Existence of harmonic measure for random walks on graphs and random environments

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"Random walks, random media, reinforcement"
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Outline of the talk

Weighted graphs

- Let (Γ, a) a weighted graph. Γ is a countably infinite set and a is a symmetric function :

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- We will write $x \sim y$ if $a(x, y) > 0$.
- We will always assume that $(\Gamma, \sim)$ is an infinite connected graph, locally finite countable graph without multiple edges.

Random walks

- The random walk $(X_n)_n$ on the weighted graph (Γ, a) is the Markov chain on Γ with transition probabilities given by

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- We denote by P_x the law of the random walk starting at the vertex $x \in \Gamma$. The corresponding expectation is denoted by E_x.
- The random walk admits reversible measures which are proportional to the measure $\pi(\cdot)$.

Notations

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- For $u : \overline{A} \to \mathbb{R}$ the Laplacian is defined by

$$\mathcal{L}u(x) := \sum_{y \sim x} p(x, y)\,[u(y) - u(x)], \quad x \in A.$$

- A function $u : \overline{A} \to \mathbb{R}$ is *harmonic* in A if for all $x \in A$,

$$(\mathcal{L}u)(x) = 0.$$

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- The *Green function* of the random walk is defined by

$$G(x,y) := \sum_{j=0}^{\infty} p(x,y,j), \quad x,y \in \Gamma$$

where $p(x,y,j) := P_x(X_j = y)$ are the transition probabilities of the walk.

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- The Green function of the random walk in $B \subset \Gamma$ is defined by

$$G_B(x,y) := \sum_{j=0}^{\infty} p_B(x,y,j), \quad x,y \in \overline{B}$$

where $p_B(x,y,j) := P_x(X_j = y;\ \forall i \le j \ \ X_i \in B)$.

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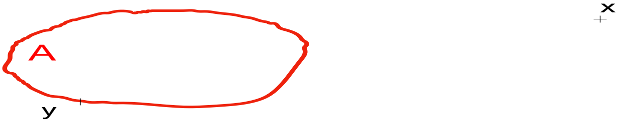

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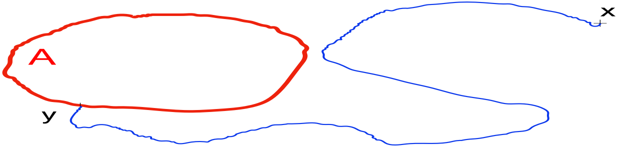

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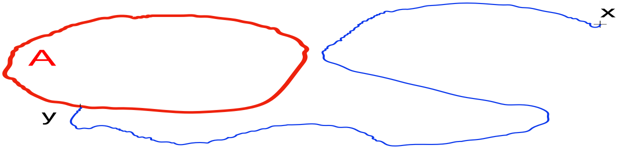

- The *hitting distribution* of a set A starting from $x \in \Gamma$ is given by :

$$H_A(x, y) := P_x(X_{\tau_A} = y).$$

hitting distribution

$$\begin{array}{rcccl} H_A(x,y) & : & \mathbb{Z}^d \times A & \to & [0;1] \\ & & (x,y) & \mapsto & H_A(x,y) \quad := P_x(X_{\tau_A} = y) \end{array}$$

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- For fixed $x \in \Gamma$, $H_A(x,.)$ is a positive measure on A with total mass $P_x(\tau_A < +\infty)$.
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- For fixed $x \in \Gamma$, $H_A(x,.)$ is a positive measure on A with total mass $P_x(\tau_A < +\infty)$.
  $(supp(H_A(x,.) \subset \partial A)$
- If $P_x(\tau_A < +\infty) > 0$, we may define a probability measure on A by conditioning that the random walk hits A,

$$\overline{H}_A(x,y) := P_x(X_{\tau_A} = y | \tau_A < +\infty).$$

Harmonic measure

- The *harmonic measure* on a finite subset A of Γ is the hitting distribution from infinity, if it exists,

$$\mathbf{H}_A(y) := \lim_{D(x,A)\to\infty} \overline{H}_A(x,y), \quad y \in A.$$

where D denote the graph distance between two vertices $x, y \in \Gamma$. It is the minimal length of a path from x to y in the graph $(\Gamma, \sim)$.

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$\Rightarrow$ Our goal is to prove the existence of the harmonic measure for all finite subsets of various weighted graphs.

