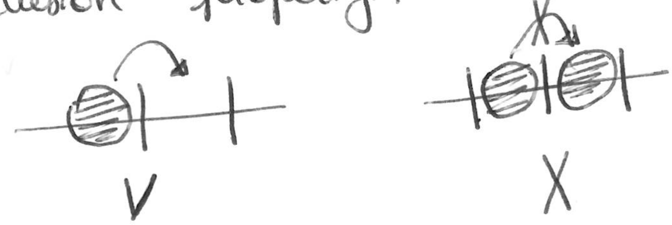
Particle systems: analytic and integrable aspects.

I ASEP & related models.

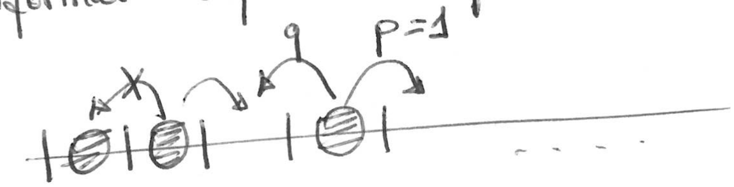
1 ASEP $\equiv$ ASymmetric Exclusion process

is a process of particles on $\mathbb{Z} + 1/2$.

The exclusion property:



- Informal definition for ASEP (p, q) $1 =$ [redacted] $p > q \geq 0$



- Formal definition:

↳ State space : $\{0,1\}^{\mathbb{Z}+1/2}$

↳ Process : Markov jump process $\xi_t \in \{0,1\}^{\mathbb{Z}+1/2}$

By definition $\xi_t^i = 1$ if and only if there is a particle at posit° i. ‖ ↳ Filtrat°: $\mathcal{F}_t = \sigma(\xi_s, t \geq s \geq 0)$

↳ Clocks: Define a family of Poisson processes

(Recall N_t ~~standard~~ Poisson process with intensity λ

$\iff$ N_t is the unique càdlàg process

s.t $\begin{cases} \Delta N_s \in \{0,1\} & \forall s \in \mathbb{R}_+ \\ N_t - \lambda t & \text{is an } \mathcal{F}_t\text{-martingale} \end{cases}$

$\iff$ $N_t = \sum_{k=0}^{+\infty} \mathbb{1}_{\{T_k \leq t\}}$ where $T_k = \sum_{i=0}^{k} \lambda \mathcal{E}_i$ [redacted])

$\uparrow$ iid

Clocks for positive jumps; parameter $p=1$

$\left(N_t^{+,x} ; t \geq 0, x \in \mathbb{Z}+1/2\right)$ ⇝ 1 clock per posit°.

Clocks for negative jumps; parameter $q<1$

$(N_t^{-,x} ; t \geq 0 ; x \in \mathbb{Z}+1/2)$

$\begin{array}{cc} 1 & 0 \\ x & x+1 \end{array}$

↳ Non exclusive posit° at time t:

$$A_t^{(+)} = \{x \in \mathbb{Z}+1/2 \mid \xi_t^{x+1} - \xi_t^{x} = -1\}$$

$$A_t^{(-)} = \{x \in \mathbb{Z}+1/2 \mid \xi_t^{(x+1)} - \xi_t^{x} = -1\}$$

↳ Next jump time conditionally to $\mathcal{F}_t$:

$$\tau_t^{+/-} := \inf\{s \geq t \mid \exists x \in A_t^{+/-}, \Delta N_s^{+/-,x} = 1\}$$

$$= \min_{x \in A_t^{+/-}} \inf\{s \geq t \mid \Delta N_s^{+/-,x} = 1\}$$

$$⇝ \quad \tau_t^{+/-} - t \mid \mathcal{F}_t = \min_{x \in A_t^{+/-}} \underbrace{\inf\{s \geq 0 \mid \Delta N_{t+s}^{+/-,x} = 1\} \mid \mathcal{F}_t}_{\substack{\overset{\mathcal{L}}{=} \mathcal{E}_x \text{ if } + \\ \overset{\mathcal{L}}{=} p\,\mathcal{E}_{x} \text{ if } -}}$$

Note this is useful for simulat°

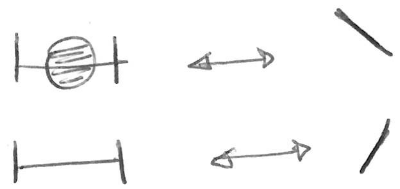
$$\overset{\mathcal{L}}{=} \begin{cases} \dfrac{\mathcal{E}}{|A_t^+|} & \text{if } + \\ \dfrac{\cancel{p}\, q\, \mathcal{E}}{|A_t^-|} & \text{if } - \end{cases}$$

⚠ In order for the process to be well-defined, one needs $|A_t^+|, |A_t^-| < +\infty$ for every $t \in \mathbb{R}_+$

⇝ Common initial condition: $\xi_0 = \mathbb{1}_{\{x<0\}}$ 0000

[2] ASEP as growth process.

