Chapter 5

Three applications of optimality conditions

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Part1

An exact controllability problem

Exact controllability of the wave equation

The notation : Ω is a bounded open subset in $\mathbb{R}^N$, its boundary Γ is of class C^2, $T > 0$, $Q = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$. For initial data $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, and for terminal data $(z_0, z_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, we look for $u \in L^2(\Sigma)$ so that the solution y to

$$(WE) \qquad \begin{aligned} & y'' - \Delta y = 0 \quad \text{in } Q, \qquad y = u \quad \text{on } \Sigma, \\ & y(0) = y_0 \quad \text{and} \quad y'(0) = y_1 \quad \text{in } \Omega, \end{aligned}$$

satisfies $y(T) = z_0$ and $y'(T) = z_1$.

Since the semigroup corresponding to the wave equation is a group, the wave equation is well posed with terminal conditions, and the controllability problem is equivalent to the null controllability problem.

Indeed if z is the solution to

$$z'' - \Delta z = 0 \quad \text{in } Q, \qquad z = 0 \quad \text{on } \Sigma,$$

$$z(T) = z_0 \quad \text{and} \quad z'(T) = z_1 \quad \text{in } \Omega,$$

and (ζ, u) obeys

$$\zeta'' - \Delta \zeta = 0 \quad \text{in } Q, \qquad \zeta = u \quad \text{on } \Sigma,$$

$$\zeta(0) = y_0 - z(0) \quad \text{and} \quad \zeta'(0) = y_1 - z'(0) \quad \text{in } \Omega,$$

$$\zeta(T) = 0 \quad \text{and} \quad \zeta'(T) = 0 \quad \text{in } \Omega,$$

then $y = z + \zeta$ is the solution to (WE) and it satisfies

$$y(T) = 0 \quad \text{and} \quad y'(T) = 0 \quad \text{in } \Omega.$$

The Hilbert Uniqueness Method

The H.U.Method due to Lions, consists in finding $u \in L^2(\Sigma)$ of minimal norm which solves the null controllability problem.

Penalized problem

$$(P_\varepsilon) \quad \text{inf}\{J_\varepsilon(y,u) \mid (y,u) \text{ obeys } (WE),\ u \in L^2(\Sigma)\},$$

the functionals J_ε is defined by

$$J_\varepsilon(y,u) \\ = \frac{1}{2\varepsilon}\int_\Omega |y(T)|^2 + \frac{1}{2\varepsilon}\|y'(T)\|^2_{H^{-1}(\Omega)} + \frac{1}{2}\int_\Sigma u^2 \,.$$

The method

- Characterize the solution of (P_ε)
- Estimates on y_ε, u_ε
- Passage to the limit

New regularity results for the wave equation Let θ be the solution to

$$\theta'' - \Delta\theta = g \quad \text{in } Q,\ \theta = 0 \quad \text{on } \Sigma,$$

$$\theta(0) = \theta_0, \quad \theta'(0) = \theta_1 \quad \text{in } \Omega.$$

Theorem. The solution θ satisfies the following estimates

$$\|\theta\|_{C([0,T];H_0^1(\Omega))} + \|\theta\|_{C^1([0,T];L^2(\Omega))} + \left\|\frac{\partial\theta}{\partial n}\right\|_{L^2(\Sigma)}$$
$$\leq C\Big(\|\theta_0\|_{H_0^1(\Omega)} + \|\theta_1\|_{L^2(\Omega)} + \|g\|_{L^1(0,T;L^2(\Omega))}\Big).$$

Inverse inequality

Theorem. There exist $T_0 > 0$ and R_0 such that for all $T > T_0$ the following estimate holds

$$(T-T_0)^{1/2}\Big(\|\theta_0\|^2_{H_0^1(\Omega)}+\|\theta_1\|^2_{L^2(\Omega)}\Big)^{1/2} \leq R_0\left\|\frac{\partial\theta}{\partial n}\right\|_{L^2(\Sigma)}.$$

Characterization of y_ε, u_ε

Theorem. The solution y_ε, u_ε to (P_ε) is characterized by

$$u_\varepsilon = \frac{\partial p_\varepsilon}{\partial n},$$

where p_ε is the solution to the adjoint equation corresponding to y_ε:

$$p_\varepsilon'' - \Delta p_\varepsilon = 0 \quad \text{in } Q, \quad p_\varepsilon = 0 \quad \text{on } \Sigma,$$

$$p_\varepsilon(T) = \frac{1}{\varepsilon}(-\Delta)^{-1} y_\varepsilon'(T), \quad p_\varepsilon'(T) = -\frac{1}{\varepsilon} y_\varepsilon(T) \quad \text{in } \Omega.$$

Estimates on y_ε, u_ε

With an integration by parts between y_ε and p_ε we get

$$\left\|\frac{\partial p_\varepsilon}{\partial n}\right\|^2_{L^2(\Sigma)} + \frac{1}{\varepsilon}\|y_\varepsilon(T)\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\|y'_\varepsilon(T)\|^2_{H^{-1}(\Omega)}$$
$$= \Big\langle p_\varepsilon(0), y_1 \Big\rangle_{H^1_0, H^{-1}} - \Big(p'_\varepsilon(0), y_0\Big)_{L^2(\Omega)}.$$

With the inverse inequality and Young inequality we obtain

$$\left\|\frac{\partial p_\varepsilon}{\partial n}\right\|^2_{L^2(\Sigma)} + \frac{1}{\varepsilon}\|y_\varepsilon(T)\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\|y'_\varepsilon(T)\|^2_{H^{-1}(\Omega)}$$
$$\leq C\Big(\|y_0\|^2_{L^2(\Omega)} + \|y_1\|^2_{H^{-1}(\Omega)}\Big).$$

Thus

$$\|u_\varepsilon\|_{L^2(\Sigma)} + \|p'_\varepsilon(0)\|_{L^2(\Omega)} + \|p_\varepsilon(0)\|_{H^1_0(\Omega)} \le C.$$

Passage to the limit

$$p'_\varepsilon(0) \rightharpoonup p_1 \quad \text{weakly in } L^2(\Omega),$$
$$p_\varepsilon(0) \rightharpoonup p_0 \quad \text{weakly in } H^1_0(\Omega),$$
$$u_\varepsilon \rightharpoonup \bar{u} \quad \text{weakly in } L^2(\Sigma),$$
$$y_\varepsilon \rightharpoonup \bar{y} \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)),$$
$$y'_\varepsilon \rightharpoonup \bar{y}' \quad \text{weakly}^* \text{ in } L^\infty(0,T;H^{-1}(\Omega)),$$
$$p_\varepsilon \rightharpoonup \bar{p} \quad \text{weakly}^* \text{ in } L^\infty(0,T;H^1_0(\Omega)),$$
$$p'_\varepsilon \rightharpoonup \bar{p}' \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)),$$

$\bar{u} = \frac{\partial \bar{p}}{\partial n}$, $\bar{y}$ is the solution to (WE) corresponding to $\bar{u}$, $\bar{y}(T) = 0$, $\bar{y}'(T) = 0$, and $\bar{p}$ is the solution to

$$\bar{p}'' - \Delta \bar{p} = 0 \quad \text{in } Q, \qquad \bar{p} = 0 \quad \text{on } \Sigma,$$

$$\bar{p}(0) = p_0 \text{ and } \bar{p}'(0) = p_1 \quad \text{in } \Omega.$$

Since $\bar{u} = \frac{\partial \bar{p}}{\partial n}$, we have

$$\int_\Sigma \left| \frac{\partial \bar{p}}{\partial n} \right|^2 = \left\langle \bar{p}(0), y_1 \right\rangle_{H_0^1, H^{-1}} - \left(\bar{p}'(0), y_0 \right)_{L^2(\Omega)}.$$

Uniqueness of $\bar{u}$

For any $(p_0, p_1) \in H_0^1(\Omega) \times L^2(\Omega)$, set

$$\Lambda(p_0, p_1) = (y'(0), -y(0)),$$

where y is the solution to

$$y'' - \Delta y = 0 \quad \text{in } Q, \quad y = \frac{\partial p}{\partial n} \quad \text{on } \Sigma,$$
$$y(T) = 0 \text{ and } \; y'(T) = 0 \quad \text{in } \Omega.$$

and

$$p'' - \Delta p = 0 \quad \text{in } Q, \quad p = 0 \quad \text{on } \Sigma,$$
$$p(0) = p_0 \text{ and } \; p'(0) = p_1 \quad \text{in } \Omega.$$

Theorem.

(i) Λ is bounded from $H_0^1(\Omega) \times L^2(\Omega)$ into $H^{-1}(\Omega) \times L^2(\Omega)$,

(ii) $\Lambda = \Lambda^*$,

(iii) Λ is an isomorphism from $H_0^1(\Omega) \times L^2(\Omega)$ onto $H^{-1}(\Omega) \times L^2(\Omega)$.

Proof. (i)

$$(p_0, p_1) \longmapsto \frac{\partial p}{\partial n}$$

belongs to $\mathcal{L}((H_0^1(\Omega) \times L^2(\Omega); L^2(\Sigma))$, and

$$\frac{\partial p}{\partial n} \longmapsto (y'(0), -y(0))$$

belongs to $\mathcal{L}(L^2(\Sigma); H^{-1}(\Omega) \times L^2(\Omega))$.

(ii) Set

$$\Lambda(q_0, q_1) = (z'(0), -z(0)),$$

where z is the solution to

$$z'' - \Delta z = 0 \quad \text{in } Q, \quad z = \frac{\partial q}{\partial n} \quad \text{on } \Sigma,$$
$$z(T) = 0 \text{ and } z'(T) = 0 \quad \text{in } \Omega.$$

and

$$q'' - \Delta q = 0 \quad \text{in } Q, \quad q = 0 \quad \text{on } \Sigma,$$
$$q(0) = q_0 \text{ and } q'(0) = q_1 \quad \text{in } \Omega.$$

With an integration by parts between q and y we obtain

$$\int_\Sigma \frac{\partial p}{\partial n}\frac{\partial q}{\partial n} = \Big\langle q_0, y'(0)\Big\rangle_{H_0^1, H^{-1}} - \Big(q_1, y(0)\Big)_{L^2(\Omega)}$$

$$= \Big\langle \Lambda(p_0, p_1), (q_0, q_1)\Big\rangle.$$

Similarly we have

$$\int_\Sigma \frac{\partial p}{\partial n}\frac{\partial q}{\partial n} = \Big\langle \Lambda(q_0, q_1), (p_0, p_1)\Big\rangle.$$

Thus

$$\Lambda = \Lambda^*.$$

(iii) Since

$$\int_\Sigma \left|\frac{\partial p}{\partial n}\right|^2 = \left\langle \Lambda(p_0, p_1), (p_0, p_1) \right\rangle$$

with the direct and inverse inequalities it follows that Λ is injective. But $\Lambda = \Lambda^*$, thus Λ is an isomorphism from $H_0^1(\Omega) \times L^2(\Omega)$ onto $H^{-1}(\Omega) \times L^2(\Omega)$.

Consequence

If we set

$$(p_0, p_1) = \Lambda^{-1}(y_1, -y_0),$$

and if p is the solution of

$$p'' - \Delta p = 0 \quad \text{in } Q, \qquad p = 0 \quad \text{on } \Sigma,$$

$$p(0) = p_0 \text{ and } \; p'(0) = p_1 \quad \text{in } \Omega.$$

then $u = \frac{\partial p}{\partial n}$ is the solution of minimal norm to the null controllability problem.

It is the minimal norm solution because (y, u, p) solves the optimality system of the minimal norm null controllability problem.

Algorithm

Find $(p_0, p_1) = \Lambda^{-1}(y_1, -y_0)$, by solving the minimization problem

$$\inf\{F(p_0, p_1) \mid (p_0, p_1) \in H_0^1(\Omega) \times L^2(\Omega)\},$$

where

$$F(p_0, p_1) =$$
$$\frac{1}{2}\Big\langle \Lambda(p_0, p_1), (p_0, p_1) \Big\rangle - \Big\langle p_0, y_1 \Big\rangle_{H_0^1, H^{-1}} + \Big(p_1, y_0\Big)_{L^2(\Omega)}.$$

This problem can be solved, after discretization, by a conjugate gradient method.

Part2

A Stabilization problem

Setting of the problem

Consider the equation

$$(E) \qquad\qquad y' = Ay + Bu, \qquad y(0) = y_0,$$

where the unbounded operator $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup on Y, denoted by $(e^{tA})_{t\geq 0}$. The operator B belongs to $\mathcal{L}(U;Y)$.

We suppose that $(e^{tA})_{t\geq 0}$ is unstable.

We look for $u \in L^2(0, \infty; U)$, in a feedback form

$$u(t) = Ky(t)$$

so that the closed loop system

$$y' = (A + BK)y, \qquad y(0) = y_0,$$

is exponentially stable on Y. That is

$$\|e^{t(A+BK)}\|_{\mathcal{L}(Y)} \leq Ce^{-\lambda t} \qquad \text{for all } t \geq 0,$$

and for some $\lambda > 0$.

Definition. The pair (A, B) is said to be stabilizable iff there exists $K \in \mathcal{L}(Y; U)$ such that $(e^{t(A+BK)})_{t\geq 0}$ is exponentially stable on Y.

Remark. If the system (E) is null controllable then the pair (A, B) is stabilizable.

