EXISTENCE OF THE HARMONIC MEASURE FOR RANDOM WALKS ON GRAPHS AND IN RANDOM ENVIRONMENTS

DANIEL BOIVIN AND CLÉMENT RAU

ABSTRACT. We give a sufficient condition for the existence of the harmonic measure from infinity of transient random walks on weighted graphs. In particular, this condition is verified by the random conductance model on $\mathbb{Z}^d$, $d \geq 3$, when the conductances are i.i.d. and the bonds with positive conductance percolate. The harmonic measure from infinity also exists for random walks on supercritical clusters of $\mathbb{Z}^2$. This is proved using results of Barlow (2004) and Barlow and Hambly (2009).

Keywords: Harmonic measure; supercritical percolation clusters; Harnack inequality; Green kernel; random conductance model.

Subject Classification: 60J05, 60K35, 60K37

1. Introduction and results

In [23], Hunt gave a probabilistic formulation of the harmonic measure of a closed set in euclidean space as the hitting distribution of the set by a d-dimensional Brownian motion started at infinity. A recent account can be found in [30, section 3.4] for instance.

In this paper, we investigate infinite weighted graphs for which it is possible to define the harmonic measure of a finite set as the hitting distribution of the set by the random walk on the graph starting at infinity. The existence of the harmonic measure for random walks goes back to Spitzer [35]. It also appears in Lawler in [27, chapter 2] for the simple symmetric random walk on $\mathbb{Z}^d$ and it is extended to a wider class of random walks in the recent book by Lawler and Limic [26, section 6.5].

From these results, one might expect that the existence of the harmonic measure for a Markov chain on $\mathbb{Z}^d$, $d \geq 2$, relies on its Green function asymptotics. The goal of this paper is to show that actually, the existence of the harmonic measure is a fairly robust result in the sense that it exists for a random walk on a weighted graph as soon as the Green function satisfies weak estimates. These imply a weak form of a Harnack inequality. In particular, it is verified by a large family of fractal-like graphs and by random conductance models on $\mathbb{Z}^d$, $d \geq 3$, given by a sequence of i.i.d. conductances as soon as there is percolation of the positive conductances. This is done using recent estimates of Andres, Barlow, Deuschel and Hambly [3].

In the recurrent case, although we do not give a general sufficient condition, we show the existence of the harmonic measure for the random walk on the supercritical cluster of $\mathbb{Z}^2$. To do so, we construct the Green kernel of the random walk by using the parabolic Harnack inequality of Barlow and Hambly [8]. The Gaussian estimates of [6] and [8] as well as an argument from [15] then provide the needed estimates.

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The results of [3] for the random conductance model are part of a long series of works which go back to homogenization of divergence form elliptic operators with random coefficients and to the investigation of the properties of the supercritical percolation cluster.

Some highlights of the properties of the random walk on the supercritical percolation cluster of $\mathbb{Z}^d$ is the proof of the Liouville property for bounded harmonic functions (see Kaimanovich [24] and [11]) and the proof of the transience of the walk when $d \geq 3$ by Grimmett, Kesten and Zhang [22].

In [6], Barlow proved upper and lower gaussian estimates for the probability transitions of a random walk on the supercritical percolation cluster. These are then used to prove a Harnack inequality [6, Theorem 3]. The Liouville property for positive harmonic functions on the percolation cluster follows as well as an estimate of the mean-square displacement of the walk.

Barlow's upper gaussian estimates were also used to prove the invariance principle for the random walk on supercritical percolation clusters by [34], [29], [12]. An extensive survey of the random conductance model was recently completed by Biskup [13].

Here we show the existence of the harmonic measure for random walks on the supercritical percolation cluster. In the transient case, it turns out that its existence follows from Green function estimates which apply widely to random walks on graphs.

In the case of the two-dimensional percolation cluster, we need both the elliptic and the parabolic Harnack inequalities of [6] and [8].

Whenever the harmonic measure from infinity exists, one can study external diffusion-limited aggregates. Their growth is determined by the harmonic measure which can also be interpreted as the distribution of an electric field on the surface of a grounded conductor with fixed charge of unity. Recent simulations by physicists of the harmonic measure in $\mathbb{Z}^d$ can be found in [1] and of percolation and Ising clusters in [2]. Analytic predictions for the harmonic measure of two dimensional clusters are given by Duplantier in [19] and [20]. See also the survey paper [5].

In contrast, for the internal diffusion-limited aggregates of random walks on percolation clusters, the limiting shape is described in [33] and [18].

1.1. **Reversible random walks.** A weighted graph (Γ, a) is given by a countably infinite set Γ and a symmetric function

$$a : \Gamma \times \Gamma \to [0; \infty[$$

which verifies $a(x, y) = a(y, x)$ for all $x, y \in \Gamma$ and

$$\pi(x) := \sum_{y \in \Gamma} a(x, y) > 0 \quad \text{for all} \quad x \in \Gamma.$$

The weight $a(x, y)$ is also called the conductance of the edge connecting x and y as the weighted graph can be interpreted as an electrical or thermic network.

Given a weighted graph (Γ, a), we will write $x \sim y$ if $a(x, y) > 0$. We will always assume that $(\Gamma, \sim)$ is an infinite, locally finite countable graph without multiple edges. A path of length n from x to y is a sequence $x_0, x_1, \ldots, x_n$ in Γ such that $x_0 = x$, $x_n = y$ and $x_{i-1} \sim x_i$ for all $1 \leq i \leq n$. The weighted graph (Γ, a) is said to be connected if $(\Gamma, \sim)$ is a connected graph, that is, for all $x, y \in \Gamma$ there is a path of finite length from x to y. The graph distance between two vertices $x, y \in \Gamma$ will be denoted by $D(x, y)$. It is the minimal length of a path from x to y in the graph $(\Gamma, \sim)$. The ball centered at $x \in \Gamma$ of radius R will be denoted by $B(x, R) := \{y \in \Gamma;\ D(x, y) < R\}$.

The random walk on the weighted graph (Γ, a) is the Markov chain on Γ with transition probabilities given by

$$p(x,y) := \frac{a(x,y)}{\pi(x)}, \quad x, y \in \Gamma. \tag{1.1}$$

We denote by P_x the law of the random walk starting at the vertex $x \in \Gamma$. The corresponding expectation is denoted by E_x. The random walk admits reversible measures which are proportional to the measure $\pi(\cdot)$.

For $A \subset \Gamma$, we have the following definitions

$\partial A := \{y \in \Gamma;\ y \notin A$ and there is $x \in A$ with $x \sim y\}$ and $\overline{A} := \partial A \cup A$,

$\tau_A := \inf\{k \geq 1;\ X_k \in A\}$ and $\overline{\tau}_A := \inf\{k \geq 0;\ X_k \in A\}$

with the convention that $\inf \emptyset = \infty$,

$D(x,A) := \inf\{D(x,y); y \in A\}$,

for a bounded function u on A, $\operatorname{osc}_A u := \sup_{x,y\in A} |u(x) - u(y)|$,

and for $u : \overline{A} \to \mathbb{R}$, $\quad Pu(x) := \sum_{y\sim x} p(x,y)u(y), \quad x \in A.$

A function $u : \overline{A} \to \mathbb{R}$ is *harmonic* in A if $Pu = u$ on A.

The *Green function* of the random walk is defined by

$$G(x,y) := \sum_{j=0}^{\infty} p(x,y,j), \quad x, y \in \Gamma \tag{1.2}$$

where $p(x,y,j) := P_x(X_j = y)$ are the transition probabilities of the walk. Note that $G(\cdot, y)$ is harmonic in $\Gamma \setminus \{y\}$.

For irreducible Markov chains, if $G(x,y) < \infty$ for some $x, y \in \Gamma$ then $G(x,y) < \infty$ for all $x, y \in \Gamma$. The random walk is *recurrent* if $G(x,y) = \infty$ for some $x, y \in \Gamma$ otherwise we say that the walk is *transient*.

The minimum of a and b and the maximum of a and b are respectively denoted by $a \wedge b$ and by $a \vee b$.

1.2. **Results on the existence of the harmonic measure.** Let $(X_j; j \in \mathbb{N})$ be a random walk on a connected weighted graph (Γ, a).

The *hitting distribution* of a set A by the random walk starting at $x \in \Gamma$ is given by

$$\overline{H}_A(x,y) := P_x(X_{\tau_A} = y | \tau_A < +\infty), \quad y \in A,$$

or, whenever the graph is recurrent, by

$$H_A(x,y) := P_x(X_{\tau_A} = y), \quad y \in A.$$

The *harmonic measure* on a finite subset A of Γ is the hitting distribution from infinity, if it exists,

$$\mathbf{H}_A(y) := \lim_{D(x,A)\to\infty} \overline{H}_A(x,y), \quad y \in A. \tag{1.3}$$

Our goal is to prove the existence of the harmonic measure for all finite subsets of various weighted graphs. The proof of the existence of the harmonic measure given in [26, section 6.5] for random walks on $\mathbb{Z}^d$, relies on a Harnack inequality and on Green function estimates.

For transient graphs, we show in Theorem I that a weak form of the Green function estimates is a sufficient condition for the existence of the harmonic measure.

As it happens for Brownian motion and for simple random walks (see for instance [30], [27]), the harmonic measure can be expressed in terms of capacities.

Let $A \subset B$ be finite subsets of Γ. The *capacity* of A with respect to B is defined by

$$\mathrm{Cap}_B(A) := \sum_{x \in A} \pi(x) P_x(\tau_{B^c} < \tau_A). \tag{1.4}$$

The *escape probability* of a set A is defined by $\mathrm{Es}_A(x) := P_x(\tau_A = \infty)$ and the capacity of a finite subset $A \subset \Gamma$ is defined by

$$\mathrm{Cap}(A) := \sum_{x \in A} \pi(x) \mathrm{Es}_A(x).$$

Our first result is the existence of the harmonic measure for transient graphs with a Green function which verifies the following weak estimates.