Motivations to study Harmonic measure

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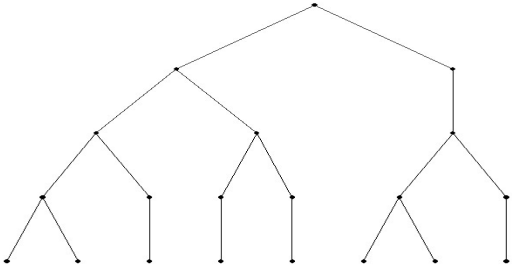
$\Rightarrow$ Physical interpretation : distribution/spread of charge on an object .

Examples where Harmonic measure does not exist

Infinite tree



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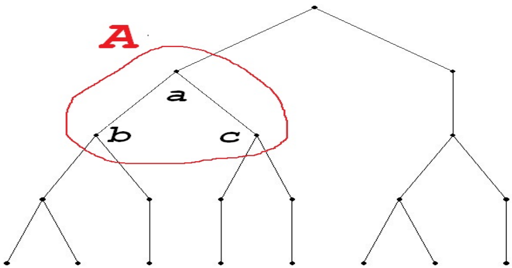

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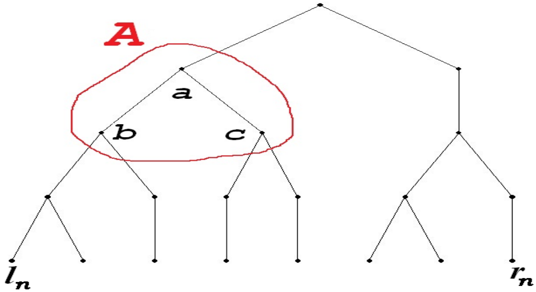

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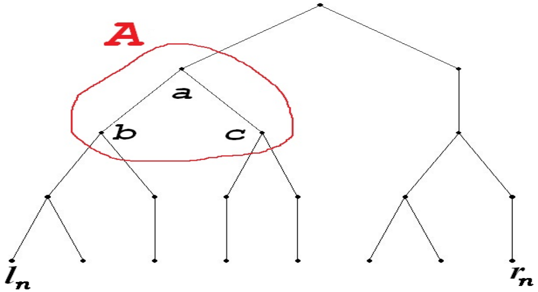

Consider for example, $\bar{H}_A(l_n, a)$ and $\bar{H}_A(r_n, a)$.

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Theorem (G. Lawler)

Let A a finite subset of $\mathbb{Z}^d$, for all y in A, we have :

$$\lim_{D(0,x)\to\infty} \bar{H}_A(x,y) = \mathbf{H_A(y)} \quad \textbf{exists}.$$

Moreover,

$$\mathbf{H_A(y)} = \lim_{\mathbf{n}\to\infty} \frac{\mathbf{P_y}(\tau_\mathbf{A} > \tau_{\partial\mathbf{B(0,n)}})}{\sum_{\mathbf{y'}\in\mathbf{A}} \mathbf{P_{y'}}(\tau_\mathbf{A} > \tau_{\partial\mathbf{B(0,n)}})}.$$

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$$\mathbb{P}(X_{n+1} - X_n = \mathbf{e_i}) = \mathbb{P}(X_{n+1} - X_n = -\mathbf{e_i})$$

and

$$\mathbb{P}(X_{n+1} = X_n) = 1 - 2\sum_{i=1..d} \mathbb{P}(X_{n+1} - X_n = \mathbf{e_i})$$

Tools of the proof of Lawler

2 majors ingredients :

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Definition

We say that a weighted graph (Γ, a) *satisfies* **H(K,M)**, *the Harnack inequality with shrinking parameter* $M > 1$, *if there is a constant* $K < \infty$ *such that for all* $x \in \Gamma$ *and* $R > 0$, *and for any non-negative harmonic function* u *on* $B(x, MR)$,

$$\max_{B(x,R)} u \leq K \min_{B(x,R)} u.$$

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- (Elliptic) Harnack inequality
- for $d \geq 3$, precise estimates of G,
  for $d = 2$, precise estimates of g, where

$$g(x) := \lim_n [G_{B(0,n)}(0) - G_{B(0,n)}(x)],$$

and so of $G_{B(0,n)}$.

Transient case

The main result for transient graphs is the existence of the harmonic measure for random walks with a Green function which verify the following estimate.