Example of a stabilizable system with a bounded control operator.

$$\begin{aligned} & y_t - \Delta y + \vec{V} \cdot \nabla y = \chi_\omega u \quad && \text{in } Q, \\ & y(0) = y_0 && \text{in } \Omega, \\ & \partial_\nu y = 0 && \text{on } \Sigma. \end{aligned}$$

Example of a stabilizable system with an unbounded control operator.

$$
\begin{aligned}
& y_t - \Delta y + \vec{V} \cdot \nabla y = 0 && \text{in } Q, \\
& y(0) = y_0 && \text{in } \Omega, \\
& \partial_\nu y = 0 && \text{on } \Sigma \setminus \Sigma_c, \\
& \partial_\nu y = u && \text{on } \Sigma_c.
\end{aligned}
$$

Theorem. Let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup on Y. The semigroup $(S(t))_{t\geq 0}$ is exponentially stable if and only if

$$\int_0^\infty \|S(t)y_0\|_Y^2 < \infty \qquad \text{for all } y_0 \in Y.$$

To solve the stabilization problem we look for the solution to the control problem (P)

$$\text{inf}\{J(y,u) \mid (y,u) \text{ obeys } (E),\ u \in L^2(0,\infty;U)\},$$

with

$$J(y,u) = \frac{1}{2}\int_0^\infty \|y(t)\|_Y^2 + \frac{1}{2}\int_0^\infty \|u(t)\|_U^2.$$

The optimal control can be written in feedback form

$$\bar{u}(t) = K\bar{y}(t).$$

The LQR problem with a finite time horizon

Consider the problem $(P(0,T,y_0))$

$$\inf\{J_T(y,u) \mid (y,u) \text{ obeys } (E),\ u \in L^2(0,T;U)\},$$

with

$$J_T(y,u) = \frac{1}{2}\int_0^T \|y(t)\|_Y^2 + \frac{1}{2}\int_0^T \|u(t)\|_U^2.$$

We know that this problem admits a unique solution $(\bar{y},\bar{u})$ characterized by the optimality system

$$\bar{y}' = A\bar{y} - BB^*p, \qquad y(0) = y_0,$$
$$-p' = A^*p + \bar{y}, \qquad p(T) = 0,$$
$$\bar{u} = -B^*p(t).$$

To find $\bar{u}$ in feedback form

$$\bar{u}(t) = K\bar{y}(t),$$

we study the family of problems $(P(s,T,\zeta))$

$$\inf\{J_{s,T}(y,u) \mid (y,u) \text{ obeys } (E_{s,\zeta}),\ u \in L^2(s,T;U)\},$$

with

$$J_{s,T}(y,u) = \frac{1}{2}\int_s^T \|y(t)\|_Y^2 + \frac{1}{2}\int_s^T \|u(t)\|_U^2,$$

and

$$(E_{s,\zeta}) \qquad y' = Ay + Bu, \qquad y(s) = \zeta.$$

The solution (y^s_ζ, u^s_ζ) to $(P(s,T,\zeta))$ is characterized by

$$\begin{aligned}
&\frac{dy^s_\zeta}{dt} = Ay^s_\zeta - BB^* p^s_\zeta, \qquad y^s_\zeta(s) = \zeta,\\
&-\frac{dp^s_\zeta}{dt} = A^* p^s_\zeta + y^s_\zeta, \qquad p^s_\zeta(T) = 0,\\
&u^s_\zeta(t) = -B^* p^s_\zeta(t).
\end{aligned}$$

By linearity we have

$$(y^s_{\beta\zeta_1+\zeta_2}, p^s_{\beta\zeta_1+\zeta_2}, u^s_{\beta\zeta_1+\zeta_2}) = \beta(y^s_{\zeta_1}, p^s_{\zeta_1}, u^s_{\zeta_1}) + (y^s_{\zeta_2}, p^s_{\zeta_2}, u^s_{\zeta_2}).$$

Thus the mapping

$$P(s) \ : \ \zeta \longmapsto p^s_\zeta(s)$$

is linear from Y into itself.

For all $t \in [0, T[$, $P(t) = P(t)^* \geq 0$

With an IBP between the solution p^s_ζ to

$$-p' = A^* p + y^s_\zeta, \qquad p(T) = 0,$$

and the solution y^s_ξ to

$$y' = Ay - BB^* p^s_\xi, \qquad y(s) = \xi,$$

we obtain

$$\Big(P(s)\zeta, \xi\Big)_Y = \int_s^T \Big(y^s_\zeta, y^s_\xi\Big)_Y + \int_s^T \Big(B^* p^s_\zeta, B^* p^s_\xi\Big)_U,$$

for all $\zeta \in Y$ and all $\xi \in Y$.

For all $t \in [0, T[$, $P(t) \in \mathcal{L}(Y)$

From the previous identity

$$\frac{1}{2}\Big(P(s)\zeta, \zeta\Big)_Y = J_{s,T}(y^s_\zeta, u^s_\zeta) \leq J_{s,T}(e^{(t-s)A}\zeta, 0)$$
$$\leq K\|\zeta\|^2_Y.$$

Thus

$$\|P(t)^{1/2}\|_{\mathcal{L}(Y)} \leq K^{1/2}, \qquad \|P(t)\|_{\mathcal{L}(Y)} \leq K.$$

$t \mapsto \Big(P(t)\zeta, \xi\Big)_Y$ is continuous

From the dynamic programming principle

$$p_\zeta^s(t) = P(t) y_\zeta^s(t) \quad \text{for all } t \in [s, T].$$

From the Duhamel formula and the DPP

$$\|y_\zeta^s(t)\| \leq \|e^{(t-s)A}\zeta\| + \int_s^T \|e^{(t-\tau)A} BB^* P(\tau) y_\zeta^s(\tau)\| d\tau.$$

Thus

$$\|y_\zeta^s\|_{C([s,T];Y)} \leq C\|\zeta\|_Y.$$

Next

$$\|p_\zeta^s\|_{C([s,T];Y)} \leq C\|\zeta\|_Y.$$

It can be shown that

$$\lim_{h\to 0}\|y_\zeta^{s+h} - y_\zeta^s\|_{C([(s+h)\wedge s,T];Y)} = 0,$$

and

$$\lim_{h\to 0}\|p_\zeta^{s+h} - p_\zeta^s\|_{C([(s+h)\wedge s,T];Y)} = 0.$$

From which we deduce that $t \mapsto \Big(P(t)\zeta, \xi\Big)_Y$ is continuous.

$P(\cdot)$ is the solution to a Differential Riccati Equation

Definition. We denote by $C_s([0,T];\mathcal{L}(Y))$ the space of mapping P from $[0,T]$ to $\mathcal{L}(Y)$ such that $t \mapsto P(t)\zeta$ belongs to $C([0,T];Y)$ for all $\zeta \in Y$.

We know that $P \in C_s([0,T];\mathcal{L}(Y))$. We are going to prove that P is the solution to the Differential Riccati Equation

$$P^*(t) = P(t) \quad \text{and} \quad P(t) \geq 0,$$

$$P'(t) + A^*P(t) + P(t)A - P(t)BB^*P(t) + I = 0,$$

$$P(T) = 0.$$

Definition. A function $P \in C_s([0,T];\mathcal{L}(Y))$ is a solution to the DRE on $(0,T)$ if, and only if, for every $(\zeta,\xi) \in D(A) \times D(A)$ the function $(P(\,\cdot\,)\zeta,\xi)$ belongs to $W^{1,1}(0,T)$ and satisfies

$$
\begin{aligned}
&P^*(t) = P(t) \quad \text{and} \quad P(t) \geq 0 \quad \text{for all } t \in [0,T],\\
&\frac{d}{dt}(P(t)\zeta,\xi) + (P(t)\zeta, A\xi) + (P(t)A\zeta,\xi)\\
&-(P(t)BB^*P(t)\zeta,\xi) + (\zeta,\xi) = 0,\\
&(P(T)\zeta,\xi) = 0.
\end{aligned}
$$

Theorem. The function P is the unique solution to the Differential Riccati Equation on $(0,T)$.

For $(\zeta, \xi) \in D(A) \times D(A)$, consider the two systems

$$z' = Az - BB^*p, \qquad z(s) = \zeta,$$
$$-p' = A^*p + z, \qquad p(T) = 0,$$

and

$$y' = Ay - BB^*q, \qquad y(s) = \xi,$$
$$-q' = A^*q + y, \qquad q(T) = 0.$$

In the previous notation we have $(z, p) = (y^s_\zeta, p^s_\zeta)$ and $(y, q) = (y^s_\xi, p^s_\xi)$.

Let us denote by $\frac{d^+}{ds}(P(s)\zeta,\xi)$ the right hand side derivative of the mapping $s \mapsto (P(s)\zeta,\xi)$. We prove that for every $(\zeta,\xi) \in D(A) \times D(A)$, we have

$$\frac{d^+}{dt}(P(t)\zeta,\xi) + (P(t)\zeta, A\xi) + (P(t)A\zeta,\xi)$$
$$-(P(t)BB^*P(t)\zeta,\xi) + (\zeta,\xi) = 0,$$

for all $t \in [0,T[$. For $(\zeta,\xi) \in D(A) \times D(A)$, the solutions z and y satisfy

$$z(t) = e^{(t-s)A}\zeta - \int_s^t e^{(t-\tau)A}BB^*p(\tau)\,d\tau,$$

and

$$y(t) = e^{(t-s)A}\xi - \int_s^t e^{(t-\tau)A}BB^*q(\tau)\,d\tau.$$

Thus we have

$$\lim_{h\searrow 0}\|\frac{1}{h}(z(s+h)-z(s))-A\zeta+BB^*p(s)\|_Y=0,$$

and

$$\lim_{h\searrow 0}\|\frac{1}{h}(y(s+h)-y(s))-A\xi+BB^*q(s)\|_Y=0.$$

Using a previous identity we obtain

$$(P(s+h)z(s+h),y(s+h))-(P(s)z(s),y(s))$$
$$=\int_{s+h}^{s}\left((z(t),y(t))+(B^*p(t),B^*q(t))\right)dt,$$

and

$$\text{lim}_{h\searrow 0}\Big((P(s+h)z(s+h),y(s+h))-(P(s)z(s),y(s))\Big)/h$$
$$=-(z(s),y(s))-(B^*p(s),B^*q(s)).$$

We also have

$$\Big(P(s+h)z(s+h),y(s+h)\Big)-\Big(P(s)z(s),y(s)\Big)$$
$$=\Big(P(s+h)z(s+h),y(s+h)-y(s)\Big)$$
$$+\Big(z(s+h)-z(s),P(s+h)y(s)\Big)$$
$$+\Big((P(s+h)-P(s))z(s),y(s)\Big).$$

Dividing by h and passing to the limit when h tends to zero, we obtain

$$
\begin{aligned}
&-(\zeta,\xi) - (B^*P(s)\zeta, B^*P(s)\xi) \\
&= (P(s)\zeta, A\xi) + (A\zeta, P(s)\xi) - 2(B^*P(s)\zeta, B^*P(s)\xi) \\
&+\frac{d^+}{ds}(P(s)\zeta,\xi),
\end{aligned}
$$

that is

$$
\begin{aligned}
&\frac{d^+}{ds}(P(s)\zeta,\xi) + (P(s)\zeta, A\xi) + (A\zeta, P(s)\xi) + (\zeta,\xi) \\
&-(B^*P(s)\zeta, B^*P(s)\xi) = 0.
\end{aligned}
$$

Since the mapping

$$s \mapsto (P(s)\zeta,\xi)$$

is continuous on $[0,T]$, and the mapping

$$s \mapsto \frac{d^+}{ds}(P(s)\zeta,\xi) = -(\zeta,\xi) + (B^*P(s)\zeta, B^*P(s)\xi)$$
$$-(P(s)\zeta, A\xi) - (A\zeta, P(s)\xi)$$

is bounded and continuous on $[0,T[$, we can affirm that

$$s \mapsto (P(s)\zeta,\xi)$$

is of class C^1 on $[0,T]$. Thus P is a solution to the Differential Riccati Equation.

Theorem. The solution $(\bar{y},\bar{u})$ to problem $(P(0,T,y_0))$ is characterized by

$$\bar{u}(t) = -B^*P(t)\bar{y}(t),$$
$$\bar{y}' = A\bar{y} - BB^*P\bar{y}, \qquad \bar{y}(0) = y_0.$$

Remark Setting $Q(t) = P(T-t)$, where P is the solution to the previous DRE, we can show that Q is the solution to

$$Q^*(t) = Q(t) \quad \text{and} \quad Q(t) \geq 0,$$

$$Q'(t) = A^*Q(t) + Q(t)A - Q(t)BB^*Q(t) + I,$$

$$Q(0) = 0.$$

Generalization to the problem

$$\text{inf}\{J_T(y,u) \mid (y,u) \text{ obeys } (E),\ u \in L^2(0,T;U)\},$$

with

$$J_T(y,u)$$
$$= \frac{1}{2}\int_0^T \|Cy(t)\|_Z^2 + \frac{1}{2}\|Dy(T)\|_{Z_T}^2 + \frac{1}{2}\int_0^T \|u(t)\|_U^2.$$

The solution $(\bar{y},\bar{u})$ is characterized by

$$\bar{u}(t) = -B^*P(t)\bar{y}(t),$$
$$\bar{y}' = A\bar{y} - BB^*P\bar{y}, \qquad \bar{y}(0) = y_0,$$

where P is the solution to

$$P^*(t) = P(t) \quad \text{and} \quad Q(t) \geq 0,$$

$$-P'(t) = A^*P(t) + P(t)A - P(t)BB^*P(t) + C^*C,$$

$$P(T) = D^*D.$$

The LQR problem with an infinite time horizon

Now we consider the control problem (P)

$$\text{inf}\{J(y,u) \mid (y,u) \text{ obeys } (E),\ \ u \in L^2(0,\infty;U)\},$$

with

$$J(y,u) = \frac{1}{2}\int_0^\infty \|y(t)\|_Y^2 + \frac{1}{2}\int_0^\infty \|u(t)\|_U^2,$$

and

$$(E) \qquad\qquad y' = Ay + Bu, \qquad\qquad y(0) = y_0.$$

Finite cost condition For every $y_0 \in Y$, there exists u_{y_0} s.t.

$$J(y(y_0, u_{y_0}), u_{y_0}) < \infty.$$

Remark If (A, B) is stabilizable then the (FCC) is satisfied. The converse proposition is true.