Definition 1.1. *We say that a weighted graph* (Γ, a) *satisfies the Green function estimates* $(\mathbf{GE}_\gamma)$ *for some* $\gamma > 0$ *if there are constants* $0 < C_i \leq C_s < \infty$ *and if for all* $z \in \Gamma$*, there exists* $R_z < \infty$ *such that for all* $x, y \in \Gamma$ *with* $D(x,y) \geq R_x \wedge R_y$ *we have*

$$\frac{C_i}{D(x,y)^\gamma} \leq G(x,y) \leq \frac{C_s}{D(x,y)^\gamma}. \tag{$\mathbf{GE}_\gamma$}$$

This condition is a weak version of [36, Definition 1] where γ is called a Greenian index. It is used by Telcs [36] to give an upper bound for the probability transitions of a Markov chain in terms of the growth rate of the volume and of the Greenian index.

Note that a graph which verifies $(\mathbf{GE}_\gamma)$ for some $\gamma > 0$ is connected and transient. We will show that $(\mathbf{GE}_\gamma)$ also implies the existence of the harmonic measure.

Theorem I. *Let* (Γ, a) *be a weighted graph which verifies* $(\mathbf{GE}_\gamma)$ *for some* $\gamma > 0$.

Then for any finite subset $A \subset \Gamma$ *the harmonic measure on* A *exists. That is, for all* $y \in A$*, the limit (1.3) exists.*

Moreover, we have:

$$\lim_{D(x,A) \to \infty} \overline{H}_A(x,y) = \lim_{m \to +\infty} H_A^m(y),$$

where, for m *large enough,*

$$H_A^m(y) = \frac{\pi(y) P_y(\tau_A > \tau_{\partial B(x_0,m)})}{\mathrm{Cap}_m(A)}$$

where $\mathrm{Cap}_m(A)$ *is the capacity of* A *with respect to* $B(x_0, m)$ *for some* $x_0 \in \Gamma$*. The limit does not depend on the choice of* x_0*.*

In the following corollaries, we describe some weighted graphs where the harmonic measure from infinity exists.

A weighted graph (Γ, a) is said to be *uniformly elliptic* if there is a constant $c \geq 1$ such that for all edges e,

$$c^{-1} \leq a(e) \leq c. \tag{1.5}$$

Consider the lattice $\mathbb{Z}^d$, $d \geq 2$, where $x \sim y$ if $|x-y|_1 = 1$ where $|\cdot|_1$ is the ℓ_1-distance.

Corollary 1.2. *Let $(\mathbb{Z}^d, a)$, $d \geq 3$, be a uniformly elliptic graph.*

Then for all finite subsets A of $\mathbb{Z}^d$ and for all $y \in A$, the limit (1.3) exists.

Moreover, we have:

$$\lim_{|x| \to +\infty} \overline{H}_A(x,y) = \lim_{m \to +\infty} H_A^m(y),$$

where $H_A^m(y) = \dfrac{\pi(y) P_y(\tau_A > \tau_{\partial B(0,m)})}{\mathrm{Cap}_m(A)}$.

Indeed, by [16, Proposition 4.2] the Green function of a uniformly elliptic graph $(\mathbb{Z}^d, a)$, $d \geq 3$, verifies the estimates $(\mathbf{GE}_\gamma)$ with $\gamma = d-2$. The existence of the harmonic measure then follows from Theorem I.

The harmonic measure also exists for a large class of fractal like graphs with some regularity properties. Various examples are given in [9] and the references therein.

A weighted graph (Γ, a) verifies the condition (p_0) if there is a constant $c > 0$ such that for all vertices $x \sim y$,

$$p(x,y) > c. \tag{p_0}$$

The volume of a ball $B(x,R)$ is defined by $V(x,R) := \sum_{y \in B(x,R)} \pi(y)$.

A weighted graph (Γ, a) has *polynomial volume growth with exponent* $\alpha > 0$ if there is a constant $c > 1$ such that for all $x \in \Gamma$ and for all $R \geq 1$,

$$c^{-1} R^\alpha \leq V(x,R) \leq c R^\alpha. \tag{V_α}$$

Note that the condition (V_α) implies the volume doubling condition of [21] for any $\alpha > 0$.

A weighted graph (Γ, a) satisfies the *resistance estimate with exponent* $\beta > 0$ if there are constants $c > 1$ and $M > 1$ such that for all $x \in \Gamma$ and for all $R \geq 1$,

$$c^{-1} \frac{V(x,R)}{R^\beta} \leq \mathrm{Cap}_{B(x,MR)}(B(x,R)) \leq c \frac{V(x,R)}{R^\beta}. \tag{RE_β}$$

A weighted graph (Γ, a) satisfies $\mathbf{H}(K)$, *the Harnack inequality with positive constant K and shrinking parameter $M > 1$*, if for all $x \in \Gamma$ and $R \geq 1$, and for any non-negative harmonic function u on $B(x, MR)$,

$$\max_{B(x,R)} u \leq K \min_{B(x,R)} u.$$

Grigor'yan and Telcs [21, Theorem 3.1] proved that if a weighted graph verifies (p_0), (V_α), (RE_β) for $\alpha > 0$ and $\beta \geq 2$ and the Harnack inequality $\mathbf{H}(K)$ then it verifies sub-gaussian estimates. These imply that if $\alpha > \beta \geq 2$ then the walk is transient and the estimates $(\mathbf{GE}_\gamma)$ hold with $\gamma = \alpha - \beta$. Hence we obtain the following corollary to theorem I.

Corollary 1.3. *Let (Γ, a) be a weighted graph which verifies (p_0), (V_α), (RE_β) for $\alpha > \beta \geq 2$ and the Harnack inequality $\mathbf{H}(K)$. Then for all finite subsets $A \subset \Gamma$ and $y \in A$ the limit (1.3) exists.*

The harmonic measure from infinity also exists for random walks in random environment and in particular for the random walk on the supercritical percolation cluster. Before stating this result, we give a brief description of the percolation model. See [25] for more details.

Denote by $\mathbb{E}^d$ the set of edges of the lattice $\mathbb{Z}^d$, $d \geq 2$, where $x \sim y$ if $|x-y|_1 = 1$.

Assume that $(a(e); e \in \mathbb{E}^d)$ are i.i.d. non-negative random variables on a probability space $(\Omega, \mathbb{P})$. Call a bond e open if $a(e) > 0$ and closed if $a(e) = 0$. Let $p = \mathbb{P}(a(e) > 0)$. By percolation theory, there exists a critical value $p_c = p_c(\mathbb{Z}^d) \in]0;1[$ such that for $p < p_c$, $\mathbb{P}$ almost surely, all open clusters of ω are finite and for $p > p_c$, $\mathbb{P}$ almost surely, there is a unique infinite cluster of open edges which is called the supercritical cluster. It will be denoted by $\mathcal{C}_\infty = \mathcal{C}_\infty(\omega)$. The edges of this graph are the open edges of the cluster and the endpoints of these edges are the vertices of the graph.

For $x, y \in \mathcal{C}_\infty(\omega)$, we will write $x \sim y$ if the edge with endpoints x and y is open. The transition probabilities of the random walk on $\mathcal{C}_\infty(\omega)$ are given by (1.1). The law of the paths starting at $x \in \mathcal{C}_\infty(\omega)$ will be denoted by P_x^ω. The random walk on the supercritical percolation cluster corresponds to the case of Bernoulli random variables. In this case, we will write $\mathbb{P}_p$ instead of $\mathbb{P}$.

$D_\omega(x,y)$ will denote the graph distance between x and y in the graph $\mathcal{C}_\infty(\omega)$ and the ball centered at $x \in \mathcal{C}_\infty(\omega)$ of radius R will be denoted by $B_\omega(x,R) = \{y \in \mathcal{C}_\infty(\omega);\ D_\omega(x,y) < R\}$.

The existence of the harmonic measure for $\mathbb{Z}^d$, $d \geq 3$, with i.i.d. conductances, is given in corollary 1.4 below. It follows from the Green function estimates of [3, Theorem 1.2 (a)]. A weaker condition which might hold even if the conductances are not i.i.d. is given in [7, Theorem 6.1].

Corollary 1.4. *Let $(\mathbb{Z}^d, a)$, $d \geq 3$, be a weighted graph where the weights $(a(e); e \in \mathbb{E}^d)$ are i.i.d. non-negative random variables on a probability space $(\Omega, \mathbb{P})$ which verify*

$$\mathbb{P}(a(e) > 0) > p_c(\mathbb{Z}^d).$$

Then there exist positive constants C_i, C_s, which depend on $\mathbb{P}$ and d, and $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$ such that for each $\omega \in \Omega_1$, $(\mathbf{GE}_\gamma)$ holds in $\mathcal{C}_\infty(\omega)$ with the constants C_i and C_s and with $\gamma = d-2$.

For any finite subset A of $\mathcal{C}_\infty$ and for all $y \in A$, the limit (1.3) exists.

Moreover, we have:

$$\lim_{|x| \to +\infty, x \in \mathcal{C}_\infty} \overline{H}_A(x,y) = \lim_{m \to +\infty} H_A^m(y),$$

where $H_A^m(y) = \frac{\pi(y) P_y^\omega(\tau_A > \tau_{\partial B_\omega(x_0,m)})}{\mathrm{Cap}_m(A)}$ for some $x_0 \in \mathcal{C}_\infty$ and for m large enough.

In [3], both the constant speed random walk and the variable speed random walk are considered. From the expression of their generators one immediately sees that they have the same harmonic functions as the discrete time random walk considered here. Moreover, since they are a time change of each other, the Green function is the same. Hence, by [3, Theorem 1.2 a] the Green function of the random walk on $\mathcal{C}_\infty(\omega) \subset \mathbb{Z}^d$, $d \geq 3$, verifies the estimates $(\mathbf{GE}_\gamma)$ with $\gamma = d-2$. The existence of the harmonic measure then follows from Theorem I.

The harmonic mesure from infinity also exists for some recurrent graphs. We will show its existence for uniformly elliptic graphs $(\mathbb{Z}^2, a)$ and for two-dimensional supercritical percolation clusters.

Theorem II. *Let $(\mathbb{Z}^2, a)$ be a weighted graph where the weights $(a(e); e \in \mathbb{E}^2)$ are i.i.d. random variables on a probability space $(\Omega, \mathbb{P}_p)$ which verify*

$$p = \mathbb{P}_p(a(e) = 1) = 1 - \mathbb{P}_p(a(e) = 0) > p_c(\mathbb{Z}^2).$$

Then $\mathbb{P}_p$ almost surely, for any finite subset A of $\mathcal{C}_\infty(\omega)$ and for all $y \in A$, the limit (1.3) exists.

