Exit time for anchored expansion

Thierry Delmotte\(^1\) Rau Clément\(^1\)

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Colloque "Marches aléatoires"
Orsay le 15, 16 et 17 sept 09
Outline of the talk

1. Introduction, why anchored expansion?
2. New results
3. Some ideas of the proof.
4. Applications
5. Open questions
1 Introduction, why anchored expansion?

- Random walks
- Links between geometry and random walks.
- What is anchored expansion.
- what we know
- what we don’t know
Let $G = (V(G), E(G))$ be a graph, we consider a random walk $(X_n)_{n \geq 0}$ on $G$ with transitions probability $p(., .)$. 

**Random walks**

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- **Assumption**: there exists a reversible measure $m$ for $X$. 
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**Assumption:** there exists a reversible measure $m$ for $X$.

We let $a(x, y) = m(x)p(x, y)$. 

**Random walks**

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**Random walks**

Links between geometry and random walks.
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Exit time for anchored expansion
Random walks

Example: Simple Random walk.

$$p(x, y) = \frac{1 \{ (x, y) \in E(G) \}}{\nu(x)},$$

where $\nu(x)$ is the number of neighbours of $x$ in $G$. 
Example: Simple Random walk.

\[ p(x, y) = \frac{1_{\{(x, y) \in E(G)\}}}{\nu(x)}, \]

where \( \nu(x) \) is the number of neighbours of \( x \) in \( G \).

In this case,

\[ m = \nu \text{ and } a = 1. \]
Links between geometry and RW

$x$ \to \text{Diagram} \to y$

Behaviours related to the quantity $|\partial A|/|A|$. 

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Exit time for anchored expansion
Links between geometry and RW

Behaviours related to the quantity $|\partial A|$, $|A|$. 

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Exit time for anchored expansion
Links between geometry and RW

\[ |\partial A| |A| \]

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Exit time for anchored expansion
Links between geometry and random walks.

What is anchored expansion?

what we know
what we don’t know

Behaviour related to the quantity $\frac{|\partial A|}{|A|}$.
One tools to control random walk : IS

Let $\mathcal{I}S_F$ the inequality : for all set $A$, \[ \frac{a(\partial A)}{F(m(A))} \geq c \]
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One tools to control random walk : IS

Let \( IS_\mathcal{F} \) the inequality : for all set \( A \)
\[
\frac{a(\partial A)}{\mathcal{F}(m(A))} \geq c
\]

Proposition (Coulhon 99)

Let \( G \) a graph such that \( IS_\mathcal{F} \) is satisfied. Assume that the function \( f : t \rightarrow t/\mathcal{F}(t) \) is increasing and that \( m_0 = \inf_{V(G)} m > 0 \), then :

\[
\sup_{x,y} \frac{p_{2n}(x,y)}{m(y)} \leq 2u(n),
\]

where \( u : \mathbb{R} \rightarrow]0; 1/m_0] \) satisfies

\[
\begin{cases} 
  u(0) = \frac{1}{m_0} \\
  u' = -\frac{u}{2g(1/u)} \\
\end{cases}
\]

with

\( g(x) = 4(f(4x)/c)^2 \)
**Some examples**

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Isoperimetry: for all subset $A$ of $\partial A|A|^{1/d} \geq C$ implies $P_{x}(X_{n} = y) \leq c n^{d/2}$.

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Exit time for anchored expansion
Some examples

Isoperimetry:

\[
\text{for all subset } A \quad |\partial A| \leq |A|^{1 - 1/d} \implies P_x(X_n = y) \leq c^n d/2.
\]
Some examples

Isoperimetry:

for all subset $A$ \[ \frac{|\partial A|}{|A|^{1-1/d}} \geq C \]
Some examples

Isoperimetry:

For all subset $A$,

$$\frac{\left| \partial A \right|}{|A|^{1-1/d}} \geq C$$

implies

$$\mathbb{P}_x(X_n = y) \leq \frac{C}{n^{d/2}}.$$
Some examples

Isoperimetry:

for all subset $A$, $|\partial A| \geq C$ implies $P_x(X_n = y) \leq e^{-cn}$
Some examples

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\text{for all subset } A, \quad |\partial A| \geq C \implies P_x(X_n = y) \leq e^{-cn}
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Isoperimetry:

for all subset $A$ \[ \frac{|\partial A|}{|A|} \geq C \]
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Isoperimetry:

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Stability of IS?