Theorem. Suppose that the (FCC) is satisfied. Then (P) admits a unique solution. This solution $(\bar{y}, \bar{u})$ obeys

$$\bar{u}(t) = -B^* P \bar{y}(t),$$

where P is the minimal solution to the ARE

$$P^* = P \geq 0,$$

$$A^* P + PA - PBB^* P + I = 0.$$

Moreover

$$J(\bar{y}, \bar{u}) = \frac{1}{2}\Big(Py_0, y_0\Big)_Y.$$

Definition. An operator $P \in \mathcal{L}(Y)$ is a solution to the ARE iff

$$P^* = P \geq 0,$$

$$(P\zeta, A\xi) + (PA\zeta, \xi) - (PBB^*P\zeta, \xi) + (\zeta, \xi) = 0.$$

An operator $P \in \mathcal{L}(Y)$ is a minimal solution if it is a solution and if

$$P \leq Q \qquad \text{for any solution } Q.$$

Theorem. The ARE admits a unique minimal solution.

Proof.

Consider the problem

$$(Q(s,T,\zeta)) \qquad \text{inf}\{I(s,T;\zeta,u) \mid u \in L^2(s,T;U)\},$$

with

$$I(s,T;\zeta,u) = \frac{1}{2}\int_s^T \|y^s_{\zeta,u}(t)\|^2_Y \, dt + \frac{1}{2}\int_s^T \|u(t)\|^2_U \, dt,$$

and $y^s_{\zeta,u}$ is the solution to

$$y' = Ay + Bu, \qquad y(s) = \zeta.$$

For every $\zeta \in Y$ let u_ζ be the solution to $(Q(s, \infty, \zeta))$. Let P_{min} be the solution to the differential Riccati equation

$$P = P^* \geq 0, \quad P(0) = 0,$$
$$P' = A^*P + PA - PBB^*P + I.$$

Let us prove that, for every $\zeta \in Y$, the mapping $t \mapsto (P(t)\zeta, \zeta)$ is nondecreasing. Let $0 < T_1 < T_2$, we know that

$$\inf(Q(0, T_1, \zeta)) = \tfrac{1}{2}(P(T_1)\zeta, \zeta),$$
$$\inf(Q(0, T_2, \zeta)) = \tfrac{1}{2}(P(T_2)\zeta, \zeta),$$

and

$$\begin{aligned}
&\mathsf{inf}(Q(0,T_2,\zeta)) \\
&= \mathsf{inf}_{u\in L^2(0,T_1;U)}\Big\{I(0,T_1,\zeta,u) + \mathsf{inf}(Q(T_1,T_2,z^0_{\zeta,u}(T_1)))\Big\} \\
&\geq \mathsf{inf}(Q(0,T_1,\zeta)).
\end{aligned}$$

Thus the mapping $t \mapsto (P(t)\zeta,\zeta)$ is nondecreasing.

On the other hand

$$(P(t)\zeta,\zeta) \leq 2I(0,t;\zeta,u_\zeta) \leq 2J(z(\zeta,u_\zeta),u_\zeta) < \infty.$$

Thus the limit $\lim_{t\to\infty}(P(t)\zeta,\zeta)$ exists and is finite for every $\zeta \in Y$. Since

$$(P(t)\zeta,\xi) = \frac{1}{4}(P(t)(\zeta+\xi),\zeta+\xi) - \frac{1}{4}(P(t)(\zeta-\xi),\zeta-\xi),$$

applying the Banach-Steinhaus theorem to the family of operator $(P(t)\zeta, \cdot)$, we deduce that $\sup_{t\geq 0}|(P(t)\zeta, \cdot)| < \infty$. Next, still with the Banach-Steinhaus theorem, we obtain $\sup_{t\geq 0}|(P(t)\cdot, \cdot)| < \infty$. Therefore there exists an operator $P^{\infty}_{min} \in \mathcal{L}(Y)$ such that

$$\lim_{t\to\infty}(P(t)\zeta, \zeta) = (P^{\infty}_{min}\zeta, \zeta).$$

Since $P(t) = P^*(t) \geq 0$ it follows that $P^{\infty}_{min} = (P^{\infty}_{min})^* \geq 0$.

For every $\zeta \in D(A)$, we have

$$\frac{d}{dt}(P(t)\zeta, \zeta)$$
$$= (P\zeta, A\zeta) + (PA\zeta, \zeta) - (PBB^*P\zeta, \zeta) + (\zeta, \zeta).$$

The mapping $t \mapsto (P(t)\zeta, \zeta)$ is of class C^1, the right hand side of the equation admits a limit when t tends to infinity, thus the limit of $\frac{d}{dt}(P(t)\zeta, \zeta)$ exists and is necessarily zero. This means that P_{min}^{∞} is a solution to the ARE. To prove that P_{min}^{∞} is a minimal solution, we suppose that $\hat{P}$ is an other solution. Observe that $\hat{P}$ is also the solution to the differential Riccati equation

$$P = P^* \geq 0, \quad P(0) = \hat{P},$$

$$P' = A^*P + PA - PBB^*P + I.$$

Since $\hat{P}(0) \geq P_{min}(0) = 0$, we have $P_{min}(t) \leq \hat{P}(t) = \hat{P}$. Passing to the limit when t tends to infinity, we prove that $P_{min}^{\infty} \leq \hat{P}$.

Theorem. The unique solution $(\bar{y}, \bar{u})$ to problem (P) satisfies the feedback formula

$$\bar{u}(t) = -B^* P^\infty_{min} \bar{y}(t),$$

where P^∞_{min} is the minimal solution to ARE, and $\bar{y}$ is the solution to

$$y' = Ay - BB^* P^\infty_{min} y, \qquad y(0) = y_0.$$

Moreover the optimal cost is given by

$$J(\bar{y}, \bar{u}) = \frac{1}{2} (P^\infty_{min} y_0, y_0)_Y.$$

The Algebraic Riccati Equation admits a unique solution.

Proof. Let $\bar{y}$ be the solution to

$$\bar{y}' = A\bar{y} - BB^* P^\infty_{min}\bar{y}, \qquad \bar{y}(0) = y_0.$$

The solution to problem

$$\inf\Big\{\frac{1}{2}\int_0^T \Big(\|y_u\|_Y^2 + \|u\|_U^2\Big) + \frac{1}{2}(P^\infty_{min} y_u(T), y_u(T))_Y$$
$$\mid u \in L^2(0,T;U)\Big\},$$

where y_u is the solution to equation

$$y' = Ay + Bu, \qquad y(0) = y_0,$$

is given by $(\hat{y}, \hat{u}) = (\hat{y}, -B^* P\hat{y})$, where P solves the

Riccati equation

$$P = P^* \geq 0, \quad P(T) = P^\infty_{min},$$
$$-P' = A^*P + PA - PBB^*P + I,$$

and $\hat{y}$ satisfies $\hat{y}' = A\hat{y} - BB^*P\hat{y}$, and $\hat{y}(0) = y_0$. Still the previous part, we have

$$(P(0)y_0, y_0) = \int_0^T \Big(\|\hat{y}\|^2 + \|\hat{u}\|^2\Big) + (P^\infty_{min}\hat{y}(T), \hat{y}(T)).$$

But P^∞_{min} is the unique solution to the above DRE. Consequently we have $(\hat{y}, \hat{u}) = (\bar{y}, \bar{u})$, and for every $T > 0$

$$(P^\infty_{min}y_0, y_0) = \int_0^T \Big(\|\bar{y}\|^2 + \|\bar{u}\|^2\Big) + (P^\infty_{min}\bar{y}(T), \bar{y}(T)).$$

When T tends to infinity we obtain

$$2J(\bar{y}, \bar{u}) \leq (P^{\infty}_{min} y_0, y_0).$$

Considering the problem

$$\inf\Big\{ \frac{1}{2} \int_0^T \Big(\|y_u\|_Y^2 + \|u\|_U^2 \Big) \mid u \in L^2(0, T; U) \Big\},$$

we also have

$$(P_{min}(T) y_0, y_0) \leq \int_0^T \Big(\|\bar{y}\|^2 + \|\bar{u}\|^2 \Big) \leq 2J(\bar{y}, \bar{u}),$$

and

$$(P_{min}(T) y_0, y_0) \leq \int_0^T \Big(\|y_u\|^2 + \|u\|^2 \Big) \leq 2J(y_u, u),$$

for all u, where P_{min} is the above DRE. By passing to the limit when T tends to infinity it yields

$$(P_{min}^{\infty} y_0, y_0) \leq \int_0^{\infty} \Big(\|\bar{y}\|^2 + \|\bar{u}\|^2 \Big) dt \leq 2J(\bar{y}, \bar{u}),$$

and

$$(P_{min}^{\infty} y_0, y_0) \leq 2J(y_u, u), \qquad \text{for all } \; u \in L^2(0, \infty; U).$$

Thus $(P_{min}^{\infty} y_0, y_0) = 2J(\bar{y}, \bar{u}) = 2\text{inf}(P)$, and $(\bar{y}, \bar{u})$ is the unique solution to problem (P).

Lemma. If P is a solution to the ARE, then the operator $A - BB^*P$ with domain $D(A)$ is the generator of an exponentially stable semigroup on Y.

Proof. Let $\zeta \in Y$, let y be the solution to

$$y(0) = \zeta, \quad y' = Ay - BB^*Py.$$

First suppose that $\zeta \in D(A)$. Let $(u_n)_n$ be a sequence in $C^1([0,\infty);U) \cap L^2(0,\infty;U)$ converging to $-B^*Py$ in $L^2(0,\infty;U)$. Let y_n be the solution to the equation

$$y(0) = \zeta, \quad y' = Ay + Bu_n.$$

With the ARE, we deduce

$$\frac{d}{dt}(Py_n(t), y_n(t)) = 2(Ay_n + Bu_n, Py_n)$$

$$= -(y_n, y_n) - (B^*Py_n, B^*Py_n) + 2(B^*Py_n, B^*Py_n)$$
$$+2(u_n, B^*Py_n).$$

Therefore we have

$$(Py_n(t), y_n(t)) + \int_0^t \Big(\|y_n\|^2 + \|B^*Py_n\|^2\Big)$$
$$= (P\zeta, \zeta) + \int_0^t \Big(2\|B^*Py_n\|^2 + 2(u_n, B^*Py_n)\Big).$$

By passing to the limit when n tends to infinity, we obtain

$$
\begin{aligned}
&\int_0^t \left(\|y\|^2 + \|B^*Py\|^2\right) \\
&\le (Py(t), y(t)) + \int_0^t \left(\|y\|^2 + \|B^*Py\|^2\right) \\
&= (P\zeta, \zeta).
\end{aligned}
$$

By a density argument this inequality also holds for every $\zeta \in Y$ and we have

$$
\int_0^\infty \left(\|y\|^2 + \|B^*Py\|^2\right) \le (P\zeta, \zeta).
$$

Lemma. Let P and Q be two solutions to the ARE. Suppose that the operator $A - BB^*P$ with domain $D(A)$, is the generator of an exponentially stable semigroup in Y. Then $P \geq Q$.