An expression for the value of the limit (1.3) is given in equation (4.51).

Theorem III. *Let $(\mathbb{Z}^2, a)$ be a uniformly elliptic weighted graph. Then for all finite subsets $A \subset \mathbb{Z}^2$ and for all $y \in A$, the limit (1.3) exists.*

Various forms of Harnack inequality that will be used in the context of transient or recurrent graphs are gathered in section 2. The proof of theorem I is given in section 3 while Theorem II and III are proved in section 4. Section 5 contains the proof of the annulus Harnack inequality that is used in the proof of Theorem II.

Note that on a bipartite graph with two infinite components, there are finite sets for which the harmonic mesure from infinity does not exist. In the last section, we construct a weighted graph which is not "finitely-partite" and where there is a finite set A for which the harmonic mesure from infinity does not exist.

It would be interesting to investigate the links between the Poisson boundary of a graph and the existence of the harmonic measures. In particular, the triviality of the Poisson boundary does not imply the existence of the harmonic measure as is shown by the lamplighter group $\mathbb{Z}^2 \wr \mathbb{Z}/2\mathbb{Z}$. See [32] and the references therein.

2. Harnack inequalities

The condition $\mathbf{H}(K)$ is the usual form of the Harnack inequality on a graph. In our context, we will work with the weaker form of the Harnack inequality given below.

Definition 2.1. *We say that a weighted graph (Γ, a) satisfies $\mathbf{wH}(K)$, the weak Harnack inequality, with the positive constant K if for all $x \in \Gamma$ and for all $R \geq 1$ there is $M_{x,R} \geq 2$ such that for any non-negative harmonic function u on $B(x, M_{x,R}R)$,*

$$\max_{B(x,R)} u \leq K \min_{B(x,R)} u.$$

Whenever the Green function estimates $(\mathbf{GE}_\gamma)$ hold for some $\gamma > 0$, the weak Harnack inequality $\mathbf{wH}(K)$ is verified. It will be essential to prove Theorem I.

Proposition 2.2. *Let (Γ, a) be a weighted graph which verifies $(\mathbf{GE}_\gamma)$ for some $\gamma > 0$. Then the graph is connected, transient and $\mathbf{wH}(K)$ holds with $K = 10\dfrac{C_s}{C_i}$.*

The proof is given in section 3. By proposition 2.2 and by corollary 1.4, the random walk on the supercritical percolation cluster of $\mathbb{Z}^d$, $d \geq 3$, verifies $\mathbf{wH}(K)$. Barlow [6, Theorem 3] showed that the supercritical percolation cluster verifies another variant of Harnack inequality. Given below is a Harnack inequality under the form that will be most useful to us. It is an immediate consequence of Theorem 5.11, proposition 6.11 and of (0.5) of Barlow's work [6].

Harnack Inequality for the percolation cluster [6]. *Let $d \geq 2$ and let $p > p_c(\mathbb{Z}^d)$. There exists $c_1 = c_1(p,d)$ and $\Omega_1 \subset \Omega$ with $\mathbb{P}_p(\Omega_1) = 1$, and $R_0(x,\omega)$ such that $3 \leq R_0(x,\omega) < \infty$ for each $\omega \in \Omega_1$, $x \in \mathcal{C}_\infty(\omega)$.*

If $R \geq R_0(x,\omega)$ and if $D(x,z) \leq \frac{1}{3} R \ln R$ and if $u : \overline{B(z,R)} \to \mathbb{R}$ is non-negative and harmonic in $B(z,R)$, then

$$\max_{B(z,R/2)} u \leq c_1 \min_{B(z,R/2)} u. \tag{2.1}$$

Moreover, there are positive constants c_2, c_3 and ε which depend on p and d such that the tail of $R_0(x,\omega)$ satisfies

$$\mathbb{P}_p(x \in \mathcal{C}_\infty, R_0(x,\cdot) \geq n) \leq c_2 \exp(-c_3 n^\varepsilon). \tag{2.2}$$

In the proof of Theorem I, we will need a regularity property of harmonic functions which is a consequence of the weak Harnack inequality.

Lemma 2.3. *Let (Γ, a) be a weighted graph which verifies $\mathbf{wH}(K)$ with shrinking parameters $(M_{x,R}; x \in \Gamma, R \geq 1)$ where $M_{x,R} \geq 2$ for all $x \in \Gamma$ and $R \geq 1$.*

Then for all $x \in \Gamma$, $R \geq 1$, $M > M_{x,R}$ and for any harmonic function u on $B(x, MR)$,

$$\operatorname*{osc}_{B(x,R)} u \leq \Big(\frac{K-1}{K+1}\Big) \operatorname*{osc}_{B(x,MR)} u. \tag{2.3}$$

Proof. Let $x \in \Gamma$, $R \geq 1$ and $M > M_{x_0,R}$. Let u be a harmonic function on $B(x, MR)$.

Set $v = u - \min_{B(x,MR)} u$.

Then by $\mathbf{wH}(K)$, since v is non-negative and harmonic in $B(x, MR)$,

$$\max_{B(x,R)} v \leq K \min_{B(x,R)} v.$$

Hence

$$\max_{B(x,R)} u - \min_{B(x,MR)} u \leq K\big(\min_{B(x,R)} u - \min_{B(x,MR)} u \big). \tag{2.4}$$

Set $\tilde{v} = (\max_{B(x,MR)} u) - u$ and proceed similarly. By $\mathbf{wH}(K)$, since $\tilde{v}$ is non-negative and harmonic in $B(x, MR)$,

$$\max_{B(x,R)} \tilde{v} \leq K \min_{B(x,R)} \tilde{v}.$$

Hence

$$\max_{B(x,MR)} u - \min_{B(x,R)} u \leq K\big(\max_{B(x,MR)} u - \max_{B(x,R)} u \big). \tag{2.5}$$

Adding (2.4) and (2.5), we obtain,

$$\operatorname*{osc}_{B(x,R)} u + \operatorname*{osc}_{B(x,MR)} u \leq K\big(\operatorname*{osc}_{B(x,MR)} u - \operatorname*{osc}_{B(x,R)} u \big).$$

∎

Since the Harnack inequality for the supercritical cluster (2.1) holds with the shrinking parameter $M = 2$, we can proceed further as in [31, p.109] to obtain the following regularity result.

Proposition 2.4. *Let $d \geq 2$ and let $p > p_c(\mathbb{Z}^d)$. Let Ω_1 and $R_0(x,\omega)$ be given by the Harnack inequality for the supercritical cluster (2.1). Then there exist positive constants ν and c such that for each $\omega \in \Omega_1$, $x_0 \in \mathcal{C}_\infty(\omega)$ if $R \geq R_0(x_0,\omega)$ and u is a non-negative harmonic function on $B_\omega(x_0,R)$ then, for all $x, y \in B_\omega(x_0,R/2)$,*

$$|u(x) - u(y)| \leq c\ \left(\frac{D(x,y)}{R}\right)^\nu \max_{B(x_0,R)} u.$$

We will also need a Harnack inequality in the annulus of the two-dimensional supercritical percolation cluster. To obtain this inequality we will use the tail estimates (2.2) of [6], a percolation result due to Kesten [25] and the following estimates of Antal and Pisztora [4, Theorem 1.1 and Corollary 1.3].

For $d \geq 2$ and $p > p_c(\mathbb{Z}^d)$, there is a constant $\mu = \mu(p,d) \geq 1$ such that

$$\limsup_{|x|_1 \to \infty} \frac{1}{|x|_1} \ln \mathbb{P}_p[x_0, x \in \mathcal{C}_\infty, D(x_0,x) > \mu |x|_1] < 0 \tag{2.6}$$

and, $\mathbb{P}_p$ almost surely, for $x_0 \in \mathcal{C}_\infty$ and for all $x \in \mathcal{C}_\infty$ such that $D(x_0,x)$ is sufficiently large

$$D(x_0,x) \leq \mu |x - x_0|_1. \tag{2.7}$$

Proposition 2.5. *Let $p > p_c(\mathbb{Z}^2)$. There is a positive constant such that $\mathbb{P}_p$-a.s., for all $x_0 \in \mathcal{C}_\infty$ and $r > 0$, if m is large enough,*
then for any non-negative function u harmonic in $B(x_0, 3\mu m) \setminus B(x_0,r)$,

$$\max_{x; D(x_0,x)=m} u(x) \leq C \min_{x; D(x_0,x)=m} u(x)$$

where μ is the constant that appears in (2.7).

The proof of this Harnack inequality is postponed to section 5.

3. Proofs for transient graphs

In this section, we prove proposition 2.2 and Theorem I.

To prove proposition 2.2, we use lemma 3.1 below which shows how to obtain a Harnack inequality from a Harnack inequality for the Green function in an annulus. This idea appeared in [14] and was used in the context of random walks on graphs by Telcs [37, p. 37]. In lemma 3.1 below, we state it slightly differently and we provide a different proof.

Let (Γ, a) be a weighted graph. The Green function of the random walk in $B \subset \Gamma$ is defined by

$$G_B(x,y) := \sum_{j=0}^{\infty} p_B(x,y,j), \quad x, y \in B$$

where $p_B(x,y,j) := P_x(X_j = y, j < \overline{\tau}_{B^c})$ are the transition probabilities of the walk with Dirichlet boundary conditions.

The Green function with Dirichlet boundary condition can be expressed in terms of the Green function of the graph. We recall this property that will be useful in the proof of proposition 2.2.

For a finite subset B of Γ,

$$G_B(x,z) = G(x,z) - \sum_{y\in\partial B} H_{\partial B}(x,y)G(y,z), \quad x,z \in B. \tag{3.1}$$

Lemma 3.1. *Let $B_0 \subset B_1 \subset B_2$ be finite subsets of Γ such that $\overline{B}_i \subset B_{i+1}$, $i = 0,1$.*

Let u be a non-negative function on $\overline{B}_2$ which is harmonic in B_2.

Then

$$\max_{B_0} u \le K \min_{B_0} u \tag{3.2}$$

where

$$K := \max_{x\in B_0}\max_{y\in B_0}\max_{z\in\partial^{\mathrm{int}}B_1} \frac{G_{B_2}(x,z)}{G_{B_2}(y,z)} \tag{3.3}$$

and $\partial^{\mathrm{int}}B_1$ is the inner boundary of B_1, that is, $\partial^{\mathrm{int}}B_1 = \{z \in B_1; \text{there is } x \in \partial B_1, x \sim z\}$.

