Exit time for anchored expansion

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INSTITUT de MATHEMATIQUES de TOULOUSE

Outline of the talk

1. Introduction, why anchored expansion ?
   - Random walks
   - Links between geometry and random walks.
   - What is anchored expansion.
   - what we know
   - what we don't know

Random walks

- Let $G = (V(G), E(G))$ be a graph, we consider a random walk $(X_n)_{n\geq 0}$ on G with transitions probability $p(.,.)$.

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- Assumption : there exists a reversible measure m for X.
- We let $a(x,y) = m(x)p(x,y)$.

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Example : Simple Random walk.

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where $\nu(x)$ is the number of neighbours of x in G.

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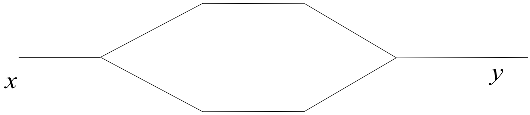
In this case,

$$m = \nu \text{ and } \; a = 1.$$

Links between geometry and RW



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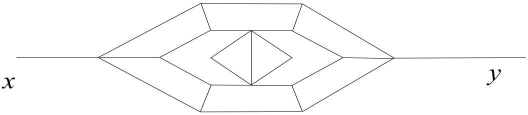

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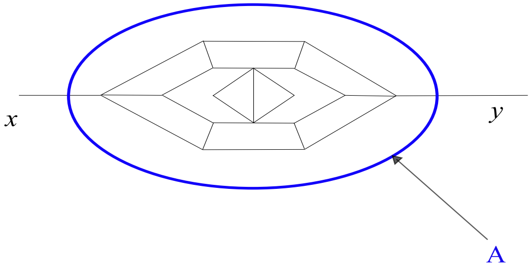

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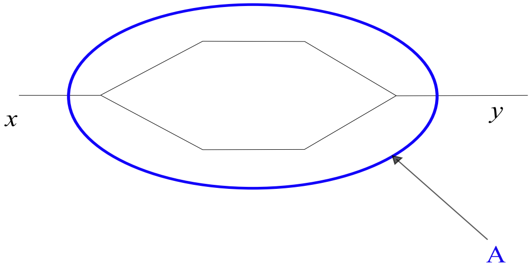

Behaviour related to the quantity $\frac{|\partial A|}{|A|}$.

One tools to control random walk : IS

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Proposition (Coulhon 99)

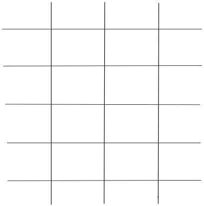
Let G a graph such that $IS_{\mathcal{F}}$ is satisfied. Assume that the function $\mathfrak{f} : t \to t/\mathcal{F}(t)$ is increasing and that $m_0 = \inf_{V(G)} m > 0$, then :

$$\sup_{x,y} \frac{p_{2n}(x,y)}{m(y)} \leq 2u(n),$$

where $u : \mathbb{R} \to]0; 1/m_0]$ *satisfies* $\begin{cases} u(0) = 1/m_0 \\ u' = -\frac{u}{2\mathfrak{g}(1/u)} \end{cases}$ *with*

$\mathfrak{g}(x) = 4(\mathfrak{f}(4x)/c)^2$

Some examples



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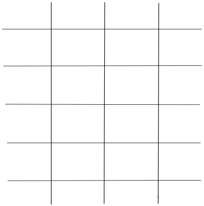
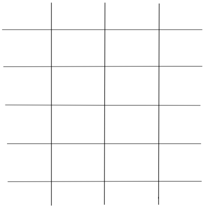


Isoperimetry :

Some examples



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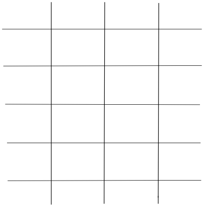

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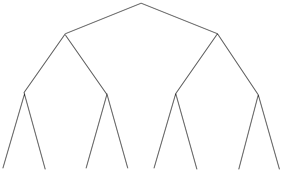
$$\mathbb{P}_x(X_n = y) \leq \frac{c}{n^{d/2}}.$$

Some examples



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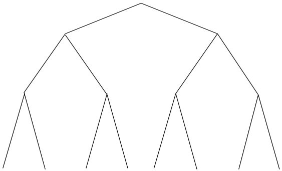

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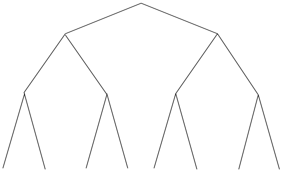

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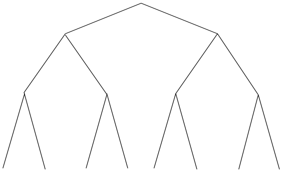

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$$\mathbb{P}_x(X_n = y) \leq e^{-cn}$$

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Anchored isoperimetry

definition.