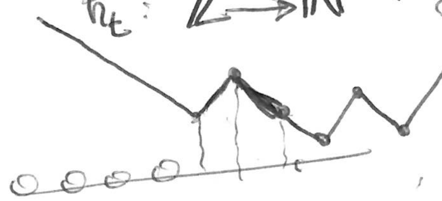
Applying the correspondence



one transforms :

$h_t : \mathbb{Z} \to \mathbb{N}$ height funct°

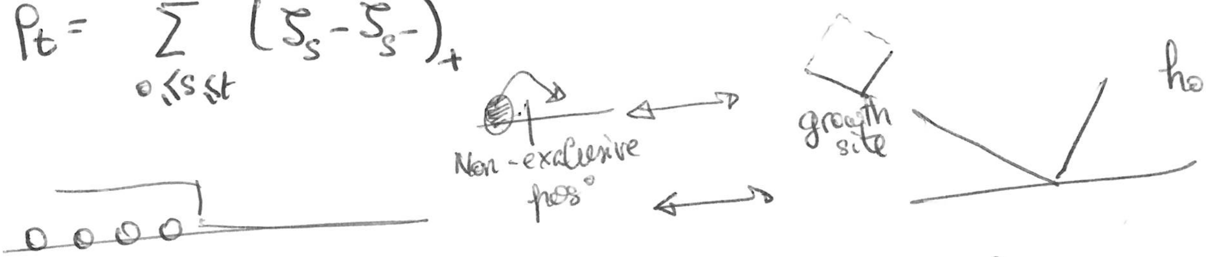
$\xi_t \in \{0,1\}^{\mathbb{Z}+1/2}$ $\longleftrightarrow$



+ Flux at 0 = Number of particles that crossed 0 $\longleftrightarrow$ $h_t(0) = 2$

$$\rho_t = \sum_{0 \le s \le t} \left(\xi_s - \xi_{s^-} \right)_+^{x=0}$$

$h_0(x) = |x|$



<u>Rmk</u>: There exists a discrete time analogue where every $\begin{cases} \text{growth site grows with probability } p \\ \text{grown site deflates } \text{———} \ q < p \end{cases}$

<u>Rmk</u>: So far, we are aiming at having a dissymetry $(q < p)$. As such ASEP is intrisically an "out-of-equilibrium" dynamic

<u>Conjecture</u> [KPZ]

Under fairly general asymptions on the growth dynamic, with dissymmetry there should be a $1:2:3$ scaling

$$\mathcal{H}_\varepsilon(t,x) = \varepsilon^{\frac{1}{2}} h_{\lfloor t/\varepsilon^{3/2} \rfloor}(x/\varepsilon) - \varepsilon^{-1} a_t$$

so that $\mathcal{H}_\varepsilon \longrightarrow \mathcal{H}$ where $\mathcal{H}$ solves the SPDE:

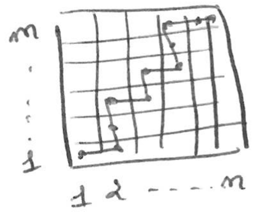
$$\frac{\partial \mathcal{H}}{\partial t} = \frac{\nu^2}{2} \frac{\partial^2 \mathcal{H}}{\partial x^2} + \frac{\lambda}{2} \left(\frac{\partial \mathcal{H}}{\partial x} \right)^2 + \sigma \dot{W}$$

$\boxed{3}$ TASEP as last passage percolat°.

TASEP = Totally ASEP.

i.e $q = 0$,

Consider the grid $\mathbb{N}_*^2$ where every point (i,j) is identified with the box $(i,j) + [-1, 0]$



- Environment $(\omega_{ij} \, ; \, (i,j) \in (\mathbb{N}^*)^2)$ as iid with law $\mathcal{L}(\omega)$.
- The set of admissible paths:

$$\Pi = \coprod_{\ell, k, n, m} \Pi^{n,m}_{k,\ell} \qquad \begin{cases} \Pi^{n,m} := \{ \pi \text{ up-right from } (1,1) \text{ to } (n,m) \} \\ \Pi^{n,m}_{\ell,k} = \{ \text{ — — — — } (k,\ell) \text{ to } (n,m) \} \end{cases}$$

- Energy of a path: $\pi \in \Pi$ is

$$E(\pi) = \sum_{(i,j) \in \pi} \omega_{ij}$$

- DLPP functional (Directed Last Passage Percolation)

$$L^{n,m}_{k,\ell} := \max_{\pi \in \Pi^{n,m}_{k,\ell}} E(\pi)$$

for shorter notat° $L(n,m) = L^{n,m}_{1,1}$.