Proof. Since P and Q are two solutions to ARE, we can verify that

$$(P-Q)(A-BB^*P)$$

$$+(A-BB^*P)^*(P-Q)+(P-Q)BB^*(P-Q)=0.$$

From this identity, we deduce:

$$\frac{d}{dt}\Big((P-Q)e^{t(A-BB^*P)}\zeta, e^{t(A-BB^*P)}\zeta\Big)$$
$$= -\|B^*(P-Q)e^{t(A-BB^*P)}\zeta\|^2,$$

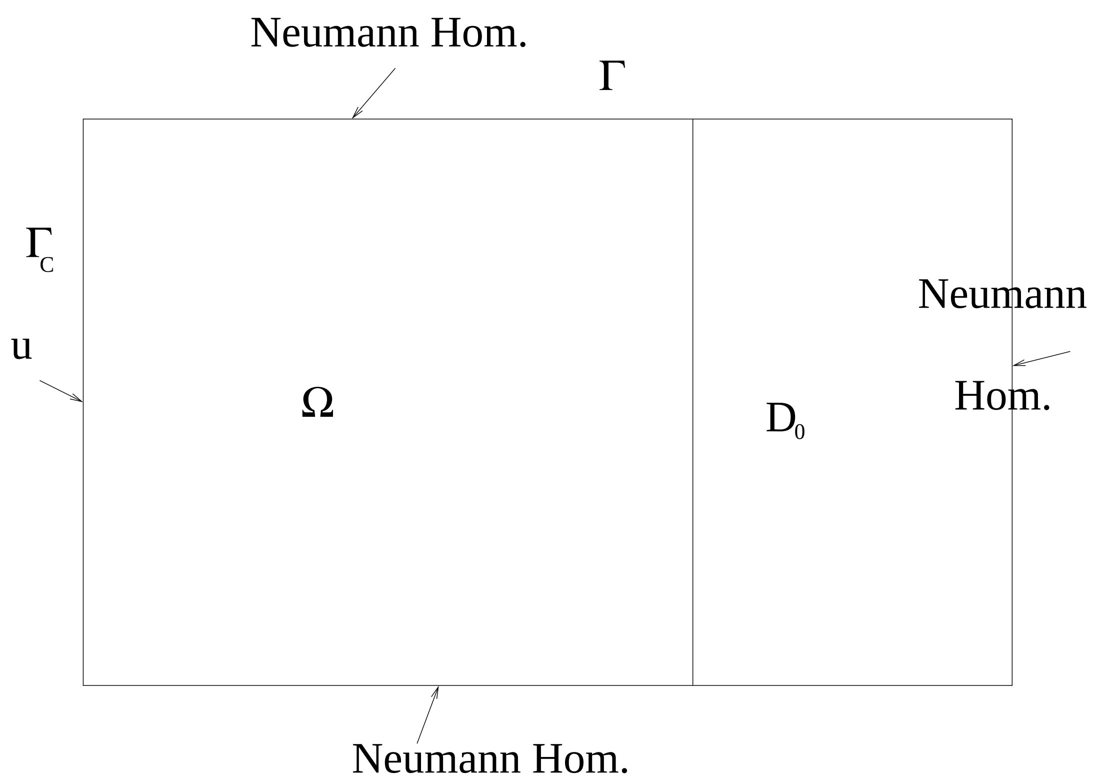
for all $\zeta \in D(A)$. By integrating this equality between

0 and T, we obtain

$$
\begin{aligned}
&\Big((P-Q)\zeta,\zeta\Big)\\
&=\Big((P-Q)e^{T(A-BB^*P)}\zeta,e^{T(A-BB^*P)}\zeta\Big)\\
&+\int_0^T\|B^*(P-Q)e^{t(A-BB^*P)}\zeta\|^2dt\\
&\geq\Big((P-Q)e^{T(A-BB^*P)}\zeta,e^{T(A-BB^*P)}\zeta\Big).
\end{aligned}
$$

By passing to the limit when T tends to infinity, we obtain $\Big((P-Q)\zeta,\zeta\Big)\geq 0$ for all $\zeta\in D(A)$, that is $P\geq Q$.

Stabilization of a convection-diffusion equation



Notations

$$Q = \Omega \times (0, \infty) \qquad \text{Space-time domain}$$

$$\Omega = (0, 10) \times (0, 10)$$

$$\Sigma = \Gamma \times (0, \infty) \qquad \text{Lateral boundary}$$

$$\Sigma_c = \Gamma_c \times (0, \infty) \qquad \text{Control boundary}$$

$$\Gamma_o = \{1\} \times (0, 10) \qquad \text{Observation on a boundary}$$

$$\Gamma_c = \{0\} \times (0, 10)$$

State equation

$$\begin{aligned} & y_t - \Delta y + \vec{V} \cdot \nabla y - cy = 0 && \text{in } Q, \\ & y(0) = y_0 && \text{in } \Omega, \\ & \partial_\nu y = 0 && \text{on } \Sigma \setminus \Sigma_c, \\ & \partial_\nu y = u && \text{on } \Sigma_c. \end{aligned}$$

Cost functional

$$I_1(y,u) = \frac{1}{2} \int_0^\infty \int_{\Gamma_c} u^2 + \int_0^\infty \int_\Omega y^2.$$

$$I_2(y,u) = \frac{1}{2} \int_0^\infty \int_{\Gamma_c} u^2 + \int_0^\infty \int_{\Gamma_o} y^2 \,.$$

Setting

$$Ay = \Delta y - \vec{V} \cdot \nabla y + cy,$$
$$D(A) = \{y \in H^2(\Omega) \mid \partial_\nu y = 0\}$$
$$\langle Bu, \phi \rangle = \int_{\Gamma_c} u\phi,$$

The state equation can be written in the following form

$$\frac{d}{dt}\int_\Omega y(t)\phi = \int_\Omega y(t)(\Delta\phi + \mathrm{div}(\vec{V}\phi)) + \int_{\Gamma_c} u\phi,$$

for all ϕ in

$$D(A^*) = \{\phi \in H^2(\Omega) \mid \partial_\nu \phi + \vec{V} \cdot \vec{n}\phi = 0 \text{ on } \Gamma\},$$

with

$$A^*\phi = \Delta\phi + \mathrm{div}(\vec{V}\phi),$$

thus we have

$$y' = Ay + Bu, \qquad y(0) = y_0.$$

Problem (P) admits a unique solution $(\bar{y}, \bar{u})$ which is characterized by

$$\bar{u}(t) = -B^* P \bar{y}(t),$$

where

P is the solution to the ARE

$$P = P^* \geq 0, \qquad A^*P + PA - PBB^*P + 2C^*C = 0,$$

B^* is the trace operator on Γ_c,

in example 1

$$C = C^* = I,$$

in example 2

$$C \qquad \text{is the trace operator on} \qquad \Gamma_0,$$

and

$$\langle C^* y, \phi \rangle = \int_{\Gamma_0} y\, \phi.$$

Remark. A null controllability result can be proved for the convection-diffusion equation. Thus the pair (A, B) is stabilizable.

Similarly we can prove that the pair (A, C) is detectable.

Thus the ARE admits a unique solution.

Algorithms

Numerical resolution of the finite dimensional Riccati equation

$$P = P^* \geq 0, \qquad A^*P + PA - PBB^*P + C^*C = 0.$$

Hypotheses

(H) The pair (A, B) is stabilizable, and the pair (A, C) is detectable.

Methods based on the computation of the eigenvalues of the matrix

$$H = \begin{bmatrix} A & -BB^* \\ CC^* & -A^* \end{bmatrix}$$

The spectrum of H is symmetric w.r. to the origin and H has no eigenvalues with a null real part.

Algorithm 1.

1 - Compute the eigenvalues and the eigenvectors of H by the QR-method.

2 - Select the eigenvectors corresponding to eigenvalues with a negative real part. Let V_1 be the matrix whose columns correspond to these vectors:

$$V_1 = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix}.$$

3 - Solve $V_{11}^* P = V_{21}^*$ to calculate P.

Algorithm 2.

1 - Write the real Schur decomposition of H

$$T = U^* H U$$

2 - Use orthogonal transformations to reorder the matrix T so that the quasi-triangular bloc T_{11} has eigenvalues with a negative real part.

3 - Solve $U_{11}^* P = U_{21}^*$ to compute P, where $U_1 = \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix}$, are the Schur-vectors corresponding to T_{11}.

Numerical tests

$$\Omega = (0,10) \times (0,10), \qquad \Delta t = 0.01, \qquad T = 10,$$

mesh size $= \; 1$

The equation

$$y_t - \Delta y + \vec{V} \cdot \nabla y - cy = 0,$$

with

$$\vec{V} = [10 \quad 3]^T, \qquad c = 0 \quad \text{or} \quad c = 3.$$

Control boundary

$$\Gamma_c = \{0\} \times (0, 10) \qquad \text{or} \qquad \Gamma_c = \{0\} \times (3, 6).$$

