"Reminder" on continous-time Markov chain (Chapter 0) I

Let E be a countable set.

- A process $X = (X_t;\ t \geq 0)$ with values in E is a family of E-valued random variables indexed by $t \in \mathbb{R}_+$
- X is a (homogenous) Markov Process when.

$$\mathbb{P}(X_t = y \mid X_s = x,\ X_{t_1} = x_1,\ \dots,\ X_{t_{n-1}} = x_{n-1})$$

$$= \mathbb{P}(X_t = y \mid X_s = x) \qquad \forall\ t_0 = s < t_1 < \dots < t_n = t$$
$$x_0 = x, \dots, x_n = y \text{ in } E$$

$$= \mathbb{P}(X_{t-s} = y \mid X_0 = x)$$

- A first description :

Proposition : $T_x = \inf\{t \geq 0 \mid X_t \neq x\} = \sup\{t \geq 0 \mid X_t = x\}$ holding time at x, starting at x.

$$\boxed{X \text{ Markov} \implies \forall x,\ T_x \text{ is an exponential r.v}}$$

Proof: $\mathbb{P}_x(T_x \geq t+s \mid T_x \geq t)$

$= \mathbb{P}_x(X_u = x\ \forall 0 \leq u \leq t+s \mid X_u = x,\ \forall\ 0 \leq u \leq t)$

simplify $\downarrow$ $= \mathbb{P}_x(X_{t+u} = x\ \forall 0 \leq u \leq s \mid X_u = x,\ \forall\ 0 \leq u \leq t)$

Markov $\downarrow$ $= \mathbb{P}(X_{t+u} = x\ \forall\ 0 \leq u \leq s \mid X_t = x)$

Time-homogeneity $\downarrow$ $= \mathbb{P}(X_u = x\ \forall\ 0 \leq u \leq s \mid X_0 = x)$

$= \mathbb{P}(T_x \geq s)$

This is the property of absence of memory which characterizes the $\mathcal{E}(x)$ variable with a certain parameter λ_x.

$$\mathcal{E}(x) = \frac{\mathcal{E}}{\lambda_x} \quad \& \quad \mathbb{P}(\mathcal{E} \in dx) = e^{-x}\,\mathbb{1}_{\mathbb{R}_+}(x)\,dx.$$

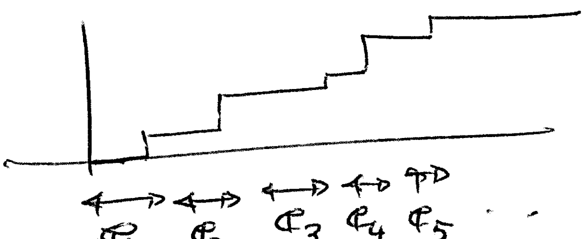
In order to continue, we need to assume some trajectorial regularity for X.
Customary choice : X is càdlàg ("continu à droite, limité à gauche").

Consider the transition matrix (discrete time) :

$$\mathbb{P} = (p_{x,y})_{(x,y)\in E\times E} \quad \& \quad p_{x,y} = \mathbb{P}(X_{T_x^-} = x,\ X_{T_x} = y)$$

defines an "Embedded Markov Chain".

<u>Example</u>: Let $(N_t\ ;\ t\geq 0)$ be the standard Poisson process ie $N_t = \sum_{i=1}^{+\infty} \mathbb{1}_{\{\sum_{i=1}^{k} \mathcal{E}_i \leq t\}}$



This is the natural continuous-time Markov process whose embedded chain lives on $E = \mathbb{N}$ & $p_{x,x+1} = 1$.

- <u>Description via clocks</u>:

We have seen that in order to generate $X = (X_t\ ;\ t\geq 0)$ one waits a time $\lambda_x \mathcal{E}$ then jumps following the transition of the embedded Markov chain.
An equivalent description is possible.

- $q_{x,y} := p_{x,y} \times \lambda_x$

$(T_x^y\ ;\ y\neq x) \equiv$ ind. $\mathcal{E}$ variables with parameter $\frac{1}{q_{x,y}}$

$\hookrightarrow T_x^y = \frac{\mathcal{E}(x,y)}{q_{x,y}}$.

• $T_x = \min_y T_x^y = \dfrac{\mathcal{E}}{\sum_y q_{x,y}} = \dfrac{\mathcal{E}}{\lambda_x}$

Exercise:

$$\boxed{\frac{\mathcal{E}}{\lambda} \wedge \frac{\mathcal{E}'}{\mu} \overset{\mathcal{L}}{=} \frac{\mathcal{E}}{\lambda+\mu}}$$

Hint: $\mathbb{P}\left(\frac{\mathcal{E}}{\lambda} \wedge \frac{\mathcal{E}'}{\mu} \geq t\right) = \mathbb{P}\left(\frac{\mathcal{E}}{\lambda} \geq t\right) \mathbb{P}\left(\frac{\mathcal{E}'}{\mu} \geq t\right)$

In the end, each site $x \in E$ has Poisson clocks $(T_x^y)_{y \neq x}$. The first one to ring gives the next transition via:

Proposition: $\mathbb{P}\left(\min_{\tilde{y}} \dfrac{\mathcal{E}(x,\tilde{y})}{q_{x,\tilde{y}}} = \overbrace{\dfrac{\mathcal{E}(x,y)}{q_{x,y}}}^{T_x^y} \right) = P_{x,y}$

Exercise: Prove it for two exponential variables (3 points).