Proof. Let $\widetilde{u}$ be the non-negative function defined on $B_1 \cup \partial B_2$ by $\widetilde{u} = u$ on B_1 and $\widetilde{u} = 0$ on ∂B_2. Let $\eta := \overline{\tau}_{B_1} \wedge \overline{\tau}_{\partial B_2}$ and let

$$w(x) = E_x\left[\widetilde{u}(X_\eta)\right], \quad x \in \overline{B}_2. \tag{3.4}$$

Then $w \ge 0$ on $\overline{B}_2$, $w = 0$ on ∂B_2 and w is harmonic on $B_2 \setminus \partial^{\mathrm{int}}B_1$. Moreover, since u is harmonic in B_2, $u = w$ on B_1 and $u \ge w$ on ∂B_2, by the maximum principle (see for instance [28, p. 19]),

$$u \ge w \qquad \text{on } \overline{B}_2. \tag{3.5}$$

For $z \in \partial^{\mathrm{int}}B_1$, set $f = (I-P)w$. Then by the maximum principle,

$$w(x) = \sum_{z\in\partial^{\mathrm{int}}B_1} G_{B_2}(x,z)f(z), \quad x \in \overline{B}_2. \tag{3.6}$$

Note that for all $z \in \partial^{\mathrm{int}}B_1$, $f(z) = w(z) - Pw(z) \ge u(z) - Pu(z) = 0$ by (3.5).

Then (3.2) follows from (3.6) and the fact that $w = u$ on B_0. ■

Remark. It is possible extend lemma 3.1 to a Harnack inequality in an annulus. If u is a non-negative function on $\overline{B}_2$ which is harmonic in B_2 except at a vertex $x_0 \in B_0$ where $Hu \ge 0$ then $\max_{\partial B_0} u \le K \min_{\partial B_0} u$ where $K := \max_{x\in B_0}\max_{y\in B_0}\max_{z\in\partial^{\mathrm{int}}B_1\cup\{x_0\}} \frac{G_{B_2}(x,z)}{G_{B_2}(y,z)}$.

Proof of proposition 2.2. Let $x_0 \in \Gamma$ and $R \ge 1$ be given. Let γ and $R_x, x \in \Gamma$, be given by $(\mathbf{GE}_\gamma)$.

Let M_1 be large enough so that

$$M_1 > 3 + \frac{1}{R}\max_{x\in B(x_0,R)} R_x \quad \text{and} \quad \frac{(M_1-2)^{-\gamma}}{(M_1+3)^{-\gamma}} < 5. \tag{3.7}$$

Then let M be large enough so that

$$M > \frac{1}{R}\max_{z\in\partial B(x_0,M_1R)} R_z \quad \text{and} \quad C_s M^{-\gamma} < \frac{1}{2}C_i(M_1+3)^{-\gamma}. \tag{3.8}$$

For these values of M and M_1, to apply lemma 3.1 with

$$B_0 = B(x_0,R),\ B_1 = B(x_0,M_1R) \ \text{ and } \ B_2 = B(x_0,(M+M_1)R),$$

we need the following estimates.

For $x, y \in B_0$, $z \in \partial^{\text{int}} B_1$ and $z' \in \partial B_2$,

$$G_{B_2}(x,z) \le G(x,z) \le C_s D(x,z)^{-\gamma} < C_s (M_1 - 2)^{-\gamma} R^{-\gamma}, \tag{3.9}$$

$$G(y,z) \ge C_i D(y,z)^{-\gamma} > C_i (M_1+3)^{-\gamma} R^{-\gamma} \quad \text{and} \tag{3.10}$$

$$G(z',z) \le C_s D(z',z)^{-\gamma} < C_s M^{-\gamma} R^{-\gamma}. \tag{3.11}$$

Then by (3.1), (3.8), (3.10) and (3.11),

$$G_{B_2}(y,z) \ge C_i (M_1+3)^{-\gamma} R^{-\gamma} - C_s M^{-\gamma} R^{-\gamma} > \frac{1}{2} C_i (M_1+3)^{-\gamma} R^{-\gamma}. \tag{3.12}$$

Then by (3.7), (3.9) and (3.12),

$$\max_{x,y\in B_0} \max_{z\in\partial^{\text{int}} B_1} \frac{G(x,z)}{G(y,z)} \le 10 C_s / C_i.$$

That is, $\mathbf{wH}(K)$ holds with the constant $K = 10C_s/C_i$. ∎

Lemma 3.2. *Let (Γ, a) be a weighted graph which verifies $(\mathbf{GE}_\gamma)$ for some $\gamma > 0$.*

Let γ and R_{x_0}, $x_0 \in \Gamma$, be given by $(\mathbf{GE}_\gamma)$. Then for all $r \ge R_{x_0}$ and $\rho > (2C_s/C_i)^{1/\gamma} r$

$$P_x(\tau_{B(x_0,r)} = \infty) > C_i/(2C_s) \tag{3.13}$$

for all $x \in \partial B(x_0, \rho)$.

Proof. Fix $x_0 \in \Gamma$ and let $r \ge R_{x_0}$.

For $R > r$, let $\eta = \inf\{j \ge 0,\ X_j \in B(x_0,r) \cup \partial B(x_0,R)\}$.

Since $G(\cdot, x_0)$ is harmonic in $\Gamma \setminus \{x_0\}$, $G(X_{n\wedge\eta}, x_0)$, $n \in \mathbb{N}$, is a martingale with respect to E_x for $x \in B(x_0,R) \setminus B(x_0,r)$.

By the optional sampling theorem, for $x \in \partial B(x_0,\rho)$ where $R > \rho > r$,

$$\begin{aligned} G(x,x_0) &= P_x[X_\eta \in \partial B(x_0,R)]\ E_x[G(X_\eta,x_0) | X_\eta \in \partial B(x_0,R)] \\ &\quad + (1 - P_x[X_\eta \in \partial B(x_0,R)]) E_x[G(X_\eta,x_0) | X_\eta \in B(x_0,r)] \end{aligned}$$

or equivalently,

$$\begin{aligned} P_x[X_\eta \in \partial B(x_0,R)] \big(E_x(G(X_\eta,x_0) | X_\eta \in B(x_0,r)) - E_x(G(X_\eta,x_0) | X_\eta \in \partial B(x_0,R)) \big) \\ = E_x(G(X_\eta,x_0) | X_\eta \in B(x_0,r)) - G(x,x_0). \end{aligned}$$

By $(\mathbf{GE}_\gamma)$, if $R > \rho > (2C_s/C_i)^{1/\gamma} r$ then

$$E_x(G(X_\eta,x_0) | X_\eta \in B(x_0,r)) - G(x,x_0) > C_i r^{-\gamma} - C_s \rho^{-\gamma} > (C_i/2) r^{-\gamma}$$

and

$$E_x(G(X_\eta,x_0) | X_\eta \in B(x_0,r)) - E_x(G(X_\eta,x_0) | X_\eta \in \partial B(x_0,R)) < C_s r^{-\gamma} - C_i R^{-\gamma} < C_s r^{-\gamma}.$$

Hence,

$$P_x[X_\eta \in \partial B(x_0,R)] > C_i/(2C_s).$$

∎

We can now state the main lemma to prove Theorem I.

Lemma 3.3. *Let (Γ, a) be a weighted graph which verifies* $(\mathbf{GE}_\gamma)$ *for some $\gamma > 0$. Let $K = 10C_s/C_i$.*

Fix $x_0 \in \Gamma$. Let A be a finite subset of Γ.

Then there is an increasing sequence of balls $(B_k,\ k \in \mathbb{N})$, centered at x_0, with $A \subset B_0$ and $B_k \uparrow \Gamma$ as $k \to \infty$ and such that for all $y \in A$ and $z \in \partial B_k$,

$$P_y(X_{\tau_A \wedge \tau_{\partial B_k}} = z | \tau_A > \tau_{\partial B_k}) = H_{\partial B_k}(x_0, z)\left[1 + O\left(\left(\frac{K-1}{K+1}\right)^k\right)\right]. \tag{3.14}$$

Proof. Let R_{x_0} be given by $(\mathbf{GE}_\gamma)$. Let $r_A \geq R_{x_0}$ be such that $A \subset B(x_0, r_A)$.

Fix $\rho_A > (2C_s/C_i)^{1/\gamma} r_A$. Then by lemma 3.2 for all $x \in \partial B(x_0, \rho_A)$ and $R > \rho_A$,

$$P_x(\tau_A > \tau_{\partial B(x_0,R)}) > P_x(\tau_{\partial B(x_0,r_A)} = \infty) > C_i/(2C_s). \tag{3.15}$$

By proposition 2.2, since Γ satisfies $(\mathbf{GE}_\gamma)$, it satisfies the weak Harnack inequality $\mathbf{wH}(K)$ with $K = 10C_s/C_i$.

Therefore, we set $M_0 = 2$ and we construct a sequence $(M_k; k \in \mathbb{N})$ such that for all $k \geq 1$, if u is a non-negative and harmonic function in

$$B_k := B(x_0, M_k M_{k-1} \cdots M_1 M_0 \rho_A)$$

then

$$\max_{B_{k-1}} u \leq K \min_{B_{k-1}} u. \tag{3.16}$$

Then by lemma 2.3, for all $k \geq 1$, if u is harmonic function in B_k then

$$\operatorname*{osc}_{B_0} u \leq \left(\frac{K-1}{K+1}\right)^k \operatorname*{osc}_{B_k} u. \tag{3.17}$$

For $k \geq 1$ and $z \in \partial B_k$, consider the function

$$f(x) = P_x(X_{\tau_{\partial B_k}} = z) = H_{\partial B_k}(x, z), \quad x \in \Gamma.$$

Since f is harmonic in B_{k-1}, by (3.17),

$$\operatorname*{osc}_{B_0} f \leq \left(\frac{K-1}{K+1}\right)^{k-1} \operatorname*{osc}_{B_{k-1}} f. \tag{3.18}$$

Furthermore, since f is non-negative and harmonic in B_k, by (3.16), we have that

$$\operatorname*{osc}_{B_{k-1}} f \leq \max_{B_{k-1}} f \leq K \min_{B_{k-1}} f \leq K f(x_0). \tag{3.19}$$

Therefore, by (3.18) and (3.19), for all $x \in B_0$ and $z \in \partial B_k$,

$$|P_x(X_{\tau_{\partial B_k}} = z) - H_{\partial B_k}(x_0, z)| \leq H_{\partial B_k}(x_0, z) K \left(\frac{K-1}{K+1}\right)^{k-1}$$

and in particular,

$$P_x(X_{\tau_{\partial B_k}} = z) = H_{\partial B_k}(x_0, z)[1 + O(\left(\frac{K-1}{K+1}\right)^k)] \tag{3.20}$$

where the constant in $O(\cdot)$ depends only on K.

