From last time

Exercise: $\begin{cases} X \leq Y & a.s \\ X \overset{\mathcal{L}}{=} Y \end{cases} \implies X = Y$ a.s

Solution: $\forall t \in \mathbb{R}$, $\mathbb{1}_{\{X \leq t\}} - \mathbb{1}_{\{Y \leq t\}} \geq 0$

But $0 = \mathbb{E}\left(\mathbb{1}_{\{X \leq t\}} - \mathbb{1}_{\{Y \leq t\}}\right) = \mathbb{P}(X \leq t) - \mathbb{P}(Y \leq t)$

Hence $\forall t \in \mathbb{R}$, (fixed) $\mathbb{1}_{\{X \leq t\}} = \mathbb{1}_{\{Y \leq t\}}$ a.s $\uparrow$ null set depend on t.

$\rightsquigarrow$ a.s $\forall t \in \underline{\underline{\mathbb{Q}}}$, $\mathbb{1}_{\{X \leq t\}} = \mathbb{1}_{\{Y \leq t\}}$

$\rightsquigarrow$ $X = Y$ a.s.

Corollary of proof: We assume $\mathbb{E}|\omega| < +\infty$

Either $s(x, y) = +\infty$ everywhere

or $s(x, y) < +\infty$ & is continuous

3 Finiteness criterion

First, let us notice that $s(x, y) = +\infty$ is possible and can be easily constructed using heavy tailed environments

For example pick $\begin{cases} \omega \overset{\mathcal{L}}{=} \text{Pareto}(\alpha) \\ \mathbb{P}(\omega \in dx) = \mathbb{1}_{\{x \geq 1\}} \dfrac{1}{x^{1+\alpha}} \dfrac{1}{\alpha} \\ \mathbb{P}(\omega \geq x) = \dfrac{1}{x^\alpha} \ ; x \geq 1, \ \alpha > 0 \end{cases}$

Lemma: $\left(\max\limits_{1 \leq i, j \leq n} \omega_{ij}\right) \dfrac{1}{n^{2/\alpha}}$ converges in law, as $n \to +\infty$ to a non trivial limit that is positive.

Proof: The proof uses the standard computat° from extremal statistics:

$$\mathbb{P}\left(\max_{1 \leq i,j \leq n} \omega_{ij} \cdot \frac{1}{n^{2/\alpha}} \geq x \right)$$

$$= 1 - \left(1 - \mathbb{P}(\omega \geq x\, n^{2/\alpha}) \right)^{n^2}$$

$$= 1 - \left(1 - \frac{1}{x^\alpha n^2} \right)^{n^2} \sim 1 - e^{-1/x^\alpha}$$

$\boxtimes$

As such, if $1 < \alpha < 2$, we have $\mathbb{E}|\omega| < +\infty$

but no asymptotic shape because of the following:

$$\frac{L(n,n)}{n} \geq \frac{\mathcal{E}(\pi_0)}{n} \quad \text{where } \pi_0 \equiv \text{Path passing from the maximal energy}$$

$$\geq \frac{\max_{1 \leq i,j \leq n} \omega_{ij}}{n} \qquad (\omega_{ij} \geq 0)$$

$$= \underbrace{n^{-2/\alpha} \max_{1 \leq i,j \leq n} \omega_{ij}}_{\longrightarrow X \text{ limit in law}} \quad \underbrace{n^{\frac{2}{\alpha}-1}}_{\longrightarrow +\infty}$$

$$\rightsquigarrow \quad \underline{\underline{g(x,y) = +\infty}}$$

Now let us state a criterion that gives finiteness

Let $F(x) = \mathbb{P}(\omega \leq x)$ be the cumulative distribution function of ω. WLOG, we can assume $\mathbb{E}(\omega) = 0$.

The main proposition of this section is

Proposition: If $\int_0^{+\infty} \sqrt{1-F(x)}\, dx < +\infty$

then $S(\alpha, \beta) < +\infty$.

In particular, this is true if $\mathbb{E}\,|\omega|^{2+\varepsilon} < +\infty$

Proof: $(1-F(x))^{1/2} = \mathbb{P}(\omega \geq x)^{1/2} \overset{\text{Markov}}{\leq} \left(\dfrac{\mathbb{E}\,|\omega|^{2+\varepsilon}}{x^{2+\varepsilon}}\right)^{1/2}$

$$= \frac{\left(\mathbb{E}\,|\omega|^{2+\varepsilon}\right)^{1/2}}{x^{1+\varepsilon/2}} \quad \text{integrable.}$$

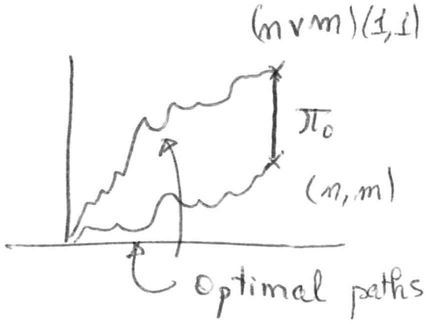
1/ Reductions: We assume $(1-F(x))^{1/2}$ is integrable.