Isoperimetric inequality is not stable under perturbations.
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ex: Bernouilli percolation process of $\mathbb{Z}^d$ destroy it.
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$\Rightarrow$ new notion more stable !!!
Isoperimetric inequality is not stable under perturbations.

ex: Bernouilli percolation process of $\mathbb{Z}^d$ destroy it.

⇒ new notion more stable !!!

Anchored isoperimetry
Definition

Let $\mathcal{F}$ a positive increasing function defined on $\mathbb{R}_+$. Let $G$ a graph with bounded valence and $o \in G$. We say that $G$ satisfies an anchored (or rooted) $\mathcal{F}$-isoperimetric inequality in $o$ if there exists a constant $C_{IS} > 0$ such that for any connected set $A$ which contains $o$ we have:

$$\frac{|\partial A|}{\mathcal{F}(|A|)} \geq C_{IS}. \quad (1)$$

$\partial A$ is equal to the set $\{(x, y) \in E(G); x \in A \text{ and } y \notin A\}$ and $|A|$ stands for the cardinal of $A$.

We will write $G$ satisfies $AIS_\mathcal{F}$. 

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When $\mathcal{F} = id$, there is an equivalent version of this definition which can be said as follow:
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**Definition**

$G$ satisfies an anchored (or rooted) isoperimetric inequality if

$$\lim_{n \to \infty} \inf \left\{ \frac{|\partial S|}{|S|} ; \text{ } S \text{ connected, } v \in S \text{ and } |S| \geq n \right\} := i^*(G)$$

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This definition does not depend on the choice of the fixed vertex $v$. 
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is strictly positif.

This definition does not depend on the choice of the fixed vertex $v$ whereas in the previous definition, the constant $C_{IS}$ depends on the point $o$. 
⇒ Now, the work is to examine what anchored isoperimetric inequality implies for random walk.
Results known.

**Theorem**

(Thomassen 92) Let $G$ a graph satisfying a AIS$_F$, then

$$\sum_k \frac{1}{F(k)^2} < \infty \Rightarrow \text{the simple random walk on } G \text{ is transient.}$$
Results known.

**Theorem (Virag 00)**

Let $G$ a graph (with bounded geometry) satisfying $\text{AIS}_{id}$ (strong anchored isoperimetric inequality ($\mathcal{F} = \text{id}$)), then

1. There exists a constant $c > 0$ such that
   \[
   \liminf_n \frac{|X_n|}{n} \geq c \ i^*(G)^7 \text{ a.s}
   \]

2. For all $x \in G$ there exists $N$ such that for all $n \geq N$ and for all $y \in G$ one has:
   \[
   p_n(x, y) \leq e^{-\alpha n^{1/3}}
   \]

where $\alpha = c' \ i^*(G)^2$
Results known.

Theorem (Chen and Peres 05)

Consider a $p-$Bernoulli percolation on a graph $G$ with constant $i^*(G) > 0$, if $p < 1$ is sufficiently close to 1 then, almost surely on the event that the open cluster $H$ containing $0$ is infinite, we have $i^*(H) > 0$
Results known.

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A refinement of the argument due to Gabor Pete shows that the conclusion holds for all $p > \frac{1}{1+i^*(G)}$. 
For a graph $G$, we replace each edge $e \in E(G)$ by a path that consists in $L_e$ new edges, where the random variable $(L_e)_{e \in E(G)}$ are independant with law $\nu$. 
Results known.

For a graph $G$, we replace each edge $e \in E(G)$ by a path that consists in $L_e$ new edges, where the random variable $(L_e)_{e \in E(G)}$ are independent with law $\nu$.

