S_w|^{-2|\gamma|} \prod_{|\gamma_1| + \ldots + |\gamma_m| = |\gamma|} \partial_{\boldsymbol{y}}^{\gamma_j} \left(|\partial_{\boldsymbol{y}} S_w|^2 \right).$$

By using (49), we deduce that

$$\left| \partial_{\boldsymbol{y}}^{\alpha} \left(\frac{\partial_{\boldsymbol{y}} S_w(t, s, \boldsymbol{x}, \boldsymbol{y})}{(t-s)|\partial_{\boldsymbol{y}} S_w(t, s, \boldsymbol{x}, \boldsymbol{y})|^2} \right) \right| \leq C_{T,s,W,R,\tilde{R},\alpha,} |\boldsymbol{x}|^{-1}. \tag{50}$$

Let us now set

$$\Upsilon = \frac{\partial_{\boldsymbol{y}} S_w}{(t-s)|\partial_{\boldsymbol{y}} S_w|^2}.$$

From [32], we have the following result: $\forall m \in \mathbb{N}$, the following equality holds

$$(L_w^*)^m = \sum_{\alpha_0, \ldots, \alpha_p} C_{\alpha_0, \ldots, \alpha_p, \beta} |\partial_{\boldsymbol{y}} S_w|^{-2m} (\partial_{\boldsymbol{y}} S_w)^{\alpha_0} \left(\partial_{\boldsymbol{y}}^{\alpha_1} \Upsilon \right) \ldots \left(\partial_{\boldsymbol{y}}^{\alpha_p} \Upsilon \right) \partial_{\boldsymbol{y}}^{\beta},$$

where the summation on the multi-index $(\alpha_0, \ldots, \alpha_p)$ is such that

$$\begin{cases} |\alpha_0| + p - 2m = -m, \\ |\alpha_1| \geq 2, \ldots, |\alpha_p| \geq 2, \\ |\alpha_1| + \ldots + |\alpha_p| - p + |\beta| = m. \end{cases}$$

This gives the inequality (48) by using (50).

We now prove the inequality (47). For all $(w_t)_{t \in [s, s+T]}, (\tilde{w}_t)_{t \in [s, s+T]} \in \mathcal{C}^{\gamma} \cap B_W(\mathcal{C}^0)$, we have: $\forall \varphi \in \mathcal{C}_0^{\infty}(\mathbb{R}^d)$,

$$\begin{aligned} (L_w^* - L_{\tilde{w}}^*) \varphi = & -\partial_{\boldsymbol{y}} . \left(\frac{\partial_{\boldsymbol{y}} S_w - \partial_{\boldsymbol{y}} S_{\tilde{w}}}{(t-s)|\partial_{\boldsymbol{y}} S_w|^2} \varphi(\boldsymbol{y}) \right) \\ & + \partial_{\boldsymbol{y}} . \left(\frac{\partial_{\boldsymbol{y}} S_{\tilde{w}}}{(t-s)|\partial_{\boldsymbol{y}} S_w|^2 |\partial_{\boldsymbol{y}} S_{\tilde{w}}|^2} \left(|\partial_{\boldsymbol{y}} S_w|^2 - |\partial_{\boldsymbol{y}} S_{\tilde{w}}|^2 \right) \varphi(\boldsymbol{y}) \right). \end{aligned} \tag{51}$$

Moreover, we also have the following expression: $\forall \alpha \in \mathbb{N}^d$,

$$\partial_{\boldsymbol{y}}^{\alpha} \left(\frac{\partial_{\boldsymbol{y}} S_w - \partial_{\boldsymbol{y}} S_{\tilde{w}}}{(t-s)|\partial_{\boldsymbol{y}} S_w(t, s, \boldsymbol{x}, \boldsymbol{y})|^2} \right) = \frac{1}{(t-s)} \sum_{\beta \leq \alpha} C_{\alpha,\beta} \partial_{\boldsymbol{y}}^{\alpha - \beta} \left(\partial_{\boldsymbol{y}} S_w - \partial_{\boldsymbol{y}} S_{\tilde{w}} \right) \partial_{\boldsymbol{y}}^{\beta} \left(|\partial_{\boldsymbol{y}} S_w|^{-2} \right).$$

The following inequality is a consequence of proposition 7

$$\left| \partial_{\boldsymbol{y}}^{\alpha} \left(\frac{\partial_{\boldsymbol{y}} S_w - \partial_{\boldsymbol{y}} S_{\tilde{w}}}{(t-s)|\partial_{\boldsymbol{y}} S_w|^2} \right) \right| \leq C_{T,s,W,R,\tilde{R},\alpha} |\boldsymbol{x}|^{-1} \|w - \tilde{w}\|_{\dot{\mathcal{C}}^0([s,s+T])}.$$

The inequality (47) is obtained by using the previous estimate, (51) and proposition 7. □

Following the arguments from [16], we now prove the strong continuity of the family of operators $(U_w(t,s))_{t \in [s, s+T]}$ at $t = s$.

Proposition 8. *Let $\psi_s \in L^2$. Then, we have*

$$\lim_{t\to s} \|U_w(t,s)\psi_s(\boldsymbol{x}) - \psi_s(\boldsymbol{x})\|_{L^2} = 0.$$

Proof. Let $\psi_s \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ and $\vartheta_R \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ a cut-off function, *i.e.*

$$\vartheta_R(\boldsymbol{x}) = \begin{cases} 1, & |\boldsymbol{x}| \leq R-1 \\ 0, & |\boldsymbol{x}| \geq R \end{cases}.$$

We set, $\forall t \in]s, s+T]$ and $\forall \boldsymbol{x} \in \mathbb{R}^d$, for $R > 0$ large enough,

$$\begin{aligned} U_w(t,s)\psi_s(\boldsymbol{x}) &= \frac{a_w(t,s)}{(2\pi i(t-s))^{d/2}} \int_{\mathbb{R}^d} e^{iS_w(t,s,\boldsymbol{x},\boldsymbol{y})}(1-\vartheta_R(\boldsymbol{x}))\psi_s(\boldsymbol{y})d\boldsymbol{y} \\ &\quad + \frac{a_w(t,s)}{(2\pi i(t-s))^{d/2}} \int_{\mathbb{R}^d} e^{iS_w(t,s,\boldsymbol{x},\boldsymbol{y})}\vartheta_R(\boldsymbol{x})\psi_s(\boldsymbol{y})d\boldsymbol{y} \\ &= I_1(t,s) + I_2(t,s). \end{aligned}$$

Thanks to lemma 6 and taking $\tilde{R} > 0$ such that $\text{supp}(\psi_s) \subset B_{\tilde{R}}(\mathbb{R}^d)$, we have, $\forall j \in \mathbb{N}$,

$$\begin{aligned} |I_1(t,s)| &= \left| \frac{a_w(t,s)}{(2\pi i(t-s))^{d/2}} \frac{(t-s)^j}{i^j} \int_{\mathbb{R}^d} (L_w^*)^j \psi_s(\boldsymbol{y}) e^{iS_w(t,s,\boldsymbol{x},\boldsymbol{y})}(1-\vartheta_R(\boldsymbol{x}))d\boldsymbol{y} \right| \\ &\leq C_{T,s,w,R,\tilde{R}}(t-s)^{j-d/2}(1+|\boldsymbol{x}|)^{-j} \sum_{|\alpha|\leq j} \|\partial_{\boldsymbol{y}}^\alpha \psi_s\|_{L^1}. \end{aligned}$$

Consequently, this leads to the estimate: for $j = \lceil d/2 \rceil + 1$

$$\|I_1(t,s)\|_{L^2} \leq C_{T,s,w,R,\tilde{R}}(t-s)^{1+\lceil d/2\rceil - d/2} \|(1+|\boldsymbol{x}|)^{-\lceil d/2\rceil -1}\|_{L^2} \sum_{|\alpha|\leq \lceil d/2\rceil+1} \|\partial_{\boldsymbol{y}}^\alpha \psi_s\|_{L^1},$$

which yields $\|I_1(t,s)\|_{L^2} \underset{t\to s}{\to} 0$.