Definition

We say that a weighted graph (Γ, a) *satisfies the Green function estimate* $\mathbf{GE}_\gamma$ *for* $\gamma > 0$ *if there are constants* $0 < C_i \leq C_s < \infty$ *and if for all* $z \in \Gamma$*, there exists* $R_z < \infty$ *such that for all* $x, y \in \Gamma$ *with* $D(x, y) \geq R_x \wedge R_y$ *we have :*

$$\frac{C_i}{D(x,y)^\gamma} \leq G(x,y) \leq \frac{C_s}{D(x,y)^\gamma}. \qquad (\mathbf{GE}_\gamma)$$

Our main result is the following :

Theorem

Let (Γ, a) be a weighted graph which verifies $\mathbf{GE}_\gamma$ *for some* $\gamma > 0$*. Then for any finite subset* $A \subset \Gamma$ *the harmonic measure on A exists.(That is, for all* $y \in A$*, the limit exists.)*
Moreover, we have :

$$\lim_{D(x,A)\to\infty} \overline{H}_A(x,y) = \lim_{m\to+\infty} H_A^m(y),$$

where, for m large enough,

$$H_A^m(y) = \frac{\pi(y)P_y(\tau_A > \tau_{\partial B(x_0,m)})}{\mathrm{Cap}_m(A)}$$

The limit does not depend on the choice of x_0*.*

1. The *capacity* of A with respect to B, for $A \subset B \subset \Gamma$, is defined by

$$\mathrm{Cap}_B(A) := \sum_{x \in A} \pi(x) P_x(\overline{\tau}_{B^c} < \tau_A).$$

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Recall that

$$\begin{aligned} \tau_A &:= \inf\{k \geq 1;\ X_k \in A\}, \\ \overline{\tau}_A &:= \inf\{k \geq 0;\ X_k \in A\} \end{aligned}$$

2. $\mathrm{Cap}_m(A)$ is the capacity of A with respect to $B(x_0, m)$ for some $x_0 \in \Gamma$.

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Definition

We say that a weighted graph (Γ, a) *satisfies* **wH(**K**)**, *the weak Harnack inequality, if there is a constant* $1 \leq K < \infty$ *such that for all* $x \in \Gamma$ *and for all* $R > 0$ *there is* $M_{x,R} \geq 2$ *such that for all* $M > M_{x,R}$ *and for any non-negative harmonic function* u *on* $B(x, MR)$,

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We will prove that the Green function estimates $\mathbf{GE}_\gamma$ imply the weak Harnack inequality.

Proposition

Let (Γ, a) *be a weighted graph which verifies* $\mathbf{(GE_\gamma)}$ *for some* $\gamma > 0$*. Then the graph is connected, transient and* $\mathbf{wH(}K\mathbf{)}$ *holds with* $K = 2^\gamma \frac{C_s}{C_i}$.

Some applications

Corollary

Let $(\mathbb{Z}^d, a)$, $d \geq 3$, be a uniformly elliptic graph. Then for all finite subsets A of $\mathbb{Z}^d$ and for all $y \in A$, the limit exists.
Moreover, we have :

$$\lim_{|x| \to +\infty} \overline{H}_A(x, y) = \lim_{m \to +\infty} H_A^m(y),$$

where $H_A^m(y) = \frac{\pi(y) P_y(\tau_A > \tau_{\partial B(0,m)})}{\mathrm{Cap}_m(A)}$.

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It follows from Delmotte's estimates.

Existence of the harmonic measure for $\mathbb{Z}^d$, $d \geq 3$, with i.i.d. conductances

Corollary

Let $(\mathbb{Z}^d, a)$*,* $d \geq 3$*, be a weighted graph where the weights* $(a(e); e \in^d)$ *are i.i.d. non-negative random variables on a probability space* $(\Omega, \mathbb{P})$ *which verify*

$$\mathbb{P}(a(e) > 0) > p_c(\mathbb{Z}^d).$$

For any finite subset A of $\mathcal{C}_\infty$ *and for all* $y \in A$*, harmonic measure exists. Moreover, we have :*

$$\lim_{|x| \to +\infty, x \in \mathcal{C}_\infty} \overline{H}_A(x, y) = \lim_{m \to +\infty} H_A^m(y).$$

Where $H_A^m(y) = \frac{\pi(y)P_y^{\omega}(\tau_A > \tau_{\partial B_{\omega}(x_0,m)})}{\mathrm{Cap}_m(A)}$ for some $x_0 \in \mathcal{C}_{\infty}$ and for m large enough.

Where $H_A^m(y) = \frac{\pi(y)P_y^\omega(\tau_A > \tau_{\partial B_\omega(x_0,m)})}{\mathrm{Cap}_m(A)}$ for some $x_0 \in \mathcal{C}_\infty$ and for m large enough.
It follows from the Green function estimates of Andres, Barlow, Deuschel, Hambly.

Recurrent Case

Percolation cluster.