Let $G^\nu$ the graph obtained in this way, we call it a random stretch of $G$. 
Results known.

$\nu$ has an exponential tail if $\nu([l; +\infty]) \leq e^{-\epsilon l}$ for $\epsilon > 0$ and $l$ large enough.
\( \nu \) has an exponential tail if \( \nu([l; +\infty]) \leq e^{-\epsilon l} \) for \( \epsilon > 0 \) and \( l \) large enough.

**Theorem (Chen and Peres 05)**

Suppose that \( G \) is an infinite graph of bounded degree and \( i^*(G) > 0 \). If \( \nu \) has an exponential tail then \( i^*(G^\nu) > 0 \) a.s.
Question 1: does anchored isoperimetry is a good tool?
what we don’t know and we will be happy to know

- **Question 1**: does anchored isoperimetry is a good tool?
- **Question 2**: does a general anchored isoperimetric inequality imply an upper bound of $p_n(x, y)$?
what we don’t know and we will be happy to know

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- **Question 2**: does a general anchored isoperimetric inequality imply an upper bound of $p_n(x, y)$?
- **Question 3**: does the sub tree of Thomassen satisfy an anchored isoperimetric inequality?
what we don’t know and we will be happy to know

- **Question 1**: does anchored isoperimetry is a good tool?
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- **Question 3**: does the sub tree of Thomassen satisfy an anchored isoperimetric inequality?
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- Stability under percolation
- Transience or recurrence
- \(\lambda_1(A)\)
- Transition kernel
- Speed

\(\IS\) and \(\AIS\) denote stability under percolation, transience or recurrence, the first eigenvalue \(\lambda_1\) of the Laplacian, transition kernel, and speed, respectively. For \(F = \text{id}\) (Virag), the transition kernel can be computed and is given by:

\[
\liminf_{n \to \infty} |X_n|_n \geq c_i(G)
\]

Cheeger

Transition kernel:

\[p^n(x,y) \leq u(n)\]

For \(F = \text{id}\) (Virag):

\[u \text{ is a solution of an ED}\]

\[p^n(x,y) \leq e^{-n^{1/3}}\]

Coulhon

Speed:

\[\liminf_{n \to \infty} |X_n|_n \geq c_i(G)\]

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Exit time for anchored expansion
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- Stability under percolation: Not in general! ok if $p$ is close to 1 (Chen, Peres, Pete)
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#### Transition kernel

- $\mathcal{IS}_F$
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#### Speed

- $\mathcal{IS}_F$
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## Introduction

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*what we know*

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Chen, Peres, (Pete)

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<td><strong>IS$_F$</strong></td>
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<td><strong>EXIT TIME FOR ANCHORED EXPANSION</strong></td>
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## Summary

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- Stability under percolation:
  - Not in general!
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- Transience or recurrence:
  - Computation $g(1/x)$ integrable in 0
  - If $\sum_k \mathcal{F}(k)^{-2} < \infty$ Thomassen

- $\lambda_1(A)$:
  - $\lambda_1(A) \geq C_{IS}^2 \frac{\mathcal{F}(|A|)^2}{|A|^2}$
  - Cheeger

- Transition kernel:
  - $p_n(x, y) \leq u(n)$
  - $u$ sol of an ED (Coulhon)
  - For $\mathcal{F} = id$ (Virag)
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- Speed:
  - Computation
  - ...

---

**Thierry Delmotte, Rau Clément**

Exit time for anchored expansion
Summary

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| Transcience or recurrence | $\lambda_1(A)$ | $\lambda_1(A) \geq C_{IS}^2 \frac{F(|A|)^2}{|A|^2}$ Cheeger |
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| Speed | Computation ... | For $F = \text{id}$ (Virag) $\lim \inf \frac{|X_n|}{n} \geq c_i(G)^7$ |
|-------|-----------------|-------------------------|
### Exit or occupation time

