Existence of harmonic measure for random walks on graphs and random environments

Daniel Boivin¹ Rau Clément²

¹LMBA, Université de Bretagne occidentale

²IMT, Université Paul Sabatier de Toulouse

MEMEMO Meeting : "Random walks, random media, reinforcement" Agay le 11-15 mai 2012

프 🖌 🖌 프 🕨

Outline of the talk



Introduction about harmonic measure

2 Results

- Sketch of the proof of Theorem, transient case
- Overview of the proof in the recurrent case

5 Open questions

Introduction Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Example where Harmonic measure does not exist Example where Harmonic measure exists



- Weighted graphs
- Random walks
- hitting distribution
- Harmonic measure
- Examples where Harmonic measure does not exist
- Example where Harmonic measure exists

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions

Weighted graphs

Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

 Let (Γ, a) a weighted graph. Γ is a countably infinite set and a is a symmetric function :

$$a: \Gamma \times \Gamma \rightarrow [0; \infty[.$$

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions

Weighted graphs

Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

 Let (Γ, a) a weighted graph. Γ is a countably infinite set and a is a symmetric function :

$$a: \Gamma \times \Gamma \rightarrow [0; \infty[.$$

We let

$$\pi(x):=\sum_{y\in\Gamma}a(x,y)>0 \ \ ext{for all} \ \ x\in\Gamma.$$

• We will write $x \sim y$ if a(x, y) > 0.

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions

Weighted graphs

Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

 Let (Γ, a) a weighted graph. Γ is a countably infinite set and a is a symmetric function :

$$a: \Gamma \times \Gamma \rightarrow [0; \infty[.$$

We let

$$\pi(x):=\sum_{y\in\Gamma}a(x,y)>0 \ \ ext{for all} \ \ x\in\Gamma.$$

- We will write $x \sim y$ if a(x, y) > 0.
- We will always assume that (Γ, ~) is an infinite connected graph, locally finite countable graph without multiple edges.

ヘロト 人間 ト 人 ヨ ト 人 ヨ ト

Introduction Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Example where Harmonic measure does not exist Example where Harmonic measure exists

Random walks

 The random walk (X_n)_n on the weighted graph (Γ, a) is the Markov chain on Γ with transition probabilities given by

$$p(x,y) := rac{a(x,y)}{\pi(x)}, \quad x,y \in \Gamma.$$

Introduction Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Open que terre Open que terre

Random walks

 The random walk (X_n)_n on the weighted graph (Γ, a) is the Markov chain on Γ with transition probabilities given by

$$p(x,y) := rac{a(x,y)}{\pi(x)}, \quad x,y \in \Gamma.$$

We denote by P_x the law of the random walk starting at the vertex x ∈ Γ. The corresponding expectation is denoted by E_x.

Introduction Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

Random walks

 The random walk (X_n)_n on the weighted graph (Γ, a) is the Markov chain on Γ with transition probabilities given by

$$p(x,y) := rac{a(x,y)}{\pi(x)}, \quad x,y \in \Gamma.$$

- We denote by P_x the law of the random walk starting at the vertex x ∈ Γ. The corresponding expectation is denoted by E_x.
- The random walk admits reversible measures which are proportional to the measure $\pi(\cdot)$.

イロト 不得 とくほ とくほとう

Introduction Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Open questions Overview of the proof in the recurrent case Open questions Open questions

Notations

Let

$$\overline{A}:=\partial A\cup A,$$

with

 $\partial A := \{y \in \Gamma; y \notin A \text{ and there is } x \in A \text{ with } x \sim y\}.$

(ロ) (同) (目) (日) (日) (の)

Introduction Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

Notations

Let

$$\overline{\boldsymbol{A}}:=\partial\boldsymbol{A}\cup\boldsymbol{A},$$

with

$$\partial A := \{ y \in \Gamma; y \notin A \text{ and there is } x \in A \text{ with } x \sim y \}.$$

• For $u: \overline{A} \to \mathbb{R}$ the Laplacian is defined by

$$\mathcal{L}u(x) := \sum_{y \sim x} p(x, y) [u(y) - u(x)], \quad x \in A.$$

• A function $u : \overline{A} \to \mathbb{R}$ is *harmonic* in A if for all $x \in A$,

 $(\mathcal{L}u)(x)=0.$

・ロト ・ 同ト ・ ヨト ・ ヨト

Introduction Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure exists

Notations

• The Green function of the random walk is defined by

$$G(x,y):=\sum_{j=0}^{\infty}p(x,y,j), \hspace{1em} x,y\in \Gamma$$

where $p(x, y, j) := P_x(X_j = y)$ are the transition probabilities of the walk.

イロト イポト イヨト イヨト

э.

 Introduction Results
 Weighted graphs

 Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions
 Mail Shifting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

Notations

• The Green function of the random walk is defined by

$$G(x,y):=\sum_{j=0}^{\infty}p(x,y,j), \quad x,y\in \Gamma$$

where $p(x, y, j) := P_x(X_j = y)$ are the transition probabilities of the walk.

 The Green function of the random walk in B ⊂ Γ is defined by

$$G_B(x,y) := \sum_{j=0}^{\infty} p_B(x,y,j), \quad x,y \in \overline{B}$$

where $p_B(x, y, j) := P_x(X_j = y; \forall i \leq j \ X_i \in B)$.

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions

hitting distribution

Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

• Let $A \subset \Gamma$, we let $\tau_A := \inf\{k \ge 1; X_k \in A\}$

Daniel Boivin , Rau Clément Existence of the harmonic measure for random walks on graphs a

イロン 不同 とくほ とくほ とう

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

hitting distribution

• Let $A \subset \Gamma$, we let $\tau_A := \inf\{k \ge 1; X_k \in A\}$



 $_{+}^{\mathbf{X}}$

イロン 不同 とくほ とくほ とう

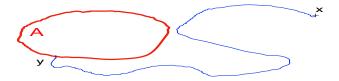
э

Daniel Boivin , Rau Clément Existence of the harmonic measure for random walks on graphs a

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

hitting distribution

• Let $A \subset \Gamma$, we let $\tau_A := \inf\{k \ge 1; X_k \in A\}$



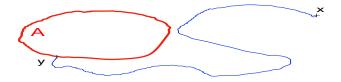
イロン 不同 とくほ とくほ とう

3

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

hitting distribution

• Let $A \subset \Gamma$, we let $\tau_A := \inf\{k \ge 1; X_k \in A\}$



The hitting distribution of a set A starting from x ∈ Γ is given by :

$$H_A(x,y) := P_x(X_{\tau_A} = y).$$

ъ

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions weignied graphs Random walks hitting distribution Harmonic measure Example where Harmonic measure does not exist Example where Harmonic measure exists

hitting distribution

$$\begin{array}{rcccc} H_{\mathcal{A}}(x,y) & : & \mathbb{Z}^d \times \mathcal{A} & \to & [0;1] \\ & & (x,y) & \mapsto & H_{\mathcal{A}}(x,y) & := \mathcal{P}_x(X_{\tau_{\mathcal{A}}}=y) \end{array}$$

Daniel Boivin , Rau Clément Existence of the harmonic measure for random walks on graphs a

イロン 不同 とくほ とくほ とう

3

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

hitting distribution

$$\begin{array}{rcl} H_{\mathcal{A}}(x,y) & : & \mathbb{Z}^d \times \mathcal{A} & \to & [0;1] \\ & & (x,y) & \mapsto & H_{\mathcal{A}}(x,y) & := P_x(X_{\tau_{\mathcal{A}}} = y) \end{array}$$

• For fixed $y \in A$, $H_A(., y)$ is a harmonic function on \overline{A}^c .

イロト 不得 とくほと くほとう

3

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

hitting distribution

$$\begin{array}{rcl} H_A(x,y) & : & \mathbb{Z}^d \times A & \to & [0;1] \\ & & (x,y) & \mapsto & H_A(x,y) & := P_x(X_{\tau_A} = y) \end{array}$$

- For fixed $y \in A$, $H_A(., y)$ is a harmonic function on \overline{A}^c .
- For fixed x ∈ Γ, H_A(x,.) is a positive measure on A with total mass P_x(τ_A < +∞). (supp(H_A(x,.) ⊂ ∂A)

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

hitting distribution

$$\begin{array}{rcl} H_{A}(x,y) & : & \mathbb{Z}^{d} \times A & \to & [0;1] \\ & & (x,y) & \mapsto & H_{A}(x,y) & := P_{x}(X_{\tau_{A}} = y) \end{array}$$

• For fixed $y \in A$, $H_A(., y)$ is a harmonic function on \overline{A}^c .

- For fixed $x \in \Gamma$, $H_A(x, .)$ is a positive measure on A with total mass $P_x(\tau_A < +\infty)$. ($supp(H_A(x, .) \subset \partial A$)
- If P_x(τ_A < +∞) > 0, we may define a probability measure on A by conditioning that the random walk hits A,

$$\overline{H}_{\mathcal{A}}(x,y) := \mathcal{P}_{x}(X_{\tau_{\mathcal{A}}} = y | \tau_{\mathcal{A}} < +\infty).$$

ヘロン 人間 とくほ とくほ とう

э.