Theorem

Let $(\mathbb{Z}^2, a)$ be a weighted graph where the weights $(a(e); e \in^2)$ are i.i.d. random variables on a probability space $(\Omega, \mathbb{P}_p)$ which verify

$$p = \mathbb{P}_p(a(e) = 1) = 1 - \mathbb{P}_p(a(e) = 0) > p_c(\mathbb{Z}^2).$$

Then $\mathbb{P}_p$ almost surely, for any finite subset A of $\mathcal{C}_\infty()$ and for all $y \in A$, the harmonic measure exists.

Recurrent Case

$\mathbb{Z}^2$ elliptic.

Theorem

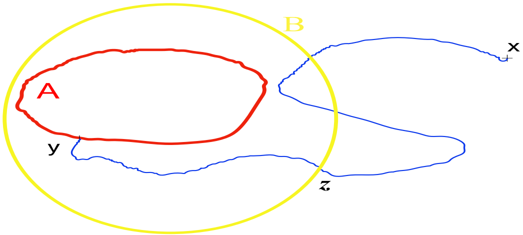
If $(\mathbb{Z}^2, a)$ is a uniformly elliptic weighted graph then for all finite subsets $A \subset \mathbb{Z}^2$ and for all $y \in A$, the limit exists.

3 Sketch of the proof of Theorem, transient case

- 1st Step : bound of $\bar{H}_A$
- 2nd Step : Replace set A by a box
- 3rd Step : Replace exit time of an annulus by exit time of a box
- 4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$
- 5th Step : Gathering the estimate
- Modification of step 4, with a weak assumption

1st Step : bound of $\bar{H}_A$

- Let $A \subset B$ be finite subsets of Γ. Recall that $H_A(x, y) = P_x(X_{\tau_A} = y)$.



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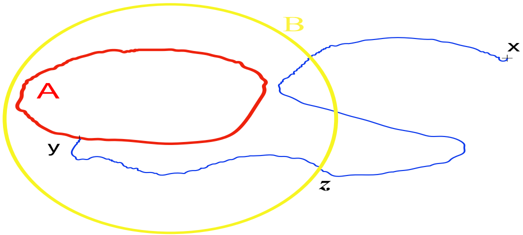

For all $x \in B^c$ and $y \in A$, we have :

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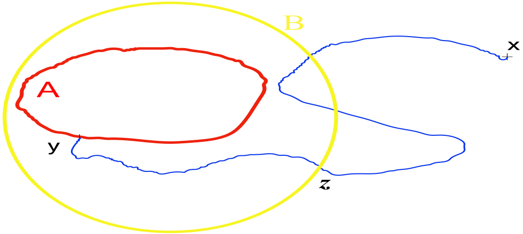

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- Then, by summing over y we get :

$$\overline{H}_A(x,y) = \frac{\sum_{z\in\partial B} G_{A^c}(x,z)H_{A\cup\partial B}(z,y)}{\sum_{z\in\partial B} G_{A^c}(x,z)P_z(\tau_A < \tau_{\partial B})}$$

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- So,
$$\min_{z \in \partial B} \frac{H_{A \cup \partial B}(z,y)}{P_z(\tau_A < \tau_{\partial B})} \leq \overline{H}_A(x,y) \leq \max_{z \in \partial B} \frac{H_{A \cup \partial B}(z,y)}{P_z(\tau_A < \tau_{\partial B})}.$$

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By reversibility and since $P_z(\tau_A < \tau_{\partial B}) = \sum_{\tilde{y} \in A} H_{A \cup \partial B}(z, \tilde{y})$,
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$$\min_{z\in\partial B} \frac{\pi(y)H_{A\cup\partial B}(y,z)}{\sum_{\tilde{y}\in A}\pi(\tilde{y})H_{A\cup\partial B}(\tilde{y},z)} \leq \overline{H}_A(x,y) \leq \max_{z\in\partial B} \frac{\pi(y)H_{A\cup\partial B}(y,z)}{\sum_{\tilde{y}\in A}\pi(\tilde{y})H_{A\cup\partial B}(\tilde{y},z)}$$

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$$\min_{z \in \partial B} \frac{\pi(y) H_{A \cup \partial B}(y, z)}{\sum_{\tilde{y} \in A} \pi(\tilde{y}) H_{A \cup \partial B}(\tilde{y}, z)} \leq \overline{H}_A(x, y) \leq \max_{z \in \partial B} \frac{\pi(y) H_{A \cup \partial B}(y, z)}{\sum_{\tilde{y} \in A} \pi(\tilde{y}) H_{A \cup \partial B}(\tilde{y}, z)}$$

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1st Step : bound of $\bar{H}_A$

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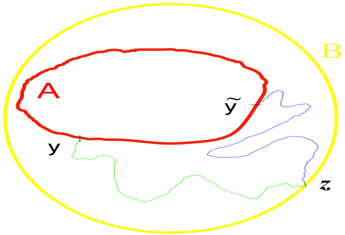
$$\frac{\pi(y) H_{A \cup \partial B}(y, z)}{\sum_{\tilde{y} \in A} \pi(\tilde{y}) H_{A \cup \partial B}(\tilde{y}, z)}.$$

$\Rightarrow$ Study of $H_{A \cup \partial B}(y, z)$ for $y \in A,\ z \in \partial B$ and B "big"...