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For $A \subset G$, let $\tau_A$ the exit time of $A$ for $X$:

$$\tau_A = \inf\{k \geq 0 ; X_k \notin A\}$$
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For $A \subset G$, let $\tau_A$ the exit time of $A$ for $X$:

$$\tau_A = \inf\{ k \geq 0 ; X_k \notin A \}$$

and when $X$ is transient let $l_A$ the occupation time of $A$ by:

$$l_A = \text{card}\{ k \in \mathbb{N} ; X_k \in A \}.$$
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Indeed, when transitions kernel are known, it is possible to estimate $\mathbb{E}(\tau_A)$
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Indeed, when transitions kernel are known, it is possible to estimate $\mathbb{E}(\tau_A)$

**ex:** In $\mathbb{Z}^d$, $IS_d \Rightarrow \rho_n(x, y) \leq c/n^{d/2} \Rightarrow E(\tau_A) \leq C|A|^{2/d}$.
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Indeed, when transitions kernel are known, it is possible to estimate $\mathbb{E}(\tau_A)$

ex : In $\mathbb{Z}^d$, $IS_d \Rightarrow p_n(x, y) \leq c/n^{d/2} \Rightarrow E(\tau_A) \leq C|A|^{2/d}$.

How to proceed without transitions kernel estimate?
Indeed, when transitions kernel are known, it is possible to estimate $E(\tau_A)$
ex: $\mathbb{Z}^d$, $IS_d \Rightarrow \rho_n(x, y) \leq c/n^{d/2} \Rightarrow E(\tau_A) \leq C|A|^{2/d}$.

How to proceed without transitions kernel estimate?

What’s happening for $AIS_F$?
2 New results

- General statement
- Consequences
New results.

Theorem

Let $G$ satisfying $AIS_F$, then for any $A \subset G$ containing $0$ we have:

$$E_0(\tau_A) \leq 2 \int_0^{G_A(0)} v(s) \, ds,$$

where $v$ is solution of the differential equation

$$\begin{cases} v(0) = m(A) \\ v' = -(C_{IS} F(v))^2. \end{cases}$$

In fact this estimate holds for $E_0(l_A)$ when $X$ is transient.
New results

Examples :

- if \( \mathcal{F}(x) = x^{1 - \frac{1}{d}} \), \( d \geq 3 \) we have :

  \[
  \mathbb{E}_o(\tau_A) \leq \mathbb{E}_o(l_A) \leq c(d) m(A)^{\frac{2}{d}},
  \]

- if \( \mathcal{F}(x) = x^{1/2} \), \( d = 2 \) we have :

  \[
  \mathbb{E}_o(\tau_A) \leq \mathbb{E}_o(l_A) \leq c(d) m(A)^{\frac{2}{d}}.
  \]
New results

Examples:

- if $\mathcal{F}(x) = x^{1 - \frac{1}{d}}$, $(d \geq 3)$ we have:
  \[\mathbb{E}_0(\tau_A) \leq \mathbb{E}_0(l_A) \leq c(d) m(A)^{\frac{2}{d}},\]

- if $\mathcal{F}(x) = x^{\frac{1}{2}}$, $(d = 2)$ we have:
  \[\mathbb{E}_0(\tau_A) \leq c m(A),\]
New results

Examples:

- If $\mathcal{F}(x) = x^{1 - \frac{1}{d}}$, $(d \geq 3)$ we have:
  \[
  \mathbb{E}_o(\tau_A) \leq \mathbb{E}_o(l_A) \leq c(d) m(A) \frac{2}{d},
  \]

- If $\mathcal{F}(x) = x^{\frac{1}{2}}$, $(d = 2)$ we have:
  \[
  \mathbb{E}_o(\tau_A) \leq c m(A),
  \]

- If $\mathcal{F}(x) = x$ we have:
  \[
  \mathbb{E}_o(\tau_A) \leq \mathbb{E}_o(l_A) \leq c \ln(m(A)),
  \]
New results

Let $G_A(.,.)$ the Green function associated to random walk killed outside $A$
New results