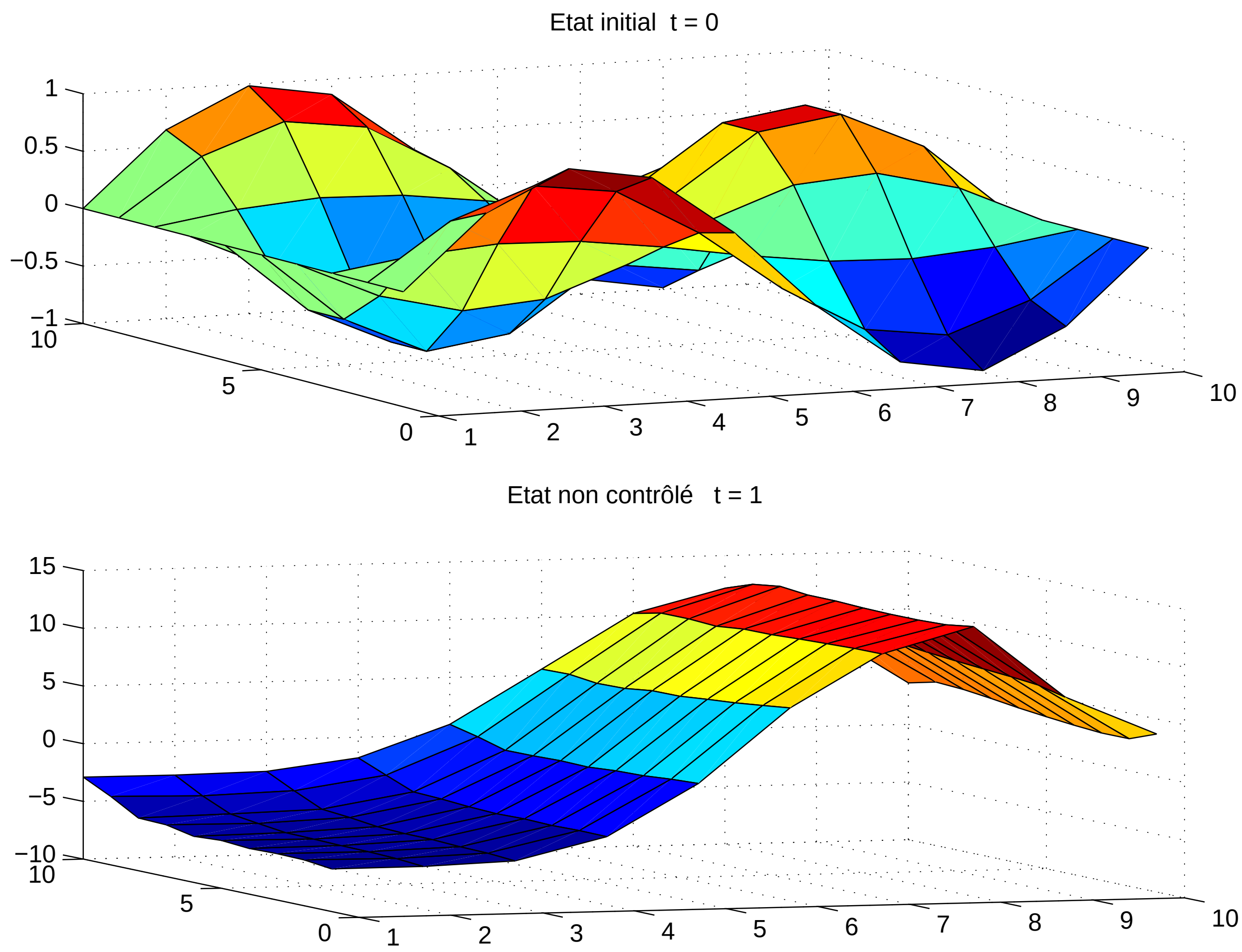
Cost functional

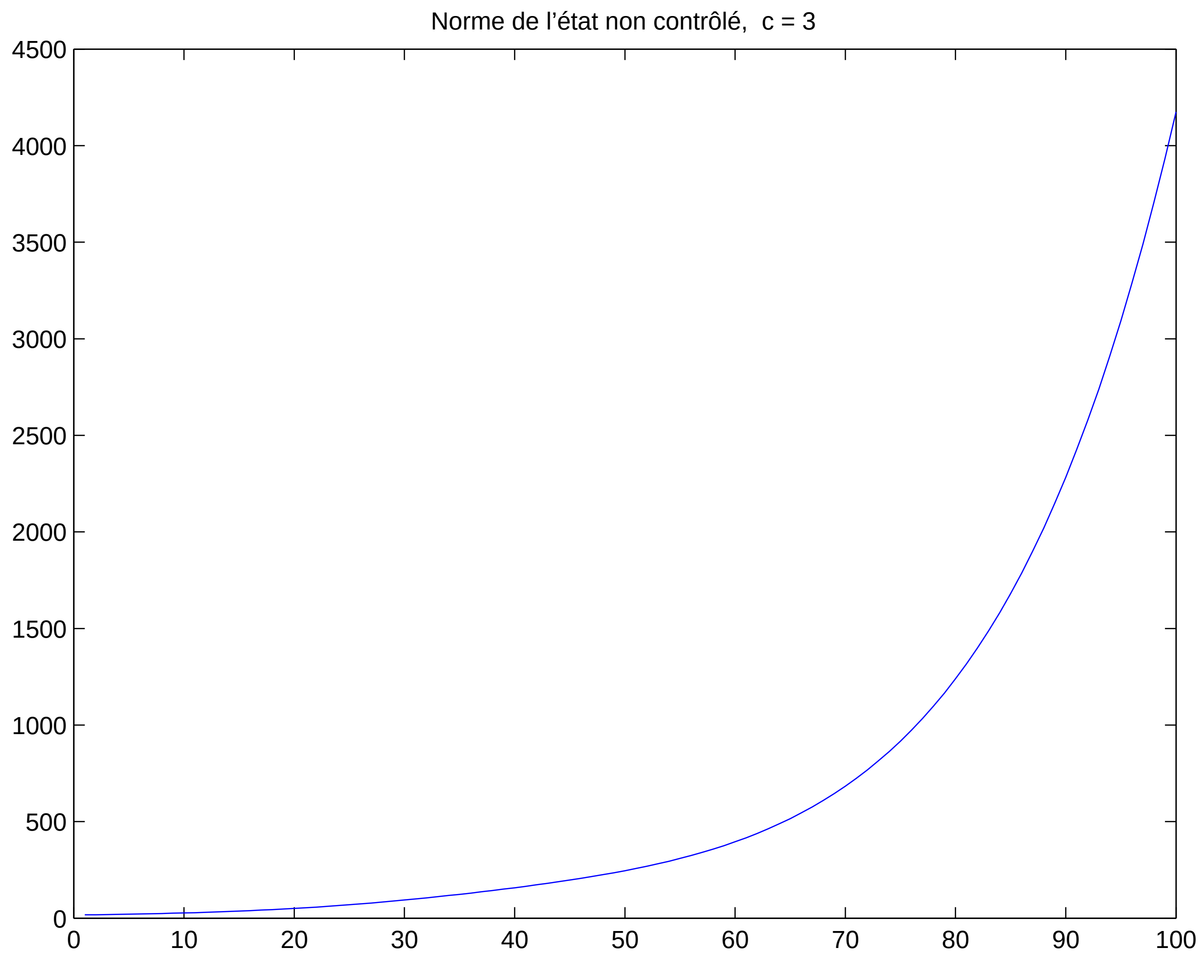
$$I_1(y, u) = \frac{1}{2} \int_0^\infty \int_{\Gamma_c} u^2 + \int_0^\infty \int_\Omega y^2$$

or

$$I_2(y, u) = \frac{1}{2} \int_0^\infty \int_{\Gamma_c} u^2 + \int_0^\infty \int_{\Gamma_o} y^2, \qquad \Gamma_o = \{1\} \times (0, 10).$$

Without control $c = 3$, $y_0 = cos(2\pi x_1/10)sin(2\pi x_2/10)$

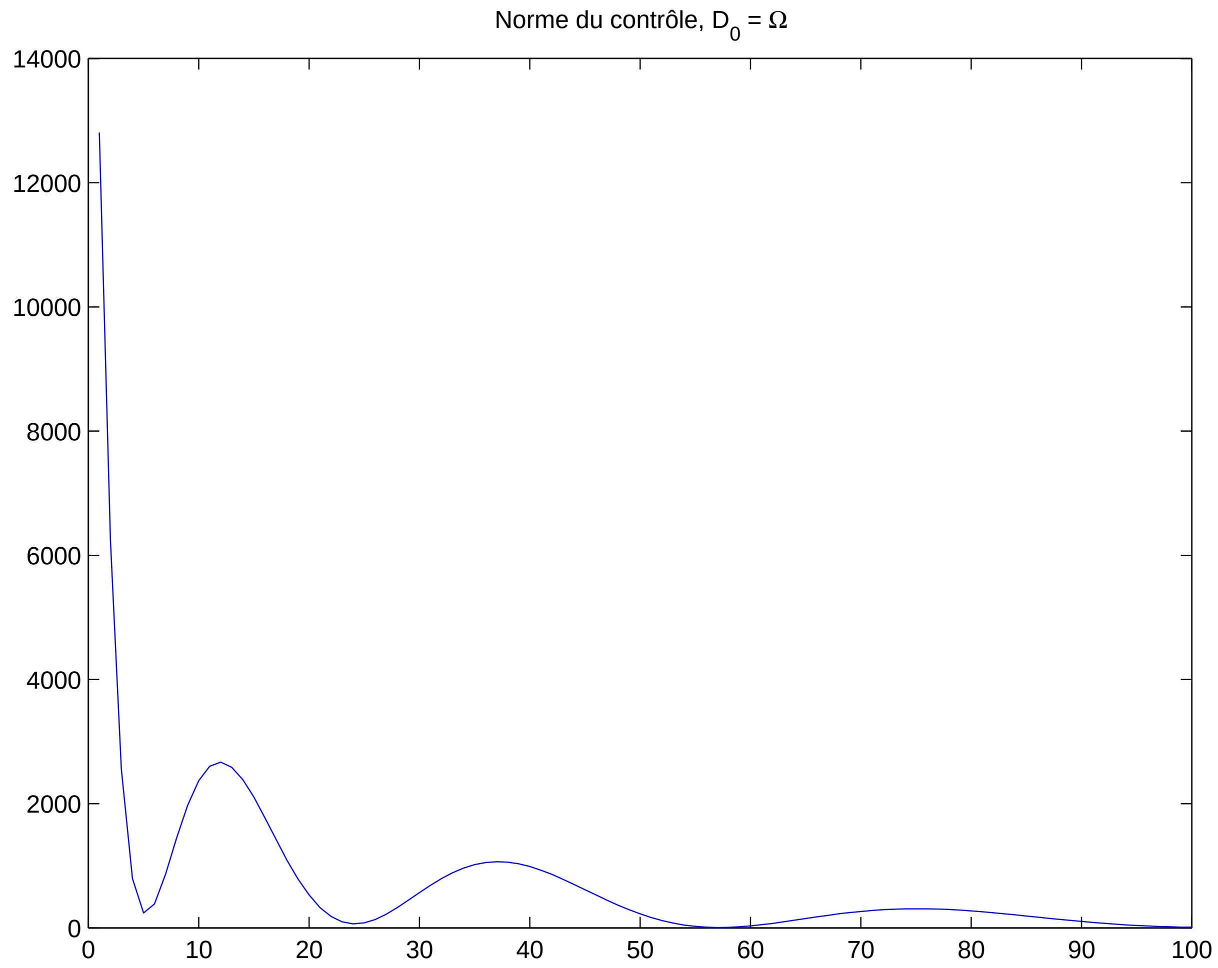


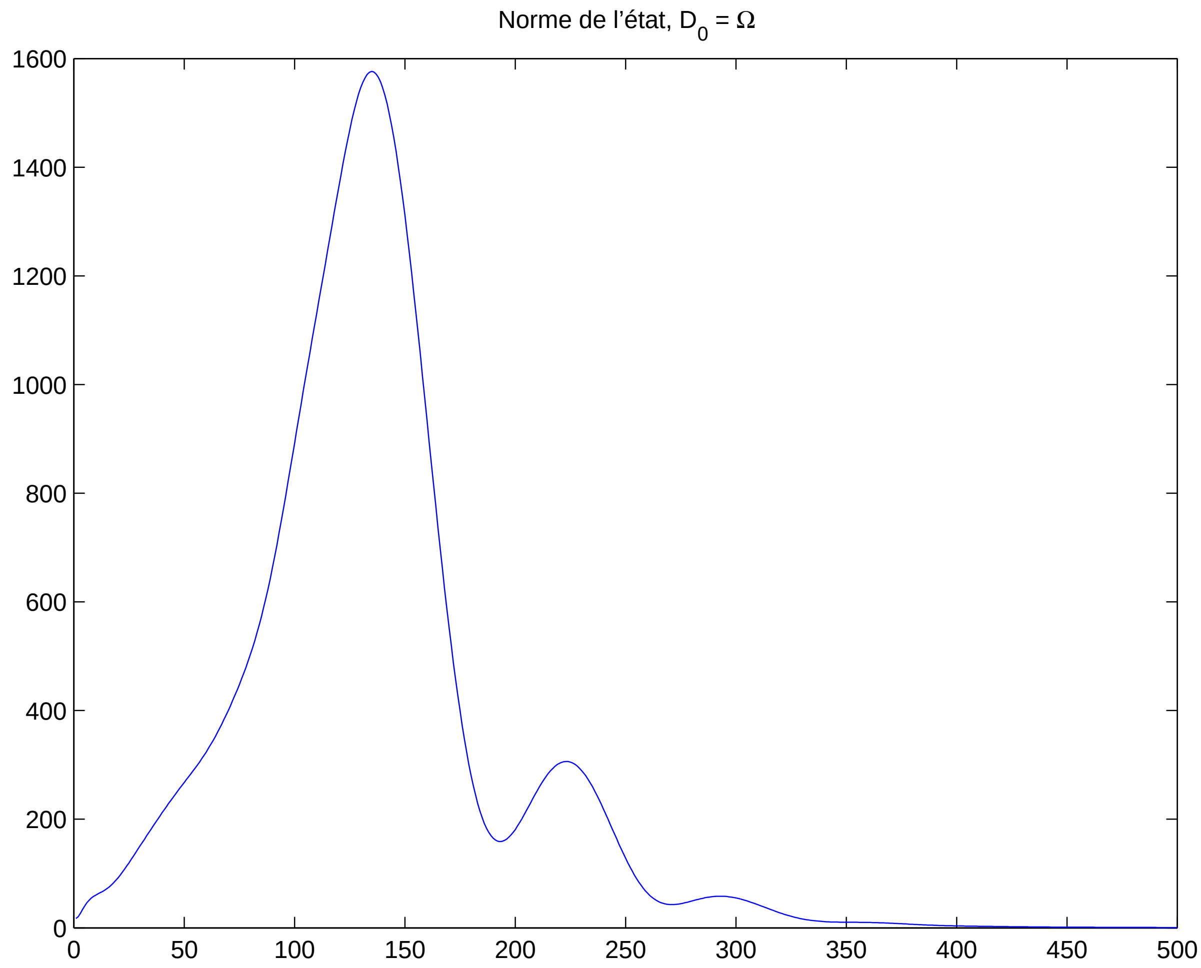

Norme de l'état non contrôlé, c = 3
4500
4000
3500
3000
2500
2000
1500
1000
500
0
0
10
20
30
40
50
60
70
80
90
100

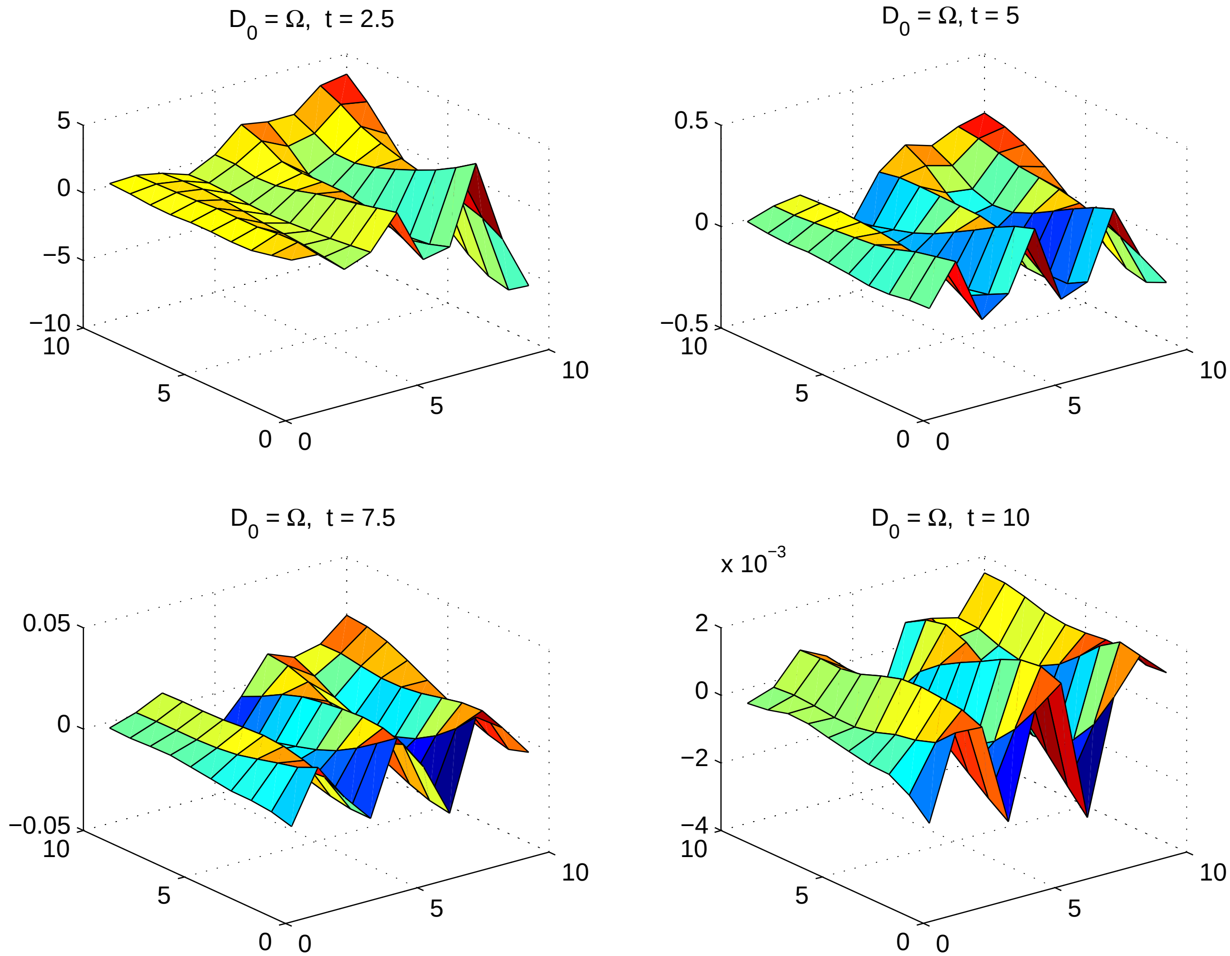
Control on $\Gamma_c = \{0\} \times (0, 10)$