Concerning the integral $I_2(t,s)$, we use the stationary phase method [14]. We solve the following equation with respect to the variable $\boldsymbol{y}$

$$\partial_{\boldsymbol{y}} S_w(t,s,\boldsymbol{x},\boldsymbol{y}) = 0,$$

which is equivalent to

$$\eta_w(t,s,\boldsymbol{x},\boldsymbol{y}) = 0. \tag{52}$$

By using the diffeomorphism $\Theta_{2,t,s,w}$ introduced in proposition 3, we can make a change of variables and obtain the solution $\boldsymbol{y} = \bar{\boldsymbol{y}}_w(t,s,\boldsymbol{x},0)$ to the equation (52). The stationary phase method gives us the following expansion

$$U_w(t,s)\psi_s(\boldsymbol{x}) = \left|\det((t-s)\partial_{\boldsymbol{y}}^2 S(t,s,\boldsymbol{x},\bar{\boldsymbol{y}}_w))\right|^{-1/2} e^{iS_w(t,s,\boldsymbol{x},\bar{\boldsymbol{y}}_w)} \left(a_w(t,s)\psi_s(\bar{\boldsymbol{y}}_w) + (t-s) q\left(t,s,\frac{\boldsymbol{x}}{t-s}\right)\right),$$

where, $\forall k \in \mathbb{N}$, there exist $K_k \in \mathbb{N}$ and $C_k > 0$ such that for all $\alpha \in \mathbb{N}^d$, $|\alpha| < k$, and $t \in]s, s+T]$,

$$\left| \partial_{\boldsymbol{x}}^\alpha q\left(t,s,\frac{\boldsymbol{x}}{t-s}\right)\right| \leq C_k a_w(t,s) \max_{|\beta|\leq K_k} \sup_{y\in\mathbb{R}^d} |\partial_{\boldsymbol{y}}^\beta \psi_0(\boldsymbol{y})|.$$

We remark that, by continuity, we have $\lim_{t\to s} a_w(t,s) = 1$ and, thanks to the partial formulation of the action from proposition 7, $\forall k, j \in \{1, ..., d\}$, we obtain

$$\lim_{t\to s}(t-s)\partial_{\boldsymbol{y}_j}\partial_{\boldsymbol{y}_k} S_w(t,s,\boldsymbol{x},\bar{\boldsymbol{y}}_w) = 1.$$

We now prove that the function $\bar{\boldsymbol{y}}_w(t,s,\boldsymbol{x},0)$ converges to $\boldsymbol{x}$. This allows us to deduce the limit

$$\lim_{t\to s}\psi_0(\bar{\boldsymbol{y}}_w(t,s,\boldsymbol{x},0)) = \psi_0(\boldsymbol{x}),$$

and also to show that $S(t,s,\boldsymbol{x},\bar{\boldsymbol{y}}_w)$ converges towards a constant when $t\to s$. From corollary 1, we have, $\forall t \in [s, s+T]$, $\forall \boldsymbol{x} \in \mathbb{R}^d$,

$$\boldsymbol{x} = \bar{\boldsymbol{y}}_w(t,s,\boldsymbol{x},0) + (M_{11}(t-s) + M_{21}(w_t - w_s))\,\bar{\boldsymbol{y}}_w(t,s,\boldsymbol{x},0) + (t-s)^{\varrho} r_1(t,s)\bar{\boldsymbol{y}}_w(t,s,\boldsymbol{x},0), \tag{53}$$

which yields the estimate, for $T > 0$ small enough, $\forall t \in [s, s+T]$,

$$|\bar{\boldsymbol{y}}_w(t,s,\boldsymbol{x},0)| \le C_{T,s,w}|\boldsymbol{x}|.$$

By assumption 1 and by using (53), we deduce the following inequality, for $0 < \varepsilon \le 1/2$,

$$|\bar{\boldsymbol{y}}_w(t,s,\boldsymbol{x},0) - \boldsymbol{x}| \le C_{T,s,w}(t-s)^{1/2+\varepsilon}(1+|\boldsymbol{x}|).$$

Concerning the expansion of the action given in proposition 7, it follows that we have

$$|S_w(t,s,\boldsymbol{x},\bar{\boldsymbol{y}}_w(t,s,\boldsymbol{x},0))| \le C_{T,s,w}(t-s)^{2\varepsilon}(1+|\boldsymbol{x}|^2).$$

This yields the limit $\lim\limits_{t\to s} S_w(t,s,\boldsymbol{x},\bar{\boldsymbol{y}}(t,s,\boldsymbol{x},0)) = 0$, which is locally uniform with respect to $\boldsymbol{x}$, ending hence the proof. □

We obtain the conservation of the L^2-norm and the uniqueness of the solution to the problem (3) in the L^2 space thanks to a classical regularization argument. The result is stated in the following proposition.

Proposition 9. *Let $\psi_0 \in L^2$ and $(w_t)_{t\in[s,s+T]} \in \mathcal{C}^1$. Then, for any solution $\psi \in L^\infty([0,T], L^2)$ of the problem (3), we have, $\forall t \in [s, s+T]$,*

$$\|\psi(t,\boldsymbol{x})\|_{L^2_{\boldsymbol{x}}} = \|\psi_0\|_{L^2_{\boldsymbol{x}}}.$$

Following the argument from [32], the uniqueness of the solution to (3) leads to the property that a solution $\psi_1(t,\boldsymbol{x}) = U_w(t,s)\psi_0(\boldsymbol{x})$ and a solution $\psi_2(t,\boldsymbol{x}) = U_w(t-r,s)U_w(r,s)\psi_0(\boldsymbol{x})$, which are such that $\psi_1(r,\boldsymbol{x}) = \psi_2(r,\boldsymbol{x})$ for all $r \in [s, s+T]$, are equal for all $t \ge r+s$. Therefore, we have, $\forall r \in [s, s+T[$, $\forall t \in [s+r, s+T]$,

$$U_w(t-r,s)U_w(r,s) = U_w(t,s),$$

which allows us to conclude that, for $(w_t)_{t\in[s,s+T]} \in \mathcal{C}^1$, $(U_w(t,s))_{t\in[s,s+T]}$ is a strongly continuous isometric propagator of L^2.

We now show the continuity of the operator U_w with respect to the trajectory $(w_t)_{t\in[s,s+T]}$. This result allows us to state that the propagator $(U_w(t,s))_{t\in[s,s+T]}$ can be extended to the case of a γ-Hölder trajectory and that, by definition 2, $U_w(t,s)\psi_s(\boldsymbol{x})$ is a solution to the problem (3).

Proposition 10. *Let $W>0$. For all $(w_t)_{t\in[s,s+T]}$ and $(\tilde{w}_t)_{t\in[s,s+T]} \in \mathcal{C}^\gamma \cap B_W(\mathcal{C}^0)$, we have: $\forall t \in]s,s+T]$, $\forall \psi_s \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,*

$$\sup_{t\in[s,s+T]} \|U_w(t,s)\psi_s(\boldsymbol{x}) - U_{\tilde{w}}(t,s)\psi_s(\boldsymbol{x})\|_{L^2} \leq C_{T,s,W}\left(\|\psi_s\|_{W^{1,j}} + \|\psi_s\|_{L^2}\right)\|w-\tilde{w}\|_{\mathcal{C}^\gamma([s,s+T])}, \tag{54}$$

with $j = \lceil d/2 \rceil + 3$.

Proof. We have, $\forall t \in]s,s+T]$, $\forall \boldsymbol{x} \in \mathbb{R}^d$,

$$\begin{aligned} U_w(t,s)\psi_s(\boldsymbol{x}) - U_{\tilde{w}}(t,s)\psi_s(\boldsymbol{x}) &= \frac{a_w(t,s)-a_{\tilde{w}}(t,s)}{(2\pi i(t-s))^{d/2}} \int_{\mathbb{R}^d} \psi_s(\boldsymbol{y}) e^{iS_w(t,s,\boldsymbol{x},\boldsymbol{y})} d\boldsymbol{y} \\ &\quad + \frac{a_{\tilde{w}}(t,s)}{(2\pi i(t-s))^{d/2}} \int_{\mathbb{R}^d} \psi_s(\boldsymbol{y}) \left(e^{iS_w(t,s,\boldsymbol{x},\boldsymbol{y})} - e^{iS_{\tilde{w}}(t,s,\boldsymbol{x},\boldsymbol{y})}\right) d\boldsymbol{y} \\ &= I_1(t,s) + I_2(t,s). \end{aligned}$$

For the integral $I_1(t,s)$, it follows from theorem 4 that, $\forall t \in [s,s+T]$,

$$\left\| \frac{a_w(t,s)-a_{\tilde{w}}(t,s)}{(2\pi i(t-s))^{d/2}} \int_{\mathbb{R}^d} \psi_s(\boldsymbol{y}) e^{iS_w(t,s,\boldsymbol{x},\boldsymbol{y})} d\boldsymbol{y} \right\|_{L^2} \leq C_{T,s,W} \left|a_w(t,s) - a_{\tilde{w}}(t,s)\right| \|\psi_0\|_{L^2}.$$

