PARTICLE SYSTEMS: ANALYTIC AND INTEGRABLE ASPECTS

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Abstract. This is a summary for the benefit of the students attending the Master 2 program at the Institute of Mathematics in Toulouse (IMT).

It is here to structure the written notes.

Contents

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Key words and phrases. ASEP, Growth processes, Directed last passage percolation, Shape theorems, Talagrand's concentration, Determinantal point processes.

Notations

If M is a matrix, its entries are denoted by $M^{i,j}$, its adjoint is $M^\dagger$ and $\mathrm{Tr}(M)$ is the trace. Matrices are denoted by capital letters or by Greek letters. The complex conjugate of z is $\overline{z}$.

1. Panorama: ASEP and related models

1.1. **Definition of ASEP.** ASEP stands for ASymmetric Exclusion Process. It is a Markov process of particles on $\mathbb{Z}+\frac{1}{2}$, which satisfy the exclusion property i.e no two particles can occupy the same position.

Informally ASEP(p=1,q) with $q<1$ is the particle system on $\mathbb{Z}+\frac{1}{2}$, where particles jump to the right with rate $p=1$ and to the left with rate $q<1$. Because the jump rate to the left is slower, there is an out-of-equilibrium flow to the right.

Remark 1.1 (Reminder on Poisson processes)**.** *By definition the Poisson process with rate λ (abbreviated $PP(\lambda)$) is the càdlàg process N defined for all $t \geq 0$ by:*

$$N_t := \sum_{k=0}^{\infty} \mathbb{1}_{\{T_k \leq \lambda t\}} \ ,$$

where the jump times are: $T_k = \frac{1}{\lambda}\sum_{i=0}^k \mathbf{e}_i$. Here the sequence $\mathbf{e}$ are i.i.d standard exponential random variables.

As customary, we write

$$N_{t,t+s} := N_{t+s} - N_{t^-} \ .$$

$$\Delta N_t := N_t - N_{t^-} \ .$$

The formal definition is as follows. ASEP(p, $q<1$) is defined as the unique Markov process $(\zeta_t, t\geq 0)$ with:

- State space $\{0,1\}^{\mathbb{Z}+\frac{1}{2}}$. By definition $\zeta_t^i = 1$ if and only if there is a particle at position i.
- Filtration $\mathcal{F}_t := \sigma\left(\zeta_s; s \leq t\right)$ is the filtration generated by the coordinate process.
- Clocks as follows. Recall that clocks give the stopping times where jumps can occur. There are two families of clocks. On the one hand, there are the clocks for positive jumps:

  $$\left(N_t^{+,x}; t\geq 0, x \in \mathbb{Z}+\frac{1}{2}\right) \ ,$$

  which are independent $PP(p=1)$, one for every site. On the other hand, we have the clocks for negative jumps:

  $$\left(N_t^{-,x}; t\geq 0, x \in \mathbb{Z}+\frac{1}{2}\right) \ ,$$

  which are independent $PP(q<1)$, one again for every site.

  When a clock "rings", i.e $N^{\pm,x}$ jumps for a certain x, two possibilities occur: either the site is contains a particle which is allowed to jump and the jump occurs; either the site is empty or contains a blocked particle. In this latter case, nothing happens.

Let us discuss whether this process is defined for all times. It turns out, one needs special initial conditions. To that endeavor, consider the sites where particles are allowed to move at time $t \in \mathbb{R}_+$:

$$A_t^{\pm} := \left\{ x \in \mathbb{Z} + \frac{1}{2} \mid \zeta_t^{x \pm 1} - \zeta_t \right\} .$$

The superscript $\pm$ indicates that we are either considering positive or negative jumps. The time of the next jump event, either to the right after t is:

$$\begin{aligned} \tau_t^{\pm} &:= \left\{ s \geq t \mid \exists x \in A_t^{\pm}, \Delta N_s^{\pm, x} \right\} \\ &= \min_{A_t^{\pm}} \inf \left\{ s \geq t \mid \Delta N_s^{\pm, x} \right\} . \end{aligned}$$

Because of the absence of memory of the exponential random variable:

$$(\mathbf{e} \mid \mathbf{e} > t) \stackrel{\mathcal{L}}{=} t + \mathbf{e} ,$$

Exercise 1.2. *Prove it!*

one finds that

$$\left(\tau_t^{\pm} - t \mid \mathcal{F}_t \right) \stackrel{\mathcal{L}}{=} \begin{cases} \frac{\mathbf{e}}{|A_t^+|} & \text{if } + \\ \frac{q\mathbf{e}}{|A_t^-|} & \text{if } - \end{cases}$$

At this point, we deduce that there are no accumulations of jumps if and only if $|A_t^+|, |A_t^-| < \infty$ for all $t \geq 0$. This imposes serious contrains on the admissible initial conditions. The initial condition we will consider is:

$$\xi_0 := \mathbb{1}_{\{x<0\}} ,$$

which is the case where all particles are packed to the left.