It suffices to show that $\displaystyle\sup_{(n,m)\geq 0} \frac{\mathbb{E}\, L^{n,m}}{n+m} < +\infty$

because of the L^1 convergence in the shape theorem.

Now, because ~~$L^{n,m} + (n \vee m - n \wedge m)$~~

$$L^{n,m} + \sum_{(i,j)\in\pi_0} \omega_{ij} \leq L^{n\vee m\,(1,1)}$$



we have $\displaystyle\frac{\mathbb{E}\,L^{n,m}}{n+m} \leq \frac{\mathbb{E}\,L^{n\vee m(1,1)}}{n\vee m}\ \underbrace{\frac{n\vee m}{n+m}}_{\leq 1}$

& we only need to prove $\displaystyle\sup_{N\in\mathbb{N}} \frac{\mathbb{E}\,L(N(1,1))}{N} < +\infty$.

2/ Comparison to Bernoulli:

Let us write $\mathbb{E}_{\mathcal{L}(\omega)} \dfrac{L(N(1,1))}{N}$ when $\mathcal{L}(\omega)$ is taken as the law of the environment.

A useful remark is the following $\longrightarrow$

IV

Remark: If ω_1 & ω_2 can be coupled so that $\omega_1 \leq \omega_2$

then $\mathbb{E}_{\mathcal{L}(\omega_1)} \frac{L(N(1,1))}{N} \leq \mathbb{E}_{\mathcal{L}(\omega_2)} \frac{L(N(1,1))}{N}$

Consequences: $\mathbb{E}_{\mathcal{L}(\omega)} \frac{L(N(1,1))}{N} \leq \mathbb{E}_{\mathcal{L}(\omega^+)} \frac{L(N(1,1))}{N}$

$$\varphi_N(p) := \mathbb{E}_{Ber(p)} \frac{L(N(1,1))}{N}$$

≤ 1 and decreasing in p

because of the coupling

$Ber(p) = \mathbb{1}_{\{U \leq p\}}$.

Now
$$\mathbb{E} \frac{L(N(1,1))}{N} = \mathbb{E} \frac{1}{N} \max_{\pi : (1,1) \to N(1,1)} \sum_{(i,j)\in\pi} \omega_{ij}$$

$$\leq \mathbb{E} \frac{1}{N} \max_{\pi : (1,1) \to N(1,1)} \sum_{(i,j)\in\pi} \underbrace{\omega_{ij}^+}_{\int_0^{+\infty} \mathbb{1}_{\{\omega_{ij} \geq x\}} dx}$$

$$= \mathbb{E} \frac{1}{N} \max_{\pi : (1,1) \to N(1,1)} \int_0^{+\infty} dx \left(\sum_{(i,j)\in\pi} \mathbb{1}_{\{\omega_{ij} \geq x\}} \right)$$

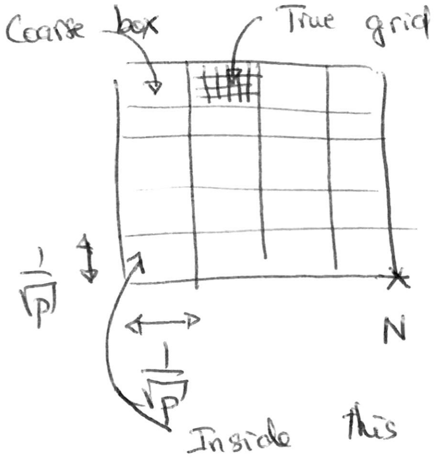
$$\leq \int_0^{+\infty} dx \, \frac{1}{N} \mathbb{E} \max_{\pi : (1,1) \to N(1,1)} \int_0^{+\infty} dx \, \mathbb{1}_{\{\omega_{ij} \geq x\}}$$

$$= \int_0^{+\infty} dx \, \mathbb{E}_{Ber(1-F(x))} \frac{L\, N(1,1)}{N}$$

3/ A coarse graining argument

Every path $\pi : (1,1) \to N(1,1)$ leads to a path $\tilde{\pi} : (1,1) \to \sqrt{p} N(1,1)$ on a coarser grid $\frac{1}{\sqrt{p}} \times \frac{1}{\sqrt{p}}$