Observation on $D_0 = \Omega$

$$c = 3, \qquad y_0(x_1, x_2) = cos(2\pi x_1/10) sin(2\pi x_2/10)$$


Norme du contrôle, D0 = Ω
14000
12000
10000
8000
6000
4000
2000
0
0
10
20
30
40
50
60
70
80
90
100


Norme de l'état, $D_0 = \Omega$
1600
1400
1200
1000
800
600
400
200
0
0
50
100
150
200
250
300
350
400
450
500

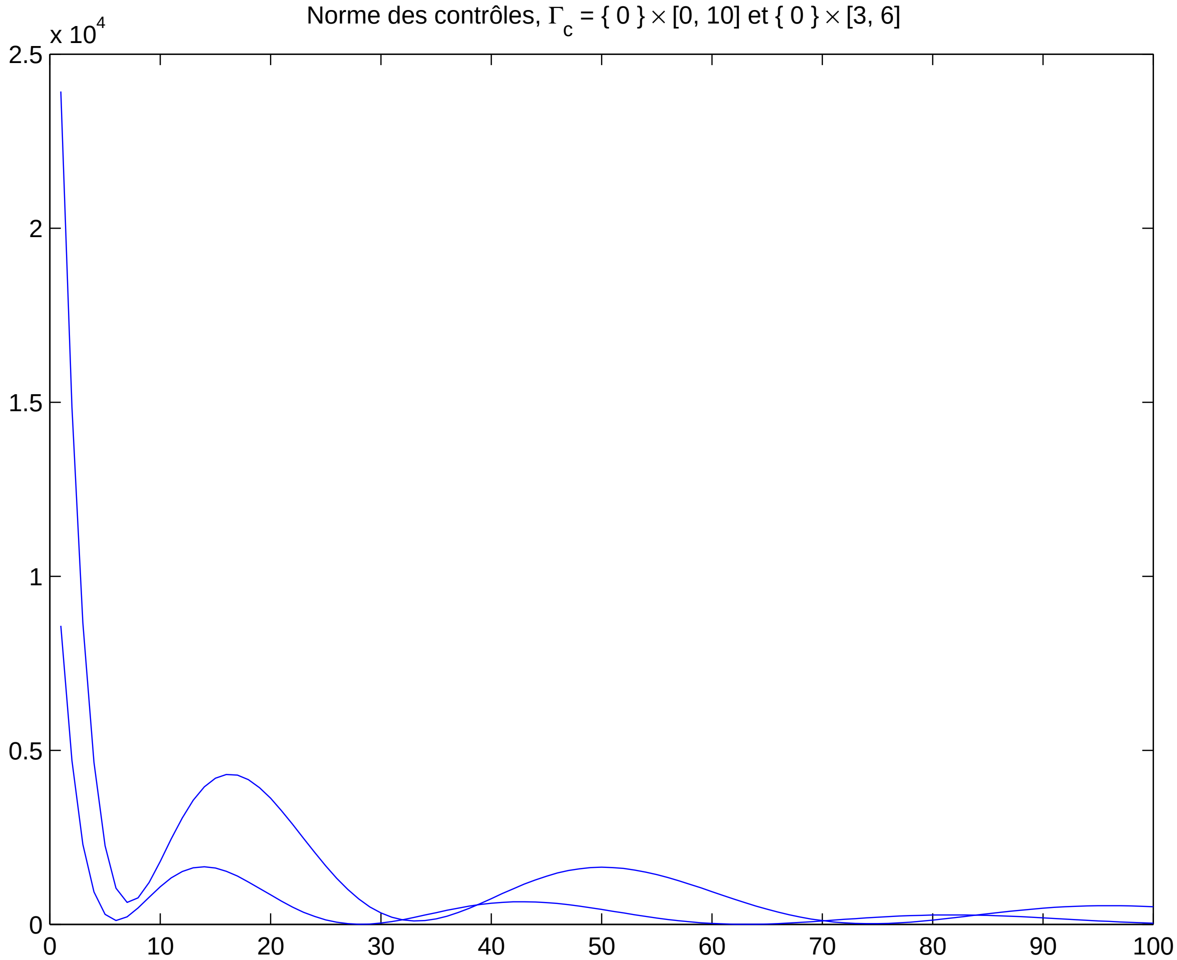

D0 = Ω, t = 2.5
5
0
−5
−10
10
5
0
0
5
10
D0 = Ω, t = 5
0.5
0
−0.5
10
5
0
0
5
10
D0 = Ω, t = 7.5
0.05
0
−0.05
10
5
0
0
5
10
D0 = Ω, t = 10
x 10−3
2
0
−2
−4
10
5
0
0
5
10

Control on $\Gamma_c = \{0\} \times (0, 10)$

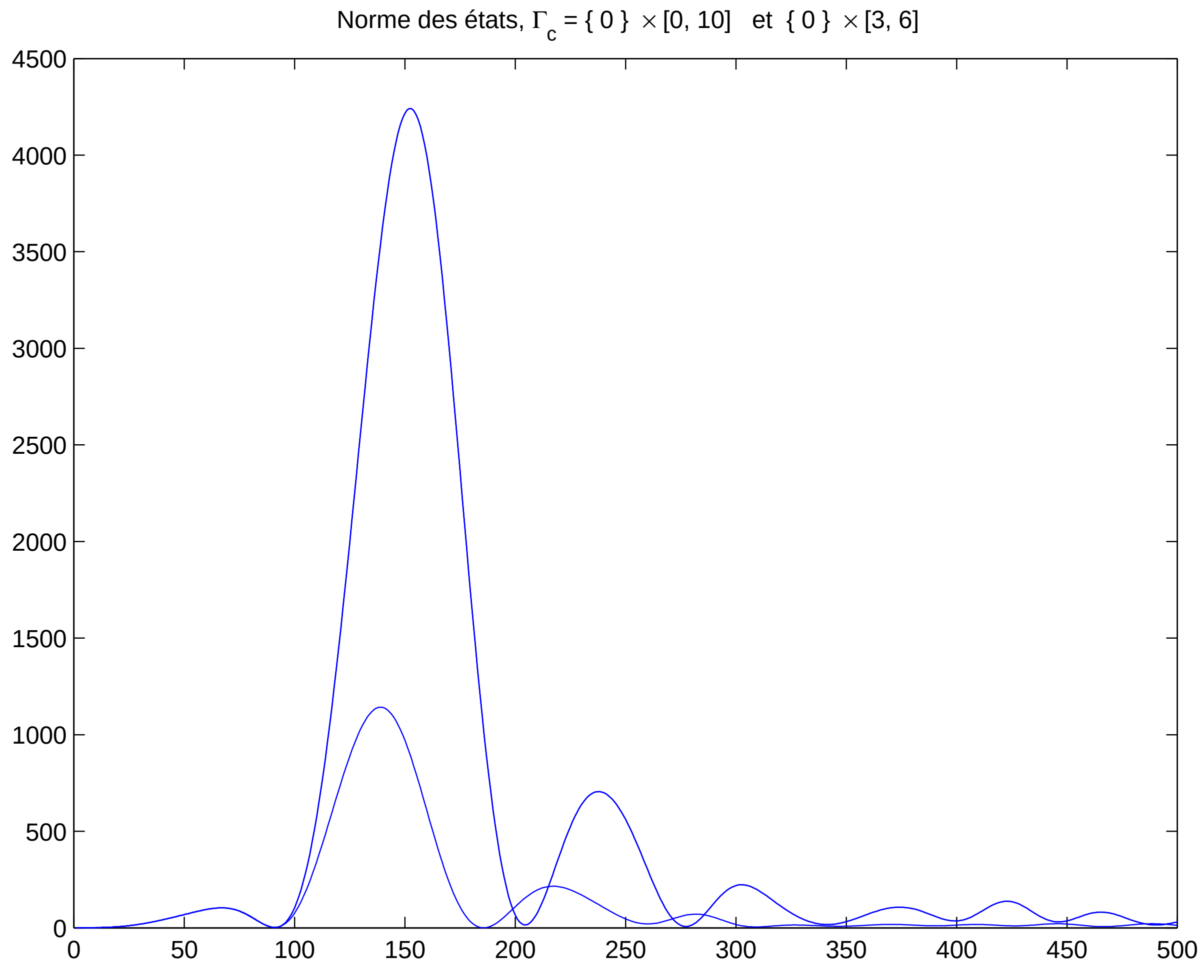
and on $\Gamma_c = \{0\} \times (3, 6)$

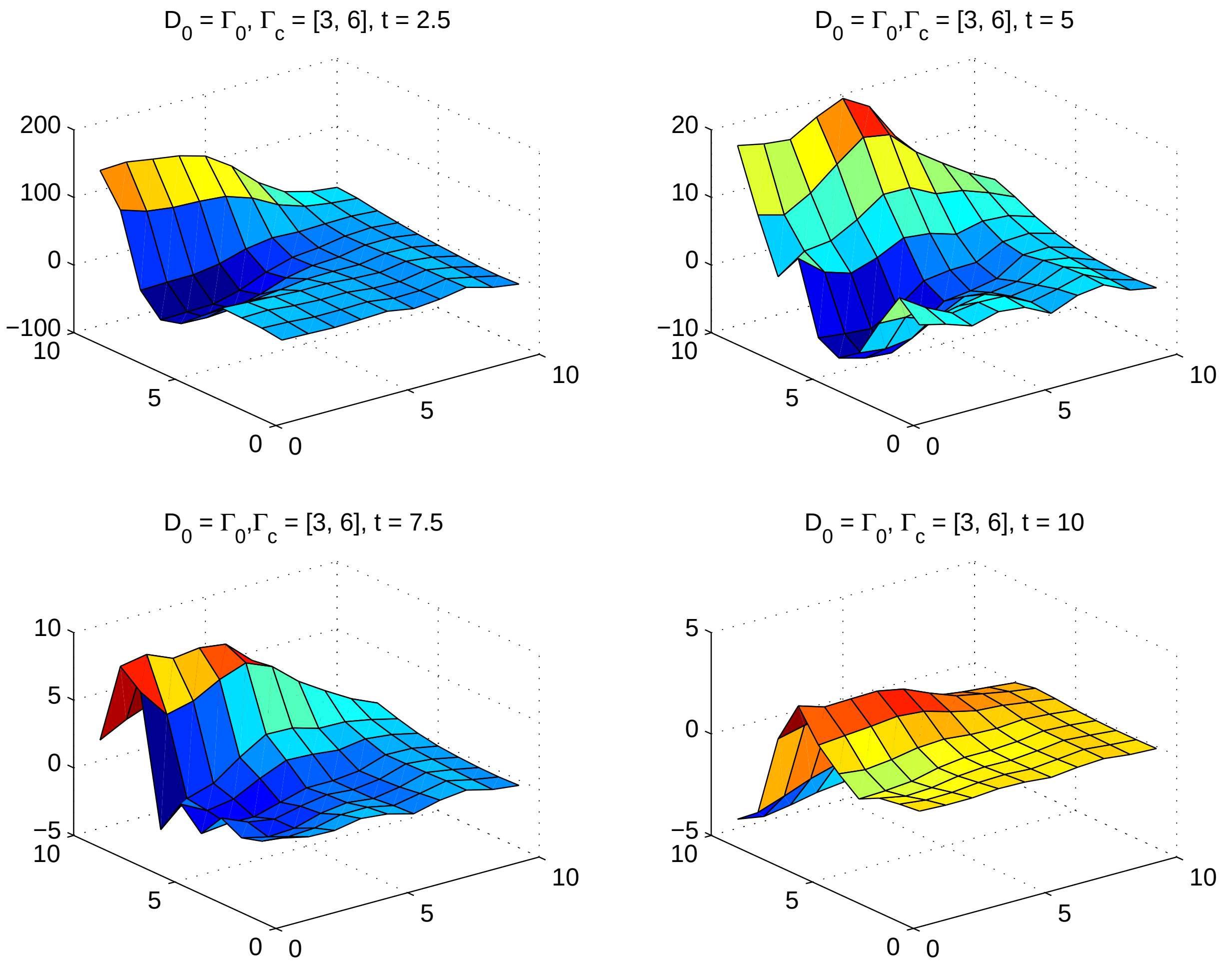
Observation on $D_0 = \Gamma_o = \{1\} \times (0, 10)$