Exercise 1.3. *Prove by recurrence that if ξ_0 is the "packed" initial condition, then indeed, there are no accumulation of jumps.*

1.2. **ASEP as a growth process.** In this subsection, we cast the ASEP $(\zeta_t; t \geq 0)$ into the corner growth model $(h_t; t \geq 0)$. The new process $(h_t; t \geq 0)$ is an randomly growing interface and $h_t : \mathbb{Z} \to \mathbb{N}$, represents the height at time t. The state state space consists of piece-wise linear functions with slope $\pm$. Slope is constant on intervals of the form $[n, n+1]$, $n \in \mathbb{N}$.

The correspondence is as follows. For every site $x \in \mathbb{Z} + \frac{1}{2}$, we put in correspondence a particle ($\zeta_t^x = 1$) with a negative slope ($\Delta h_t^x = -1$), and a hole is correspondence with a positive slope.

Remark 1.4. *There is a discrete time analogue where every growth site grows with probability p, while every already grown site deflates with probability $q < p$.*

So far, we are always aiming at having a disymmetry ($q < p$). As such there is, on average, a positive flux of particules going from the left to the right. ASEP is intrisically an "out-of-equilibrium" dynamic! It makes no sense to study the possible convergences to stationary measures, given our initial condition. Nevertheless, there could be initial conditions which allow convergence to an equilibrium.

Conjecture 1.5 (KPZ conjecture)**.** *Under fairly general assumptions on the growth dynamic, with a smoothing mechanism, noise and dissymmetry there should be an 1:2:3 scaling:*

$$\mathcal{H}_\varepsilon(t, x) = \varepsilon^{\frac{1}{2}} h_{\lfloor t/\varepsilon^{\frac{3}{2}} \rfloor}(x/\varepsilon) - \varepsilon^{-1} a_t$$

so that $\mathcal{H}_\varepsilon \stackrel{\varepsilon\to 0}{\longrightarrow} \mathcal{H}$*, where* $\mathcal{H}$ *solves the SPDE (Stochastic Partial Differential Equation):*

$$\frac{\partial \mathcal{H}}{\partial t} = \frac{\nu^2}{2}\frac{\partial^2 \mathcal{H}}{\partial x^2} + \frac{\lambda}{2}\left(\frac{\partial \mathcal{H}}{\partial x}\right)^2 + \sigma \cdot W \ .$$

One should think of this conjecture as an idealized version of invariance principle for random growth processes, in the spirit of Donsker's theorem, which concerns random walks:

Theorem 1.6 (Donsker)**.** *If* $X_1, X_2, \dots$ *are iid random variables with* $\mathbb{E}X_1 = 0$ *and* $\mathbb{E}X_1^2 = 1$*, and:*

$$h(n) = \sum_{i=1}^n X_i \ ,$$

$$\mathcal{H}_\varepsilon(x) = \varepsilon^{\frac{1}{2}} h(x/\varepsilon) \ .$$

Then, $\mathcal{H}_\varepsilon$ *converges in law as* $\varepsilon \to 0$ *i.e for the topology of weak convergence of measures on the Polish space of continuous functions. The limit is a universal object called Brownian motion.*

1.3. **TASEP as last passage percolation.** TASEP stands for Totally ASymmetric Exclusion Process. This is the case where asymmetry is total by imposing no jumps to the left. A definition in one line is:

$$\text{TASEP} := \text{ASEP}(p = 1, q = 0) \ .$$

1.4. **TASEP as queues in tandem.** Consider infinitely many customers labelled $\{1, 2, \dots\}$, and infinitely many servers labelled by $\{1, 2, \dots\}$ as well.

The service follows the FIFO (First-In-First-Out) policy and customers follow each other sequencially.

The servicing time is iid with law $\mathcal{L}(\omega)$.

Proposition 1.7. *Let* $\tau_{k,l}$ *be the time when customer* l *leaves the* k*-th server, and* $\omega_{k,l}$ *be his servicing time. Then, we have the equality between processes:*

$$\left(\tau_{k,l}; (k,l) \in \mathbb{N}_*^2\right) \stackrel{\mathcal{L}}{=} \left(L^{k,l}; (k,l) \in \mathbb{N}_*^2\right) \ .$$

Proof. Show that τ and L satisfy exactly the same recurrence. □

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2. Analytical aspects

2.1. **The law of large numbers: Existence of asymptotic shapes.** The result is that, upon observing a specific direction $(x, y) \in \mathbb{R}_+^2$, $L(\lfloor Nx \rangle, \lfloor Ny \rfloor)$ grows linearly.