Note that $A \subset \overline{B}(x_0, \rho) \subset B_0$. Then by (3.20), for $x \in \partial B(x_0, \rho_A)$ and $k \geq 0$,

$$\begin{aligned} P_x(X_{\tau_{\partial B_k}} = z | \tau_A < \tau_{\partial B_k}) &= \sum_{y \in A} P_x(X_{\tau_A} = y | \tau_A < \tau_{\partial B_k}) P_y(X_{\tau_{\partial B_k}} = z) \\ &= H_{\partial B_k}(x_0, z) \left[1 + O\Big(\Big(\frac{K-1}{K+1} \Big)^k \Big) \right]. \end{aligned} \tag{3.21}$$

For $x \in \partial B(x_0, \rho_A)$ and $z \in \partial B_k$, $k \geq 0$, we have that

$$P_x(X_{\tau_{\partial B_k}} = z) = P_x(X_{\tau_{\partial B_k} \wedge \tau_A} = z) + P_x(X_{\tau_{\partial B_k}} = z | \tau_A \leq \tau_{\partial B_k})(1 - P_x(\tau_A > \tau_{\partial B_k})).$$

Then,

$$\begin{aligned} P_x(X_{\tau_{\partial B_k} \wedge \tau_A} = z) &= P_x(X_{\tau_{\partial B_k}} = z) - P_x(X_{\tau_{\partial B_k}} = z | \tau_A \leq \tau_{\partial B_k}) \\ &\quad + P_x(X_{\tau_{\partial B_k}} = z | \tau_A \leq \tau_{\partial B_k}) P_x(\tau_A > \tau_{\partial B_k}) \\ &= H_{\partial B_k}(x_0, z) P_x(\tau_A > \tau_{\partial B_k}) [1 + O\Big(\Big(\frac{K-1}{K+1} \Big)^k \Big)] \end{aligned} \tag{3.22}$$

by (3.20), (3.21) and by the lower estimate (3.15).

But every path from A to ∂B_k must go through some vertex of $\partial B(x_0, \rho_A)$. So, for $y \in A$ and $z \in \partial B_k$,

$$\begin{aligned} P_y(X_{\tau_{\partial B_k} \wedge \tau_A} = z) &= \sum_{x \in \partial B(x_0, \rho_A)} P_y(X_{\tau_{\partial B(x_0, \rho_A)} \wedge \tau_A} = x) P_x(X_{\tau_{\partial B_k} \wedge \tau_A} = z) \\ &\overset{(3.22)}{=} H_{\partial B_k}(x_0, z) [1 + O\Big(\Big(\frac{K-1}{K+1} \Big)^k \Big)] \\ &\qquad \times \sum_{x \in \partial B(x_0, \rho_A)} P_y(X_{\tau_{\partial B(x_0, \rho_A)} \wedge \tau_A} = x) P_x(\tau_A > \tau_{\partial B_k}) \\ &= H_{\partial B_k}(x_0, z) [1 + O\Big(\Big(\frac{K-1}{K+1} \Big)^k \Big)] P_y(\tau_A > \tau_{\partial B_k}). \end{aligned}$$

■

As in Lawler [27, p. 49], using a last exit decomposition, we obtain the following representation of the hitting distribution in a weighted graph (Γ, a).

Let $A \subset B$ be finite subsets of Γ. Then for all $x \in B^c$ and $y \in A$,

$$H_A(x, y) = \sum_{z \in \partial B} G_{A^c}(x, z) H_{A \cup \partial B}(z, y), \tag{3.23}$$

$$\overline{H}_A(x, y) = \frac{\sum_{z \in \partial B} G_{A^c}(x, z) H_{A \cup \partial B}(z, y)}{\sum_{z \in \partial B} G_{A^c}(x, z) P_z(\tau_A < \tau_{\partial B})}$$

and

$$\min_{z \in \partial B} \frac{H_{A \cup \partial B}(z, y)}{P_z(\tau_A < \tau_{\partial B})} \leq \overline{H}_A(x, y) \leq \max_{z \in \partial B} \frac{H_{A \cup \partial B}(z, y)}{P_z(\tau_A < \tau_{\partial B})}.$$

Then by reversibility, $\pi(z) H_{A \cup \partial B}(z, y) = \pi(y) H_{A \cup \partial B}(y, z)$ and

$P_z(\tau_A < \tau_{\partial B}) = \sum_{\tilde{y}\in A} H_{A\cup\partial B}(z,\tilde{y})$. Hence,

$$\min_{z\in\partial B} \frac{\pi(y)H_{A\cup\partial B}(y,z)}{\sum_{\tilde{y}\in A}\pi(\tilde{y})H_{A\cup\partial B}(\tilde{y},z)} \le \overline{H}_A(x,y) \le \max_{z\in\partial B} \frac{\pi(y)H_{A\cup\partial B}(y,z)}{\sum_{\tilde{y}\in A}\pi(\tilde{y})H_{A\cup\partial B}(\tilde{y},z)} \tag{3.24}$$

We complete the proof of Theorem I with the help of (3.24).

Proof of Theorem I. Let A be a finite subset of Γ and let $x_0 \in \Gamma$.

Let $r_A \ge R_{x_0}$ be such that $A \subset B(x_0, r_A)$.

Let $(B_k; k \in \mathbb{N})$ be an increasing sequence of balls given by lemma 3.3.

By equation (3.14), for all $y \in A$, $k \ge 2$ and $z \in \partial B_k$,

$$\pi(y)H_{A\cup\partial B_k}(y,z) = H_{\partial B_k}(x_0,z)[1 + O(\Big(\frac{K-1}{K+1}\Big)^k)]\pi(y)P_y(\tau_A > \tau_{\partial B_k}). \tag{3.25}$$

By summing over $y \in A$ the equation (3.25) gives,

$$\sum_{y\in A}\pi(y)P_y(X_{\tau_{\partial B_k}\wedge\tau_A} = z) = H_{\partial B_k}(x_0,z)[1 + O(\Big(\frac{K-1}{K+1}\Big)^k)]\sum_{y\in A}\pi(y)P_y(\tau_A > \tau_{\partial B_k}). \tag{3.26}$$

Since (Γ, a) is connected, both sides of (3.26) are positive. So we can divide (3.25) by (3.26). And a short calculation shows that

$$\frac{\pi(y)H_{A\cup\partial B_k}(y,z)}{\sum_{\tilde{y}\in A}\pi(\tilde{y})P_{\tilde{y}}(X_{\tau_{\partial B_k}\wedge\tau_A} = z)} = \frac{\pi(y)P_y(\tau_A > \tau_{\partial B_k})}{\sum_{\tilde{y}\in A}\pi(\tilde{y})P_{\tilde{y}}(\tau_A > \tau_{\partial B_k})}[1 + O(\Big(\frac{K-1}{K+1}\Big)^k)]$$

where the constant in $O(\cdot)$ still depends only on K.

By (3.24), we have that for all $v \notin B_k$,

$$\min_{z\in\partial B_k} \frac{\pi(y)H_{A\cup\partial B_k}(y,z)}{\sum_{\tilde{y}\in A}\pi(\tilde{y})P_{\tilde{y}}(X_{\tau_{\partial B_k}\wedge\tau_A} = z)} \le \overline{H}_A(v,y) \le \max_{z\in\partial B_k} \frac{\pi(y)H_{A\cup\partial B_k}(y,z)}{\sum_{\tilde{y}\in A}\pi(\tilde{y})P_{\tilde{y}}(X_{\tau_{\partial B_k}\wedge\tau_A} = z)}$$

So for all $v \notin B_k$ we get:

$$\overline{H}_A(v,y) = \frac{\pi(y)P_y(\tau_A > \tau_{\partial B_k})}{\mathrm{Cap}_{B_k}(A)}[1 + O(\Big(\frac{K-1}{K+1}\Big)^k)] \tag{3.27}$$

As v goes to $+\infty$ in an arbitrary way, we can let $k \to \infty$ as well. Hence, by (3.27), we obtain that $\lim_{v\to+\infty}\overline{H}_A(v,y)$ exists and

$$\lim_{v\to+\infty}\overline{H}_A(v,y) = \frac{\pi(y)P_y(\tau_A > +\infty)}{\sum_{\tilde{y}\in A}\pi(\tilde{y})P_{\tilde{y}}(\tau_A > +\infty)}.$$

■

4. The two-dimensional supercritical percolation cluster

In this section, we prove the existence of the harmonic measure for the random walk on a supercritical percolation cluster of $\mathbb{Z}^2$. The proof of Theorem III, for the uniformly elliptic random walk on $\mathbb{Z}^2$, is similar but with many simplifications since we can use the estimates of [16] instead of Barlow's estimates.

The error term in the local central limit theorem for the simple random walk on $\mathbb{Z}^2$ is $O(k^{-2})$ (see [27, (1.10)] for instance). For the random walk on a supercritical percolation cluster of $\mathbb{Z}^2$, Barlow and Hambly [8, (1.4)] proved a local central limit theorem with an error term which is $O(k^{-1})$. Because of this difficulty, we first construct the Green kernel using the parabolic Harnack inequality [8, (3.2)]. Then we proceed as in Černý [15, section 3] to estimate the Green function in a finite ball. Finally, using a Harnack inequality in an annulus, we obtain Green kernel estimates which are sufficient to prove the existence of the harmonic measure although they are weaker than the estimates which hold for the simple random walk on $\mathbb{Z}^2$ (see [27, Theorem 1.6.2]).