Weighted graphs Random walks hitting distribution **Harmonic measure** Examples where Harmonic measure does not exist Example where Harmonic measure exists

Harmonic measure

 The harmonic measure on a finite subset A of Γ is the hitting distribution from infinity, if it exists,

 $\mathbf{H}_{A}(y) := \lim_{D(x,A) \to \infty} \overline{H}_{A}(x,y), \quad y \in A.$

where *D* denote the graph distance between two vertices $x, y \in \Gamma$. It is the minimal length of a path from *x* to *y* in the graph (Γ, \sim) .

イロト (四) (正) (正) (正) のへで

Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

Harmonic measure

 The harmonic measure on a finite subset A of Γ is the hitting distribution from infinity, if it exists,

$$\mathbf{H}_{\mathcal{A}}(y) := \lim_{D(x,\mathcal{A})\to\infty} \overline{H}_{\mathcal{A}}(x,y), \quad y\in \mathcal{A}.$$

where *D* denote the graph distance between two vertices $x, y \in \Gamma$. It is the minimal length of a path from *x* to *y* in the graph (Γ, \sim) .

• Independance of the direction and the way how *x* goes through ∞ .

イロン 不得 とくほ とくほ とう

1

Weighted graphs Random walks hitting distribution **Harmonic measure** Examples where Harmonic measure does not exist Example where Harmonic measure exists

Harmonic measure

 The harmonic measure on a finite subset A of Γ is the hitting distribution from infinity, if it exists,

$$\mathbf{H}_{\mathcal{A}}(y) := \lim_{D(x,\mathcal{A})\to\infty} \overline{H}_{\mathcal{A}}(x,y), \quad y\in \mathcal{A}.$$

where *D* denote the graph distance between two vertices $x, y \in \Gamma$. It is the minimal length of a path from *x* to *y* in the graph (Γ, \sim) .

• Independance of the direction and the way how *x* goes through ∞ .

 \Rightarrow Our goal is to prove the existence of the harmonic measure for all finite subsets of various weighted graphs.

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weignieu graphs Random walks hitting distribution **Harmonic measure** Examples where Harmonic measure does not exist Example where Harmonic measure exists

Motivations to study Harmonic measure

The *harmonic measure* on a finite subset A of Γ is the hitting distribution from infinity, if it exists,

$$\mathbf{H}_{\mathcal{A}}(y) := \lim_{D(x,\mathcal{A}) \to \infty} \overline{H}_{\mathcal{A}}(x,y), \quad y \in \mathcal{A}.$$

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution **Harmonic measure** Examples where Harmonic measure does not exist Example where Harmonic measure exists

Motivations to study Harmonic measure

The *harmonic measure* on a finite subset A of Γ is the hitting distribution from infinity, if it exists,

$$\mathbf{H}_{\mathcal{A}}(y) := \lim_{D(x,\mathcal{A})\to\infty} \overline{H}_{\mathcal{A}}(x,y), \quad y\in \mathcal{A}.$$

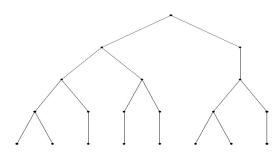
 \Rightarrow Physical interpretation : distribution/spread of charge on an object .

・ロト ・ ア・ ・ ヨト ・ ヨト

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

Examples where Harmonic measure does not exist

Infinite tree

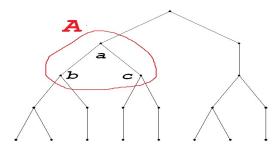


イロト イポト イヨト イヨト

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

Examples where Harmonic measure does not exist

Infinite tree

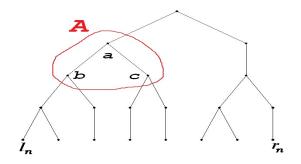


イロト イポト イヨト イヨト

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

Examples where Harmonic measure does not exist

Infinite tree

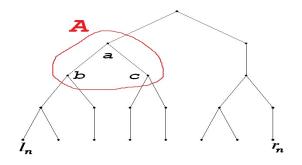


イロト イポト イヨト イヨト

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

Examples where Harmonic measure does not exist

Infinite tree



Consider for example, $\overline{H}_A(I_n, a)$ and $\overline{H}_A(r_n, a)$.

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure **Examples where Harmonic measure does not exist** Example where Harmonic measure exists

Examples where Harmonic measure does not exist

• Other example : Cayley graph of the Lamplighter group

Daniel Boivin , Rau Clément Existence of the harmonic measure for random walks on graphs a

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure **Examples where Harmonic measure does not exist** Example where Harmonic measure exists

Examples where Harmonic measure does not exist

• Other example : Cayley graph of the Lamplighter group

$$\mathbb{Z}^2 \wr \frac{\mathbb{Z}}{2\mathbb{Z}}$$

Daniel Boivin , Rau Clément Existence of the harmonic measure for random walks on graphs a

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure **Examples where Harmonic measure does not exist** Example where Harmonic measure exists

Examples where Harmonic measure does not exist

• Other example : Cayley graph of the Lamplighter group

$$\mathbb{Z}^2 \wr \frac{\mathbb{Z}}{2\mathbb{Z}}$$

Daniel Boivin , Rau Clément Existence of the harmonic measure for random walks on graphs a

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

Examples where Harmonic measure exists

The harmonic measure exists in \mathbb{Z}^d for the simple random walk.

ヘロト 人間 ト ヘヨト ヘヨト

Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

Examples where Harmonic measure exists

The harmonic measure exists in \mathbb{Z}^d for the simple random walk.

Theorem (G. Lawler)

Let A a finite subset of \mathbb{Z}^d , for all y in A, we have :

$$\lim_{D(0,x)\to\infty}\bar{H}_{A}(x,y)=\mathsf{H}_{\mathsf{A}}(y) \quad \text{exists}.$$

Moreover,

$$\mathbf{H}_{\mathbf{A}}(\mathbf{y}) = \lim_{\mathbf{n} \to \infty} \frac{\mathbf{P}_{\mathbf{y}}(\tau_{\mathbf{A}} > \tau_{\partial \mathbf{B}(\mathbf{0},\mathbf{n})})}{\sum_{\mathbf{y}' \in \mathbf{A}} \mathbf{P}_{\mathbf{y}'}(\tau_{\mathbf{A}} > \tau_{\partial \mathbf{B}(\mathbf{0},\mathbf{n})})}$$

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

Examples where Harmonic measure exists

Remark

Lawler and Limic have proved that harmonic measure exists in \mathbb{Z}^d for a wider class of random walks,

Daniel Boivin , Rau Clément Existence of the harmonic measure for random walks on graphs a

ヘロト 人間 ト ヘヨト ヘヨト

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

Examples where Harmonic measure exists

Remark

Lawler and Limic have proved that harmonic measure exists in \mathbb{Z}^d for a wider class of random walks, like :

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

Examples where Harmonic measure exists

Remark

Lawler and Limic have proved that harmonic measure exists in \mathbb{Z}^d for a wider class of random walks, like :

$$\mathbb{P}(X_{n+1} - X_n = \mathbf{e_i}) = \mathbb{P}(X_{n+1} - X_n = -\mathbf{e_i})$$

and
$$\mathbb{P}(X_{n+1} = X_n) = 1 - 2\sum_{i=1...d} \mathbb{P}(X_{n+1} - X_n = \mathbf{e_i})$$

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

Tools of the proof of Lawler

2 majors ingredients :

• (Elliptic) Harnack inequality

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

Tools of the proof of Lawler

2 majors ingredients :

• (Elliptic) Harnack inequality

Definition

We say that a weighted graph (Γ , a) satisfies **H(K,M)**, the Harnack inequality with shrinking parameter M > 1, if there is a constant $K < \infty$ such that for all $x \in \Gamma$ and R > 0, and for any non-negative harmonic function u on B(x, MR),

 $\max_{B(x,R)} u \leq K \min_{B(x,R)} u.$

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

Tools of the proof of Lawler

2 majors ingredients :

- (Elliptic) Harnack inequality
- for $d \ge 3$, precise estimates of G,

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

Tools of the proof of Lawler

2 majors ingredients :

- (Elliptic) Harnack inequality
- for *d* ≥ 3, precise estimates of *G*, for *d* = 2, precise estimates of *g*, where

$$g(x) := \lim_{n} [G_{B(0,n)}(0) - G_{B(0,n)}(x)],$$

and so of $G_{B(0,n)}$.

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists



Results

- Transient case
- Recurrent case

<ロト <回 > < 注 > < 注 > 、

Results Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Weighted graphs Random walks hitting distribution Harmonic measure Examples where Harmonic measure does not exist Example where Harmonic measure exists

Daniel Boivin , Rau Clément Existence of the harmonic measure for random walks on graphs a

◆□> ◆□> ◆豆> ◆豆> ・豆 ・ のへで

Transient case Recurrent case

Transient case

The main result for transient graphs is the existence of the harmonic measure for random walks with a Green function which verify the following estimate.