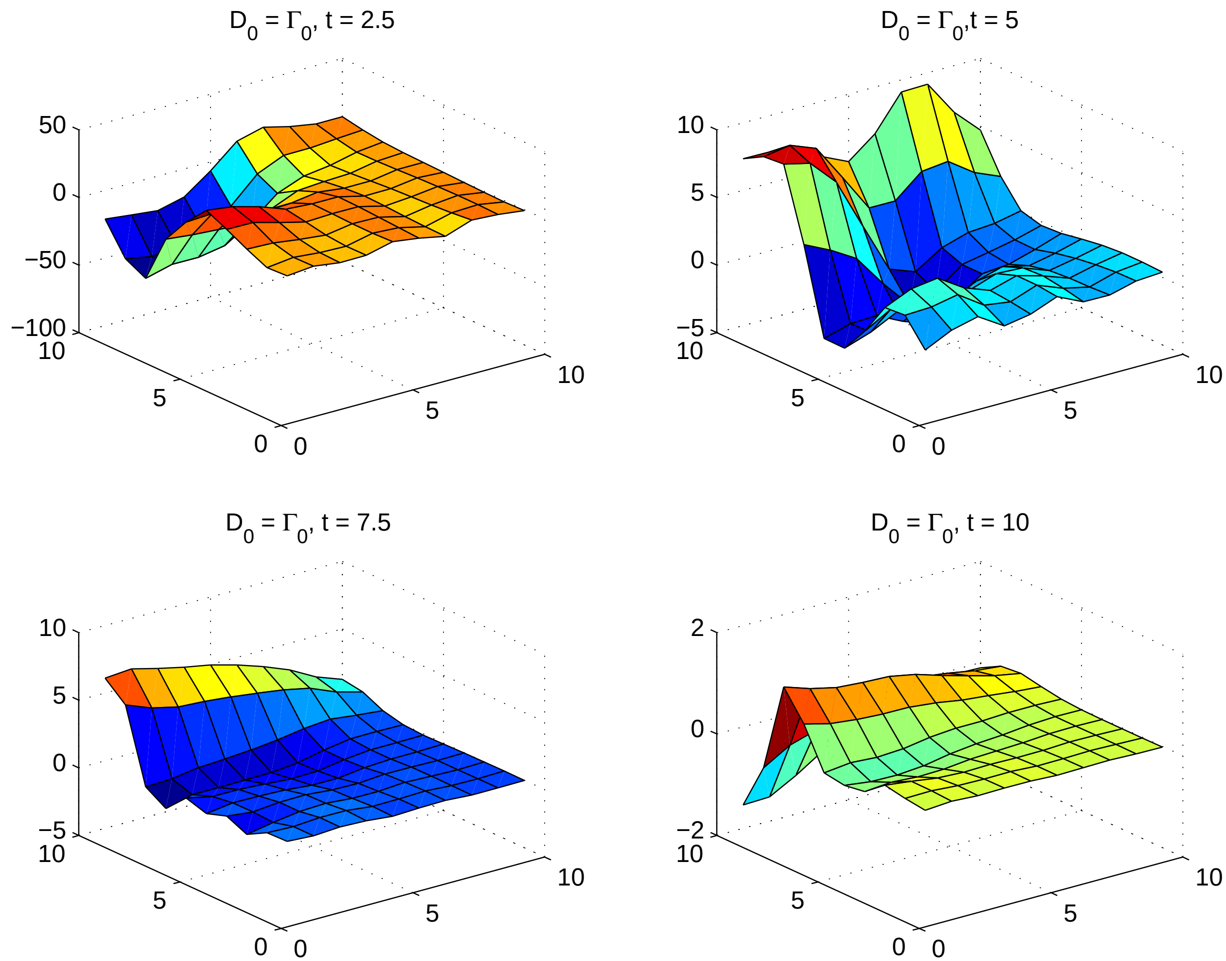
$$c = 3, \qquad y_0(x_1, x_2) = cos(2\pi x_1/10)sin(2\pi x_2/10)$$



Norme des contrôles, Γ_c = { 0 } × [0, 10] et { 0 } × [3, 6]
x 10^4
2.5
2
1.5
1
0.5
0
0
10
20
30
40
50
60
70
80
90
100


Norme des états, Γ_c = { 0 } × [0, 10]   et { 0 } × [3, 6]
4500
4000
3500
3000
2500
2000
1500
1000
500
0
0
50
100
150
200
250
300
350
400
450
500


D0 = Γ0, Γc = [3, 6], t = 2.5
D0 = Γ0,Γc = [3, 6], t = 5
D0 = Γ0,Γc = [3, 6], t = 7.5
D0 = Γ0, Γc = [3, 6], t = 10

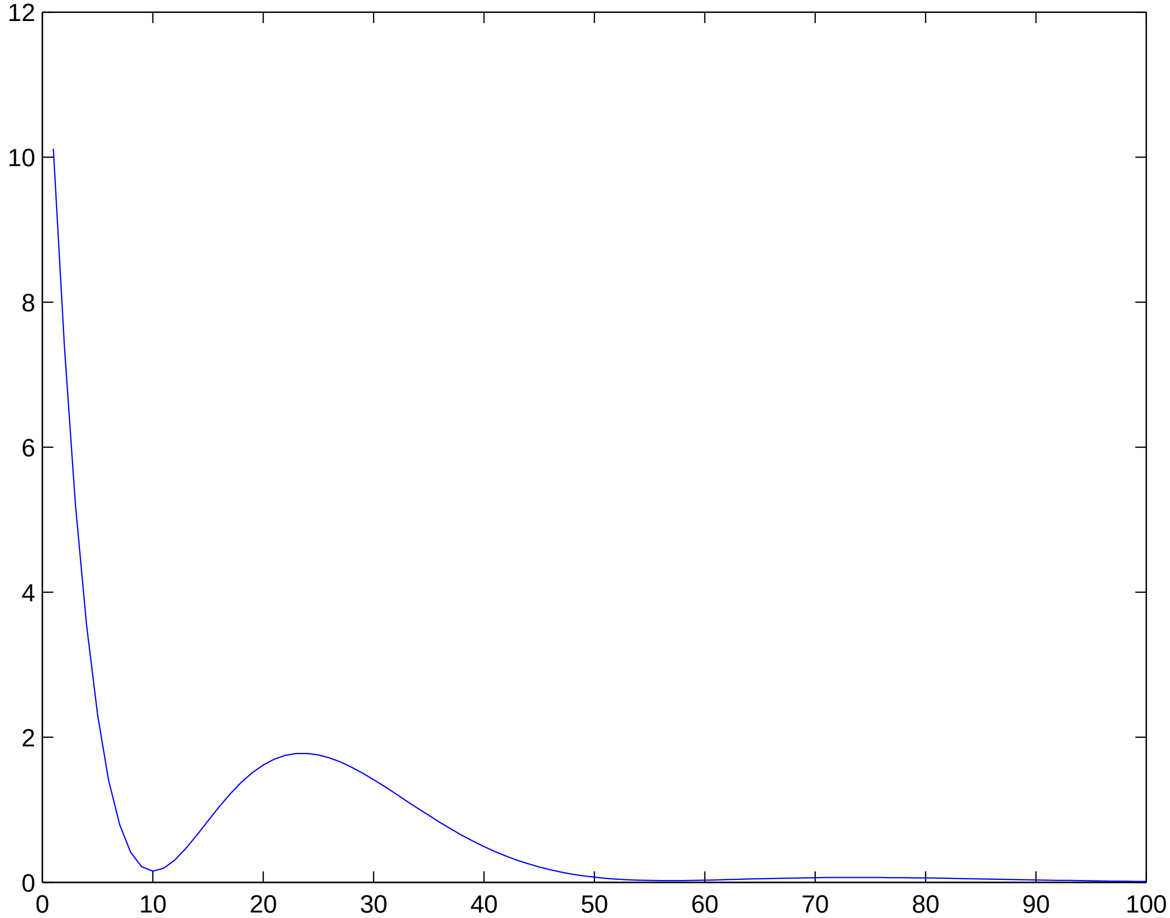

D0 = Γ0, t = 2.5
50
0
−50
−100
10
5
0
0
5
10
D0 = Γ0,t = 5
10
5
0
−5
10
5
0
0
5
10
D0 = Γ0, t = 7.5
10
5
0
−5
10
5
0
0
5
10
D0 = Γ0, t = 10
2
0
−2
10
5
0
0
5
10

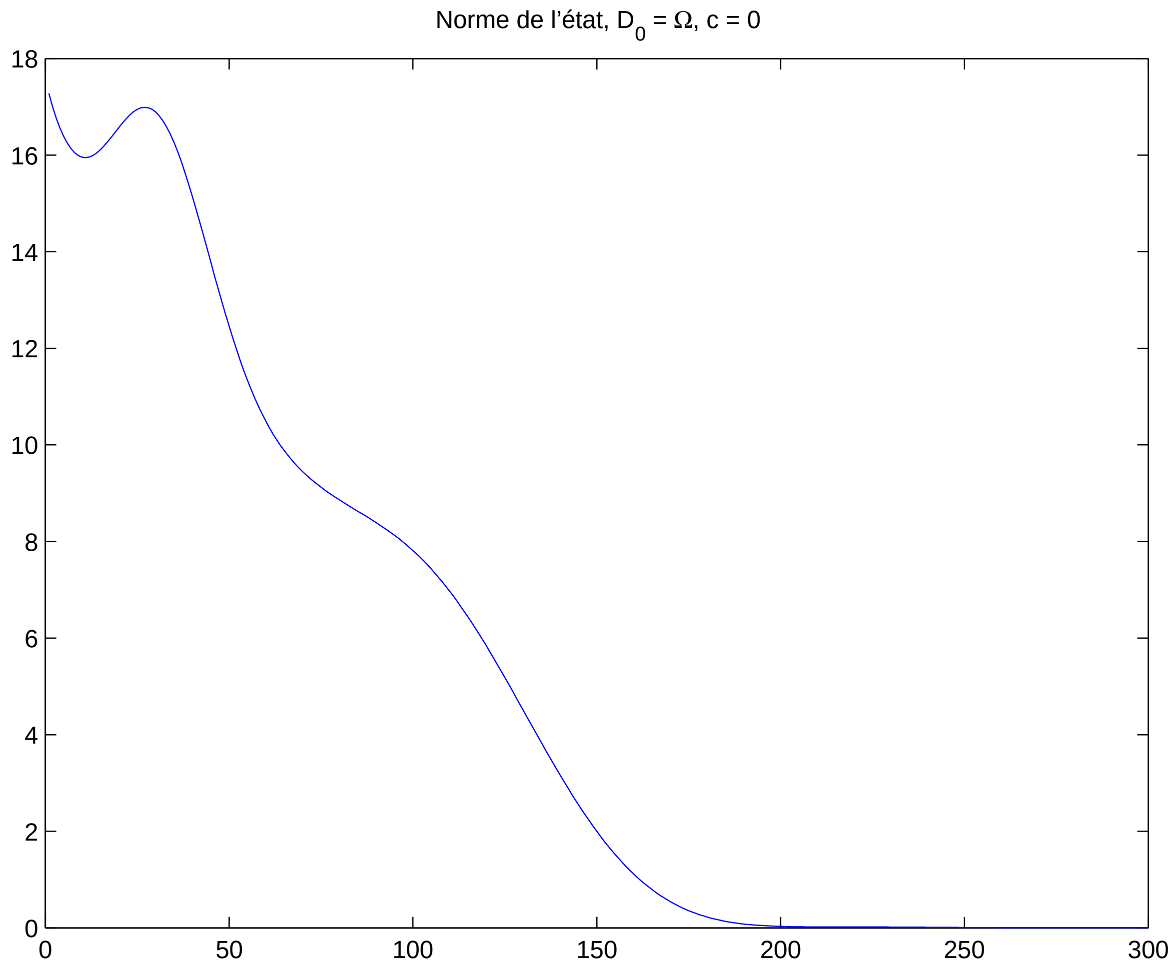
Control on $\Gamma_c = \{0\} \times (0, 10)$