1st Step : bound of $\bar{H}_A$

Compare $H_{A\cup\partial B}(y,z)$ and $H_{A\cup\partial B}(\tilde{y},z)$ for $y,\tilde{y}\in A$ and $z\in\partial B$



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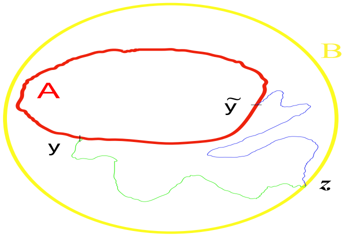

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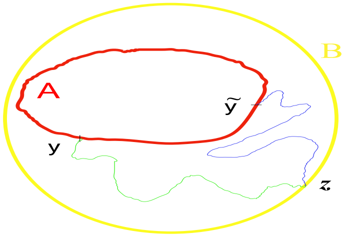

$\Rightarrow$ Harnack inequality...

2nd Step : Replace set A by a box

Let $x_0 \in \Gamma$ and $r_A > 0$ such that $A \subset B(x_0, r_A)$.

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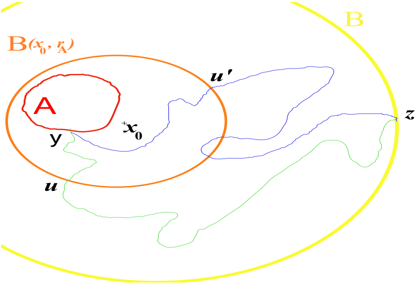

$$P_y(X_{\tau_{\partial B} \wedge \tau_A} = z) = \sum_{u \in \partial B(x_0, r_A)} P_y(X_{\tau_{\partial B(x_0, r_A)} \wedge \tau_A} = u) P_u(X_{\tau_{\partial B} \wedge \tau_A} = z)$$

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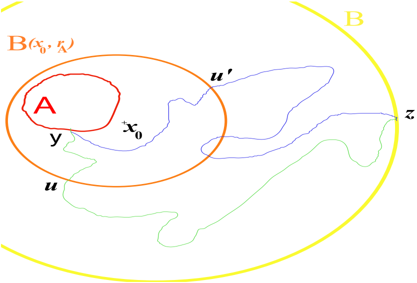

$\Rightarrow$ Study of $P_u(X_{\tau_{\partial B} \wedge \tau_A} = z)$ for $u \in \partial B(x_0, r_A)$.

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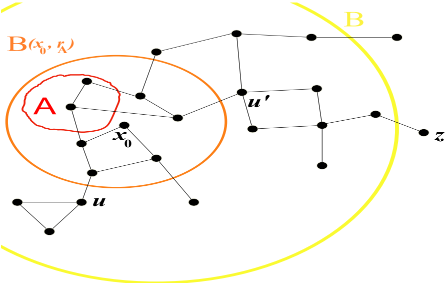

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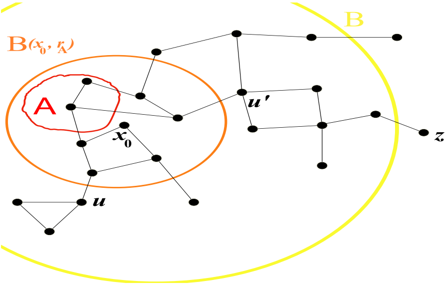

Pb : if some u are not connected to ∂B in $B - A$,

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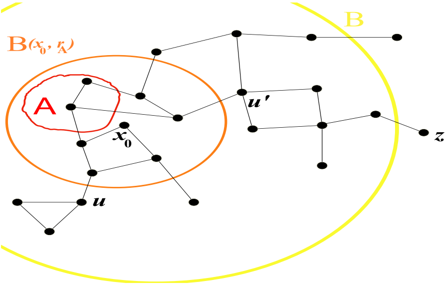


Pb : if some u are not connected to ∂B in $B - A$, no chance to compare $P_u(X_{\tau_{\partial B} \wedge \tau_A} = z)$ for all u.