Introduction

Definition

We say that a weighted graph (Γ , *a*) satisfies the Green function estimate **GE**_{γ} for $\gamma > 0$ if there are constants $0 < C_i \le C_s < \infty$ and if for all $z \in \Gamma$, there exists $R_z < \infty$ such that for all $x, y \in \Gamma$ with $D(x, y) \ge R_x \land R_y$ we have :

$$rac{C_i}{D(x,y)^\gamma} \leq G(x,y) \leq rac{C_s}{D(x,y)^\gamma}$$

 (\mathbf{GE}_{γ})

э

Transient case Recurrent case

Our main result is the following :

Theorem

Let (Γ, a) be a weighted graph which verifies \mathbf{GE}_{γ} for some $\gamma > 0$. Then for any finite subset $A \subset \Gamma$ the harmonic measure on A exists.(That is, for all $y \in A$, the limit exists.) Moreover, we have :

Introduction

$$\lim_{D(x,A)\to\infty}\overline{H}_A(x,y)=\lim_{m\to+\infty}H^m_A(y),$$

where, for m large enough,

$$H^m_A(y) = rac{\pi(y) P_y(au_A > au_{\partial B(x_0,m)})}{\operatorname{Cap}_m(A)}$$

The limit does not depend on the choice of x_0 .

Daniel Boivin, Rau Clément

Existence of the harmonic measure for random walks on graphs a

Transient case Recurrent case

The capacity of A with respect to B, for A ⊂ B ⊂ Γ, is defined by

$$\operatorname{Cap}_{B}(A) := \sum_{x \in A} \pi(x) P_{x}(\overline{\tau}_{B^{c}} < \tau_{A}).$$

イロト 不得 とくほ とくほとう

Transient case Recurrent case

The capacity of A with respect to B, for A ⊂ B ⊂ Γ, is defined by

$$\operatorname{Cap}_{B}(A) := \sum_{x \in A} \pi(x) P_{x}(\overline{\tau}_{B^{c}} < \tau_{A}).$$

Recall that

$$\tau_A := \inf\{k \ge 1; \ X_k \in A\},\\ \overline{\tau}_A := \inf\{k \ge 0; \ X_k \in A\}$$

② Cap_{*m*}(*A*) is the capacity of *A* with respect to *B*(x_0, m) for some $x_0 \in \Gamma$.

Transient case Recurrent case

One main step in the proof of Theorem I, is to work with some Harnack inequality. We will use the following weak Harnack inequality :

Introduction

ヘロト 人間 ト ヘヨト ヘヨト

э

Transient case Recurrent case

One main step in the proof of Theorem I, is to work with some Harnack inequality. We will use the following weak Harnack inequality :

Introduction

Definition

We say that a weighted graph (Γ , a) satisfies **wH**(K), the weak Harnack inequality, if there is a constant $1 \le K < \infty$ such that for all $x \in \Gamma$ and for all R > 0 there is $M_{x,R} \ge 2$ such that for all $M > M_{x,R}$ and for any non-negative harmonic function u on B(x, MR),

$$\max_{B(x,R)} u \leq K \min_{B(x,R)} u.$$

<ロ> <同> <同> <三> <三> <三> <三> <三</p>

Transient case Recurrent case

One main step in the proof of Theorem I, is to work with some Harnack inequality. We will use the following weak Harnack inequality :

Introduction

Definition

We say that a weighted graph (Γ , a) satisfies **wH**(K), the weak Harnack inequality, if there is a constant $1 \le K < \infty$ such that for all $x \in \Gamma$ and for all R > 0 there is $M_{x,R} \ge 2$ such that for all $M > M_{x,R}$ and for any non-negative harmonic function u on B(x, MR),

$$\max_{B(x,R)} u \leq K \min_{B(x,R)} u.$$

We will prove that the Green function estimates \mathbf{GE}_{γ} imply the weak Harnack inequality.

◆□▶ ◆□▶ ★ □▶ ★ □▶ → □ → の Q ()

Introduction

Transient case Recurrent case

Proposition

Let (Γ, a) be a weighted graph which verifies (\mathbf{GE}_{γ}) for some $\gamma > 0$. Then the graph is connected, transient and **wH**(*K*) holds with $K = 2^{\gamma} \frac{C_s}{C_i}$.

< □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □

Introduction Results

Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Transient case Recurrent case

Some applications

Corollary

Let (\mathbb{Z}^d, a) , $d \ge 3$, be a uniformly elliptic graph. Then for all finite subsets A of \mathbb{Z}^d and for all $y \in A$, the limit exists.

Moreover, we have :

$$\lim_{|x|\to+\infty}\overline{H}_A(x,y)=\lim_{m\to+\infty}H_A^m(y),$$

where
$$H^m_A(y) = \frac{\pi(y)P_y(\tau_A > \tau_{\partial B(0,m)})}{\operatorname{Cap}_m(A)}$$
.

ヘロト ヘアト ヘビト ヘビト

э.

Introduction Results

Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Transient case Recurrent case

Some applications

Corollary

Let (\mathbb{Z}^d, a) , $d \ge 3$, be a uniformly elliptic graph. Then for all finite subsets A of \mathbb{Z}^d and for all $y \in A$, the limit exists.

Moreover, we have :

$$\lim_{|x|\to+\infty}\overline{H}_{\mathcal{A}}(x,y)=\lim_{m\to+\infty}H_{\mathcal{A}}^m(y).$$

where $H_A^m(y) = \frac{\pi(y)P_y(\tau_A > \tau_{\partial B(0,m)})}{\operatorname{Cap}_m(A)}$.

It follows from Delmotte's estimates.

ヘロト ヘアト ヘビト ヘビト

Introduction Results heorem, transient case

Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Transient case Recurrent case

Existence of the harmonic measure for \mathbb{Z}^d , $d \ge 3$, with i.i.d. conductances

Corollary

Let (\mathbb{Z}^d, a) , $d \ge 3$, be a weighted graph where the weights $(a(e); e \in^d)$ are *i.i.d.* non-negative random variables on a probability space (Ω, \mathbb{P}) which verify

 $\mathbb{P}(a(e) > 0) > p_c(\mathbb{Z}^d).$

For any finite subset A of C_{∞} and for all $y \in A$, harmonic measure exists. Moreover, we have :

$$\lim_{|x|\to+\infty,x\in\mathcal{C}_{\infty}}\overline{H}_{A}(x,y)=\lim_{m\to+\infty}H_{A}^{m}(y).$$

Transient case Recurrent case

Where
$$H_A^m(y) = rac{\pi(y)P_y^{\omega}(au_A > au_{\partial B_{\omega}(x_0,m)})}{\operatorname{Cap}_m(A)}$$
 for some $x_0 \in \mathcal{C}_{\infty}$ and for *m* large enough.

Daniel Boivin , Rau Clément Existence of the harmonic measure for random walks on graphs a

◆□▶ ◆□▶ ◆三▶ ◆三▶ ・三 のへで

Transient case Recurrent case

Where
$$H_A^m(y) = \frac{\pi(y)P_y^{\omega}(\tau_A > \tau_{\partial B_{\omega}(x_0,m)})}{\operatorname{Cap}_m(A)}$$
 for some $x_0 \in \mathcal{C}_{\infty}$ and for *m* large enough.
It follows from the Green function estimates of Andres, Barlow, Deuschel, Hambly.

◆□▶ ◆□▶ ◆三▶ ◆三▶ ・三 のへで

Introduction Results

Transient case Recurrent case

Recurrent Case

Percolation cluster.

Theorem

Let (\mathbb{Z}^2, a) be a weighted graph where the weights $(a(e); e \in {}^2)$ are *i.i.d.* random variables on a probability space (Ω, \mathbb{P}_p) which verify

$$p=\mathbb{P}_{
ho}(a(e)=1)=1-\mathbb{P}_{
ho}(a(e)=0)>
ho_{c}(\mathbb{Z}^{2}).$$

Then \mathbb{P}_p almost surely, for any finite subset A of $\mathcal{C}_{\infty}()$ and for all $y \in A$, the harmonic measure exists.

ヘロト ヘアト ヘビト ヘビト

Introduction Results

Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions

Recurrent Case

Transient case Recurrent case

 \mathbb{Z}^2 elliptic.

Theorem

If (\mathbb{Z}^2, a) is a uniformly elliptic weighted graph then for all finite subsets $A \subset \mathbb{Z}^2$ and for all $y \in A$, the limit exists.