4.1. **The Green kernel and its properties.**

Lemma 4.1. *$\mathbb{P}_p$-almost surely, for all $x_0, x \in \mathcal{C}_\infty(\omega)$, the series*

$$\sum_{k=0}^{\infty} [p(x_0, x_0, k) - p(x, x_0, k)] \tag{4.28}$$

converges. The limit will be denoted by $g(x, x_0)$.

Let $G_{2n}(x, y)$ and $p_{2n}(x, y, k)$ be respectively the Green function and the probability transitions of the random walk in the ball $B_\omega(x_0, 2n)$ with Dirichlet boundary conditions. Then

$$g(x, x_0) = \lim_n \sum_{k=0}^{\infty} [p_{2n}(x_0, x_0, k) - p_{2n}(x, x_0, k)] = \lim_n [G_{2n}(x_0, x_0) - G_{2n}(x, x_0)]. \tag{4.29}$$

Proof. Let R_0 be given by the Harnack inequality for the supercritical cluster (2.2). Then as in the proof of [6, Proposition 6.1], we have that for $x \in \mathcal{C}_\infty$ and $R \geq R_0(x)$, $B(x, R)$ is very good (see [6, definition 1.7]) with $N_B \leq R^{1/(10(d+2))}$ and it is exceedingly good (see [6, definition 5.4]).

Now let $R \geq R_0(x) \vee 16$ and let $R_1 = R \ln R$. Then, since $R_1 \geq R_0$, $B = B(x, R_1)$ is very good with $N_B^{2d+4} \leq R_1^{(2d+4)/(10(d+2))} \leq R_1/(2 \ln R_1)$. Then by [8, Theorem 3.1], there exists a positive constant C_H such that the parabolic Harnack inequality [8, (3.2)] holds in $Q(x, R, R^2)$. Therefore [8, Proposition 3.2] holds with $s(x_0) = R_0(x_0) \vee 16$ and $\rho(x_0, x) = R_0(x_0) \vee 16 \vee D(x_0, x)$.

Fix $x_0 \in \mathcal{C}_\infty$ then $v(n, x) = p(x, x_0, n) + p(x, x_0, n+1)$ is a caloric function, that is, it verifies

$$v(n+1, x) = Pv(n, x), \quad (n, x) \in \mathbb{N} \times \mathcal{C}_\infty.$$

Let $k > 4D(x_0, x)^2$. Let $t_0 = k + 1$ and $r_0 = \sqrt{t_0}$. Then $v(n, x)$ is caloric in $]0, r_0^2] \times B(x_0, r_0)$, $x \in B(x_0, r_0/2)$ since $D(x_0, x) \leq \sqrt{k} < r_0/2$, and $t_0 - \rho(x_0, x)^2 \leq k \leq t_0 - 1$.

Then by the upper gaussian estimates [6, Theorem 5.7] and [8, (2.18)] and by [8, Proposition 3.2], there is $\nu > 0$ such that

$$\begin{aligned} |v(k,x) - v(k,x_0)| &\leq C \left(\frac{\rho(x_0,x)}{\sqrt{t_0}} \right)^{\nu} \sup_{Q_+} v \\ &\leq C \left(\frac{\rho(x_0,x)}{\sqrt{t_0}} \right)^{\nu} \frac{1}{r_0^2} \\ &\leq C \; \frac{\rho(x_0,x)^{\nu}}{k^{1+\nu/2}}. \end{aligned}$$

Moreover, for all $k > 4D(x_0,x)^2$,

$$|p(x,x_0,k) - p(x_0,x_0,k)| \leq C \; \frac{\rho(x_0,x)^{\nu}}{k^{1+\nu/2}}.$$

Hence (4.28) converges. Then (4.29) follows by Lebesgue dominated convergence theorem. ■

The harmonic measure will be expressed in terms of the function u_A defined below.

Definition 4.2. *$\mathbb{P}_p$-a.s., for a finite subset A of $\mathcal{C}_\infty(\omega)$ and for a fixed $x_0 \in A$, let*

$$u_A(x,x_0) := g(x,x_0) - E_x^{\omega} g(X_{\overline{\tau}_A}, x_0), \quad x \in \mathcal{C}_\infty(\omega).$$

Note that, $u_A(\cdot, x_0) = 0$ on A and

$$\begin{aligned} P^{\omega} u_A(x,x_0) &= P^{\omega} g(x,x_0) - \sum_{y \sim x} p(x,y) E_y^{\omega} g(X_{\overline{\tau}_A}, x_0) \\ &= g(x,x_0) - \mathbf{1}_{x_0}(x) - E_x^{\omega} g(X_{\tau_A}, x_0), \quad x \in \mathcal{C}_\infty(\omega). \end{aligned}$$

4.2. **Green kernel estimates.** We obtain upper and lower bounds on $\mathrm{Cap}_{B_\omega(x_0,n)}(\{x_0\})$ by the arguments of [15, section 3] with the heat kernel bounds for the discrete time random walk. They appear in [6, Theorem 1 and remark 7] and [8, Theorem 5.1] with details given in [8, section 2]). We state them below in terms of the graph distance by taking into account (2.7).

For $x, y \in \mathcal{C}_\infty$, let $\widehat{p}(x,y,k) := p(x,y,k) + p(x,y,k+1)$. Then, there are positive constants $\eta, c, c_4, c_5, c_6, c_7$ and random variables $\widehat{R}(x,\omega)$ such that

$$\mathbb{P}_p(x \in \mathcal{C}_\infty, \widehat{R}(x,\omega) \geq n) \leq c \exp(-n^{\eta}/c) \tag{4.30}$$

and if $x, y \in \mathcal{C}_\infty$ and $n \geq cD(x,y) \vee \widehat{R}(x,\omega)$, then

$$c_4 n^{-1} \exp(-c_5 D(x,y)^2/n) \leq \widehat{p}(x,y,k) \leq c_6 n^{-1} \exp(-c_7 D(x,y)^2/n). \tag{4.31}$$

Let $G_n(x,y)$ and $p_n(x,y,k)$ be respectively the Green function and the probability transitions of the random walk in the ball $B_\omega(x_0,n)$ with Dirichlet boundary conditions.

Proposition 4.3. *Let $p > p_c(\mathbb{Z}^2)$. Let c_4 and c_6 be the constants that appear in (4.31). Then $\mathbb{P}_p$-a.s. for $x_0 \in \mathcal{C}_\infty$, for all n sufficiently large,*

$$\frac{c_4}{3} \ln n \leq G_n(x_0,x_0) < 4c_6 \ln n. \tag{4.32}$$

Remark. Note from (4.32), since $\mathrm{Cap}_{B_\omega(x_0,n)}(\{x_0\}) = a(x_0)/G_n(x_0,x_0)$ (see [28, section 9.4] for instance), there is a constant $C > 1$ such that $\mathbb{P}_p$-a.s. for $x_0 \in \mathcal{C}_\infty$, for all n sufficiently large,

$$C^{-1} \leq (\ln n) \; \mathrm{Cap}_{B_\omega(x_0,n)}(\{x_0\}) \leq C. \tag{4.33}$$

Proof. We follow [15, section 3] with few modifications.

For $x_0 \in \mathcal{C}_\infty$, let $\sigma_n := \inf\{k > 0;\ X_k \notin B(x_0, n)\}$. Then we write the Green function as

$$\begin{aligned} G_n(x_0, x_0) &= E_{x_0}\Big(\sum_{0 \le k < \sigma_n} \mathbf{1}_{\{X_k = x_0\}} \Big) \\ &= \sum_{0 \le k \le n^2} p(x_0, x_0, k) + \sum_{n^2 < k} P_{x_0}(X_k = x_0,\ k < \sigma_n) \\ &\qquad - \sum_{0 \le k \le n^2} P_{x_0}(X_k = x_0, \sigma_n \le k) \\ &:= S_1 + S_2 - S_3. \end{aligned}$$

By the upper and lower bounds of (4.31), we obtain that for all n sufficiently large,

$$\frac{c_4}{2} \ln n \le S_1 \le \widehat{R}(x_0, \omega) + 2c_6 \ln n < 3c_6 \ln n. \tag{4.34}$$

For the sum S_3, by the strong Markov property and by reversibilty we have for n large enough,

$$\begin{aligned} S_3 &\le \sup_{y \in \partial B(x_0, n)} \sum_{2n \le k \le n^2} p(y, x_0, k) \\ &\le 4 \sup_{y \in \partial B_\omega(x_0, n)} \sum_{2n \le k \le n^2} p(x_0, y, k) \\ &\le 4\Big(\widehat{R}(x_0, \omega) + \sum_{2n \le k} \frac{c_6}{k} e^{-c_7 n^2/k}\ \Big) \\ &\le 4(\widehat{R}(x_0, \omega) + C\ ) \end{aligned}$$

It remains to bound S_2. As noticed in [10, lemma 3.3], by (4.30) and by Borel-Cantelli, there exists a constant $0 < C < +\infty$ such that, $\mathbb{P}_p$ almost surely, for all sufficiently large n,

$$\sup_{x \in B(x_0, n)} \widehat{R}(x, \omega) < C (\ln n)^{1/\eta}. \tag{4.35}$$

So, by the gaussian lower bound of (4.31) and by (4.35), we deduce that,

$$\sup_{x \in B(x_0, n)} \sum_{y \in B(x_0, n)} p(x, y, n^2) = \sup_{x \in B(x_0, n)} [1 - \sum_{y \notin B(x_0, n)} p(x, y, n^2)] < \rho < 1 \tag{4.36}$$

For $k \ge n^2$, let $k = an^2 + b$, with $0 \le b < n^2$ and $a \in \mathbb{N}$. By (4.35) and by the gaussian upper bound of (4.31), we deduce that,

$$\begin{aligned} p_n(x_0, x_0, k) &= \sum_{z_1, z_2, \ldots, z_{a-1} \in B(x_0, n)} p_n(x_0, z_1, n^2) p_n(z_1, z_2, n^2) \ldots p_n(z_{a-1}, x_0, n^2 + b) \\ &< \rho^{a-1} \frac{c}{n^2 + b} \\ &< \frac{c}{n^2} \exp(-k/(cn^2)). \end{aligned}$$

Using this upper bound on the transition probabilities, we have that

$$\begin{aligned} S_2 &= \sum_{n^2<k} P_{x_0}(X_k = x_0,\ k < \sigma_n) \\ &= \sum_{n^2<k} p_n(x_0, x_0, k) \\ &< \sum_{n^2<k} \frac{c}{n^2} \exp(-k/(cn^2)) < C. \end{aligned}$$

Gathering inequalities for S_1, S_2 and S_3, we obtain (4.32). ■

Lemma 4.4. *There is a constant $c_8 \geq 1$ such that, $\mathbb{P}_p$-a.s., for all $x_0 \in \mathcal{C}_\infty$ there is $\rho = \rho(x_0)$ such that if $D(x_0, x) > \rho$,*

$$c_8^{-1} \ln D(x_0, x) \leq g(x, x_0) \leq c_8 \ln D(x_0, x) \tag{4.37}$$

and

$$u_A(x, x_0) \asymp \ln D(x_0, x) \quad \textit{as} \quad D(x_0, x) \to \infty. \tag{4.38}$$

Proof. Let $x_0 \in \mathcal{C}_\infty$ and $r > 0$. Write $\tau_r := \inf\{k > 0; X_k \in B(x_0, r)\}$ and for $m \geq 1$, write $\sigma_m := \inf\{k > 0; X_k \notin B(x_0, m)\}$.