Definition

Let $\mathcal{F}$ a positive increasing function defined on $\mathbb{R}_+$. Let G a graph with bounded valence and $o \in G$. We say that G satisfies an anchored (or rooted) $\mathcal{F}$-isoperimetric inequality in o if there exists a constant $C_{\mathrm{IS}} > 0$ such that for any connected set A which contains o we have :

$$\frac{|\partial A|}{\mathcal{F}(|A|)} \geq C_{\mathrm{IS}}. \quad (1)$$

∂A is equal to the set $\{(x, y) \in E(G);\ x \in A \text{ and } y \notin A\}$ and $|A|$ stands for the cardinal of A.

We will write G satisfies $AIS_{\mathcal{F}}$.

Picture

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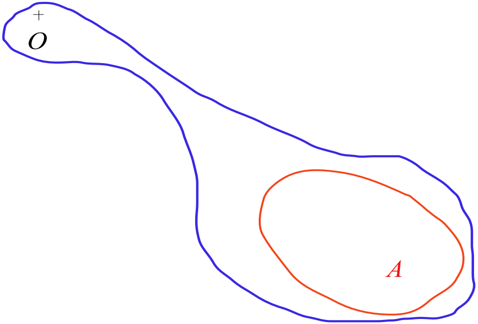

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This definition does not depend on the choice of the fixed vertex v whereas in the previous definition, the constant C_{IS} depends on the point o.

definition.

$\Rightarrow$ Now, the work is to examin what anchored isoperimetric inequality implies for random walk.

Results known.

Theorem

(Thomassen 92) Let G a graph satisfying a $AIS_{\mathcal{F}}$, then

$$\sum_k \frac{1}{\mathcal{F}(k)^2} < \infty \Rightarrow \text{ the simple random walk on G is transient.}$$

Results known.

Theorem (Virag 00)

Let G a graph (with bounded geometry) satisfying AIS_{id} (strong anchored isoperimetric inequality ($\mathcal{F} = id$)), then

1. *there exists a constant $c > 0$ such that* $\liminf_n \frac{|X_n|}{n} \geq c\, i^\star(G)^7$ *a.s*
2. *for all $x \in G$ there exists N such that for all $n \geq N$ and for all $y \in G$ one has :*

$$p_n(x, y) \leq e^{-\alpha n^{1/3}}$$

where $\alpha = c'\ i^\star(G)^2$

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Theorem (Chen and Peres 05)

Consider a $p-$Bernouilli percolation on a graph G with constant $i^\star(G) > 0$, if $p < 1$ is sufficiently close to 1 then, almost surely on the event that the open cluster H containing 0 is infinite, we have $i^\star(H) > 0$

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A refinement of the argument due to Gabor Pete shows that the conclusion holds for all $p > \frac{1}{1+i(^{\star}G)}$.

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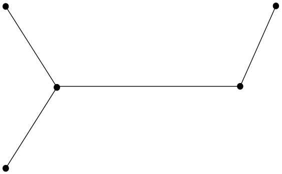
For a graph G, we replace each edge $e \in E(G)$ by a path that consists in L_e new edges, where the random variable $(L_e)_{e\in E(G)}$ are independant with law ν.



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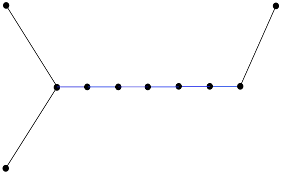

Let G^ν the graph obtained in this way, we call it a random strech of G.

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Theorem (Chen and Peres 05)

Suppose that G is an infinite graph of bounded degree and $i^\star(G) > 0$. If ν has an exponential tail then $i^\star(G^\nu) > 0$ a.s.

what we don’t know and we will be happy to know

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Transcience or recurrence		
$\lambda_1(A)$		
Transition kernel		
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and when X is transient let l_A the occupation time of A by :

$$l_A = \text{card}\{k \in \mathbb{N}; \; X_k \in A\}.$$

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What's happening for $AIS_{\mathcal{F}}$?

2 New results

- General statement
- Consequences

New results.

Theorem

Let G satisfying $AIS_{\mathcal{F}}$, then for any $A \subset G$ containing 0 we have :

$$\mathbb{E}_o(\tau_A) \leq 2 \int_0^{G_A(0)} v(s)\,\mathrm{d}s, \qquad (2)$$

where v is solution of the differential equation

$$\begin{cases} v(0) = m(A) \\ v' = -(C_{IS}\mathcal{F}(v))^2. \end{cases}$$

In fact this estimate holds for $\mathbb{E}_0(l_A)$ when X is transient.