Let us now give an estimate of the right hand side of the previous inequality. We remark that

$$a_w(t,s) - a_{\tilde{w}}(t,s) = e^{\frac{1}{2}\int_s^t \left(-\Delta_{\boldsymbol{x}} S_{\tilde{w}}(\tau,s) + \frac{d}{(\tau-s)}\right) d\tau} \left(e^{\frac{1}{2}\int_s^t (-\Delta_{\boldsymbol{x}} S_w(\tau,s) + \Delta_{\boldsymbol{x}} S_{\tilde{w}}(\tau,s)) d\tau} - 1\right). \tag{55}$$

The continuity of the amplitude function gives the following bound

$$\left| e^{\frac{1}{2}\int_s^t \left(-\Delta_{\boldsymbol{x}} S_w(\tau,s) + \frac{d}{(\tau-s)}\right) d\tau} \right| \leq C_{T,s,W}. \tag{56}$$

Thanks to proposition 7, we also have

$$-\Delta_{\boldsymbol{x}} S_w(\tau,s) + \Delta_{\boldsymbol{x}} S_{\tilde{w}}(\tau,s) = \frac{w_\tau - \tilde{w}_\tau - w_s + \tilde{w}_s}{\tau - s} \mathrm{Tr}(M_{21}) - \Delta_{\boldsymbol{x}} r_w(\tau,s) + \Delta_{\boldsymbol{x}} r_{\tilde{w}}(\tau,s). \tag{57}$$

Recalling that $(w_t)_{t\in[s,s+T]}$ and $(\tilde{w}_t)_{t\in[s,s+T]}$ are γ-Hölder functions, we remark that

$$\left| \int_s^t \left(-\Delta_{\boldsymbol{x}} S_w(\tau,s) + \Delta_{\boldsymbol{x}} S_{\tilde{w}}(\tau,s)\right) d\tau \right| \leq C_{T,s,W} |t-s|^\gamma.$$

Since, $\forall t \in]s,s+T]$,

$$\left| \int_s^t \frac{w_\tau - \tilde{w}_\tau - w_s + \tilde{w}_s}{\tau - s} d\tau \right| \leq C T^\gamma \|w-\tilde{w}\|_{\mathcal{C}^\gamma([s,s+T])},$$

we deduce, by using (55), (56) and (57), the estimate

$$\begin{aligned} |a_w(t,s) - a_{\tilde{w}}(t,s)| &\leq C_{T,s,W} \left| \int_s^t \left(-\Delta_{\boldsymbol{x}} S_w(\tau,s) + \Delta_{\boldsymbol{x}} S_{\tilde{w}}(\tau,s)\right) d\tau \right| \\ &\leq C_{T,s,W} \|w-\tilde{w}\|_{\mathcal{C}^\gamma([s,s+T])}. \end{aligned}$$

Concerning the integral $I_2(t,s)$, we use a cut-off function $\vartheta_R \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, with $R>0$ (that will be chosen later), and we set, since $\operatorname{supp}(\psi_0) \subset B(0,\tilde{R})$,

$$\begin{aligned}
I_2(t,s) =& \frac{a_{\tilde{w}}(t,s)}{(2\pi i(t-s))^{d/2}} \int_{\mathbb{R}^d} \psi_s(\boldsymbol{y}) e^{iS_{\tilde{w}}(t,s,\boldsymbol{x},\boldsymbol{y})} \left(e^{iS_w(t,s,\boldsymbol{x},\boldsymbol{y})-iS_{\tilde{w}}(t,s,\boldsymbol{x},\boldsymbol{y})} - 1 \right) \vartheta_R(\boldsymbol{x}) \vartheta_{\tilde{R}}(\boldsymbol{y}) d\boldsymbol{y} \\
&+ \frac{a_{\tilde{w}}(t,s)}{(2\pi i(t-s))^{d/2}} \int_{\mathbb{R}^d} \psi_s(\boldsymbol{y}) \left(e^{iS_w(t,s,x,y)} - e^{iS_{\tilde{w}}(t,s,x,y)} \right) (1-\vartheta_R(\boldsymbol{x})) \vartheta_{\tilde{R}}(\boldsymbol{y}) d\boldsymbol{y} \\
=& I_{21}(t,s) + I_{22}(t,s).
\end{aligned}$$

For the integral I_{21}, it follows from proposition 7 that, $\forall \alpha, \beta \in \mathbb{N}^d$,

$$\begin{aligned}
\left| \partial_x^\alpha \partial_y^\beta \left(e^{iS_w(t,s,\boldsymbol{x},\boldsymbol{y})-iS_{\tilde{w}}(t,s,\boldsymbol{x},\boldsymbol{y})} - 1 \right) \vartheta_R(\boldsymbol{x}) \vartheta_{\tilde{R}}(\boldsymbol{y}) \right| &\leq C_{T,s,W,R,\tilde{R}} \|w-\tilde{w}\|_{\mathcal{C}^0([s,s+T])} \\
&\leq C_{T,s,W,R,\tilde{R}} T^\gamma \|w-\tilde{w}\|_{\mathcal{C}^\gamma([s,s+T])}
\end{aligned}$$

Considering the phase function $(t-s)S_{\tilde{w}}(t,s,\boldsymbol{x},\boldsymbol{y})$ and applying theorem 4, this yields, $\forall t \in [s,s+T]$,

$$\|I_{21}(t,s)\|_{L^2} \leq C_{T,s,W,R,\tilde{R}} T^\gamma \|w-\tilde{w}\|_{\mathcal{C}^\gamma([s,s+T])} \|\psi_s\|_{L^2}.$$

By using (47) for the integral $I_{22}(t,s)$, we obtain that, $\forall j \in \mathbb{N}$,

$$\begin{aligned}
I_{22}(t,s) =& \frac{a_{\tilde{w}}(t,s)}{(2\pi i(t-s))^{d/2})} \frac{(t-s)^j}{i^j} \int_{\mathbb{R}^d} \left((L_w^*)^j - (L_{\tilde{w}}^*)^j \right) \psi_s(\boldsymbol{y}) e^{iS_w(t,s,\boldsymbol{x},\boldsymbol{y})} (1-\vartheta_R(\boldsymbol{x})) \vartheta_{\tilde{R}}(\boldsymbol{y}) d\boldsymbol{y} \\
&+ \frac{a_{\tilde{w}}(t,s)}{(2\pi i(t-s))^{d/2})} \frac{(t-s)^j}{i^j} \int_{\mathbb{R}^d} (L_{\tilde{w}}^*)^j \psi_s(\boldsymbol{y}) \left(e^{iS_w(t,s,\boldsymbol{x},\boldsymbol{y})} - e^{iS_{\tilde{w}}(t,s,\boldsymbol{x},\boldsymbol{y})} \right) (1-\vartheta_R(\boldsymbol{x})) \vartheta_{\tilde{R}}(\boldsymbol{y}) d\boldsymbol{y}.
\end{aligned}$$

Thanks to inequality (47), for $R>0$ large enough, we have

$$\begin{aligned}
\left| \left((L_w^*)^j - (L_{\tilde{w}}^*)^j \right) \psi_s(\boldsymbol{y}) (1-\vartheta_R(\boldsymbol{x})) \right| &= \left| (1-\vartheta_R(\boldsymbol{x})) \sum_{k=0}^{j-1} (L_w^*)^{j-k-1} (L_w^* - L_{\tilde{w}}^*) (L_{\tilde{w}}^*)^k \psi_s(\boldsymbol{y}) \right| \\
&\leq C_{T,s,W,R,\tilde{R}} (1+|\boldsymbol{x}|)^{-j} \sum_{|\alpha| \leq j} |\partial_{\boldsymbol{y}}^\alpha \psi_s(\boldsymbol{y})| T^\gamma \|w-\tilde{w}\|_{\mathcal{C}^\gamma([s,s+T])}.
\end{aligned} \tag{58}$$

Therefore, applying inequality (58) and lemma 6 leads to

$$\begin{aligned}
|I_{22}(t,s)| \leq C_{T,s,W,R,\tilde{R}} (t-s)^{j-d/2} a_{\tilde{w}}(t,s) \sum_{|\alpha| \leq j} \|\partial_{\boldsymbol{y}}^\alpha \psi_s(\boldsymbol{y})\|_{L^1} \left((1+|\boldsymbol{x}|)^{-j} + (1+|\boldsymbol{x}|)^{-j+2} \right) \\
\times T^\gamma \|w-\tilde{w}\|_{\mathcal{C}^\gamma([s,s+T])}.
\end{aligned}$$

By taking $j = \lceil d/2 \rceil + 3$, we finally obtain the inequality

$$\begin{aligned}
\|I_{22}(t,s)\|_{L^2_{\boldsymbol{x}}} \leq C_{T,s,W,R,\tilde{R}} (t-s)^{3+\lceil d/2 \rceil - d/2} a_w(t,s) \textstyle\sum_{|\alpha| \leq j} \|\partial_{\boldsymbol{y}}^\alpha \psi_s(\boldsymbol{y})\|_{L^1} \\
\times \left(\|(1+|\boldsymbol{x}|)^{-d/2-3}\|_{L^2} + \|(1+|\boldsymbol{x}|)^{-d/2-1}\|_{L^2} \right) T^\gamma \|w-\tilde{w}\|_{\mathcal{C}^\gamma([s,s+T])},
\end{aligned}$$

which ends the proof. □

We end this section by the following result that can be obtained in a similar way to [12, 32].