ヘロン 人間 とくほ とくほ とう

Transient case Recurrent case



Sketch of the proof of Theorem, transient case

Introduction

- 1st Step : bound of \bar{H}_A
- 2nd Step : Replace set A by a box
- 3rd Step : Replace exit time of an annulus by exit time of a box
- 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$
- 5th Step : Gathering the estimate
- Modification of step 4, with a weak assumption

ヘロト ヘアト ヘビト ヘビト

Introduction Results

Sketch of the proof of Theorem, transient case Overview of the proof in the recurrent case Open questions Transient case Recurrent case

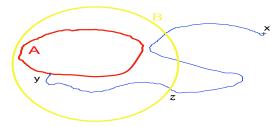
Daniel Boivin , Rau Clément Existence of the harmonic measure for random walks on graphs a

◆□▶ ◆□▶ ◆三▶ ◆三▶ ・三 のへで

1st Step : bound of \overline{H}_A 2nd Step : Replace set *A* by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

1st Step : bound of \bar{H}_A

• Let $A \subset B$ be finite subsets of Γ . Recall that $H_A(x, y) = P_x(X_{\tau_A} = y)$.

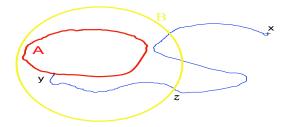


イロト イポト イヨト イヨト

1st Step : bound of \tilde{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

1st Step : bound of \bar{H}_A

• Let $A \subset B$ be finite subsets of Γ . Recall that $H_A(x, y) = P_x(X_{\tau_A} = y)$.



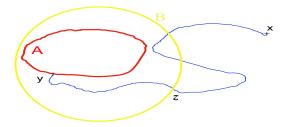
For all $x \in B^c$ and $y \in A$, we have :

$$H_{A}(x,y) = \sum_{z \in \partial B} G_{A^{c}}(x,z) H_{A \cup \partial B}(z,y).$$

1st Step : bound of \overline{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

1st Step : bound of \bar{H}_A

• Let $A \subset B$ be finite subsets of Γ . Recall that $H_A(x, y) = P_x(X_{\tau_A} = y)$.



• For all $x \in B^c$ and $y \in A$, we have :

$$H_{A}(x,y) = \sum_{z \in \partial B} G_{A^{c}}(x,z) H_{A \cup \partial B}(z,y).$$

1st Step : bound of \overline{H}_A 2nd Step : Replace set *A* by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Betimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

1st Step : bound of \bar{H}_A

For all $x \in B^c$ and $y \in A$, we have :

$$H_A(x,y) = \sum_{z \in \partial B} G_{A^c}(x,z) H_{A \cup \partial B}(z,y).$$

1st Step : bound of \overline{H}_A 2nd Step : Replace set *A* by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

1st Step : bound of \bar{H}_A

For all $x \in B^c$ and $y \in A$, we have :

$$\mathcal{H}_{\mathcal{A}}(x,y) = \sum_{z \in \partial B} G_{\mathcal{A}^c}(x,z) \mathcal{H}_{\mathcal{A} \cup \partial B}(z,y).$$

• Then, by summing over *y* we get :

$$\overline{H}_{A}(x,y) = \frac{\sum_{z \in \partial B} G_{A^{c}}(x,z) H_{A \cup \partial B}(z,y)}{\sum_{z \in \partial B} G_{A^{c}}(x,z) P_{z}(\tau_{A} < \tau_{\partial B})}$$

1st Step : bound of \overline{H}_A 2nd Step : Replace set *A* by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

1st Step : bound of \bar{H}_A

For all $x \in B^c$ and $y \in A$, we have :

$$\mathcal{H}_{\mathcal{A}}(x,y) = \sum_{z \in \partial B} \mathcal{G}_{\mathcal{A}^c}(x,z) \mathcal{H}_{\mathcal{A} \cup \partial B}(z,y).$$

• Then, by summing over *y* we get :

$$\overline{H}_{A}(x,y) = \frac{\sum_{z \in \partial B} G_{A^{c}}(x,z) H_{A \cup \partial B}(z,y)}{\sum_{z \in \partial B} G_{A^{c}}(x,z) P_{z}(\tau_{A} < \tau_{\partial B})}$$

So,

$$\min_{z\in\partial B}\frac{H_{A\cup\partial B}(z,y)}{P_z(\tau_A<\tau_{\partial B})}\leq \overline{H}_A(x,y)\leq \max_{z\in\partial B}\frac{H_{A\cup\partial B}(z,y)}{P_z(\tau_A<\tau_{\partial B})}.$$

1st Step : bound of \overline{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

1st Step : bound of \bar{H}_A

By reversibility and since $P_z(\tau_A < \tau_{\partial B}) = \sum_{\tilde{y} \in A} H_{A \cup \partial B}(z, \tilde{y})$, For all $x \in B^c$ and $y \in A$, we obtain :

1st Step : bound of \overline{H}_A 2nd Step : Replace set *A* by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

1st Step : bound of \bar{H}_A

By reversibility and since $P_z(\tau_A < \tau_{\partial B}) = \sum_{\tilde{y} \in A} H_{A \cup \partial B}(z, \tilde{y})$, For all $x \in B^c$ and $y \in A$, we obtain :

$$\min_{z \in \partial B} \frac{\pi(y) H_{A \cup \partial B}(y, z)}{\sum_{\tilde{y} \in A} \pi(\tilde{y}) H_{A \cup \partial B}(\tilde{y}, z)} \leq \overline{H}_{A}(x, y) \leq \max_{z \in \partial B} \frac{\pi(y) H_{A \cup \partial B}(y, z)}{\sum_{\tilde{y} \in A} \pi(\tilde{y}) H_{A \cup \partial B}(\tilde{y}, z)}$$

1st Step : bound of \tilde{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

1st Step : bound of \bar{H}_A

By reversibility and since $P_z(\tau_A < \tau_{\partial B}) = \sum_{\tilde{y} \in A} H_{A \cup \partial B}(z, \tilde{y})$, For all $x \in B^c$ and $y \in A$, we obtain :

$$\min_{z \in \partial B} \frac{\pi(y) H_{A \cup \partial B}(y, z)}{\sum_{\tilde{y} \in A} \pi(\tilde{y}) H_{A \cup \partial B}(\tilde{y}, z)} \leq \overline{H}_{A}(x, y) \leq \max_{z \in \partial B} \frac{\pi(y) H_{A \cup \partial B}(y, z)}{\sum_{\tilde{y} \in A} \pi(\tilde{y}) H_{A \cup \partial B}(\tilde{y}, z)}$$

$$\Rightarrow \text{ Study of}$$

 $\frac{\pi(\mathbf{y})H_{A\cup\partial B}(\mathbf{y},z)}{\sum_{\tilde{\mathbf{y}}\in A}\pi(\tilde{\mathbf{y}})H_{A\cup\partial B}(\tilde{\mathbf{y}},z)}.$

くロト (過) (目) (日)

1st Step : bound of \tilde{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

1st Step : bound of \bar{H}_A

By reversibility and since $P_z(\tau_A < \tau_{\partial B}) = \sum_{\tilde{y} \in A} H_{A \cup \partial B}(z, \tilde{y})$, For all $x \in B^c$ and $y \in A$, we obtain :

$$\min_{z \in \partial B} \frac{\pi(y) H_{A \cup \partial B}(y, z)}{\sum_{\tilde{y} \in A} \pi(\tilde{y}) H_{A \cup \partial B}(\tilde{y}, z)} \leq \overline{H}_{A}(x, y) \leq \max_{z \in \partial B} \frac{\pi(y) H_{A \cup \partial B}(y, z)}{\sum_{\tilde{y} \in A} \pi(\tilde{y}) H_{A \cup \partial B}(\tilde{y}, z)}$$

$$\Rightarrow \text{Study of}$$

$$\frac{\pi(y)H_{A\cup\partial B}(y,z)}{\sum_{\tilde{y}\in A}\pi(\tilde{y})H_{A\cup\partial B}(\tilde{y},z)}.$$

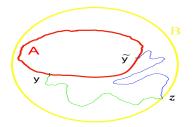
 \Rightarrow Study of $H_{A\cup\partial B}(y,z)$ for $y \in A, z \in \partial B$ and B "big"...

ヘロト ヘアト ヘビト ヘビト

1st Step : bound of \overline{H}_A 2nd Step : Replace set *A* by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

1st Step : bound of \bar{H}_A

Compare $H_{A\cup\partial B}(y,z)$ and $H_{A\cup\partial B}(\tilde{y},z)$ for $y, \tilde{y} \in A$ and $z \in \partial B$



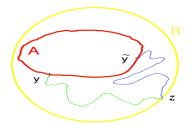
イロト イポト イヨト イヨト

ъ

1st Step : bound of \overline{H}_A 2nd Step : Replace set *A* by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

1st Step : bound of \bar{H}_A

Compare $H_{A\cup\partial B}(y,z)$ and $H_{A\cup\partial B}(\tilde{y},z)$ for $y, \tilde{y} \in A$ and $z \in \partial B$



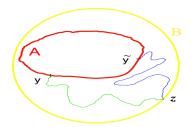
イロト イポト イヨト イヨト

ъ

1st Step : bound of \overline{H}_A 2nd Step : Replace set *A* by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

1st Step : bound of \bar{H}_A

Compare $H_{A\cup\partial B}(y,z)$ and $H_{A\cup\partial B}(\tilde{y},z)$ for $y, \tilde{y} \in A$ and $z \in \partial B$



 \Rightarrow Harnack inequality...