New results

Examples :

- if $\mathcal{F}(x) = x^{1-\frac{1}{d}}$, $(d \geq 3)$ we have :

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- if $\mathcal{F}(x) = x$ we have :

$$\mathbb{E}_o(\tau_A) \leq \mathbb{E}_o(l_A) \leq c\ \ln(m(A)),$$

New results

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Proposition

If G satisfies $AIS_{\mathcal{F}}$, then u satisfies the following differential inequation : $\begin{cases} u(0) = m(A) \\ u' \leq -(C_{IS}\mathcal{F}(u))^2. \end{cases}$

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$$u(t) = m(x \in A;\ G_A(0,x) \geq t)$$

There exists t_0 such that for all $t \geq t_0,\ G_A(0,x) \leq t_0$.

We retrieve a weak version of Virag's result

Proposition

Let G a graph satisfying AIS_{id} and let $(X_n)_n$ simple random walk on G. Then we have :

$$\mathbb{P}(\lim_n \frac{d(o, X_n)}{n} = 0) = 0.$$

3 Some ideas of the proof.

- Some properties of Green function
- Connexion with exit time

Green function

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$$G^A(x,y) = \frac{1}{m(y)} \sum_{k \geq 0} \mathbb{P}_x^A(X_k = y).$$

$$G^A(x) = G^A(o,x).$$

The discrete Laplacian is

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where P^A is the operator defined on functions which are zero outside A by :

$$P^A f(x) \quad = \quad \mathbb{E}_x(f(X_1)\, 1_{\{X_1 \in A\}}) = \sum_{y \in A} p^A(x,y) f(y)$$

Some properties of Green function

Let A a connected set that contains 0.

Proposition

G^A is harmonic on $A \setminus \{0\}$, more precisely :

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Consequence :

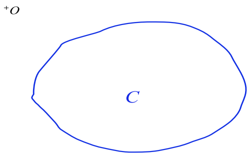
Corollary

The inward flow through any $B \subset A$ satisfies :

$$\sum_{e \in \partial B} a(e) \nabla_e G^A = 1_{\{o \in B\}}. \qquad (3)$$

Level sets of Green function

Let $A_s = \{x \in A\,;\, G^A(x) \geq s\}$ and let C a connected component of A_s. Assume $0 \notin C$



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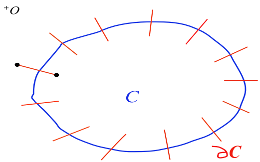

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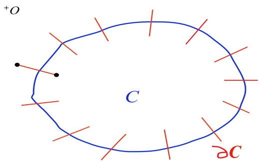

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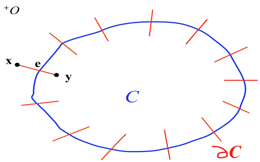

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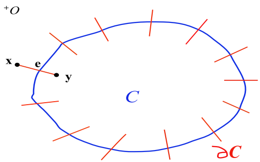

By previous corollary, $\sum_{e \in \partial C} a(e) \nabla_e G^A = 0$,
So there exist an edge $e = (x, y)$ such that $G(x) \geq G(y)$
that gives a contradiction since $C \subset A_s = \{x \in A \, ; G^A(x) \geq s\}$.

Level sets of Green function

Proposition

The level sets $A_s = \{x \in A\,; G^A(x) \geq s\}$ *are connected and contain o.*

Level sets of Green function

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The level sets $A_s = \{x \in A\,;\, G^A(x) \geq s\}$ are connected and contain o.

So if G satisfies $AIS_{\mathcal{F}}$, we can apply isoperimetric inequality to the sets A_s.

Differential equation for level sets of Green function

So we get the differential inequation : $\begin{cases} u(0) = m(A) \\ u' \leq -(C_{IS}\mathcal{F}(u))^2. \end{cases}$.

Connexion with exit time.

$$
\begin{aligned}
\mathbb{E}_o(\tau_A) &= \sum_{x\in A,\ k\geq 0} \mathbb{P}_o^A(X_k = x) \\
&= \sum_{x\in A} m(x)\ G^A(x) \\
&= \sum_{x\in A} m(x) \int_{\mathbb{R}_+} 1_{\{G^A(x)\geq t\}}\ dt \\
&= \int_{\mathbb{R}_+} m(\{x\in A;\ G^A(x)\geq t\})\ dt \\
&= \int_{\mathbb{R}_+} u(t)\ dt
\end{aligned}
$$

Minoration of the diffusion constant.