Theorem 2.1. *There is a deterministic function* $S : \mathbb{R}_+^2 \to \mathbb{R} \sqcup \{\infty\}$ *such that for all* (x, y)

$$\lim_{N\to\infty} \frac{L\left(\lfloor Nx \rfloor, \lfloor Ny \rfloor\right)}{N} = s(x, y) \ a.s \ ,$$

where the shape function s *satisfies:*

- *(Homogeneity)*

$$\forall c \geq 0, \forall (x,y),\ \ s(cx,cy) = cs(x,y)\ .$$

- *(Superadditivity)*

$$\forall (x_1,y_1,x_2,y_2),\ \ s(x_1,y_1) + s(x_2,y_2) \leq s(x_1+x_2, y_1+y_2)\ .$$

- *(Symmetry)*

$$\forall (x,y),\ \ s(x,y) = s(y,x)\ .$$

- *(Concavity)*

$$\forall t \in [0,1], (x_1,y_1,x_2,y_2),\ ts(x_1,y_1)+(1-t)s(x_2,y_2) \leq s(tx_1+(1-t)x_2, ty_1+(1-t)y_2)\ .$$

- *(Monotonicity upon correcting by the mean)*

$$s(x+h,y) = s(x,y) + h\mathbb{E}(\omega)\ .$$

Moreover, if $\mathcal{L}(\omega)$ has sufficiently many moments, the shape function s is continuous.

In fact there is an "asymptotic shape" for the ball $\frac{1}{t}\mathcal{B}(t)$ as $t \to$. This theorem is a more refined version of the previous one, since one needs some uniformity of the convergence in all possible directions $(x,y) \in \mathbb{R}_+^2$. This will be done at the end of this section 2.

2.1.1. *Kingman's superadditive ergodic theorem.* The main tool for proving Theorem 2.1.

The following theorem, with a surprising number of applications, has a long history. The form we are presenting, with two indices, is in fact due to Liggett (1985). The proof is not optimal and is simplified since we are only considering DLPP. Nevertheless, we state the theorem in its optimal form.

Theorem 2.2 (Kingman's superadditive ergodic theorem)**.** *Suppose that $(X_{m,n}, 0 \leq m < n)$ is a random sequence in L^1, which satisfies the following hypotheses:*

- *Superadditivity:*

$$X_{0,m} + X_{m,n} \leq X_{0,n}$$

- *For all $m \in \mathbb{N}$,*

$$\left(X_{nm,(n+1)m}; n \geq 1\right)$$

  is stationary and ergodic - Although we will only need iid.

- $$\mathcal{L}\left(X_{m,m+k}; k \geq 1\right)$$

  does not depend on m.

Then $\lim_{n\to\infty} \frac{X_{0,n}}{n} = \gamma$ exists almost surely and in $L^1(\Omega)$. Moreover $\gamma = \sup_{n\in\mathbb{N}} \mathbb{E}\frac{X_{0,n}}{n} \in \mathbb{R} \cup \{\infty\}$.

Proof. See scanned notes. □

2.1.2. *Proof of the shape theorem.* See scanned notes.

2.1.3. *Finiteness criterion.* First, let us notice that $s(x, y) = \infty$ is possible and an example can be easily constructed using heavy tailed environments ω. For example, we can pick $\omega \stackrel{\mathcal{L}}{=} Pareto(\alpha)$, where the Pareto distribution is defined by

$$\mathbb{P}\left(\omega \in dx\right) = \mathbb{1}_{x \geq 1} \frac{1}{x^{1+\alpha}} \frac{1}{\alpha} \ ,$$

or equivalently

$$\mathbb{P}\left(\omega \geq x\right) = \frac{1}{x^{\alpha}} \ .$$

Lemma 2.3.

$$\left(\max_{1 \leq i,j \leq n} \omega_{i,j} \right) \frac{1}{n^{2/\alpha}}$$

converges in law as $n \to \infty$ to a non-trivial limit that is positive.

Proof. Classical from extremal statistics. □

From that, one can easily bound from below the last passage time by the maximal value on the grid. For $1 < \alpha < 2$, the shape function is infinite although $\mathbb{E}\omega < \infty$.

Now let us give a finiteness criterion. Let $F(x) := \mathbb{P}(\omega \leq x)$ be the cumulative distribution function of ω. Without loss of generality, we can assume $\mathbb{E}(\omega) = 0$. The main proposition of this section is:

Proposition 2.4. *If $\int_0^\infty \sqrt{1 - F(x)} dx < \infty$, then $s(x, y) < \infty$.*

In particular, this is true if $\mathbb{E}|\omega|^{2+\varepsilon} < \infty$.

Proof. See scanned notes. □

2.2. **Concentration around the mean.**

2.2.1. *Talagrand's concentration inequality.* Talagrand's concentration inequality is a fairly general inequality

2.2.2. *Application to DLPP.*

3. Integrable aspects

Generalities.

3.1. **Combinatorics of Young tableaux.**

3.1.1. *Definitions.*

3.1.2. *Schur functions.*

3.1.3. *The Robinson-Schensted-Knuth correspondence.*

3.1.4. *Greene's Lemma.*

3.2. **The Schur process.**

3.2.1. *Definition.*

3.2.2. *Determinantal point processes.*

3.3. **Asymptotic analysis of a Fredholm determinant.**