1st Step : bound of H_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

2nd Step : Replace set A by a box

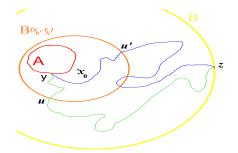
Let $x_0 \in \Gamma$ and $r_A > 0$ such that $A \subset B(x_0, r_A)$.

Daniel Boivin , Rau Clément Existence of the harmonic measure for random walks on graphs a

1st Step : bound of H_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

2nd Step : Replace set A by a box

Let $x_0 \in \Gamma$ and $r_A > 0$ such that $A \subset B(x_0, r_A)$.



$$P_{y}(X_{\tau_{\partial B} \wedge \tau_{A}} = z) = \sum_{u \in \partial B(x_{0}, r_{A})} P_{y}(X_{\tau_{\partial B}(x_{0}, r_{A})} \wedge \tau_{A}} = u) P_{u}(X_{\tau_{\partial B} \wedge \tau_{A}} = z)$$

イロト イポト イヨト イヨト

 Introduction
 1st Step : bound of H_A

 2nd Step : Replace est A by a box

 Sketch of the proof of Theorem, transient case

 Overview of the proof in the recurrent case

 Open questions

 1st Step : bound of H_A

 2nd Step : Replace est A by a box

 3rd Step : Replace est time of an annulus by exit time of a box

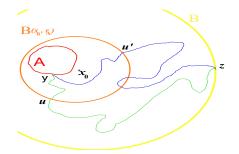
 4th Step : Estimate of $P_u(X_{\tau \partial B} = z)$

 5th Step : Gathering the estimate

 Modification of step 4, with a weak assumption

2nd Step : Replace set A by a box

Let $x_0 \in \Gamma$ and $r_A > 0$ such that $A \subset B(x_0, r_A)$.

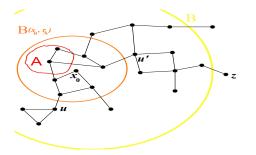


 \Rightarrow Study of $P_u(X_{\tau_{\partial B} \wedge \tau_A} = z)$ for $u \in \partial B(x_0, r_A)$.

イロン 不同 とくほ とくほ とう

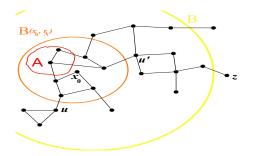
1st Step : bound of H_A **2nd Step : Replace set A by a box** 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

2nd Step : Replace set A by a box



1st Step : bound of H_A **2nd Step : Replace set A by a box** 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

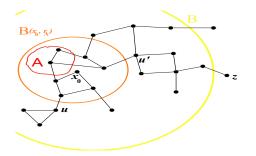
2nd Step : Replace set A by a box



Pb : if some *u* are not connected to ∂B in B - A,

1st Step : bound of H_A **2nd Step : Replace set A by a box** 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

2nd Step : Replace set A by a box



Pb : if some *u* are not connected to ∂B in B - A, no chance to compare $P_u(X_{\tau_{\partial B} \wedge \tau_A} = z)$ for all *u*.

ヘロト ヘアト ヘビト ヘビト

1st Step : bound of H_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

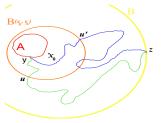
2nd Step : Replace set A by a box

A condition like

(*)
$$P_u(\tau_A > \tau_{\partial B}) > c > 0$$
,

(with c independant of B...)

enables us to remove this case.



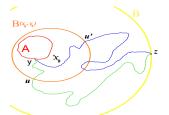
1st Step : bound of H_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

2nd Step : Replace set A by a box

A condition like

$$(*) \quad P_{\mathcal{U}}(\tau_{\mathcal{A}} > \tau_{\partial \mathcal{B}}) > \boldsymbol{c} > \boldsymbol{0},$$

(with c independant of B...)

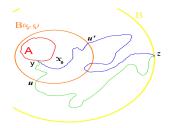


enables us to remove this case. And so, we have to study

for all
$$u \in \partial B(x_0, r_A)$$
 $\mathbb{P}_u(X_{\tau_{\partial B} \wedge \tau_A} = z)$.

Introduction
Results1st Step : Bound of H_A Sketch of the proof of Theorem, transient case
Overview of the proof in the recurrent case
Open questions2nd Step : Replace set A by a box
3rd Step : Replace exit time of an annulus by exit time of a box
4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$
5th Step : Gathering the estimate
Modification of step 4, with a weak assumption

3rd Step : Replace exit time of an annulus by exit time of a box



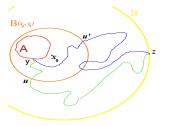
イロト イポト イヨト イヨト

Introduction
Results1st Step : bound of H_A
2nd Step : Replace exit time of an annulus by exit time of a box
3rd Step : Replace exit time of an annulus by exit time of a box
3rd Step : Seplace exit time of an annulus by exit time of a box
3rd Step : Seplace exit time of an annulus by exit time of a box
3rd Step : Seplace exit time of an annulus by exit time of a box
3rd Step : Seplace exit time of an annulus by exit time of a box
3rd Step : Seplace exit time of a box
3rd Step : Seplace exit time of an annulus by exit time of a box
3rd Step : Seplace exit time of an annulus by exit time of a box
3rd Step : Seplace exit time

3rd Step : Replace exit time of an annulus by exit time of a box

The condition

(*)
$$P_u(\tau_A > \tau_{\partial B}) > c > 0$$
,



also implies, that we can study

 $P_u(X_{\tau_{\partial B}}=z), \text{ for } u \in \partial B(x_0, r_A).$

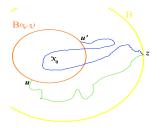
イロト 不得 とくほ とくほとう

э

1st Step : bound of \overline{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4. with a weak assumption

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

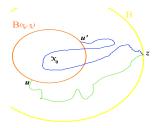
Assume graph Γ satisfies classical Harnack inequality **H(K,M)**.



1st Step : bound of \overline{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4. with a weak assumption

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

Assume graph Γ satisfies classical Harnack inequality **H(K,M)**.



Introduction
Results1st Step : bound of \bar{H}_A
2nd Step : Replace set A by a box
3rd Step : Replace set A by a box
3rd Step : Replace set time of an annulus by exit time of a box
4th Step : Estimate of $P_U(X_{\tau_{OB}} = z)$
5th Step : Gathering the estimate
Modification of step 4, with a weak assumption

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

Recall parameters meaning in Harnack inequality

Definition

We say that a weighted graph (Γ , a) satisfies **H(K,M)**, if all $x \in \Gamma$ and R > 0, and for any non-negative harmonic function u on B(x, MR),

 $\max_{B(x,R)} u \leq K \min_{B(x,R)} u.$

ヘロン 人間 とくほ とくほ とう

э.

Introduction
Results1st Step : bound of H_A
2nd Step : Replace set A by a box
3rd Step : Replace set A by a box
3rd Step : Replace exit time of an annulus by exit time of a box
4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$
5th Step : Gathering the estimate
Modification of step 4, with a weak assumption

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

Assume graph Γ satisfies classical Harnack inequality **H(K,M)**.

くロト (過) (目) (日)

Introduction
Results1st Step : bound of H_A
2nd Step : Replace set A by a box
3rd Step : Replace exit time of an annulus by exit time of a box
3rd Step : Estimate of $P_u(X_{\tau_{OB}} = z)$
5th Step : Gathering the estimate
Modification of step 4, with a weak assumption

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

Assume graph Γ satisfies classical Harnack inequality **H(K,M)**. Let

•
$$B_0 = B(x_0, r_A)$$

ヘロト 人間 ト ヘヨト ヘヨト

Introduction
Results1st Step : bound of H_A
2nd Step : Replace set A by a box
3rd Step : Replace exit time of an annulus by exit time of a box
3rd Step : Estimate of $P_u(X_{\tau_{OB}} = z)$
5th Step : Gathering the estimate
Modification of step 4, with a weak assumption

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

Assume graph Γ satisfies classical Harnack inequality **H(K,M)**. Let

$$\bullet B_0 = B(x_0, r_A)$$

•
$$B_1 = B(x_0, Mr_A)$$

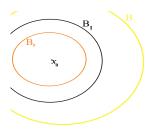
ヘロト 人間 ト ヘヨト ヘヨト

Introduction
ResultsThe Step : Dound of H_A
2 and Step : Replace set A by a box
3rd Step : Replace exit time of an annu
4th Step : Estimate of $P_u(X_{\tau\partial B} = z)$
5th Step : Gathering the estimate
Modification of step 4, with a weak ass

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

Assume graph Γ satisfies classical Harnack inequality **H(K,M)**. Let

- $B_0 = B(x_0, r_A)$
- $B_1 = B(x_0, Mr_A)$
- *B* any box such that $B_1 \subsetneq B$

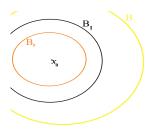


Introduction
ResultsThe Step : Dound of H_A
2 and Step : Replace set A by a box
3rd Step : Replace exit time of an annu
4th Step : Estimate of $P_u(X_{\tau\partial B} = z)$
5th Step : Gathering the estimate
Modification of step 4, with a weak ass

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

Assume graph Γ satisfies classical Harnack inequality **H(K,M)**. Let

- $B_0 = B(x_0, r_A)$
- $B_1 = B(x_0, Mr_A)$
- *B* any box such that $B_1 \subsetneq B$



1st Step : bound of \overline{H}_A 2nd Step : Replace set *A* by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4. with a weak assumption

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

Assume graph Γ satisfies classical Harnack inequality **H(K,M)**.