Assume that X is a random walk on a graph G which is now supposed to be a subgraph of $\mathbb{Z}^d$.
We suppose that X admits a reversible measure m satisfying :

$$\forall x \in G \quad m(x) \leq c.$$

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Proposition

Assume G satifies $d-$dimensionnal anchored isoperimetric inequality with constant C_{IS} and that $(\tilde{X}_k^N)_k$ converges in law to a brownian motion with matrix covariance $\sigma\, Id$, then there exists a constant $a(d) > 0$ such that

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In particular, $\sigma > 0$.

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Proof : That follows from our estimates for exit time.

Random environments.

- Consider the graph $\mathcal{L}^d = (\mathbb{Z}^d, E_d)$.

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- Assumption : $Q(\omega(e) > 0) = 1$

Random environments.

- X will design the random walk on the graph $\mathcal{L}_d$ starting from the origin with transitions probabilty given by :

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- We denote by $\mathbb{P}_0^\omega$ the law of X and by $\mathbb{E}_0^\omega$ its expectation.
- The random walk X admits reversible measures which are proportional to the measure m^ω defined by :

$$m^\omega(x) = \sum_{z\sim x}\omega(x,z).$$

In this case, we have : $a^\omega(x,y) = \omega(x,y)$.

Isoperimetry for random environments.

Proposition

Let Q be a law on environments such that $Q(\omega(e) > 0) = 1$. *There exists* $\beta_0(Q, d) > 0$ *such that Q a.s for all environment* ω, *there exists* $N_0(\omega) \in \mathbb{N}$ *such that for all connected sets A which contained* 0,

$$m^\omega(A) \geq N_0(\omega) \Rightarrow \frac{a^\omega(\partial A)}{m^\omega(A)^{1-\frac{1}{d}}} \geq \beta_0.$$

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No control for small sets.
Proof : Contour argument.

Exit or occupation time for random environments.

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Proposition

Let $d \geq 1$. There exists constants $C = C(Q, d)$ such that Q a.s for all environment ω :
for any connected subset B which contains the origin and with volume $m^\omega(B)$ large enough,

- *for $d \geq 3$,* $\mathbb{E}_0(l_B) \leq C\ m^\omega(B)^{2/d}$
- *for $d = 2$* $\mathbb{E}_0(\tau_B) \leq C\ m^\omega(B)$.

Go back to isoperimetry.

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Remark

In papers of Boukhadra or Berger-Biskup, it is proved that we can build environments where the return probability is greater than $1/n^2$. By our proposition 4.2, the d-dimensional anchored isoperimetric inequality is satisfied on these environments and so in dimension higher than 4, no one can hope to prove that in this case, the return probability is in $1/n^{d/2}$.

Percolation context.

- Consider the particular case $\omega : E_d \to \{0, 1\}$

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- Anchored isoperimetry on $\mathcal{C}$?

Anchored isoperimetry on percolation cluster.

Proposition

Let $p > p_c(d)$. There exists $\beta_0(p, d) > 0$ such that Q a.s on $\#\mathcal{C} = +\infty$, there exists $N_0(\omega) \in \mathbb{N}$, for all connected sets A of $\mathcal{C}$ which contained 0 :

$$\left(|A| \geq N_0 \Rightarrow \frac{|\partial_{\mathcal{C}g} A|}{|A|^{1-\frac{1}{d}}} \geq \beta_0, \right) \tag{4}$$

where $\partial_{\mathcal{C}g} A = \{(x, y) \in E^d;\ \omega(x, y) = 1 \text{ et } x \in A\ ;\ y \notin A\}$.

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Proof : Similary to isoperimetry on random environment,

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Proof : Similary to isoperimetry on random environment, but one more ingredient : renormalization.

Exit or occupation time on percolation cluster.

Proposition

Let $p > p_c(d)$ and $d \geq 1$. There exist constants $C = C(p, d)$ such that Q a.s on the event $\{\#\mathcal{C} = +\infty\}$:
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Remark

We retrieve a consequence of Barlow or Mathieu and Remy result's.

Indeed, the control $\mathbb{P}_0(X_k = y) \leq \nu(y) c_1 k^{-d/2} e^{\frac{-c_2 |y|_1^2}{k}}$ enables us to get upper bound of exit (or occupation) time of the correct order.

Open questions.

- Question 1 : does a general anchored isoperimetric inequality imply an upper bound of $p_n(x, y)$?

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- Question 2 : does anchored expansion is the good tool to prove an invariance principle (in random environments) ?

A suivre...