Note that for all $n > 3\mu m$, where μ is the constant that appears in (2.6), $P_.(\sigma_n < \tau_r)$ is harmonic in $B(x_0, 3\mu m) \setminus B(x_0, r)$. Then by the annulus Harnack inequality (proposition 2.5), if m is sufficiently large and $D(y, x_0) = m$, then $\sum_{x \in B(x_0,r)} \pi(x) P_x(\sigma_n < \tau_r)$

$$\begin{aligned} &= \sum_{x \in B(x_0,r)} \sum_{x'; D(x_0,x')=m} \pi(x) P_x(X(\sigma_m) = x', \sigma_m < \tau_r) P_{x'}(\sigma_n < \tau_r) \\ &\asymp P_y(\sigma_n < \tau_r) \sum_{x \in B(x_0,r)} \sum_{x'; D(x_0,x')=m} \pi(x) P_x(X(\sigma_m) = x', \sigma_m < \tau_r) \\ &\asymp P_y(\sigma_n < \tau_r) \mathrm{Cap}_m(B(x_0, r)). \end{aligned}$$

By $f_1(y, m, n) \asymp f_2(y, m, n)$ here, we mean that there is a constant $c \geq 1$ which does not depend on y, m, n nor on ω and r, and such that $\mathbb{P}_p$-a.s for m is sufficiently large and $D(y, x_0) = m$, then

$$0 < c^{-1} f_1(y, m, n) \leq f_2(y, m, n) \leq c f_1(y, m, n).$$

Then, since r is fixed, by the capacity estimates (4.33), for $m = D(y, x_0)$,

$$P_y(\sigma_n < \tau_r) \asymp \frac{\mathrm{Cap}_n(B(x_0, r))}{\mathrm{Cap}_m(B(x_0, r))} \asymp \frac{\ln m}{\ln n} = \frac{\ln D(y, x_0)}{\ln n}. \tag{4.39}$$

It follows from (4.39) and the capacity estimates (4.33) that for m sufficiently large, $D(x, x_0) = m$ and $n > 3\mu m$,

$$\begin{aligned} G_n(x_0, x_0) - G_n(x, x_0) &= G_n(x_0, x_0) - P_x(\tau_{x_0} < \sigma_n) G_n(x_0, x_0) \\ &= G_n(x_0, x_0) P_x(\tau_{x_0} > \sigma_n) \\ &\asymp \ln n \tfrac{\ln D(x, x_0)}{\ln n}. \end{aligned}$$

Then (4.37) follows by (4.29). ■

We will need to work in sets defined in terms of $g(\cdot, x_0)$. Let

$$\widetilde{B}_n := \widetilde{B}(x_0, n) := \{x \in \mathcal{C}_\infty; g(x, x_0) < \ln n\} \text{ and } \widetilde{\sigma}_n := \inf\{k \geq 0; X_k \notin \widetilde{B}(x_0, n)\}. \tag{4.40}$$

Note that by (4.37), for all n sufficiently large,

$$B(x_0, n^{1/c_8}) \subset \widetilde{B}(x_0, n) \subset B(x_0, n^{c_8}). \tag{4.41}$$

Lemma 4.5. *There is a positive constant C such that, $\mathbb{P}_p$-a.s., for any non empty finite subset A of $\mathcal{C}_\infty$ and $x_0 \in A$, if m sufficiently large and $n > (3\mu m)^{c_8}$, then*

$$\min_{y; m = D(y, x_0)} P_y(\widetilde{\sigma}_n < \tau_A) \geq C(\ln m / \ln n).$$

Proof. The lemma is a consequence of (4.39). For a finite subset A of $\mathcal{C}_\infty$ such that $x_0 \in A \subset B(x_0, r)$, for m sufficiently large and $n > (3\mu m)^{c_8}$,

$$\begin{aligned}
\min_{y; m = D(y, x_0)} P_y(\widetilde{\sigma}_n < \tau_A) &\geq \min_{y; m = D(y, x_0)} P_y(\widetilde{\sigma}_n < \tau_{B(x_0, r)}) \\
&\geq \min_{y; m = D(y, x_0)} P_y(\sigma_{B(x_0, n^{1/c_8})} < \tau_{B(x_0, r)}) \\
&\geq C(\ln m / \ln n^{1/c_8}).
\end{aligned}$$

■

The next lemma is the analogue of [26, Proposition 6.4.7]. The comparison result for D and the $|\cdot|_1$-distance of Antal and Pisztora [4], see (2.7) is used in its proof.

Proposition 4.6. *$\mathbb{P}_p$-a.s., for a finite subset A of $\mathcal{C}_\infty(\omega)$ and for $x_0 \in A$ and $x \in A^c$,*

$$u_A(x, x_0) = \lim_n (\ln n) P_x(\widetilde{\sigma}_n < \tau_A).$$

Proof. Let $R_0(z, \omega)$ be as the Harnack inequality for the supercritical cluster (2.2). By (2.6) of Antal and Pisztora, and by (4.41)

$$\sum_n \sum_{z \in \partial \widetilde{B}(x_0, n)} \mathbb{P}_p(z \in \mathcal{C}_\infty, R_0(z, \cdot) \geq n^{1/c_8}) \leq C \sum_n n^{2c_8} \exp(-c_3 n^{\varepsilon/c_8}) < \infty.$$

Therefore, by Borel-Cantelli, there is $\Omega_1 \subset \Omega$ with $\mathbb{P}_p(\Omega_1) = 1$, such that for all $\omega \in \Omega_1$ there is n_0 such that for all $n \geq n_0$ and for all $z \in \partial \widetilde{B}(x_0, n)$, $R_0(z) < n^{1/c_8}$.

Let $z \in \partial \widetilde{B}(x_0, n)$ where $n \geq n_0$. Then there is $z' \in \widetilde{B}(x_0, n)$ such that $z' \sim z$ and

$$g(z', x_0) < \ln n \leq g(z, x_0).$$

Moreover, by (4.41), $D(z, x_0) > n^{1/c_8}$. Then by Hölder's continuity property given in proposition 2.4 and by (4.37),

$$\begin{aligned}
0 \leq g(z, x_0) - \ln n &\leq g(z, x_0) - g(z', x_0) \\
&\leq c \left(\frac{1}{n^{1/c_8}}\right)^\nu \max_{B(z, n^{1/c_8})} g(\cdot, x_0) \\
&\leq \frac{c}{C} \left(\frac{1}{n^{1/c_8}}\right)^\nu \ln n.
\end{aligned} \tag{4.42}$$

By the optional stopping theorem applied to the martingale $g(X_k, x_0)$, $k \geq 0$ and for n large enough and $x \in \widetilde{B}(x_0, n) \setminus A$,

$$\begin{aligned} g(x, x_0) &= E_x\left[g(X_{\overline{\tau}_A \wedge \widetilde{\sigma}_n}, x_0)\right], \\ &= P_x(\widetilde{\sigma}_n < \tau_A) E_x\left[g(X_{\widetilde{\sigma}_n}, x_0) \mid \widetilde{\sigma}_n < \tau_A\right] \\ &\quad + P_x(\tau_A < \widetilde{\sigma}_n) E_x\left[g(X_{\tau_A}, x_0) \mid \tau_A < \widetilde{\sigma}_n\right]. \end{aligned} \tag{4.43}$$

But

$$\begin{aligned} \lim_n P_x(\tau_A < \widetilde{\sigma}_n) E_x\left[g(X_{\tau_A}, x_0) \mid \tau_A < \widetilde{\sigma}_n\right] &= \lim_n E_x\left[g(X_{\tau_A}, x_0); \tau_A < \widetilde{\sigma}_n\right] \\ &= E_x g(X_{\tau_A}, x_0) \end{aligned}$$

Therefore by (4.42) and (4.43), $u_A(x, x_0) = \lim_n (\ln n) P_x(\widetilde{\sigma}_n < \tau_A)$. ■

We can now prove the analogue of lemma 3.3 for the supercritical cluster. Theorem II will follow from this lemma and from proposition 4.6 above.

4.3. **The main lemma and the proof of Theorem II.**

Lemma 4.7. *Let $p > p_c(\mathbb{Z}^2)$. Let Ω_1 and $R_0(x, \omega)$ be as in the Harnack inequality for the percolation cluster (2.2). There is $\nu' > 0$ such that the following holds.*

Let $\omega \in \Omega_1$ and let A be a finite subset of $\mathcal{C}_\infty(\omega)$. Fix $x_0 \in A$.

Then there is $N_0 = N_0(x_0, A, \omega)$ such that for all $n > N_0$, for all $y \in A$ and $z \in \partial\widetilde{B}(x_0, n)$,

$$H_{A \cup \partial\widetilde{B}(x_0,n)}(y, z) = P_y(\tau_A > \widetilde{\sigma}_n) H_{\partial\widetilde{B}(x_0,n)}(x_0, z) \left[1 + O\Big(\frac{\ln n}{n^{\nu'}}\Big)\right] \tag{4.44}$$

where $\widetilde{B}_n$ and $\widetilde{\sigma}_n$ are as in (4.40). $\nu' > 0$ depends on the Hölder exponent given by proposition 2.4 and the constants given in (4.37) The constant in $O(\cdot)$ depends on ω and A and on the constants that appear in (2.1), (2.2) and in proposition 2.4.