Lemma 7. *For every trajectory* $(w_t)_{t\in[s,s+T]} \in \mathcal{C}^\gamma$*, we have,* $\forall t \in [s, s+T]$*,* $\boldsymbol{x} \in \mathbb{R}^d$*,* $\forall j \in \{1, ..., d\}$*,*

$$i\partial_{\boldsymbol{x}_j} U_w(t,s) = -\bar{\boldsymbol{\xi}}_{w,j}(t,s,0,0)U_w(t,s)\partial_{\boldsymbol{x}_j} + \sum_{k=1}^{d} -\frac{\partial \bar{\boldsymbol{\xi}}_{w,j}}{\partial \boldsymbol{y}_k}(t,s)U_w(t,s)\boldsymbol{x}_k + i\frac{\partial \bar{\boldsymbol{\xi}}_{w,j}}{\partial \boldsymbol{\eta}_k}(t,s)U_w(t,s)\partial_{\boldsymbol{x}_k}, \tag{59}$$

and

$$\boldsymbol{x}_j U_w(t,s) = \bar{\boldsymbol{x}}_{w,j}(t,s,0,0)U_w(t,s) + \sum_{k=1}^{d} \frac{\partial \bar{\boldsymbol{x}}_{w,j}}{\partial \boldsymbol{y}_k}(t,s)U_w(t,s)\boldsymbol{x}_k + i\frac{\partial \bar{\boldsymbol{x}}_{w,j}}{\partial \boldsymbol{\eta}_k}(t,s)U_w(t,s)\partial_{\boldsymbol{x}_k}, \tag{60}$$

Proof. It follows from proposition 6 that, $\forall t \in]s, s+T]$, $\forall j \in \{1, ..., d\}$,

$$\begin{aligned}\partial_{\boldsymbol{x}_j} S_w(t,s,\boldsymbol{x},\boldsymbol{y}) &= \bar{\boldsymbol{\xi}}_{w,j}(t,s,0,0) + \sum_{k=1}^{d} \boldsymbol{y}_k \frac{\partial \bar{\boldsymbol{\xi}}_{w,j}}{\partial \boldsymbol{y}_k}(t,s) + \bar{\boldsymbol{\eta}}_{w,k}(t,s,\boldsymbol{x},\boldsymbol{y})\frac{\partial \bar{\boldsymbol{\xi}}_{w,j}}{\partial \boldsymbol{\eta}_k}(t,s) \\ &= \bar{\boldsymbol{\xi}}_{w,j}(t,s,0,0) + \sum_{k=1}^{d} \boldsymbol{y}_k \frac{\partial \bar{\boldsymbol{\xi}}_{w,j}}{\partial \boldsymbol{y}_k}(t,s) - \partial_{\boldsymbol{y}_k} S_w(t,s,\boldsymbol{x},\boldsymbol{y})\frac{\partial \bar{\boldsymbol{\xi}}_{w,j}}{\partial \boldsymbol{\eta}_k}(t,s).\end{aligned}$$

Therefore, using an integration by parts, we deduce equation (59). Equation (60) is obtained in a similar way by using an expansion of $\boldsymbol{x} = \bar{\boldsymbol{x}}_w(t,s,\boldsymbol{y},\bar{\boldsymbol{\eta}}_w(t,s,\boldsymbol{x},\boldsymbol{y}))$. ☐

If follows from the previous lemma that the propagator $(U_w(t,s))_{t\in[s,s+T]}$ is linear from Σ^n, $n \in \mathbb{N}$, to itself: $\forall n \in \mathbb{N}$, there exists a constant $C > 0$ such that, $\forall \psi_s \in \Sigma^n$, $\forall t \in [s, s+T]$,

$$\|U_w(t,s)\psi_s\|_{\Sigma^n} \leq C_{T,W}\|\psi_s\|_{\Sigma^n}. \tag{61}$$

By using (61) and a density argument, we can extend the sequential continuity of the Itô map $\mathcal{I}_t(w) = U_w(t,s)\psi_s$ from proposition 10 to any $\psi_s \in \Sigma^n$, $n \in \mathbb{N}$. This finally proves theorem 1 which is the main result of the paper.

5 Applications: Strichartz estimates and an Avron-Herbst Formula

In this section, we begin by proving theorem 2 which states the existence and uniqueness of a solution to the *mild* equation (12). We obtain the global in time existence of solutions for L^2-subcritical nonlinearities and local in time existence for H^1-subcritical nonlinearities. Then, in the H^1-subcritical case, we extend the existence of solutions from local to global under certain assumptions on the initial data, the Hamiltonian $\mathcal{H}$ and β. This corresponds to the proof of theorem 3.

5.1 Strichartz estimates

Thanks to the results of section 4, we are able to state that ψ, the solution of problem (3), is represented by using the propagator $(U_w(t,s))_{t\in[s,s+T]}$, which is given by formula (10). This propagator is inhomogeneous, strongly continuous and isometric in L^2. Let us now recall the definition of an admissible pair (p,q).

Definition 5. *Let* p *and* $q \in \mathbb{R}^+$*. Then,* (p,q) *is called an admissible pair if*

$$\frac{2}{p} = d\left(\frac{1}{2} - \frac{1}{q}\right) \tag{62}$$

and $2 \leq p, q \leq \infty$ *with* $(p,q,d) \neq (2,\infty,2)$*.*

The proof of theorem 2 is based on the following Strichartz estimates [9, 20].

Theorem 5. *Let $T > 0$ and $s \in \mathbb{R}^+$. Let us consider $(X(t,s))_{t\in[s,s+T]}$ as an inhomogeneous, strongly continuous and isometric propagator in L^2. Furthermore, we assume that $(X(t,s))_{t\in[s,s+T]}$ satisfies the following assumption: $\forall t \in [s,s+T]$, $\forall r \in [s,t]$,*

$$X(t,r)^* X(t,s) = X(r,s), \tag{63}$$

and, for $2 \le p \le \infty$, there exists a constant $C > 0$ such that, $\forall t \in]s,s+T]$, $\forall \psi_s \in L^{p'}$,

$$\|X(t,s)\psi_s\|_{L^p} \le \frac{C}{|t-s|^{d(1/2-1/p)}} \|\psi_s\|_{L^{p'}}, \tag{64}$$

where p' is the conjugate exponent of p, i.e. $1/p + 1/p' = 1$. Then, for all admissible pair (p,q), there exists a constant $C_{1,p,q} > 0$ such that

$$\|X(\cdot,s)\psi_s\|_{L^p([s,s+T],L^q)} \le C_{1,p,q}\|\psi_s\|_{L^2}. \tag{65}$$

Moreover, let (m,ℓ) be an admissible pair, then there exists a constant $C_{2,p,q} > 0$ such that, for all $g \in L^{m'}([s,T],L^{\ell'})$,

$$\left\| \int_s^{\cdot} X(\cdot,\tau) g(\tau) d\tau \right\|_{L^p([s,s+T],L^q)} \le C_{2,p,q}\|g\|_{L^{m'}([s,s+T],L^{\ell'})}. \tag{66}$$

We can immediately see that the propagator $(U_w(t,s))_{t\in[s,s+T]}$ fulfills (64). Equation (63) also holds since, for all $(w_t)_{t\in[s,s+T]} \in \mathcal{C}^1$, we have

$$\begin{aligned} U_w^*(t,r)U_w(t,s) - U_w(r,s) &= \int_r^t \frac{d}{d\tau}\left(U_w^*(\tau,r)U_w(\tau,s)\right) d\tau \\ &= \int_r^t U_w^*(\tau,r) i\mathcal{H}^*(\tau) U_w(\tau,s) d\tau \\ &\quad - \int_r^t U_w^*(\tau,r) i\mathcal{H}(\tau) U_w(\tau,s) d\tau. \end{aligned}$$

This proves the expected result assuming that $\mathcal{H}(t,\boldsymbol{x},-i\nabla)$ is a self-adjoint operator in L^2, $\forall t \in [s,s+T]$.