This coarser grid is obtained by grouping boxes together.



box are $\frac{1}{\sqrt{p}} \times \frac{1}{\sqrt{p}} = p^{-1}$ Bernoulli's

$$\boxed{\mathrm{V}}$$

$$L(N(1,1)) = \max_{\pi : (1,1) \to N(1,1)} \mathcal{T}(\pi)$$

$$\leqslant \max_{\tilde{\pi} : (1,1) \to (\sqrt{p}\, N)(1,1)} \sum_{(i,j) \in \tilde{\pi}} X_{ij}$$

where X_{ij} is the $\underline{\underline{\text{maximal}}}$ / $\underline{\underline{\text{total}}}$ energy in the coarser box:

$$X_{ij} = \sum_{\mathcal{B}_p \in (i,j)} \mathcal{B}_p$$

$$\overset{\mathcal{L}}{=} \text{Binomial}\left(\left(\frac{1}{\sqrt{p}}\right)^2, p\right)$$

$$= \text{Binomial}\left(p^{-1}, p\right)$$

Intuition:

$\text{Binomial}(p^{-1}, p) \approx \text{Poisson}(1)$ as $p \to +\infty$

As such

$$\mathbb{E}_{Ber(p)} \frac{L(N(1,1))}{N}$$

$$\leqslant \mathbb{E}_{\text{Binomial}(p^{-1}, p)} \frac{L\,(\sqrt{p}\, N)(1,1)}{\sqrt{p}\, N} \sqrt{p}$$

$$\leqslant \sup_{N \in \mathbb{N}} \mathbb{E}_{\text{Bin}(p^{-1}, p)} \frac{L\, N(1,1)}{N} \cdot \sqrt{p}$$

$$= S_{\text{Bin}(p^{-1}, p)}(1,1)\, \sqrt{p}$$

From the second step:

$$\mathbb{E}\, \frac{L(N(1,1))}{N} \leqslant \int_0^{+\infty} dx\, \sqrt{1-F(x)}\; \overbrace{S_{\text{Bin}(p^{-1}, p)}(1,1)}^{\circledast}$$

$\uparrow\ p = 1 - F(x)$

we only have to ~~to~~ prove that

$$\circledast \simeq S_{\text{Poisson}(1)}(1,1) < +\infty.$$

as $x \to +\infty$

4/ Finiteness for Poisson (or approximate Poisson) for large deviations. $\boxed{\text{VI}}$

For general ω:

$$\mathbb{P}\left(\frac{L(N(1,1))}{2N} \geqslant x \right) \leqslant \# \{\pi : (1,1) \to N(1,1)\} \, \mathbb{P}\left(\sum_{(i,j)\in\pi_0} \omega_{ij} \geqslant 2Nx\right)$$

$$= \binom{2N}{N} \mathbb{P}\left(\sum_{i=1}^{2N} \omega_i \geqslant 2Nx \right)$$

$$\leqslant 2^{2N} e^{-2N\Lambda^*(x)} \quad \text{(Cramer bound)}$$

For $\omega \overset{\mathcal{L}}{=} \text{Poisson}(1)$, then

$$\Lambda(t) = \log \mathbb{E}\, e^{t\,\text{Poisson}(1)} \overset{\text{exercise!}}{=} e^t - 1$$

$$\Lambda^*(x) = \sup_{t\geqslant 0} tx - \Lambda(t) \underset{\text{exercise!}}{=} 1 + \log\left(\frac{x}{e}\right)$$

Hence for, say, $x = e$:

$$\mathbb{P}\left(\frac{L(N(1,1))}{2N} \geqslant e \right) \leqslant 2^{2N} e^{-2N} = \left(\frac{2}{e}\right)^{2N} \quad \text{summable.}$$

Borel-Cantelli $\rightsquigarrow$ $\underbrace{\limsup_{N\to+\infty} \frac{L(N(1,1))}{2N}}_{S_{\text{Poisson}(1)}(1,1)} \leqslant e < +\infty$

Now for p small enough, $\Lambda^*_{\text{Binomial}(p^{-1},p)}(x) \overset{p\to 0}{=} \Lambda^*(x) + o(1)$

since $\Lambda_{\text{Binomial}(p^{-1},p)}(x) = p^{-1}\log(1+p(e^x-1))$

Thus for p small enough: $= (e^x - 1)(1+o(1))$

uniformly on compact sets

$S_{\text{Binomial}(p^{-1},p)} \leqslant 2e < +\infty$ $\boxed{\text{VII}}$