Let

•
$$B_0 = B(x_0, r_A)$$

•
$$B_1 = B(x_0, Mr_A)$$

• B any box such that

 $B_1 \subsetneq B$ Let $f_B(u) = P_u(X_{\tau_{\partial B}} = z)$ defined on B_1 .

1st Step : bound of \overline{H}_A 2nd Step : Replace set *A* by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4. with a weak assumption

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

Assume graph Γ satisfies classical Harnack inequality **H(K,M)**.

Let

•
$$B_0 = B(x_0, r_A)$$

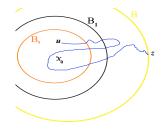
•
$$B_1 = B(x_0, Mr_A)$$

• B any box such that

 $B_1 \subsetneq B$

Let $f_B(u) = P_u(X_{\tau_{\partial B}} = z)$ defined on B_1 .

• f is positive on B₁



4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate

4th Step : Estimate of $P_{u}(X_{\tau_{\partial B}} = z)$

Assume graph Г satisfies classical Harnack inequality H(K,M).

Let

•
$$B_0 = B(x_0, r_A)$$

•
$$B_1 = B(x_0, Mr_A)$$

×,

B any box such that $B_1 \subseteq B$

Let $f_B(u) = P_u(X_{\tau_{\partial B}} = z)$ defined on B_1 .

- f is positive on B_1
- f is harmonic on B_1

so we can compare : $f_B(u)$ and $f_B(u')$ for $u, u' \in B_0$

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate

4th Step : Estimate of $P_{u}(X_{\tau_{\partial B}} = z)$

Assume graph Г satisfies classical Harnack inequality H(K,M).

Let

•
$$B_0 = B(x_0, r_A)$$

•
$$B_1 = B(x_0, Mr_A)$$

×,

B any box such that $B_1 \subseteq B$

Let $f_B(u) = P_u(X_{\tau_{\partial B}} = z)$ defined on B_1 .

- f is positive on B_1
- f is harmonic on B_1

so we can compare : $f_B(u)$ and $f_B(u')$ for $u, u' \in B_0$

1st Step : bound of \overline{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4. with a weak assumption

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

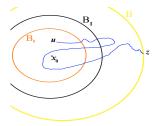
Assume graph Γ satisfies classical Harnack inequality **H(K,M)**. Let

•
$$B_0 = B(x_0, r_A)$$

•
$$B_1 = B(x_0, Mr_A)$$

• B any box such that $B \subset B$

 $B_1 \subsetneq B$ Let $f_B(u) = P_u(X_{\tau_{\partial B}} = z)$ defined on B_1 .



1st Step : bound of \overline{H}_A 2nd Step : Replace set *A* by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4. with a weak assumption

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

Assume graph Γ satisfies classical Harnack inequality **H(K,M)**. Let

•
$$B_0 = B(x_0, r_A)$$

•
$$B_1 = B(x_0, Mr_A)$$

• *B* any box such that

B_s u X_b

 $B_1 \subsetneq B$ Let $f_B(u) = P_u(X_{\tau_{\partial B}} = z)$ defined on B_1 . Harnack inequality also gives us that :

$$osc_{B_0}(f_B) \leq rac{K-1}{K+1} \ osc_{B_1}(f_B),$$

where $osc_E(f) = max_E f - min_E f$.

1st Step : bound of \overline{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

くロト (過) (目) (日)

æ

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

Let $k \ge 1$ and let

• $B_k = B(x_0, M^k r_A)$



1st Step : bound of \overline{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

Let $k \ge 1$ and let

- $B_k = B(x_0, M^k r_A)$
- *B* such that $B_k \subsetneq B$

1st Step : bound of \overline{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

Let $k \ge 1$ and let

- $B_k = B(x_0, M^k r_A)$
- *B* such that $B_k \subsetneq B$
- $f_B(u) = P_u(X_{\tau_{\partial B}} = z)$ defined on B_k , which is positive and harmonic on B_k .

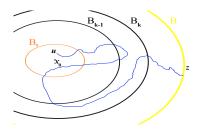
くロト (過) (目) (日)

1st Step : bound of \overline{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

Let $k \ge 1$ and let

- $B_k = B(x_0, M^k r_A)$
- *B* such that $B_k \subsetneq B$
- $f_B(u) = P_u(X_{\tau_{\partial B}} = z)$ defined on B_k , which is positive and harmonic on B_k .



ヘロト ヘ戸ト ヘヨト ヘヨト

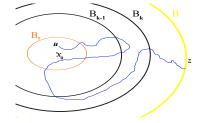
1st Step : bound of \overline{H}_A 2nd Step : Replace set *A* by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{OB}} = z)$ 5th Step : Gathering the estimate Modification of step 4. with a weak assumption

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

Let $k \ge 1$ and let

•
$$B_k = B(x_0, M^k r_A)$$

- *B* such that $B_k \subsetneq B$
- $f_B(u) = P_u(X_{\tau_{\partial B}} = z)$ defined on B_k , which is positive and harmonic on B_k . Similarly,



ヘロト ヘ戸ト ヘヨト ヘヨト

$$osc_{B_{k-1}}(f_B) \leq \frac{K-1}{K+1} osc_{B_k}(f_B).$$

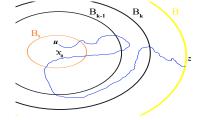
1st Step : bound of \overline{H}_A 2nd Step : Replace set *A* by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{T_{OB}} = z)$ 5th Step : Gathering the estimate Modification of step 4. with a weak assumption

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

Let $k \ge 1$ and let

•
$$B_k = B(x_0, M^k r_A)$$

- *B* such that $B_k \subsetneq B$
- $f_B(u) = P_u(X_{\tau_{\partial B}} = z)$ defined on B_k , which is positive and harmonic on B_k . Similarly,



$$osc_{B_{k-1}}(f_B) \leq rac{K-1}{K+1} \ osc_{B_k}(f_B).$$

And so,

$$osc_{B_0}(f_B) \leq \left(rac{K-1}{K+1}
ight)^k \ osc_{B_k}(f_B),$$

Introduction
Results1st Step : bound of \bar{H}_A
2nd Step : Replace set A by a box
3rd Step : Replace set A by a box
3rd Step : Replace set A by a box
3rd Step : Estimate of an annulus by exit time of a box
4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$
5th Step : Gathering the estimate
Modification of step 4, with a weak assumption

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

Finaly, for *B* big enough and for $u \in B(x_0, r_A)$, we get :

$$egin{array}{rcl} |f_B(u)-f_B(x_0)| &\leq & (rac{K-1}{K+1})^k osc_{B_k}(f_B) \ &\leq & (rac{K-1}{K+1})^k \ \max_{B_k}(f_B) \ &\leq & (rac{K-1}{K+1})^k \ K \ f_B(x_0) \end{array}$$

Daniel Boivin , Rau Clément Existence of the harmonic measure for random walks on graphs a

Introduction
Results1st Step : bound of \bar{H}_A
2nd Step : Replace set A by a box
3rd Step : Replace set A by a box
3rd Step : Replace set A is a box
3rd Step : Replace set A by a box
4th Step : Estimate of an annulus by exit time of a box
4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$
5th Step : Gathering the estimate
Modification of step 4, with a weak assumption

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

Finaly, for *B* big enough and for $u \in B(x_0, r_A)$, we get :

$$egin{array}{rcl} |f_B(u) - f_B(x_0)| &\leq & (rac{K-1}{K+1})^k osc_{B_k}(f_B) \ &\leq & (rac{K-1}{K+1})^k \ \max_{B_k}(f_B) \ &\leq & (rac{K-1}{K+1})^k \ K \ f_B(x_0) \end{array}$$

So,

$$f_B(u) = f_B(x_0)[1 + O((\frac{K-1}{K+1})^k)].$$

1st Step : bound of \overline{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4. with a weak assumption

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

$$f_B(u) = f_B(x_0)[1 + O((\frac{K-1}{K+1})^k)].$$

Daniel Boivin , Rau Clément Existence of the harmonic measure for random walks on graphs a

イロト イポト イヨト イヨト

Introduction
Results1st Step : bound of H_A
2nd Step : Replace set A by a box
3rd Step : Replace exit time of an annulus by exit time of a box
3rd Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$
5th Step : Gathering the estimate
Modification of step 4, with a weak assumption

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

$$f_B(u) = f_B(x_0)[1 + O((\frac{K-1}{K+1})^k)].$$

Taking for example $B = B_{k+1}$, this can be read :

$$P_{u}(X_{\tau_{\partial B_{k+1}}} = z) = H_{\partial B_{k+1}}(x_{0}, z)[1 + O(\left(\frac{K-1}{K+1}\right)^{k})],$$

where the constant in $O(\cdot)$ depends only on K.