By using the Strichartz estimates, we can now prove theorem 2 by Banach fixed point theorems and bounds on the L^2- and Σ^1-norms [2, 8, 12, 25]. Let us begin by defining the mapping Γ: $\forall \phi \in L^{m'}([s,s+T],L^{\ell'})$, (m,ℓ) an admissible pair, $\forall t \in [s,s+T]$, $\forall \psi_s \in L^2$,

$$\Gamma(\phi)(t,\boldsymbol{x}) = U_w(t,s)\psi_s(\boldsymbol{x}) + \beta \int_s^t U_w(t,\tau)|\phi(\tau,\boldsymbol{x})|^{2\sigma}\phi(\tau,\boldsymbol{x}) d\tau. \tag{67}$$

To begin with, we show the uniqueness of a fixed point to Γ in order to obtain local in time solutions. Here, we consider the cases of an initial data in L^2 and Σ^1.

Let $\psi_s \in L^2$ and introduce the functional space

$$X_M = \left\{\phi \in \mathcal{C}^0([s,s+T],L^2) \cap L^r([s,s+T],L^{2\sigma+2}); \|\phi\|_{X_M} \le M\right\},$$

where

$$\|\phi\|_{X_M} := \sup_{t\in[s,s+T]} \|\phi(t,\cdot)\|_{L^2} + \|\phi\|_{L^r([s,s+T],L^{2\sigma+2})} \ \text{ and } \ M = 2C_{1,r,2\sigma+2}\|\psi_s\|_{L^2}.$$

In the above relations, $(r, 2\sigma+2)$ is an admissible pair with $\sigma < \frac{2}{d}$. Moreover, in [31], X_M is proved to be a closed subset of $L^r([s,s+T],L^{2\sigma+2})$. Let (p,q) be an admissible pair. We obtain, by applying the $L^p([s,s+T],L^q)$ norm to Γ and using the Strichartz estimates from theorem 5, the following estimate

$$\begin{aligned}\|\Gamma(\phi)\|_{L^p([s,s+T],L^q)} \leq & C_{1,p,q}\|\psi_s\|_{L^2} + C_{2,p,q}|\beta| \left\| |\phi|^{2\sigma}\phi \right\|_{L^{m'}([s,s+T],L^{\ell'})} \\ \leq & C_{1,p,q}\|\psi_s\|_{L^2} + C_{2,p,q}|\beta| \, \|\phi\|^{2\sigma+1}_{L^{m'(2\sigma+1)}([s,s+T],L^{\ell'(2\sigma+1)})}.\end{aligned}$$

Therefore, choosing $(p,q) = (r, 2\sigma+2)$, $\ell' = \frac{2\sigma+2}{2\sigma+1}$ and by using an Hölder inequality, $\forall \phi \in X_M$, we obtain the inequalities

$$\begin{aligned}\|\Gamma(\phi)\|_{L^r([s,s+T],L^{2\sigma+2})} \leq & C_{1,p,q}\|\psi_s\|_{L^2} + C_{2,p,q}|\beta| T^{1-\frac{2\sigma+2}{r}}|\beta| \, \|\phi\|^{2\sigma+1}_{L^r([s,s+T],L^{2\sigma+2})} \\ \leq & C_{1,p,q}\|\psi_s\|_{L^2} + C_{2,p,q}|\beta| T^{1-\frac{2\sigma+2}{r}}|\beta| M^{2\sigma+1}.\end{aligned} \tag{68}$$

Similarly we have: $\forall \phi_1, \phi_2 \in X_M$,

$$\|\Gamma(\phi_1) - \Gamma(\phi_2)\|_{L^r([s,s+T],L^{2\sigma+2})} \leq C_{2,p,q}|\beta| T^{1-\frac{2\sigma+2}{r}} M^{2\sigma} \|\phi_1 - \phi_2\|_{L^r([s,s+T],L^{2\sigma+2})}. \tag{69}$$

Therefore, by choosing $T > 0$ such that

$$C_{2,p,q}|\beta| T^{1-\frac{2\sigma+2}{r}} M^{2\sigma} < 1/2, \tag{70}$$

we prove that Γ is a contractive application from X_M to itself, which leads to the existence and uniqueness of a solution in X_M to the problem (12).

We now let $\psi_s \in \Sigma^1$. A similar result is obtained by considering the functional space ($M > 0$)

$$Y_M = \left\{ \phi, \partial_{\boldsymbol{x}_j}\phi, \boldsymbol{x}_j\phi \in \mathcal{C}^0([s,s+T],L^2) \cap L^r([s,s+T],L^{2\sigma+2}), \ \forall j \in \{1,...,d\}; \|\phi\|_{Y_M} \leq M \right\}$$

with

$$\|\phi\|_{Y_M} := \|\phi\|_{X_M} + \sum_{j=1}^{d} \left(\|\boldsymbol{x}_j\phi\|_{X_M} + \|\partial_{\boldsymbol{x}_j}\phi\|_{X_M} \right)$$

Following a proof analogous to the one found in [31] for the X_M functional space, we note that Y_M is a closed subset of

$$\tilde{Y}^{r,2\sigma+2} := \left\{ \phi, \partial_{\boldsymbol{x}_j}\phi, \boldsymbol{x}_j\phi \in L^r([s,s+T],L^{2\sigma+2}), \ \forall j \in \{1,...,d\} \right\},$$

endowed with the norm

$$\|\phi\|_{\tilde{Y}^{r,2\sigma+2}} := \|\phi\|_{L^r([s,s+T],L^{2\sigma+2})} + \sum_{j=1}^{d} \|\boldsymbol{x}_j\phi\|_{L^r([s,s+T],L^{2\sigma+2})} + \|\partial_{\boldsymbol{x}_j}\phi\|_{L^r([s,s+T],L^{2\sigma+2})}.$$

Since the operators $\partial_{\boldsymbol{x}_j}$ and $\boldsymbol{x}_j$ do not commute with the propagator $(U_w(t,s))_{t\in[s,s+T]}$, we use lemma 7 (and theorem 5) to deduce that, $\forall \phi \in \tilde{Y}^{r', \frac{2\sigma+1}{2\sigma+2}}$,

$$\|U_w(\cdot,s)\psi_s\|_{\tilde{Y}^{r,2\sigma+2}} \leq C_{1,r,2\sigma+2}\|\psi_s\|_{\Sigma^1} \quad \text{and} \quad \left\| \int_s^{\cdot} U_w(\cdot,\tau)\phi(\tau)d\tau \right\|_{\tilde{Y}^{r,2\sigma+2}} \leq C_{2,r,2\sigma+2}\|\phi\|_{\tilde{Y}^{r', \frac{2\sigma+1}{2\sigma+2}}}. \tag{71}$$

We remark that the constants in the two previous inequality are independent of $T > 0$ since the classical orbits $\bar{\boldsymbol{\xi}}$ and $\bar{\boldsymbol{x}}$ in Lemma 7 are bounded on any interval $[s, s+T_0]$, $T_0 > 0$. We fix $M = 2C_{1,r,2\sigma+2}\|\psi_s\|_{\Sigma^1}$.

Thanks to the inequalities (71) and a Banach fixed point theorem in Y_M for $T > 0$ sufficiently small, we conclude the existence and uniqueness of a solution in Y_M to the problem (12).