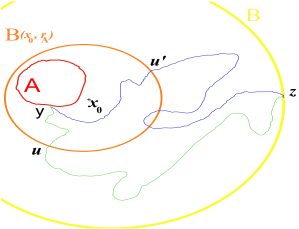
2nd Step : Replace set A by a box

A condition like

$$(*) \quad P_u(\tau_A > \tau_{\partial B}) > c > 0,$$

(with c independant of B...)

enables us to remove this case.



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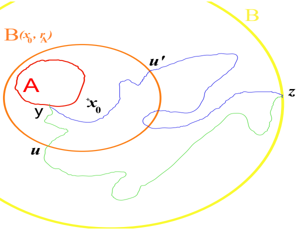
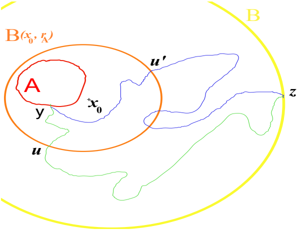


enables us to remove this case. And so, we have to study

$$\text{for all } u \in \partial B(x_0, r_A) \quad \mathbb{P}_u(X_{\tau_{\partial B} \wedge \tau_A} = z).$$

3rd Step : Replace exit time of an annulus by exit time of a box



3rd Step : Replace exit time of an annulus by exit time of a box

The condition

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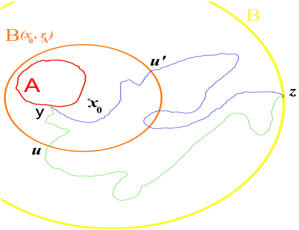
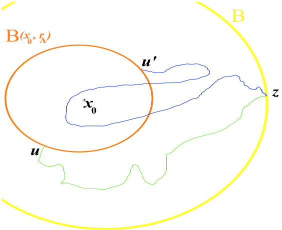


also implies, that we can study

$$P_u(X_{\tau_{\partial B}} = z), \quad \text{for } u \in \partial B(x_0, r_A).$$

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

Assume graph Γ satisfies classical Harnack inequality **H(K,M)**.



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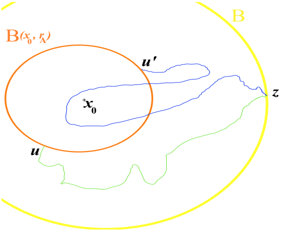

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Recall parameters meaning in Harnack inequality

Definition

We say that a weighted graph (Γ, a) *satisfies* **H(K,M)**, *if all* $x \in \Gamma$ *and* $R > 0$, *and for any non-negative harmonic function u on* $B(x, MR)$,

$$\max_{B(x,R)} u \leq K \min_{B(x,R)} u.$$

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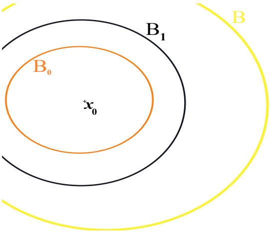

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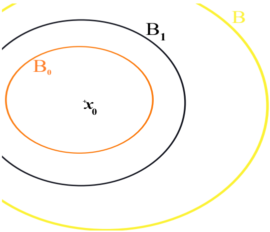

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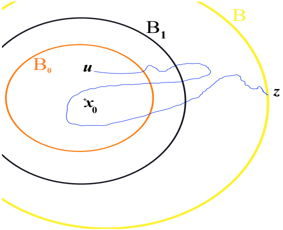

Let $f_B(u) = P_u(X_{\tau_{\partial B}} = z)$ defined on B_1.

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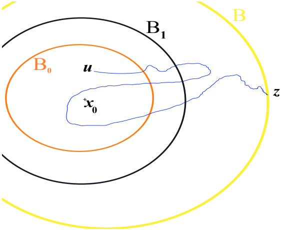

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- f is positive on B_1

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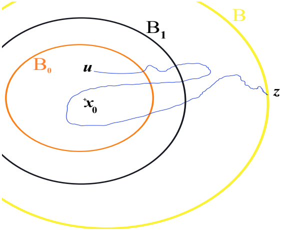

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so we can compare : $f_B(u)$ and $f_B(u')$ for $u, u' \in B_0$

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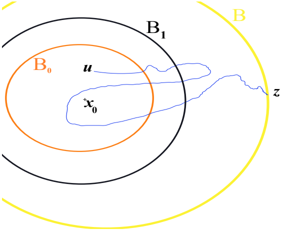

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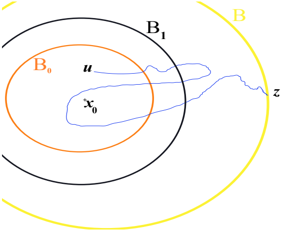

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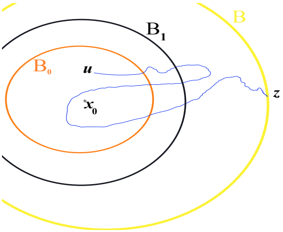

Let $f_B(u) = P_u(X_{\tau_{\partial B}} = z)$ defined on B_1.
Harnack inequality also gives us that :

$$osc_{B_0}(f_B) \leq \frac{K-1}{K+1} \, osc_{B_1}(f_B),$$

where $osc_E(f) = \max_E f - \min_E f$.