1st Step : bound of \overline{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4. with a weak assumption

5th Step : Final estimate

With condition like

(*) $P_{\mu}(\tau_{A} > \tau_{\partial B}) > c > 0$,

(with c independant of B),

Daniel Boivin , Rau Clément Existence of the harmonic measure for random walks on graphs a

ヘロト 人間 ト ヘヨト ヘヨト

э.

1st Step : bound of \overline{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{OB}} = z)$ 5th Step : Gathering the estimate Modification of step 4. with a weak assumption

5th Step : Final estimate

With condition like

$$(*) \quad P_u(\tau_A > \tau_{\partial B}) > c > 0,$$

(with *c* independant of *B*), we deduce (step 3) that : for all $u \in B(x_0, r_A)$,

$$P_u(X_{\tau_{\partial B_{k+1}}\wedge\tau_A}=z)=H_{\partial B_{k+1}}(x_0,z)P_x(\tau_A>\tau_{\partial B_{k+1}})[1+O(\left(\frac{K-1}{K+1}\right)^k)].$$

1st Step : bound of \overline{H}_A 2nd Step : Replace set *A* by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4. with a weak assumption

5th Step : Final estimate

With condition like

$$(*) \quad P_u(\tau_A > \tau_{\partial B}) > c > 0,$$

(with *c* independant of *B*), we deduce (step 3) that : for all $u \in B(x_0, r_A)$,

$$P_{u}(X_{\tau_{\partial B_{k+1}}\wedge\tau_{A}}=z)=H_{\partial B_{k+1}}(x_{0},z)P_{x}(\tau_{A}>\tau_{\partial B_{k+1}})[1+O(\left(\frac{K-1}{K+1}\right)^{k})].$$

And then by step 2, we finally get, for all $y \in A$,

$$P_{y}(X_{\tau_{\partial B_{k+1}} \wedge \tau_{\mathcal{A}}} = z) = H_{\partial B_{k+1}}(x_{0}, z) \left[1 + O\left(\left(\frac{K-1}{K+1}\right)^{k}\right)\right] P_{y}(\tau_{\mathcal{A}} > \tau_{\partial B_{k+1}})$$

ヘロト ヘアト ヘビト ヘビト

1

1st Step : bound of \overline{H}_A 2nd Step : Replace set *A* by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4. with a weak assumption

5th Step : Final estimate

This can be read, for all $y \in A$,

$$H_{A\cup\partial B_{k+1}}(y,z) = H_{\partial B_{k+1}}(x_0,z)[1+O((\frac{K-1}{K+1})^k)]P_y(\tau_A > \tau_{\partial B_{k+1}}).$$

イロト イポト イヨト イヨト

æ

1st Step : bound of \overline{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau \partial B} = z)$ 5th Step : Gathering the estimate Modification of step 4 with a weak assumption

5th Step : Final estimate

This can be read, for all $y \in A$,

$$H_{A\cup\partial B_{k+1}}(y,z)=H_{\partial B_{k+1}}(x_0,z)[1+O(\left(\frac{K-1}{K+1}\right)^k)]P_y(\tau_A>\tau_{\partial B_{k+1}}).$$

And then,

$$\frac{\pi(\mathbf{y})H_{A\cup\partial B_{k+1}}(\mathbf{y}, \mathbf{z})}{\sum_{\tilde{\mathbf{y}}\in A}\pi(\tilde{\mathbf{y}})P_{\tilde{\mathbf{y}}}(X_{\tau_{\partial B_{k+1}}\wedge\tau_{A}}=\mathbf{z})} = \frac{\pi(\mathbf{y})P_{\mathbf{y}}(\tau_{A} > \tau_{\partial B_{k+1}})}{\sum_{\tilde{\mathbf{y}}\in A}\pi(\tilde{\mathbf{y}})P_{\tilde{\mathbf{y}}}(\tau_{A} > \tau_{\partial B_{k+1}})} \times [1 + O(\left(\frac{K-1}{K+1}\right)^{k})]$$

イロト イポト イヨト イヨト

æ

1st Step : bound of \overline{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4. with a weak assumption

5th Step : Final estimate

But, remember that,

$\min_{z \in \partial B} \frac{\pi(y) \mathcal{H}_{A \cup \partial B}(y, z)}{\sum_{\tilde{y} \in \mathcal{A}} \pi(\tilde{y}) \mathcal{H}_{A \cup \partial B}(\tilde{y}, z)} \leq \overline{\mathcal{H}}_{\mathcal{A}}(x, y) \leq \max_{z \in \partial B} \frac{\pi(y) \mathcal{H}_{A \cup \partial B}(y, z)}{\sum_{\tilde{y} \in \mathcal{A}} \pi(\tilde{y}) \mathcal{H}_{A \cup \partial B}(\tilde{y}, z)}$

ヘロト 人間 とくほ とくほ とう

э.

1st Step : bound of \overline{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4. with a weak assumption

5th Step : Final estimate

So, from

$$\frac{\pi(\mathbf{y})H_{\mathbf{A}\cup\partial B_{k+1}}(\mathbf{y}, \mathbf{z})}{\sum_{\tilde{\mathbf{y}}\in\mathbf{A}}\pi(\tilde{\mathbf{y}})P_{\tilde{\mathbf{y}}}(X_{\tau_{\partial B_{k+1}}\wedge\tau_{\mathbf{A}}}=\mathbf{z})} = \frac{\pi(\mathbf{y})P_{\mathbf{y}}(\tau_{\mathbf{A}} > \tau_{\partial B_{k+1}})}{\sum_{\tilde{\mathbf{y}}\in\mathbf{A}}\pi(\tilde{\mathbf{y}})P_{\tilde{\mathbf{y}}}(\tau_{\mathbf{A}} > \tau_{\partial B_{k+1}})} \times [1 + O(\left(\frac{K-1}{K+1}\right)^{k})],$$

we obtain that $\lim_{v\to+\infty} \overline{H}_A(v, y)$ exists and

$$\lim_{\nu \to +\infty} \overline{H}_{\mathcal{A}}(\nu, y) = \frac{\pi(y) \mathcal{P}_{\mathcal{Y}}(\tau_{\mathcal{A}} > +\infty)}{\sum_{\tilde{\mathcal{Y}} \in \mathcal{A}} \pi(\tilde{\mathcal{Y}}) \mathcal{P}_{\tilde{\mathcal{Y}}}(\tau_{\mathcal{A}} > +\infty)}$$

イロト 不得 とくほ とくほとう

3

1st Step : bound of \tilde{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

イロト イポト イヨト イヨト

How is use Harnack inequality?

• $B_k = B(x_0, M^k r_A)$



1st Step : bound of \overline{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

How is use Harnack inequality?

- $B_k = B(x_0, M^k r_A)$
- *B* such that $B_k \subsetneq B$
- $f_B(u) = P_u(X_{\tau_{\partial B}} = z)$ defined on B_k ,

Introduction
Results1st Step : bound of H_A Sketch of the proof of Theorem, transient case
Overview of the proof in the recurrent case
Open questions2nd Step : Replace exit time of an annulus by e
4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ State in the stimate
Modification of step 4, with a weak assumption

How is use Harnack inequality?

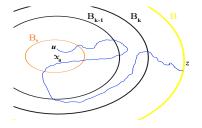
- $B_k = B(x_0, M^k r_A)$
- *B* such that $B_k \subsetneq B$
- $f_B(u) = P_u(X_{\tau_{\partial B}} = z)$ defined on B_k , which is positive and harmonic on B_k .

1st Step : bound of \overline{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

How is use Harnack inequality?

•
$$B_k = B(x_0, M^k r_A)$$

- *B* such that $B_k \subsetneq B$
- $f_B(u) = P_u(X_{\tau_{\partial B}} = z)$ defined on B_k , which is positive and harmonic on B_k .



イロト イポト イヨト イヨト

So, by Harnack inequality

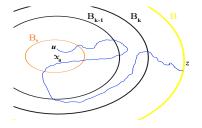
$$osc_{B_{k-1}}(f_B) \leq rac{K-1}{K+1} \ osc_{B_k}(f_B).$$

1st Step : bound of \overline{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

How is use Harnack inequality?

•
$$B_k = B(x_0, M^k r_A)$$

- *B* such that $B_k \subsetneq B$
- $f_B(u) = P_u(X_{\tau_{\partial B}} = z)$ defined on B_k , which is positive and harmonic on B_k .