We are now in position to prove the first part of theorem 2. Let $\psi_s \in L^2$. With the help of the Strichartz estimates from theorem 5, it follows from equation (12) that

$$\|\psi\|_{L^\infty([s,s+T_0],L^2)} + \|\psi\|_{L^r([s,s+T_0],L^{2\sigma+2}} \leq 2C_{1,r,\sigma}\|\psi_s\|_{L^2} + 2C_{2,r,\sigma}|\beta|\|\psi\|_{L^\theta([s,s+T_0],L^{2\sigma+2})}, \quad (72)$$

where $\theta = \frac{d\sigma}{2(\sigma+1)}$. Since $\sigma < 2/d$, we have $1/\theta > 1/r$ and, thus,

$$\|\psi\|_{L^\theta([s,s+T_0],L^{2\sigma+2})} \leq T_0^{1/\theta-1/r}\|\psi\|_{L^r([s,s+T_0],L^{2\sigma+2})}. \quad (73)$$

Therefore, by using the inequalities (72) and (73) and choosing $T_0 > 0$ small enough, we are able to bound the solution ψ by the L^2-norm of ψ_s on any time interval of length T_0. This leads to the existence of global in time solutions to the problem (12) in $\mathcal{C}^0([s,\infty[,L^2) \cap L^r([s,\infty[,L^{2\sigma+2})$. Furthermore, if $\psi_s \in \Sigma^1$, we use the following inequality

$$\|\psi\|_{\tilde{Y}^{\infty,2}} + \|\psi\|_{\tilde{Y}^{r,2\sigma+2}} \leq 2C_{1,r,2\sigma+2}\|\psi_s\|_{\Sigma^1} + 2C_{2,r,2\sigma+2}|\beta|\|\psi\|_{\tilde{Y}^{r,2\sigma+2}},$$

and (73) to obtain a similar result in $\mathcal{C}^0([s,\infty[,\Sigma^1)$. The second part of theorem 2 is shown by using the inequality

$$\||\phi|^{2\sigma}\phi\|_{\tilde{Y}^{r',\frac{2\sigma+1}{2\sigma+2}}} \leq C_3T^{1-\theta}\|\phi\|^{2\sigma}_{L^\infty([s,s+T],H^1)}\|\phi\|_{\tilde{Y}^{r,2\sigma+2}}, \quad (74)$$

where $\theta = \frac{d\sigma}{2(\sigma+1)} < 1$, and applying a Banach fixed point theorem in Y_M in a similar way as for the first part of the theorem. However, since we can not obtain the boundedness on any time interval in this case, existence and uniqueness can only be established for local solutions.

5.2 An Avron-Herbst formula

We now intend to prove the existence and uniqueness of global solutions of equation (12) under the assumptions 1 and 2. The key to obtain this result is to exhibit uniform in time bounds on Σ^1 for the local solutions. As we mentioned before, since the hamiltonian is time-dependent and irregular, it is not possible to achieve such bounds by relying directly on the energy as in the deterministic case (*i.e.* with $\dot{w}_t = 0$). This problem can be overcome by using an Avron-Herbst formula [5] to deduce a formulation of the solution of equation (12) that involves the solution of a deterministic nonlinear Schrödinger equation for which the bounds easily follow.

Let us begin with the following Avron-Herbst formula in the linear case.

Proposition 11. *Let $\psi_s \in L^2$ and suppose that the assumptions 1 are verified. Moreover, set $M_{21} = 0$ and $M_{22} = 0$ in formula (2). Then there exists $T > 0$ such that*

$$U_w(t,s)\psi_s(\boldsymbol{x}) = e^{i\mathbf{A}_w(t,s)\cdot\boldsymbol{x}+ib_w(t,s)}V(t-s)\psi_s(\boldsymbol{x}-\mathbf{B}_w(t,s)),\ \forall t \in [s,s+T],\ \forall \boldsymbol{x} \in \mathbb{R}^d, \quad (75)$$

where $(V(t-s))_{t\in[s,s+T]}$ is the propagator such that $\phi(t,\boldsymbol{x}) = V(t-s)\psi_s(\boldsymbol{x})$ is the solution of the following Schrödinger equation

$$\begin{cases} i\partial_t\phi(t,\boldsymbol{x}) = -\dfrac{1}{2}\Delta\phi(t,\boldsymbol{x}) - i(M_{11}\boldsymbol{x})\cdot\nabla\phi(t,\boldsymbol{x}) + (M_{12}\boldsymbol{x})\cdot\boldsymbol{x}\phi(t,\boldsymbol{x}),\ \forall t \in [s,s+T],\ \forall \boldsymbol{x} \in \mathbb{R}^d, \\ \phi(s,\boldsymbol{x}) = \psi_s(\boldsymbol{x}), \end{cases} \quad (76)$$

the functions $\mathbf{B}_w$ *and* $\mathbf{A}_w$ *are solutions, in the sense of [13, 30], of the following system of stochastic differential equations*

$$\left\{ \begin{array}{l} \partial_t \mathbf{B}_w(t,s) = \mathbf{A}_w(t,s) + M_{11}\mathbf{B}_w(t,s) + \mathbf{V}_{12} + \mathbf{V}_{22}\dot{w}_t, \ \forall t \in [s, s+T], \\ \partial_t \mathbf{A}_w(t,s) = -M_{11}^*\mathbf{A}_w(t,s) - (M_{12} + M_{12}^*)\,\mathbf{B}_w(t,s) - \mathbf{V}_{11} - \mathbf{V}_{21}\dot{w}_t, \ \forall t \in [s, s+T], \end{array} \right. \tag{77}$$

with the initial conditions $\mathbf{B}_w(s,s) = \mathbf{A}_w(s,s) = 0$*, and the function* b_w *is given by*

$$b_w(t,s) = -\int_s^t \left[\frac{1}{2}|\mathbf{A}_w(\tau,s)|^2 - (M_{12}\mathbf{B}_w(\tau,s)) \cdot \mathbf{B}_w(\tau,s) + \mathbf{V}_{11} \cdot \mathbf{B}_w(\tau,s) + \mathbf{V}_{22} \cdot \mathbf{A}_w(\tau,s)\dot{w}_\tau \right] d\tau. \tag{78}$$

Proof. Thanks to theorem 1, we know that $\psi(t,\boldsymbol{x}) = U_w(t,s)\psi_s(\boldsymbol{x})$ is the solution of the linear Schrödinger equation (3). Suppose that $(w_t)_{t\in[s,s+T]} \in \mathcal{C}^1([s,s+T],\mathbb{R})$. By setting

$$\psi_w(t,\boldsymbol{x}) = \phi(t, \boldsymbol{x} - \mathbf{B}_w(t,s))e^{i\mathbf{A}_w(t,s)\cdot\boldsymbol{x} + ib_w(t,s)}, \tag{79}$$

we obtain

$$i\partial_t\psi_w(t,\boldsymbol{x}) = \left(i\partial_t - i\partial_t\mathbf{B}_w(t,s)\cdot\nabla - [\partial_t\mathbf{A}_w(t,s)\cdot\boldsymbol{x} + \partial_t b_w(t,s)]\right)\phi(t,\boldsymbol{x} - \mathbf{B}_w(t,s))e^{i\mathbf{A}_w(t,s)\cdot\boldsymbol{x} + ib_w(t,s)}$$

and

$$-i\nabla\psi_w(t,\boldsymbol{x}) = \left(-i\nabla + \mathbf{A}_w(t,s)\right)\phi(t,\boldsymbol{x} - \mathbf{B}_w(t,s))e^{i\mathbf{A}_w(t,s)\cdot\boldsymbol{x} + ib_w(t,s)}.$$

Since ψ is the solution of equation (3), we deduce that ϕ is the solution of the following equation

$$\left\{ \begin{array}{l} i\partial_t\phi(t,\boldsymbol{x} - \mathbf{B}_w(t,s)) = \tilde{\mathcal{H}}_1(\boldsymbol{x} - \mathbf{B}_w(t,s), -i\nabla + \mathbf{A}_w(t,s))\phi(t,\boldsymbol{x} - \mathbf{B}_w(t,s)), \ \forall t \in]s,\infty[, \ \forall \boldsymbol{x} \in \mathbb{R}^d, \\ \phi(s,\boldsymbol{x} - \mathbf{B}_w(s,s))e^{i\mathbf{A}_w(s,s)\cdot\boldsymbol{x} + ib_w(s,s)} = \psi_s(\boldsymbol{x}) \in L^2, \end{array} \right.$$

where $\tilde{\mathcal{H}}_1(\boldsymbol{x},\boldsymbol{\xi}) = \frac{1}{2}|\boldsymbol{\xi}|^2 + (M_{11}\boldsymbol{x})\cdot\boldsymbol{\xi} + (M_{12}\boldsymbol{x})\cdot\boldsymbol{x}$, if the functions $\mathbf{B}_w$, $\mathbf{A}_w$ and b_w are solutions of the equations (77) and (78).