- An optimal path π^* is s.t $L^{n,m} = E(\pi^*)$

Properties:

- $L^{n,m} = \omega_{n,m} + \max(L_{n-1,m}, L_{n,m-1})$
- (Superadditivity) $L^{k,\ell}_{1,1} + L^{n,m}_{k+1,\ell+1} \leqslant L^{n,m}_{1,1}$

Rmk: Why the name of DLPP?

- "Directed": Privileged direction up-right
- "Last passage": If w_{ij} are seen as time the max suggests that we are interested in the slowest paths.

$\rightsquigarrow$ $L^{n,m}$ is the (last) moment where all paths have led to that point.

In contrast, $\min_{\pi \in \Pi^{n,m}} \mathcal{T}(\pi)$ is refered to as "first passage" and the quantity of interest is a geodesic distance $\rightsquigarrow$ Important for random geometry (of graphs).

- "Percolation": $w_{ij} = \begin{cases} -\infty & \text{path closed} \\ * & \text{path open.} \end{cases}$

$L_1^{n,m} > -\infty \iff \exists$ open path from $(1,1)$ to (n,m).

Convent° : $L^{n,m} = 0$ if $n \leq 0$ or $m \leq 0$.

In order to see the relationship with the corner growth model, define the ball of radius t:

$$\mathcal{B}(t) := \{ (n,m) \in \mathbb{Z}^2 \mid L^{n,m} \leq t \}.$$

Proposition: [DLPP $\simeq$ Corner Growth]

Assume $w \overset{\mathcal{L}}{=} \mathcal{E}$.

After rotation of $45°$, $(\mathcal{B}(t); t \geq 0) \overset{\mathcal{L}}{=} (h_t ; t \geq 0)$

Proof: • $\partial\mathcal{B}(t)$ has state space in $\{$ Piece wise linear functo with slope $\pm 1 \}$ and so does $(h_t ; t \geq 0)$

- By definition h is Markov with rate 1 for every growth corner

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- All we need to prove is that $\mathcal{B}(t)$ is Markov with the appropriate transitions.

For every growth corner (i,j):

$$t < L_{ij} = w_{ij} + \underbrace{L_{i-1,j} \vee L_{i,j-1}}_{\mathcal{F}_t \text{ measurable as by definition } \begin{cases} L^{i-1,j} \leq t \\ L^{i,j-1} \leq t \end{cases}}$$

$\mathcal{F}_t = (\mathcal{B}(s); s \leq t) = (L^{ij}; L^{ij} \leq t)$

The remaining time is

$$\left(L^{ij} - t \mid \mathcal{F}_t \right)$$

$$\stackrel{\mathcal{L}}{=} \Big(w_{ij} - \underbrace{\left(L^{i-1,j} \vee L^{i,j-1} + t \right)}_{x} \mid \mathcal{F}_t \Big)$$

$$= \left(w_{ij} - x \mid w_{ij} > x \right)$$

$$\stackrel{\mathcal{L}}{=} \mathcal{E} \quad \text{(ABSENCE OF MEMORY!!)}.$$

Exercise [Discrete time analogue]

Assume that $w \stackrel{\mathcal{L}}{=} 1 + \mathrm{Geom}(q)$ and $p = (1-q)$ and consider the discrete time corner growth (pick growth site as growing w/ proba p).

Then $(\mathcal{B}(t)\,;\ t \in \mathbb{N}) \stackrel{\mathcal{L}}{=} (h_t,\ t \in \mathbb{N})$

4 TASEP as queues in tandem.

Consider infinitely many customers $\{1, 2, 3, \dots\}$ and infinitely many servers $\{1, 2, 3, \dots\}$.

The service follows the FIFO policy and the servicing time is iid with law $\mathcal{L}(w)$.

Customers follow each other sequencially.

Proposition. If $\begin{cases} \tau^{\ell,k} \text{ is the time when customer } \ell \text{ leaves the } k\text{-th server} \\ w_{k,\ell} \text{ is his servicing time.} \end{cases}$

Then $\left(\tau^{k,\ell} ; k,\ell \right) \stackrel{\mathcal{L}}{=} \left(L^{k,\ell} ; (k,\ell) \right)$

Proof:

$$\tau^{k,\ell} = \underbrace{w_{k,\ell}}_{\text{servicing time}} + \underbrace{\text{Entrance time for customer } \ell \text{ at the } k\text{-th server}}$$

$$= \begin{cases} \tau^{k,\ell-1} & \text{if } \ell \text{ was waiting for } \ell-1 \text{ to finish} \\ \tau^{k-1,\ell} & \text{if } k\text{-th server was free when } \ell \text{ left server } k-1. \end{cases}$$

$$= w_{k,\ell} + \tau^{k,\ell-1} \vee \tau^{k-1,\ell}$$

$\blacksquare$