イロト イポト イヨト イヨト

So, by Harnack inequality

$$osc_{B_{k-1}}(f_B) \leq rac{K-1}{K+1} \ osc_{B_k}(f_B).$$

1st Step : bound of \overline{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4. with a weak assumption

Weak Harnack inequality

Definition

We say that a weighted graph (Γ , a) satisfies **wH**(K), the weak Harnack inequality, if there is a constant $1 \le K < \infty$ such that for all $x \in \Gamma$ and for all R > 0 there is $M_{x,R} \ge 2$ such that for all $M > M_{x,R}$ and for any non-negative harmonic function u on B(x, MR),

 $\max_{B(x,R)} u \leq K \min_{B(x,R)} u.$

イロト イポト イヨト イヨト

1

1st Step : bound of \overline{H}_A 2nd Step : Replace set A by a box 3rd Step : Replace exit time of an annulus by exit time of a box 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$ 5th Step : Gathering the estimate Modification of step 4, with a weak assumption

Weak Harnack inequality

Definition

We say that a weighted graph (Γ , a) satisfies **wH**(K), the weak Harnack inequality, if there is a constant $1 \le K < \infty$ such that for all $x \in \Gamma$ and for all R > 0 there is $M_{x,R} \ge 2$ such that for all $M > M_{x,R}$ and for any non-negative harmonic function u on B(x, MR),

 $\max_{B(x,R)} u \leq K \min_{B(x,R)} u.$

Assuming Weak Harnack inequality, we replace $B(x_0, M^k r_A)$ by $B(x_0, M_k M_{k-1}...M_1 r_A)$ such that

$$M_i = M(x_0, M_{i-1}M_{i-2}...M_1r_A).$$

・ロト ・ 同ト ・ ヨト ・ ヨト … ヨ

Introduction
Results1st Step : bound of \overline{H}_A
2nd Step : Replace set A by a box
3rd Step : Replace exit time of an annulus by exit time of a box
d Sketch of the proof of Theorem, transient case
Overview of the proof in the recurrent case
Open questions1st Step : bound of \overline{H}_A
2nd Step : Replace set A by a box
d Step : Replace exit time of an annulus by exit time of a box
4th Step : Estimate of $P_u(X_{T_{OB}} = z)$
5th Step : Gathering the estimate
Modification of step 4, with a weak assumption

Condition (*)

Estimate (GE_{γ}) gives us **wH**(K)

Daniel Boivin , Rau Clément Existence of the harmonic measure for random walks on graphs a

イロト イポト イヨト イヨト

æ

Introduction
Results1st Step : bound of \bar{H}_A
2nd Step : Replace set A by a box
3rd Step : Replace set A by a box
4th Step : Cathering the estimate
Modification of step 4, with a weak assumptionCondition (*)

Estimate (GE_{γ}) gives us **wH**(K) and the wanted condition

(*) $P_u(\tau_A > \tau_{\partial B}) > c > 0.$

Lemma

Let (Γ, a) be a weighted graph which verifies (\mathbf{GE}_{γ}) . Set $\theta = (2\frac{C_s}{C_i})^{\frac{1}{\gamma}}$. Then for all $x_0 \in \Gamma$, $M > \theta$, $R \ge R_{x_0}$ and $x \in \partial B(x_0, \theta R)$, we have : for all $A \subset B(x_0, R)$,

$$P_{x}(\tau_{A} > \tau_{\partial B(x_{0},MR)}) > \frac{C_{i}}{2C_{s}}.$$
(1)

くロト (過) (目) (日)

ъ



イロン 不同 とくほ とくほ とう

3

Supercritical cluster percolation in dimension 2

Replace G by g

Daniel Boivin , Rau Clément Existence of the harmonic measure for random walks on graphs a

Supercritical cluster percolation in dimension 2

- Replace G by g
- Estimate of g.

Supercritical cluster percolation in dimension 2

- Replace G by g
- Estimate of *g*. Tools :
 - Harnack inequality (Parabolic and Elliptic in a annulus)

Supercritical cluster percolation in dimension 2

- Replace G by g
- Estimate of g. Tools :
 - Harnack inequality (Parabolic and Elliptic in a annulus)
 - Capacity of a box of supercritical percolation of $\mathbb{Z}^2.$ (Kesten Grid)

Supercritical cluster percolation in dimension 2

- Replace G by g
- Estimate of g. Tools :
 - Harnack inequality (Parabolic and Elliptic in a annulus)
 - Capacity of a box of supercritical percolation of \mathbb{Z}^2 . (Kesten Grid) we proved that there is a constant $C \ge 1$ such that \mathbb{P}_p -a.s. for $x_0 \in \mathcal{C}_\infty$, for all *n* sufficiently large,

$$C^{-1} \leq \ln(n) \ Cap_{B_{\omega}(x_0,n)}(\{x_0\}) \leq C.$$

ヘロト 人間 ト ヘヨト ヘヨト

Supercritical cluster percolation in dimension 2

- Replace G by g
- Estimate of g. Tools :
 - Harnack inequality (Parabolic and Elliptic in a annulus)
 - Capacity of a box of supercritical percolation of \mathbb{Z}^2 . (Kesten Grid) we proved that there is a constant $C \ge 1$ such that \mathbb{P}_p -a.s. for $x_0 \in \mathcal{C}_\infty$, for all *n* sufficiently large,

$$C^{-1} \leq \ln(n) \operatorname{Cap}_{B_{\omega}(x_0,n)}(\{x_0\}) \leq C.$$

• Antal and Pisztora estimate of the chemical distance.

ヘロア 人間 アメヨア 人口 ア

Supercritical cluster percolation in dimension 2

- Replace G by g
- Estimate of g. Tools :
 - Harnack inequality (Parabolic and Elliptic in a annulus)
 - Capacity of a box of supercritical percolation of \mathbb{Z}^2 . (Kesten Grid) we proved that there is a constant $C \ge 1$ such that \mathbb{P}_p -a.s. for $x_0 \in \mathcal{C}_\infty$, for all *n* sufficiently large,

$$C^{-1} \leq \ln(n) \ Cap_{B_{\omega}(x_0,n)}(\{x_0\}) \leq C.$$

- Antal and Pisztora estimate of the chemical distance.
- A last trick in a bound of H_A .

ヘロト ヘアト ヘビト ヘビト

Supercritical cluster percolation in dimension 2

A last trick in a bound of H_A . Let $\sigma_m := \inf\{k > 0; X_k \notin B(x_0, m)\}.$

Daniel Boivin , Rau Clément Existence of the harmonic measure for random walks on graphs a

Supercritical cluster percolation in dimension 2

A last trick in a bound of
$$H_A$$
.
Let $\sigma_m := \inf\{k > 0; X_k \notin B(x_0, m)\}$.

$$H_A(x, y) = \frac{\pi(x)H_A(x, y)}{\pi(x)\sum_{y' \in A}H_A(x, y')}$$

$$= \frac{\pi(y)P_y(\widetilde{\sigma}_n < \tau_A)}{\sum_{y' \in A}\pi(y')P_{y'}(\widetilde{\sigma}_n < \tau_A)} \left[1 + O\left(n^{-\nu'}\right)\right]$$

$$= \frac{(\ln n)\pi(y)P_y(\widetilde{\sigma}_n < \tau_A)}{(\ln n)\sum_{y' \in A}\pi(y')P_{y'}(\widetilde{\sigma}_n < \tau_A)} \left[1 + O\left(n^{-\nu'}\right)\right]$$

Supercritical cluster percolation in dimension 2

A last trick in a bound of
$$H_A$$
.
Let $\sigma_m := \inf\{k > 0; X_k \notin B(x_0, m)\}$.

$$H_A(x, y) = \frac{\pi(x)H_A(x, y)}{\pi(x)\sum_{y' \in A}H_A(x, y')}$$

$$= \frac{\pi(y)P_y(\widetilde{\sigma}_n < \tau_A)}{\sum_{y' \in A}\pi(y')P_{y'}(\widetilde{\sigma}_n < \tau_A)} \left[1 + O\left(n^{-\nu'}\right)\right]$$

$$= \frac{(\ln n)\pi(y)P_y(\widetilde{\sigma}_n < \tau_A)}{(\ln n)\sum_{y' \in A}\pi(y')P_{y'}(\widetilde{\sigma}_n < \tau_A)} \left[1 + O\left(n^{-\nu'}\right)\right]$$

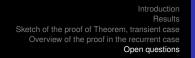
Fact : for $x \in A^c$,

$$\lim_{n} (\ln n) P_{x}(\widetilde{\sigma}_{n} < \tau_{A}) \text{ exists}$$



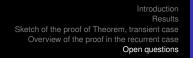
Daniel Boivin , Rau Clément Existence of the harmonic measure for random walks on graphs a

ヘロト 人間 とくほとくほとう





 Question 1 : Links between Martin/Poisson boundary and the existence of the harmonic measure? (Existence of harmonic measure Triviality of the boundary)



Open questions.

- Question 1 : Links between Martin/Poisson boundary and the existence of the harmonic measure? (Existence of harmonic measure Triviality of the boundary)
- Question 2 : Is there a connection between existence of harmonic measure and invariance principle (in random environments) ?

ヘロト 人間 ト ヘヨト ヘヨト

Thanks for your attention ... !

Daniel Boivin , Rau Clément Existence of the harmonic measure for random walks on graphs a

くロト (過) (目) (日)

э