Let $G_A(.,.)$ the Green function associated to random walk killed outside $A$ and let

$$A_t = \{ x \in A; \, G_A(0, x) \geq t \}$$

and

$$u(t) = m(A_t)$$
Let $G_A(.,.)$ the Green function associated to random walk killed outside $A$ and let

$$A_t = \{ x \in A; \ G_A(0, x) \geq t \}$$

and

$$u(t) = m(A_t)$$

**Proposition**

If $G$ satisfies $\text{AIS}_F$, then $u$ satisfies the following differential inequality:

$$\begin{cases}
  u(0) = m(A) \\
  u' \leq -(\text{CIS}_F(u))^2.
\end{cases}$$
We retrieve Thomassen’s result

Assume \( \int_{1}^{+\infty} \frac{du}{F(u)^2} < +\infty \) for \( F \) continuous on \( \mathbb{R}_+ \).
We retrieve Thomassen’s result

Assume \( \int_{1}^{+\infty} \frac{du}{F(u)^2} < +\infty \) for \( F \) continuous on \( \mathbb{R}_+ \).

Integrating \( \begin{cases} u(0) = m(A) \\ u' \leq -(C_{IS}F(u))^2 \end{cases} \) gives us that

\[
\int_{u(t)}^{u(0)} \frac{ds}{F(s)^2} \geq C_{IS}^2 t.
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So

\[
\lim_{t \to +\infty} u(t) = 0 \text{ uniformly in } A.
\]
We retrieve Thomassen’s result

Assume $\int_1^{+\infty} \frac{du}{F(u)^2} < +\infty$ for $F$ continuous on $\mathbb{R}_+$. Integrating

\[
\begin{cases}
u(0) = m(A) \\ \nu' \leq -(C_{IS}F(u))^2.
\end{cases}
\]

gives us that

\[
\int_{\nu(t)}^{\nu(0)} \frac{ds}{F(s)^2} \geq C_{IS}^2 t.
\]

So

\[
\lim_{t \to +\infty} \nu(t) = 0 \text{ uniformly in } A.
\]

Since

\[
\nu(t) = m(x \in A; G_A(0, x) \geq t)
\]
We retrieve Thomassen’s result

Assume \( \int_1^{+\infty} \frac{du}{F(u)^2} < +\infty \) for \( F \) continuous on \( \mathbb{R}_+ \).

Integrating \( \begin{cases} u(0) = m(A) \\ u' \leq -(C_{IS}F(u))^2 \end{cases} \) gives us that

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So

\[
\lim_{t \to +\infty} u(t) = 0 \text{ uniformly in } A.
\]

Since

\[
u(t) = m(x \in A; \ G_A(0, x) \geq t)
\]

There exists \( t_0 \) such that for all \( t \geq t_0, \ G_A(0, x) \leq t_0 \).
We retrieve a weak version of Virag’s result

Proposition

Let $G$ a graph satisfying $AIS_{id}$ and let $(X_n)_n$ simple random walk on $G$. Then we have:

$$\mathbb{P}(\lim_{n} \frac{d(o, X_n)}{n} = 0) = 0.$$
Some ideas of the proof.

- Some properties of Green function
- Connexion with exit time
Let $A$ connected such that $0 \in A$. 

Green function
Let $A$ connected such that $0 \in A$. Consider the random walk killed outside $A$, with following transitions :

$$p^A(x, y) = \begin{cases} p(x, y) & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$
Green function

Let $A$ connected such that $0 \in A$. Consider the random walk killed outside $A$, with following transitions:

$$p^A(x, y) = \begin{cases} p(x, y) & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

$$G^A(x, y) = \frac{1}{m(y)} \sum_{k \geq 0} \mathbb{P}^A_x(X_k = y).$$

$$G^A(x) = G^A(o, x).$$
The discrete Laplacian is

\[ \Delta^A f = (\text{Id} - P^A) f, \]
The discrete Laplacian is

\[ \triangle^A f = (Id - P^A) f, \]

where \( P^A \) is the operator defined on functions which are zero outside \( A \) by:

\[ P^A f(x) = \mathbb{E}_x(f(X_1) \ 1_{\{X_1 \in A\}}) = \sum_{y \in A} p^A(x, y)f(y) \]
Let $A$ a connected set that contains $0$.