Since ψ_w is continuous with respect to $(w_t)_{t\in[s,s+T]}$, we can extend the relation (79) to the case where $(w_t)_{t\in[s,s+T]}$ is a γ-Hölder function which satisfies the assumptions 1. Furthermore, we can see that the functions $\mathbf{B}_w$ and $\mathbf{A}_w$ are exactly the classical orbits associated to the Hamiltonian operator $\mathcal{H}$ starting at $\mathbf{B}_w(s,s) = \mathbf{A}_w(s,s) = 0$ (*i.e.* they are solutions of the Hamilton equations (14)). Therefore, there existence and uniqueness, in the sense of [13, 30], follows directly. Concerning the function b_w, the only term that poses a difficulty is the last one where the noise $(\dot{w}_t)_{t\in[s,s+T]}$ appears. We can deal with it by means of an integration by parts. That is, we have

$$\begin{aligned} \int_s^t \mathbf{A}_w(\tau,s)\dot{w}_\tau d\tau = \mathbf{A}_w(t,s)w_t + \int_s^t \left(M_{11}^*\mathbf{A}_w(\tau,s) + (M_{12} + M_{12}^*)\mathbf{B}_w(\tau,s) + \mathbf{V}_{11}\right) w_\tau d\tau \\ + \int_s^t \mathbf{V}_{21} w_\tau \dot{w}_\tau d\tau, \end{aligned}$$

where the last term from the right hand side can be exactly integrated. This conclude the proof this proposition. □

Corollary 3. *Let* $\psi_s \in L^2$*. Under the same assumptions as those in proposition 11, we have,* $\forall t \in [s, s+T]$ *and* $\forall \tau \in [t, s]$,

$$U_w(t,\tau)e^{i\mathbf{A}_w(\tau,s)\cdot\boldsymbol{x} + ib_w(\tau,s)}\psi_s(\boldsymbol{x} - \mathbf{B}_w(\tau,s)) = e^{i\mathbf{A}_w(t,s)\cdot\boldsymbol{x} + ib_w(t,s)}V(t-\tau)\psi_s(\boldsymbol{x} - \mathbf{B}_w(t,s)). \tag{80}$$

Proof. Since $(V(t-s))_{t\in[s,s+T]}$ is the propagator associated to equation (76), we have, $\forall \tau \in [s,t]$,

$$V(\tau-s)V(\tau-s)^*\psi_s(\boldsymbol{x}) = \psi_s(\boldsymbol{x}).$$

Therefore, we obtain, with the help of proposition 11,

$$\begin{aligned} U_w(t,\tau)e^{i\mathbf{A}_w(\tau,s)\cdot\boldsymbol{x}+ib_w(\tau,s)}\psi_s(\boldsymbol{x}-\mathbf{B}_w(\tau,s)) &= U_w(t,\tau)e^{i\mathbf{A}_w(\tau,s)\cdot\boldsymbol{x}+ib_w(\tau,s)}V(\tau-s)V(\tau-s)^*\psi_s(\boldsymbol{x}-\mathbf{B}_w(\tau,s)) \\ &= U_w(t,s)V(\tau-s)^*\psi_s(\boldsymbol{x}) \\ &= e^{i\mathbf{A}_w(t,s)\cdot\boldsymbol{x}+ib_w(t,s)}V(t-\tau)\psi_s(\boldsymbol{x}-\mathbf{B}_w(t,s)). \end{aligned}$$

□

We are now enable to link the solution of the equation (12) to the solution of the following deterministic *mild* equation

$$\phi(t,\boldsymbol{x}) = V(t-s)\psi_s(\boldsymbol{x}) - i\beta\int_s^t V(t-\tau)|\phi(\tau,\boldsymbol{x})|^{2\sigma}\phi(\tau,\boldsymbol{x})d\tau,\ \forall t\in[s,\infty[,\ \forall \boldsymbol{x}\in\mathbb{R}^d. \tag{81}$$

This link stems from proposition 11 and the fact that the nonlinearity is gauge invariant, *i.e.* we have $f(\psi e^{ia}) = f(\psi)e^{ia}$ where $f(\psi) = |\psi|^{2\sigma}\psi$ and $a\in\mathbb{R}$. Indeed, let us consider the solution ψ of equation (12). By replacing

$$\psi(t,\boldsymbol{x}) = \phi(t,\boldsymbol{x}-\mathbf{B}_w(t,s))e^{i\mathbf{A}_w(t,s)\cdot\boldsymbol{x}+ib_w(t,s)}, \tag{82}$$

in equation (12) and thanks to proposition 11 and corollary 3, we obtain

$$\phi(t,\boldsymbol{x}-\mathbf{B}_w(t,s)) = V(t-s)\psi_s(\boldsymbol{x}-\mathbf{B}_w(t,s)) - i\beta\int_s^t V(t-\tau)|\phi(\tau,\boldsymbol{x}-\mathbf{B}_w(t,s))|^{2\sigma}\phi(\tau,\boldsymbol{x}-\mathbf{B}_w(t,s))d\tau,$$

which permits us to identify the function ϕ as the solution of equation (81). Therefore, by using formula (82), it suffices to prove that the Σ^1-norm of ϕ remains bounded at any time in order to prove theorem 3. The following proposition provides a positive answer to this last problem and concludes the proof of theorem 3.

Proposition 12. *Let $\psi_s \in \Sigma^1$ and $0<\sigma<\infty$ if d=1,2 or $0<\sigma<\frac{2}{d-2}$ if $d\geq 3$. Suppose that M_{11} is skew-adjoint and $\beta\geq 0$. Then equation (81) admits a unique solution ϕ in $\mathcal{C}^0([s,\infty[,\Sigma^1)$.*

Proof. We only sketch the proof since most of the arguments that are used can be found in [2]. Our aim is to derive an *a priori* estimate on the Σ^1-norm of ϕ, the solution of the equation (81). To begin with, we introduce the change of variables

$$X(t-s,\boldsymbol{x}) = e^{M_{11}(t-s)}\boldsymbol{x}, \quad \forall t\in[s,s+T[,$$

and remark that, since M_{11} is skew-adjoint, the jacobian matrix J_X associated to X verifies

$$\det(J_X) = e^{\mathrm{Tr}(M_{11})(t-s)} = 1.$$

Furthermore, the Laplacian is invariant with respect to this change of variables since

$$\Delta_x\phi(t,X(t-s,\boldsymbol{x})) = e^{M_{11}(t-s)}e^{M_{11}^*(t-s)}\Delta_X\phi(t,X(t-s,\boldsymbol{x})) = \Delta_X\phi(t,X(t-s,\boldsymbol{x})).$$

Hence, the function $\varphi(t,\boldsymbol{x}) = \phi(t, X(t-s,\boldsymbol{x}))$ satisfies the Schrödinger equation

$$\begin{cases} i\partial_t \varphi(t,\boldsymbol{x}) = -\dfrac{1}{2}\Delta\varphi(t,\boldsymbol{x}) + \upsilon(t-s,\boldsymbol{x})\varphi(t,\boldsymbol{x}) + \beta|\varphi|^{2\sigma}\varphi(t,\boldsymbol{x}),\ \forall t\in[s,s+T],\ \forall \boldsymbol{x}\in\mathbb{R}^d, \\ \varphi(s,\boldsymbol{x}) = \psi_s(\boldsymbol{x}), \end{cases} \tag{83}$$

where $\upsilon(t-s,\boldsymbol{x}) = (M_{12}X(t-s,\boldsymbol{x}))\cdot X(t-s,\boldsymbol{x})$. We now introduce the energy functional $\mathcal{E}$ associated to the equation (83)

$$\mathcal{E}(t,\varphi) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla\varphi(t,\boldsymbol{x})|^2 + \upsilon(t-s,\boldsymbol{x})|\varphi(t,\boldsymbol{x})|^2 + \frac{\beta}{2\sigma+2}|\varphi(t,\boldsymbol{x})|^{2\sigma+2} \right) d\boldsymbol{x}. \tag{84}$$

In general, this energy is not conserved since υ is time-dependent and we obtain that

$$\frac{d}{dt}\mathcal{E}(t,\varphi) = \int_{\mathbb{R}^d} \partial_t \upsilon(t-s,\boldsymbol{x})|\varphi(t,\boldsymbol{x})|^2 d\boldsymbol{x}. \tag{85}$$