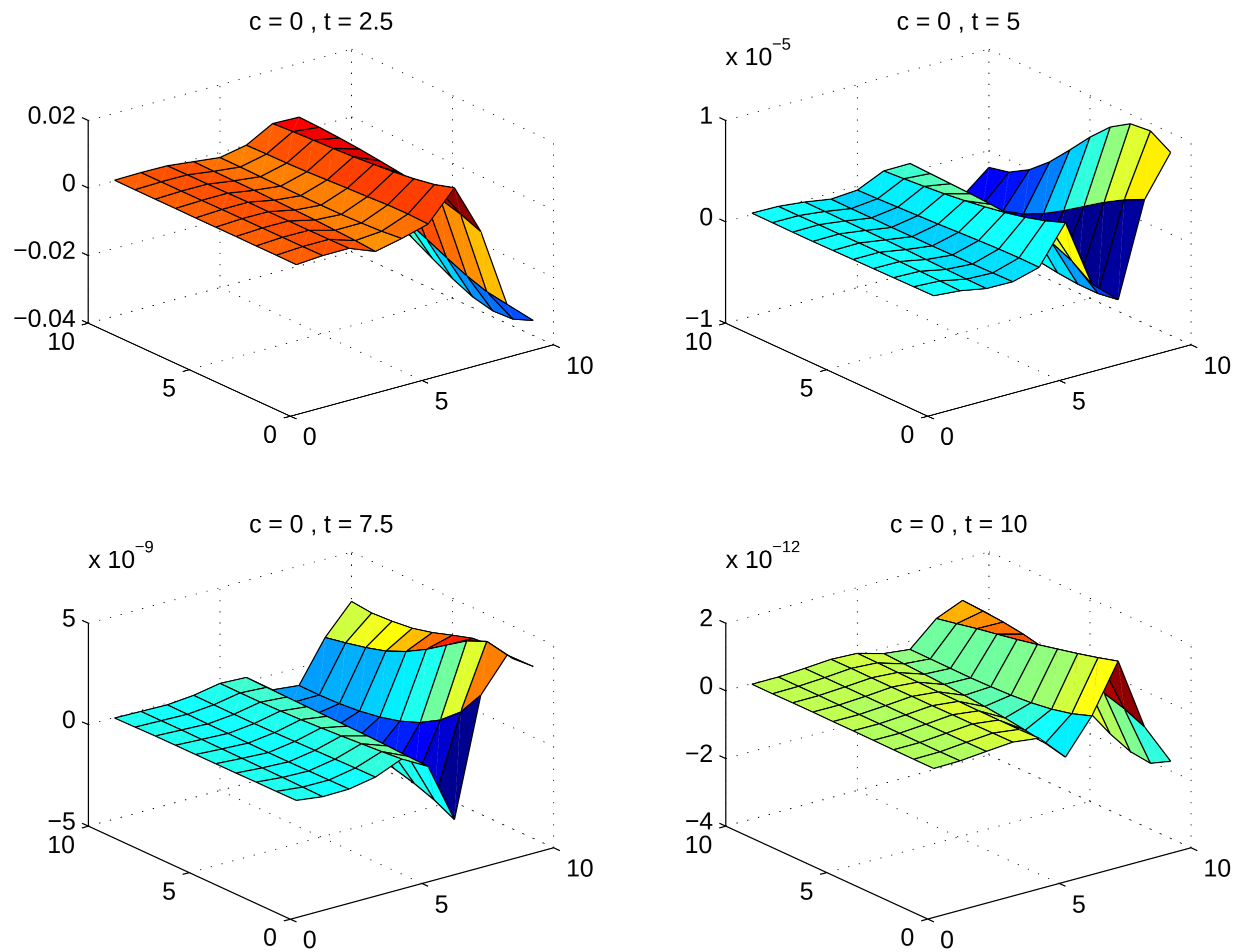
Observation on $D_0 = \Omega$

$$c = 0, \qquad y_0(x_1, x_2) = cos(2\pi x_1/10)sin(2\pi x_2/10)$$

Norme du contrôle, $D_0 = \Omega$, c = 0




Norme de l'état, D0 = Ω, c = 0
18
16
14
12
10
8
6
4
2
0
0
50
100
150
200
250
300


c = 0 , t = 2.5
c = 0 , t = 5
x 10^-5
c = 0 , t = 7.5
x 10^-9
c = 0 , t = 10
x 10^-12

Part 3

The conjugate gradient method

for solving an

optimal control problem

The conjugate gradient method

Consider the optimization problem

$$(P_1) \qquad \inf\{F(u) \mid u \in U\},$$

where U is a Hilbert space and F is a quadratic functional

$$F(u) = \frac{1}{2}(u, Qu)_U - (b, u)_U.$$

In this setting $Q \in \mathcal{L}(U)$, $Q = Q^* > 0$, $b \in U$, and $(\cdot, \cdot)_U$ denotes the scalar product in U. For simplicity we write $(\cdot, \cdot)$ in place of $(\cdot, \cdot)_U$.

The principe of the GCM:

1. Choose $u_0 \in U$ and compute

$$d_0 = -Qu_0 + b.$$

Minimize F over $C_0 = u_0 + \text{Vect}(d_0)$. Let u_1 be the solution.

2. If $d_0, \ldots, d_{k-1}, u_{k-1}$ are known, u_k is the solution of

$$(P_1) \qquad \inf\{F(u) \mid u \in C_{k-1}\},$$

where $C_{k-1} = u_{k-1} + \text{Vect}(d_0, \ldots, d_{k-1})$

Let us recall the GC algorithm:

Algorithm 1.

Initialization. Choose u_0 in U. Compute $g_0 = Qu_0 - b$. Set $d_0 = -g_0$ and $n = 0$.

Step 1. Compute

$$\rho_n = (g_n, g_n)/(d_n, Qd_n),$$

and

$$u_{n+1} = u_n + \rho_n d_n.$$

Determine

$$g_{n+1} = Qu_{n+1} - b = g_n + \rho_n Qd_n.$$

Step 2. If $\|g_{n+1}\|_U / \|g_0\|_U \leq \varepsilon$, stop the algorithm and take $u = u_{n+1}$, else compute

$$\beta_n = (g_{n+1}, g_{n+1})/(g_n, g_n),$$

and

$$d_{n+1} = -g_{n+1} + \beta_n d_n.$$

Replace n by $n+1$ and go to step 1.

The conjugate gradient method for control problems

We want to apply the CGM to problems studied in chapters 1, 3, 4. The state equation is of the form

$$(E) \qquad y' = Ay + Bu + f, \qquad y(0) = y_0,$$

and the control problem is defined by

$$(P_2) \qquad \inf\{J(y_u, u) \mid u \in L^2(0, T; U)\},$$

$$J(y,u) = \frac{1}{2}\int_0^T \|Cy(t) - z_d(t)\|_Z^2$$

$$+\frac{1}{2}\|Dy(T) - z_T\|_{Z_T}^2 + \frac{1}{2}\int_0^T \|u(t)\|_U^2.$$

We have to identify problem (P_2) with a problem of the form (P_1). Let y_u be the solution to equation (E), and set $F(u) = J(y_u, u)$. Observe that $(y_u, y_u(T)) = (\Lambda_1 u, \Lambda_2 u) + \zeta(f, y_0)$, where Λ_1 is a bounded linear operator from $L^2(0,T;U)$ to $L^2(0,T;Y)$, and Λ_2 is a bounded linear operator from $L^2(0,T;U)$ to Y. We must determine the quadratic form Q such that

$$J(y_u, u) = \frac{1}{2}(u, Qu)_U - (b, u)_U + c.$$

Since $(y_u, y_u(T)) = (\Lambda_1 u, \Lambda_2 u) + \zeta(f, y_0)$, we have

$$Q = \Lambda_1^* \widehat{C}^* \widehat{C} \Lambda_1 + \Lambda_2^* D^* D \Lambda_2 + I,$$

where $\widehat{C} \in \mathcal{L}(L^2(0,T;Y); L^2(0,T;Z))$ is defined by $(\widehat{C}y)(t) = Cy(t)$ for all $y \in L^2(0,T;Z)$, and $\widehat{C}^* \in \mathcal{L}(L^2(0,T;Z); L^2(0,T;Y))$ is the adjoint of $\widehat{C}$. In the

CGM we have to compute Qd for some $d \in L^2(0,T;U)$. Observe that $(\Lambda_1 d, \Lambda_2 d)$ is equal to $(w_d, w_d(T))$, where w_d is the solution to

$$w' = Aw + Bd, \qquad w(0) = 0.$$

Moreover, using an IBP, we can prove that $\Lambda_1^* g = B^* p_1$, where p_1 is the solution to equation

$$-p' = A^* p + g, \qquad p(T) = 0,$$

and $\Lambda_2^* p_T = B^* p_2$, where p_2 is the solution to equation

$$-p' = A^* p, \qquad p(T) = p_T.$$

Thus $\Lambda_1^* \widehat{C}^* \widehat{C} \Lambda_1 d + \Lambda_2^* D^* D \Lambda_2 d$ is equal to $B^* p$, where

p is the solution to

$$-p' = A^*p + C^*Cw_d, \qquad p(T) = D^*Dw_d(T).$$

If we apply Algorithm 1 to problem (P_2) we obtain:

Algorithm 2.

Initialization. Choose u_0 in $L^2(0, T; U)$. Denote by y^0 the solution to the state equation

$$y' = Ay + Bu_0 + f, \qquad y(0) = y_0.$$

Denote by p^0 the solution to the adjoint equation

$$-p' = A^*p + C^*(Cy^0 - z_d), \quad p(T) = D^*(Dy^0(T) - z_T).$$

Compute $g_0 = B^*p^0 + u_0$, set $d_0 = -g_0$ and $n = 0$.

Step 1. To compute Qd_n, we calculate w_n the solution to equation

$$w' = Aw + Bd_n, \qquad w(0) = 0.$$

We compute p_n the solution to equation

$$-p' = A^*p + C^*Cw_n, \qquad p(T) = D^*Dw_n(T).$$

We have $Qd_n = B^*p_n + d_n$. Set $\bar{g}_n = B^*p_n + d_n$. Compute

$$\rho_n = -(g_n, g_n)/(\bar{g}_n, g_n),$$

and

$$u_{n+1} = u_n + \rho_n d_n.$$

Determine

$$g_{n+1} = g_n + \rho_n \bar{g}_n.$$

Step 2. If $\|g_{n+1}\|_{L^2(0,T;U)}/\|g_0\|_{L^2(0,T;U)} \leq \varepsilon$, stop the algorithm and take $u = u_{n+1}$, else compute

$$\beta_n = (g_{n+1}, g_{n+1})/(g_n, g_n),$$

and

$$d_{n+1} = -g_{n+1} + \beta_n d_n.$$

Replace n by $n+1$ and go to step 1.

Algorithms for discrete problems

For numerical computations, we have to write discrete approximations to control problems. Suppose that equation

$$y' = Ay + Bu + f, \qquad y(0) = y_0,$$

is approximated by an implicit Euler scheme

$$(DE) \qquad \begin{array}{l} y^0 = y_0, \\ \text{for } n = 1, \ldots, M, \quad y^n \text{ is the solution to} \\ \frac{1}{\Delta t}(y^n - y^{n-1}) = Ay^n + Bu^n + f^n, \end{array}$$

where $f^n = \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} f(t)\,dt$, $u^n = \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} u(t)\,dt$, $t_n = n\Delta t$, and $T = M\Delta t$. To approximate the functional $J(y,u)$ we set

$$J_M(y,u) = \frac{1}{2}\Delta t \sum_{n=1}^{M} \|Cy^n - z_d^n\|_Z^2$$

$$+\frac{1}{2}\|Dy^M - z_T\|_{Z_T}^2 + \frac{1}{2}\Delta t \sum_{n=1}^{M} \|u^n\|_U^2,$$

with $y = (y^0, \ldots, y^M)$, $u = (u^1, \ldots, u^M)$, $z_d^n = \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} z_d(t)\,dt$. We can define a discrete control problem associated with (P_2) as follows:

(P_M) $\inf\{J_M(y,u) \mid u \in U^M,\ (y,u)$ satisfies $(DE)\}$.

To apply the CGM to problem (P_M), we have to compute the gradient of the mapping $u \mapsto J_M(y_u, u)$, where y_u is the solution to (DE) corresponding to u. Set $F_M(u) = J_M(y_u, u)$. We have

$$F'_M(\bar{u})u = \Delta t \sum_{n=1}^{M} (C\bar{y}^n - z_d^n, Cw_u^n)_Z$$

$$+(D\bar{y}^M - y_T, Dw_u^M)_{Z_T} + \Delta t \sum_{n=1}^{M} (\bar{u}^n, u^n)_U,$$

where $\bar{y} = y_{\bar{u}}$ and $w = (w^0, \ldots, w^M) \in Y^{M+1}$ is defined by

$$(G) \qquad \begin{array}{l} w^0 = 0, \\ \text{for } n = 1, \ldots, M, \quad w^n \text{ is the solution to} \\ \frac{1}{\Delta t}(w^n - w^{n-1}) = Aw^n + Bu^n. \end{array}$$

To find the expression of $F'_M(\bar{u})$, we have to introduce an adjoint equation. Let $p = (p^0, \dots, p^M)$ be in Y^{M+1}, or in $D(A^*)^{M+1}$ if we want to justify the calculations. Taking a weak formulation of the different equations in (G), we can write

$$\frac{1}{\Delta t}((w^n - w^{n-1}), p^{n-1})_Y - (w^n, A^* p^{n-1})_Y$$
$$= (Bu^n, p^{n-1})_Y = (u^n, B^* p^{n-1})_U.$$

Now, by adding the different equalities, we find the adjoint equation by identifying

$$\Delta t \sum_{n=1}^{M} (C\bar{y}^n - y_d^n, Cw_u^n)_Z + (D\bar{z}^M - y_T, Dw_u^M)_{Z_T}$$

with

$$\Delta t \sum_{n=1}^{M} (u^n, B^* p^{n-1})_U.$$

More precisely, if $p = (p^0, \ldots, p^M)$ is defined by

$$\begin{aligned}
&p^M = D^*(D\bar{z}^M - y_T),\\
&\text{for } n = 1, \ldots, M, \quad p^n \text{ is the solution to}\\
&\frac{1}{\Delta t}(-p^n + p^{n-1}) = A^* p^{n-1} + C^*(C\bar{z}^n - y_d^n),
\end{aligned}$$

then

$$F'_M(\bar{u})u = \Delta t \sum_{n=1}^{M} (u^n, B^* p^{n-1})_U + \Delta t \sum_{n=1}^{M} (\bar{u}^n, u^n)_U.$$

Observe that the above identification is not justified since $D^*(D\bar{y}^M - y_T)$ does not necessarily belong to

$D(A^*)$. In practice, a 'space-discretization' is also performed. This means that equation (E) is replaced by a system of ordinary differential equations, the operator A is replaced by an operator belonging to $\mathcal{L}(\mathbb{R}^\ell)$, where ℓ is the dimension of the discrete space, and the above calculations are justified for the corresponding discrete problem.

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