**Proposition**

$G^A$ is harmonic on $A \setminus \{0\}$, more precisely:

$$\triangle^A G^A = \frac{\delta_0}{m(0)}$$
Let $A$ a connected set that contains $0$.

**Proposition**

$G^A$ is harmonic on $A \setminus \{0\}$, more precisely:

$$\nabla^A G^A = \frac{\delta_0}{m(0)}$$

**Consequence:**

**Corollary**

*The inward flow through any $B \subset A$ satisfies:*

$$\sum_{e \in \partial B} a(e) \nabla_e G^A = 1_{\{o \in B\}}.$$

(3)
Let $A_s = \{ x \in A ; G^A(x) \geq s \}$ and let $C$ a connected component of $A_s$. Assume $0 \notin C$.
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By previous corollary, $\sum_{e \in \partial C} a(e) \nabla_e G^A = 0$, \n
**Level sets of Green function**
Let $A_s = \{ x \in A ; G^A(x) \geq s \}$ and let $C$ a connected component of $A_s$. Assume $0 \notin C$.

By previous corollary, $\sum_{e \in \partial C} a(e) \nabla_e G^A = 0$.
So there exist an edge $e = (x, y)$ such that $G(x) \geq G(y)$. 

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**Level sets of Green function**

Thierry Delmotte, Rau Clément
Let $A_s = \{ x \in A ; G^A(x) \geq s \}$ and let $C$ a connected component of $A_s$. Assume $0 \notin C$

By previous corollary, $\sum_{e \in \partial C} a(e) \nabla_e G^A = 0$.
So there exist an edge $e = (x, y)$ such that $G(x) \geq G(y)$
that gives a contradiction since $C \subset A_s = \{ x \in A ; G^A(x) \geq s \}$. 
Proposition

The level sets $A_s = \{x \in A ; G^A(x) \geq s\}$ are connected and contain $o$. 
Level sets of Green function

Proposition

The level sets $A_s = \{x \in A : G^A(x) \geq s\}$ are connected and contain $o$.

So if $G$ satisfies $AIS_\mathcal{F}$, we can apply isoperimetric inequality to the sets $A_s$. 
So we get the differential inequation:

\[
\begin{cases}
    u(0) = m(A) \\ 
    u' \leq -(C_{IS} F(u))^2.
\end{cases}
\]
Connexion with exit time.

\[ \mathbb{E}_o(\tau_A) = \sum_{x \in A, \ k \geq 0} \mathbb{P}_o^A(X_k = x) \]

\[ = \sum_{x \in A} m(x) G^A(x) \]

\[ = \sum_{x \in A} m(x) \int_{\mathbb{R}^+} 1\{G^A(x) \geq t\} \ dt \]

\[ = \int_{\mathbb{R}^+} m(\{x \in A; \ G^A(x) \geq t\}) \ dt \]

\[ = \int_{\mathbb{R}^+} u(t) \ dt \]
Applications

- Non degeneration for invariance principle
- Exit time in random environments
- Exit time in percolation model
Assume that $X$ is a random walk on a graph $G$ which is now supposed to be a subgraph of $\mathbb{Z}^d$. We suppose that $X$ admits a reversible measure $m$ satisfying:

$$\forall x \in G \quad m(x) \leq c.$$
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Let $\tilde{X}_k^N$ the renormalized random walk defined by

$$\tilde{X}_k^N = \frac{1}{N} X_{kN^2}.$$
Minoration of the diffusion constant.

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Proposition

Assume $G$ satisfies $d$–dimensionnal anchored isoperimetric inequality with constant $C_{IS}$ and that $(\tilde{X}_k^N)_k$ converges in law to a brownian motion with matrix covariance $\sigma \text{Id}$, then there exists a constant $a(d) > 0$ such that

$$\sigma > a(d) \cdot C_{is}.$$  

In particular, $\sigma > 0$. 