However, we can still use it to bound the Σ^1-norm of φ, which directly implies the bound of the Σ^1-norm of ϕ. We start by controlling the L^2-norm of $\nabla\varphi$ with the help of $\mathcal{E}$ and the L^2-norm of $\boldsymbol{x}\varphi$. Since $\beta > 0$, we have

$$\frac{1}{2}\|\nabla\varphi(t,\boldsymbol{x})\|_{L^2}^2 \leq \mathcal{E}(t,\varphi) + \left| \int_{\mathbb{R}^d} \upsilon(t-s,\boldsymbol{x})|\varphi(t,\boldsymbol{x})|^2 d\boldsymbol{x} \right| \leq \mathcal{E}(t,\varphi) + C_1\|\boldsymbol{x}\varphi(t,\boldsymbol{x})\|_{L^2}^2,$$

where $C_1 = \|M_{12}\|e^{2\|M_{11}\|T}$. Furthermore, by integrating in time (85), we deduce that

$$\begin{aligned} \frac{1}{2}\|\nabla\varphi(t,\cdot)\|_{L^2}^2 &\leq \mathcal{E}(s,\psi_s) + \int_s^t \int_{\mathbb{R}^d} \partial_t\upsilon(r-s,\boldsymbol{x})|\varphi(r,\boldsymbol{x})|^2 d\boldsymbol{x}dr + C_1\|\boldsymbol{x}\varphi(t,\boldsymbol{x})\|_{L^2}^2 \\ &\leq \mathcal{E}(s,\psi_s) + C_1\|\boldsymbol{x}\varphi(t,\boldsymbol{x})\|_{L^2}^2 + C_2\int_s^t \|\boldsymbol{x}\varphi(r,\boldsymbol{x})\|_{L^2}^2 dr, \end{aligned} \tag{86}$$

where $C_2 = 2\|M_{12}\|\|M_{11}\|e^{2\|M_{11}\|T}$ and $|\mathcal{E}(s,\psi_s)| < +\infty$ thanks to the Sobolev embedding $\Sigma^1 \subset H^1 \subset L^{2\sigma+2}$. Hence, to obtain a bound on the Σ^1-norm of φ, we only have to estimate uniformly in time the L^2-norm of $\boldsymbol{x}\varphi$. To do this, we compute

$$\frac{d}{dt}\|\boldsymbol{x}\varphi(t,\boldsymbol{x})\|_{L^2}^2 = 2\Im\int_{\mathbb{R}^d} (\boldsymbol{x}\varphi^*(t,\boldsymbol{x}))\cdot\nabla\varphi(t,\boldsymbol{x})d\boldsymbol{x} \leq \|\boldsymbol{x}\varphi(t,\boldsymbol{x})\|_{L^2}^2 + \|\nabla\varphi(t,\boldsymbol{x})\|_{L^2}^2,$$

which yields, by using (86),

$$\frac{d}{dt}\|\boldsymbol{x}\varphi(t,\boldsymbol{x})\|_{L^2}^2 \leq C\left(1 + \|\boldsymbol{x}\varphi(t,\boldsymbol{x})\|_{L^2}^2 + \int_s^t \|\boldsymbol{x}\varphi(r,\boldsymbol{x})\|_{L^2}^2 dr\right).$$

This leads to a uniform bound in time of the L^2-norm of $\boldsymbol{x}\varphi$ with the help of a Gronwall inequality and, thus, concludes our proof. □

Acknowledgments.

This work was partially supported by the French ANR grants MicroWave NT09_460489 ("Programme Blanc" call) and ANR-12-MONU-0007-02 BECASIM ("Modèles Numériques" call).

References

[1] G. Agrawal. *Nonlinear fiber optics*. Springer, 2000.

[2] P. Antonelli, D. Marahrens, and C. Sparber. On the Cauchy problem for nonlinear Schrödinger equations with rotation. *Discrete Continuous and Dynamical Systems*, 32, 2012.

[3] V. I. Arnol'd. *Mathematical methods of classical mechanics*, volume 60. Springer, 1989.

[4] K. Asada and D. Fujiwara. On the boundedness of integral transformations with rapidly oscillatory kernels. *Journal of the Mathematical Society of Japan*, 27(4):628–639, 1975.

[5] J. E. Avron and I. W. Herbst. Spectral and scattering theory of Schrödinger operators related to the Stark effect. *Communications in Mathematical Physics*, 52(3):239–254, 1977.

[6] R. Carles. Remarks on nonlinear Schrödinger equations with harmonic potential. In *Annales Henri Poincaré*, volume 3, pages 757–772. Springer, 2002.

[7] R. Carles. Nonlinear Schrödinger equations with repulsive harmonic potential and applications. *SIAM Journal on Mathematical Analysis*, 35(4):823–843, 2003.

[8] R. Carles. Nonlinear Schrödinger equation with time dependent potential. *Communications in Mathematical Sciences*, 9(4):937–964, 2011.

[9] T. Cazenave. *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, 2003.

[10] R. Courant and D. Hilbert. *Methods of Mathematical Physics*. Number vol. 2 in Methods of Mathematical Physics. Wiley, 2008.

[11] A. de Bouard and R. Fukuizumi. Modulation analysis for a stochastic NLS equation arising in Bose-Einstein condensation. *Asymptotic Analysis*, 63(4):189–235, 2009.

[12] A. de Bouard and R. Fukuizumi. Representation formula for stochastic Schrödinger evolution equations and applications. *Nonlinearity*, 25(11):2993, 2012.

[13] H. Doss. Liens entre équations différentielles stochastiques et ordinaires. In *Annales de l'institut Henri Poincaré (B) Probabilités et Statistiques*, volume 13, pages 99–125. Gauthier-Villars, 1977.

[14] M. V. Fedoryuk. The stationary phase method and pseudodifferential operators. *Russian Mathematical Surveys*, 26(1):65–115, 1971.

[15] D. Fujiwara. On the boundedness of integral transformations with highly oscillatory kernels. *Proceedings of the Japan Academy*, 51:96–99, 1975.

[16] D. Fujiwara. A construction of the fundamental solution for the Schrödinger equation. *Journal d'Analyse Mathématique*, 35:41–96, 1979.

[17] M. E. Gehm, K. M. O'Hara, T. A. Savard, and J. E. Thomas. Dynamics of noise-induced heating in atom traps. *Physical Review A*, 58:3914–3921, Nov 1998.

[18] H. Goldstein, C. Poole, and J. Safko. Classical mechanics, 2002.

[19] W. R. Hamilton, A. A. W. Conway, and J. J. L. Synge. *The mathematical papers of sir William Rowan Hamilton*, volume 4 of *Cunningham memoirs*. Cambridge University Press, 1931.

[20] M. Keel and T. Tao. Endpoint Strichartz estimates. *American Journal of Mathematics*, pages 955–980, 1998.

[21] H. Kitada. On a construction of the fundamental solution for Schrödinger equations. *Journal of the Faculty of Science, University of Tokyo*, 27, 1980.

[22] Lev Davidovich Landau, EM Lifshitz, JB Sykes, JS Bell, and ME Rose. Quantum mechanics, non-relativistic theory: Vol. 3 of course of theoretical physics. *Physics Today*, 11:56, 1958.

[23] E. H. Lieb and R. Seiringer. Derivation of the Gross-Pitaevskii equation for rotating Bose gases. *Communications in mathematical physics*, 264(2):505–537, 2006.

[24] H. Mizutani. Strichartz estimates for Schrödinger equations with variable coefficients and unbounded potentials. *Analysis & PDE*, 6(8):1857–1898, 2014.

[25] Y.-G. Oh. Cauchy problem and Ehrenfest's law of nonlinear Schrödinger equations with potentials. *Journal of Differential Equations*, 81(2):255–274, 1989.

[26] Gennady Samorodnitsky and Murad S Taqqu. Stable non-gaussian random processes. *Econometric Theory*, 13:133–142, 1997.

[27] T. A. Savard, K. M. O'Hara, and J. E. Thomas. Laser-noise-induced heating in far-off resonance optical traps. *Physical Review A*, 56:R1095–R1098, Aug 1997.

[28] Kamal P. Singh, Anatole Kenfack, and Jan M. Rost. Femtosecond photodissociation of molecules facilitated by noise. *Physical Review A*, 77:022707, Feb 2008.

[29] Kamal P. Singh and Jan M. Rost. Optimal stochastic enhancement of photoionization. *Physical Review Letters*, 98:160201, Apr 2007.

[30] H. J. Sussmann. On the gap between deterministic and stochastic ordinary differential equations. *Annals of Probability*, 6(1):19–41, 1978.

[31] Y. Tsutsumi. L^2-solutions for nonlinear Schrödinger equations and nonlinear groups. *Funkcialaj Ekvacioj. Serio Internacia*, 30(1):115–125, 1987.

[32] K. Yajima. Schrödinger evolution equations with magnetic fields. *Journal d'Analyse Mathématique*, 56:29–76, 1991.