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

Let $k \geq 1$ and let

- $B_k = B(x_0, M^k r_A)$

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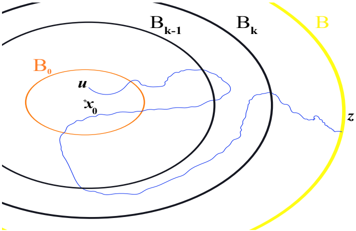

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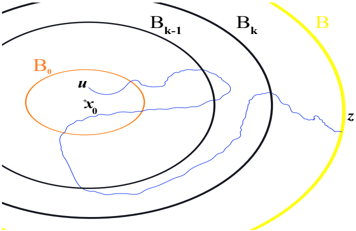

Similarly,

$$osc_{B_{k-1}}(f_B) \leq \frac{K-1}{K+1}\, osc_{B_k}(f_B).$$

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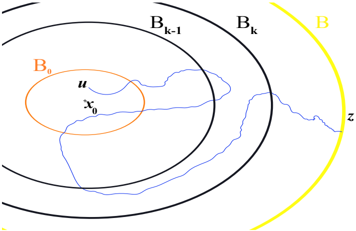

Similarly,

$$osc_{B_{k-1}}(f_B) \leq \frac{K-1}{K+1}\, osc_{B_k}(f_B).$$

And so,

$$osc_{B_0}(f_B) \leq \left(\frac{K-1}{K+1}\right)^k osc_{B_k}(f_B),$$

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

Finaly, for B big enough and for $u \in B(x_0, r_A)$, we get :

$$\begin{aligned} |f_B(u) - f_B(x_0)| &\leq (\frac{K-1}{K+1})^k osc_{B_k}(f_B) \\ &\leq (\frac{K-1}{K+1})^k \max_{B_k}(f_B) \\ &\leq (\frac{K-1}{K+1})^k \ K \ f_B(x_0) \end{aligned}$$

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&\leq (\frac{K-1}{K+1})^k \; K \, f_B(x_0)
\end{aligned}
$$

So,

$$f_B(u) = f_B(x_0)[1 + O((\frac{K-1}{K+1})^k)].$$

4th Step : Estimate of $P_u(X_{\tau_{\partial B}} = z)$

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$$f_B(u) = f_B(x_0)[1 + O((\frac{K-1}{K+1})^k)].$$

Taking for example $B = B_{k+1}$, this can be read :

$$P_u(X_{\tau_{\partial B_{k+1}}} = z) = H_{\partial B_{k+1}}(x_0, z)[1 + O(\left(\frac{K-1}{K+1}\right)^k)],$$

where the constant in $O(\cdot)$ depends only on K.

5th Step : Final estimate

With condition like

$$(*) \quad P_u(\tau_A > \tau_{\partial B}) > c > 0,$$

(with c independant of B),

5th Step : Final estimate

With condition like

$$(*) \quad P_u(\tau_A > \tau_{\partial B}) > c > 0,$$

(with c independant of B), we deduce (step 3) that :
for all $u \in B(x_0, r_A)$,

$$P_u(X_{\tau_{\partial B_{k+1}} \wedge \tau_A} = z) = H_{\partial B_{k+1}}(x_0, z) P_x(\tau_A > \tau_{\partial B_{k+1}})[1 + O(\left(\frac{K-1}{K+1}\right)^k)].$$

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And then by step 2, we finaly get, for all $y \in A$,

$$P_y(X_{\tau_{\partial B_{k+1}} \wedge \tau_A} = z) = H_{\partial B_{k+1}}(x_0, z)\left[1 + O(\left(\frac{K-1}{K+1}\right)^k)\right] P_y(\tau_A > \tau_{\partial B_{k+1}})$$

5th Step : Final estimate

This can be read, for all $y \in A$,

$$H_{A\cup\partial B_{k+1}}(y,z) = H_{\partial B_{k+1}}(x_0,z)[1 + O(\left(\frac{K-1}{K+1}\right)^k)]P_y(\tau_A > \tau_{\partial B_{k+1}}).$$