Proof:

That follows from our estimates for exit time.
Proposition

Assume $G$ satisfies $d$–dimensionnal anchored isoperimetric inequality with constant $C_{IS}$ and that $(\tilde{X}_k^N)_k$ converges in law to a brownian motion with matrix covariance $\sigma \text{Id}$, then there exists a constant $a(d) > 0$ such that

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Proof : That follows from our estimates for exit time.
Random environments.

Consider the graph $\mathcal{L}^d = (\mathbb{Z}^d, E_d)$. 
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An environment is a function $\omega : E_d \rightarrow [0; 1]$. 
Random environments.

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- An environment is a function $\omega : E_d \to [0; 1]$.
- Let $\Omega = [0, 1]^{E_d}$ be the set of environments and let $Q$ be a product probability measure on $\Omega$ such that the family $(\omega(e))_{e \in E_d}$ forms independent identically distributed random variables.
Random environments.

- Consider the graph $\mathcal{L}^d = (\mathbb{Z}^d, E_d)$.
- An environment is a function $\omega : E_d \to [0; 1]$.
- Let $\Omega = [0, 1]^{E_d}$ be the set of environments and let $Q$ be a product probability measure on $\Omega$ such that the family $(\omega(e))_{e \in E_d}$ forms independent identically distributed random variables.
- Assumption : $Q(\omega(e) > 0) = 1$
X will design the random walk on the graph $\mathcal{L}_d$ starting from the origin with transitions probability given by:

$$p^\omega(x, y) = \frac{\omega(x, y)}{\sum_{z \sim x} \omega(x, z)}.$$
Random environments.

- $X$ will design the random walk on the graph $\mathcal{L}_d$ starting from the origin with transitions probability given by:

$$p^\omega(x, y) = \frac{\omega(x, y)}{\sum_{z \sim x} \omega(x, z)}.$$ 

- We denote by $\mathbb{P}^\omega_0$ the law of $X$ and by $\mathbb{E}^\omega_0$ its expectation.
Random environments.

- $X$ will design the random walk on the graph $\mathcal{L}_d$ starting from the origin with transitions probability given by:
  \[
  p^\omega(x, y) = \frac{\omega(x, y)}{\sum_{z \sim x} \omega(x, z)}.
  \]

- We denote by $\mathbb{P}_0^\omega$ the law of $X$ and by $\mathbb{E}_0^\omega$ its expectation.
- The random walk $X$ admits reversible measures which are proportional to the measure $m^\omega$ defined by:
  \[
  m^\omega(x) = \sum_{z \sim x} \omega(x, z).
  \]

In this case, we have:
\[
a^\omega(x, y) = \omega(x, y).
\]
Proposition

Let $Q$ be a law on environments such that $Q(\omega(e) > 0) = 1$. There exists $\beta_0(Q, d) > 0$ such that $Q$ a.s for all environment $\omega$, there exists $N_0(\omega) \in \mathbb{N}$ such that for all connected sets $A$ which contained 0,

$$m^\omega(A) \geq N_0(\omega) \Rightarrow \frac{a^\omega(\partial A)}{m^\omega(A)^{1-\frac{1}{d}}} \geq \beta_0.$$
Proposition

Let $Q$ be a law on environments such that $Q(\omega(e) > 0) = 1$. There exists $\beta_0(Q, d) > 0$ such that $Q$ a.s. for all environment $\omega$, there exists $N_0(\omega) \in \mathbb{N}$ such that for all connected sets $A$ which contained $0$,

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No control for small sets.
Isoperimetry for random environments.

**Proposition**

Let $Q$ be a law on environments such that $Q(\omega(e) > 0) = 1$. There exists $\beta_0(Q, d) > 0$ such that $Q$ a.s for all environment $\omega$, there exists $N_0(\omega) \in \mathbb{N}$ such that for all connected sets $A$ which contained 0,

$$m^\omega(A) \geq N_0(\omega) \Rightarrow \frac{a^\omega(\partial A)}{m^\omega(A)^{1-\frac{1}{d}}} \geq \beta_0.$$ 

No control for small sets.

**Proof**: Contour argument.
Exit or occupation time for random environments.

Proposition

Let $d \geq 1$. There exists constants $C = C(Q,d)$ such that $Q$ a.s.
for all environment $\omega$:
for any connected subset $B$ which contains the origin and with volume $m_\omega(B)$ large enough,
for $d \geq 3$,
$$E_0(\ell_B) \leq C m_\omega(B)^{2/d}.$$ For $d = 2$,
$$E_0(\tau_B) \leq C m_\omega(B).$$

Thierry Delmotte, Rau Clément
Proposition

Let $d \geq 1$. There exists constants $C = C(Q, d)$ such that $Q$ a.s. for all environment $\omega$:

for any connected subset $B$ which contains the origin and with volume $m^\omega(B)$ large enough,

- for $d \geq 3$, $\mathbb{E}_0(l_B) \leq C m^\omega(B)^{2/d}$
- for $d = 2$, $\mathbb{E}_0(\tau_B) \leq C m^\omega(B)$. 
Remark

In papers of Boukhadra or Berger-Biskup, it is proved that we can build environments where the return probability is greater than $\frac{1}{n^2}$. By our proposition 4.2, the d-dimensional anchored isoperimetric inequality is satisfied on these environments and so in dimension higher than 4, no one can hope to prove that in this case, the return probability is in $\frac{1}{n^{d/2}}$.
Remark

In papers of Boukhadra or Berger-Biskup, it is proved that we can build environments where the return probability is greater than $1/n^2$. By our proposition 4.2, the $d$-dimensional anchored isoperimetric inequality is satisfied on these environments and so in dimension higher than 4, no one can hope to prove that in this case, the return probability is in $1/n^{d/2}$. 
Consider the particular case $\omega : E_d \rightarrow \{0, 1\}$
Percolation context.

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Anchored isoperimetry on $C$?
Proposition

Let \( p > p_c(d) \). There exists \( \beta_0(p, d) > 0 \) such that \( Q \) a.s on \( \#C = +\infty \), there exists \( N_0(\omega) \in \mathbb{N} \), for all connected sets \( A \) of \( C \) which contained 0 :

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(|A| \geq N_0 \Rightarrow \frac{|\partial_{C^g} A|}{|A|^{1 - \frac{1}{d}}} \geq \beta_0),
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where \( \partial_{C^g} A = \{(x, y) \in E^d; \omega(x, y) = 1 \text{ et } x \in A ; y \not\in A\} \).
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Proof: Similarly to isoperimetry on random environment,
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Proof: Similary to isoperimetry on random environment, but one more ingredient: renormalization.
Exit or occupation time on percolation cluster.

Proposition

Let $p > p_c(d)$ and $d \geq 1$. There exist constants $C = C(p, d)$ such that $Q$ a.s on the event $\{\#C = +\infty\}$: for any connected subset $B$ of $C$ which contains the origin and with volume large enough,
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- for $d = 2$, $\mathbb{E}_0(\tau_B) \leq C|B|$
Remark

We retrieve a consequence of Barlow or Mathieu and Remy result's.

Indeed, the control $P_{0}(X_k = y) \leq \nu(y) c_1 k^{-d/2} e^{-c_2 |y|^2} \frac{1}{k^n}$ enables us to get upper bound of exit (or occupation) time of the correct order.
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Indeed, the control $\mathbb{P}_0(X_k = y) \leq \nu(y) c_1 k^{-d/2} e^{-c_2 |y|^2_1 / k}$ enables us to get upper bound of exit (or occupation) time of the correct order.
Open questions.

**Question 1** : does a general anchored isoperimetric inequality imply an upper bound of $p_n(x, y)$?
Open questions.

- **Question 1**: does a general anchored isoperimetric inequality imply an upper bound of $\rho_n(x, y)$?
- **Question 2**: does anchored expansion is the good tool to prove an invariance principle (in random environments)?
A suivre...