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And then,

$$\frac{\pi(y)H_{A\cup\partial B_{k+1}}(y,z)}{\sum_{\tilde{y}\in A}\pi(\tilde{y})P_{\tilde{y}}(X_{\tau_{\partial B_{k+1}}\wedge\tau_A} = z)} = \frac{\pi(y)P_y(\tau_A > \tau_{\partial B_{k+1}})}{\sum_{\tilde{y}\in A}\pi(\tilde{y})P_{\tilde{y}}(\tau_A > \tau_{\partial B_{k+1}})} \times [1 + O(\left(\frac{K-1}{K+1}\right)^k)]$$

5th Step : Final estimate

But, remember that,

$$\min_{z\in\partial B} \frac{\pi(y)H_{A\cup\partial B}(y,z)}{\sum_{\tilde{y}\in A}\pi(\tilde{y})H_{A\cup\partial B}(\tilde{y},z)} \leq \overline{H}_A(x,y) \leq \max_{z\in\partial B} \frac{\pi(y)H_{A\cup\partial B}(y,z)}{\sum_{\tilde{y}\in A}\pi(\tilde{y})H_{A\cup\partial B}(\tilde{y},z)}$$

5th Step : Final estimate

So, from

$$\frac{\pi(y)H_{A\cup\partial B_{k+1}}(y,z)}{\sum_{\tilde{y}\in A}\pi(\tilde{y})P_{\tilde{y}}(X_{\tau_{\partial B_{k+1}}\wedge\tau_A} = z)} = \frac{\pi(y)P_y(\tau_A > \tau_{\partial B_{k+1}})}{\sum_{\tilde{y}\in A}\pi(\tilde{y})P_{\tilde{y}}(\tau_A > \tau_{\partial B_{k+1}})} \times [1 + O((\frac{K-1}{K+1})^k)],$$

we obtain that $\lim_{v\to+\infty}\overline{H}_A(v,y)$ exists and

$$\lim_{v\to+\infty}\overline{H}_A(v,y) = \frac{\pi(y)P_y(\tau_A > +\infty)}{\sum_{\tilde{y}\in A}\pi(\tilde{y})P_{\tilde{y}}(\tau_A > +\infty)}.$$

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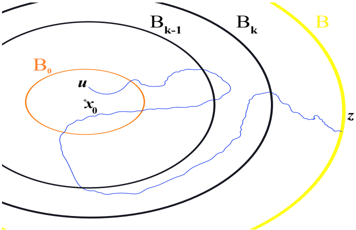

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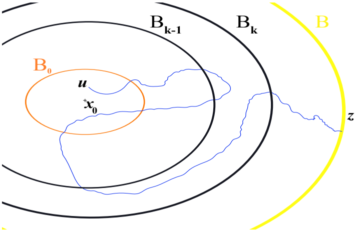

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Weak Harnack inequality

Definition

We say that a weighted graph (Γ, a) *satisfies* **wH(*K*)**, *the weak Harnack inequality, if there is a constant* $1 \leq K < \infty$ *such that for all* $x \in \Gamma$ *and for all* $R > 0$ *there is* $M_{x,R} \geq 2$ *such that for all* $M > M_{x,R}$ *and for any non-negative harmonic function* u *on* $B(x, MR)$,

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Assuming Weak Harnack inequality, we replace $B(x_0, M^k r_A)$ by $B(x_0, M_k M_{k-1} ... M_1 r_A)$ such that

$$M_i = M(x_0, M_{i-1} M_{i-2} ... M_1 r_A).$$

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Estimate **(**GE_γ**)** gives us **wH(**K**)** and the wanted condition

$$(*) \quad P_u(\tau_A > \tau_{\partial B}) > c > 0.$$

Lemma

Let (Γ, a) be a weighted graph which verifies ($\mathbf{GE}_\gamma$). Set $\theta = (2\frac{C_s}{C_i})^{\frac{1}{\gamma}}$. Then for all $x_0 \in \Gamma$, $M > \theta$, $R \geq R_{x_0}$ and $x \in \partial B(x_0, \theta R)$, we have : for all $A \subset B(x_0, R)$,

$$P_x(\tau_A > \tau_{\partial B(x_0, MR)}) > \frac{C_i}{2C_s}. \qquad (1)$$

4 Overview of the proof in the recurrent case

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Fact : for $x \in A^c$,

$$\lim_n (\ln n) P_x(\widetilde{\sigma}_n < \tau_A) \text{ exists}$$

5 Open questions

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- Question 2 : Is there a connection between existence of harmonic measure and invariance principle (in random environments) ?

Thanks for your attention ... !