Proof. Let m be sufficiently large so that $A \subset B(x_0, m)$ and so that (4.42) holds for all $n > (3\mu m)^{c_8}$.

For $R_1 > \max\{R_0(x_0, \omega), (3\mu m)/4, m\}$, let $B_1 = B(x_0, R_1)$, $B_2 = B(x_0, 2R_1)$, $B_3 = B(x_0, 4R_1)$. Set $n = (4R_1)^{c_8}$ and let $\widetilde{B}_n = \widetilde{B}(x_0, n)$ and $\widetilde{\sigma}_n$ be as in (4.40). Note that by (4.41), $B_3 \subset \widetilde{B}_n$ and (4.42) holds.

For $z \in \partial\widetilde{B}_n$, consider the function

$$f(x) = P_x(X_{\widetilde{\sigma}_n} = z), \quad x \in \mathcal{C}_\infty(\omega).$$

Since f is harmonic on B_2, by proposition 2.4, for all $u \in B_1$,

$$|f(u) - f(x_0)| \leq c\Big(\frac{D(x_0, u)}{R_1}\Big)^\nu \max_{B_2} f.$$

In particular, for $u \in \partial B(x_0, m)$,

$$|f(u) - f(x_0)| \leq c\Big(\frac{m}{R_1}\Big)^\nu \max_{B_2} f. \tag{4.45}$$

Now by considering f harmonic on B_3, by (2.1), we have that

$$\max_{B_2} f \leq c_1 f(x_0). \tag{4.46}$$

Therefore, by (4.45) and (4.46), for all $u \in \partial B(x_0, m)$,

$$P_u(X_{\widetilde{\sigma}_n} = z) = H_{\partial \widetilde{B}_n}(x_0, z)\left[1 + O\Big(\Big(\frac{m}{R_1}\Big)^{\nu}\Big)\right]. \tag{4.47}$$

On the set $\{\tau_A < \widetilde{\sigma}_n\}$, we let $\eta = \inf\{j \geq \tau_A;\ X_j \in \partial B(x_0, m)\}$.

Let $\partial B(x_0, m)[\widetilde{B}_n, A] := \{x \in \partial B(x_0, m); 0 < P_x(\widetilde{\sigma}_n < \tau_A) < 1\}$. Then using (4.47), we obtain that for all $x \in \partial B(x_0, m)[\widetilde{B}_n, A]$,

$$\begin{aligned} P_x(X_{\widetilde{\sigma}_n} = z | \tau_A < \widetilde{\sigma}_n) &= \sum_{u \in \partial B(x_0, m)} P_x(X_\eta = u | \tau_A < \widetilde{\sigma}_n) P_u(X_{\widetilde{\sigma}_n} = z) \\ &= H_{\partial \widetilde{B}_n}(x_0, z)\left[1 + O\Big(\Big(\frac{m}{R_1}\Big)^{\nu}\Big)\right]. \end{aligned} \tag{4.48}$$

Let $x \in \partial B(x_0, m)[\widetilde{B}_n, A]$. By (4.47), (4.48) and (4.42), we get from the relation

$$\begin{aligned} P_x(X_{\widetilde{\sigma}_n} = z) &= P_x(X_{\widetilde{\sigma}_n} = z | \tau_A > \widetilde{\sigma}_n) P_x(\tau_A > \widetilde{\sigma}_n) \\ &\quad + P_x(X_{\widetilde{\sigma}_n} = z | \tau_A \leq \widetilde{\sigma}_n)(1 - P_x(\tau_A > \widetilde{\sigma}_n)), \end{aligned}$$

that

$$\begin{aligned} P_x(X_{\widetilde{\sigma}_n} = z | \tau_A > \widetilde{\sigma}_n) &= H_{\partial \widetilde{B}_n}(x_0, z)\left[1 + \frac{1}{P_x(\tau_A > \widetilde{\sigma}_n)} O\Big(\Big(\frac{m}{R_1}\Big)^{\nu}\Big) + O\Big(\Big(\frac{m}{R_1}\Big)^{\nu}\Big)\right] \\ &= H_{\partial \widetilde{B}_n}(x_0, z)\left[1 + O\Big(\frac{\ln n}{\ln m}\Big(\frac{m}{R_1}\Big)^{\nu}\Big)\right] \\ &= H_{\partial \widetilde{B}_n}(x_0, z)\left[1 + O\Big(\frac{\ln n}{n^{\nu'}}\Big)\right] \end{aligned}$$

where $\nu' = \nu / c_8 > 0$ and where the constant in the last $O(\cdot)$ now depends on ω and A. This can also be written as,

$$P_x(X_{\widetilde{\sigma}_n \wedge \tau_A} = z) = H_{\partial \widetilde{B}_n}(x_0, z) P_x(\tau_A > \widetilde{\sigma}_n)\left[1 + O\Big(\frac{\ln n}{n^{\nu'}}\Big)\right]. \tag{4.49}$$

Note that every path from $y \in A$ to $\partial \widetilde{B}_n$ must go through some vertex of $\partial B(x_0, m)[\widetilde{B}_n, A]$. So, for all $y \in A$ and for all $z \in \partial \widetilde{B}_n$,

$$\begin{aligned} P_y(X_{\widetilde{\sigma}_n \wedge \tau_A} = z) &= \sum_{x \in \partial B(x_0, m)[\widetilde{B}_n, A]} P_y(X_{\tau_{\partial B(x_0, m)} \wedge \tau_A} = x) P_x(X_{\widetilde{\sigma}_n \wedge \tau_A} = z) \\ &\overset{(4.49)}{=} H_{\partial \widetilde{B}_n}(x_0, z)\left[1 + O\Big(\frac{\ln n}{n^{\nu'}}\Big)\right] \\ &\qquad \times \sum_{x \in \partial B(x_0, m)[\widetilde{B}_n, A]} P_y(X_{\tau_{\partial B(x_0, m)} \wedge \tau_A} = x) P_x(\tau_A > \widetilde{\sigma}_n) \\ &= H_{\partial \widetilde{B}_n}(x_0, z)\left[1 + O\Big(\frac{\ln n}{n^{\nu'}}\Big)\right] P_y(\tau_A > \widetilde{\sigma}_n). \end{aligned}$$

Hence (4.44) holds with $N_0 = (4 \max\{R_0(x_0, \omega), (3\mu m)/4, m\})^{c_8}$. ■

Theorem II follows from lemma 4.7.

Proof. Let $y \in A$. Let $\widetilde{B}_n$ and $\widetilde{\sigma}_n$ be as in (4.40). For $x \notin \widetilde{B}_n$, by (3.23), by reversibility of the Markov chain and by (4.44), for all $n > N_0$,

$$\begin{aligned}
\pi(x)H_A(x,y) &= \pi(x)P_x(X_{\tau_A} = y) \\
&= \pi(x) \sum_{z \in \partial \widetilde{B}_n} G_{A^c}(x,z) H_{A \cup \partial \widetilde{B}_n}(z,y) \\
&= \sum_{z \in \partial \widetilde{B}_n} G_{A^c}(z,x) \pi(y) H_{A \cup \partial \widetilde{B}_n}(y,z) \\
&= \sum_{z \in \partial \widetilde{B}_n} G_{A^c}(z,x) \pi(y) P_y(\widetilde{\sigma}_n < \tau_A) H_{\partial \widetilde{B}_n}(x_0,z) \left[1 + O\left(n^{-\nu'}\right)\right] \\
&= \pi(y) P_y(\widetilde{\sigma}_n < \tau_A) \sum_{z \in \partial \widetilde{B}_n} G_{A^c}(z,x) H_{\partial \widetilde{B}_n}(x_0,z) \left[1 + O\left(n^{-\nu'}\right)\right]. \qquad (4.50)
\end{aligned}$$

At this point for the supercritical cluster of $\mathbb{Z}^d$, $d \geq 3$, it suffices to sum over $y \in A$ and divide the equations. However, since the walk is recurrent on the supercritical percolation cluster of $\mathbb{Z}^2$, $P_y(\widetilde{\sigma}_n < \tau_A) \to 0$ as $n \to \infty$, this would lead to an indeterminate limit. But by (4.50),

$$\begin{aligned}
H_A(x,y) &= \frac{\pi(x)H_A(x,y)}{\pi(x)\sum_{y' \in A} H_A(x,y')} \\
&= \frac{\pi(y)P_y(\widetilde{\sigma}_n < \tau_A)}{\sum_{y' \in A} \pi(y') P_{y'}(\widetilde{\sigma}_n < \tau_A)} \left[1 + O\left(n^{-\nu'}\right)\right]
\end{aligned}$$

and by proposition 4.6,

$$\begin{aligned}
\lim_{x \to \infty} H_A(x,y) &= \lim_{n \to \infty} \frac{(\ln n)\pi(y)P_y(\widetilde{\sigma}_n < \tau_A)}{(\ln n)\sum_{y' \in A} \pi(y') P_{y'}(\widetilde{\sigma}_n < \tau_A)} \left[1 + O\left(n^{-\nu'}\right)\right] \\
&= \frac{\pi(y) P u_A(y, x_0)}{\sum_{y' \in A} \pi(y') P u_A(y', x_0)}. \qquad (4.51)
\end{aligned}$$

■

5. Proof of proposition 2.5

In this proof, we keep the notations of [6] except for the graph distance which will still be denoted by $D(x,y)$.

For a cube Q of side n, let $Q^+ := A_1 \cap \mathbb{Z}^d$ and $Q^{\oplus} := A_2 \cap \mathbb{Z}^2$ where A_1 and A_2 are the cubes in $\mathbb{R}^2$ with the same center as Q and with side length $\frac{3}{2}n$ and $\frac{6}{5}n$ respectively. Note that $Q \subset Q^{\oplus} \subset Q^+$.

$\mathcal{C}(x)$ is the connected open cluster that contains x. $\mathcal{C}_Q(x)$, which will be called the open Q cluster, is the set of vertices connected to x by an open path within Q. And $\mathcal{C}^{\vee}(Q)$ is the largest open Q cluster (with some rule for breaking ties).

Set $\alpha_2 = (11(d+2))^{-1}$ with $d = 2$.

Proof. By [6, lemma 2.24] and by Borel-Cantelli, for all $x \in \mathbb{Z}^2$, there is N_x such that for all $n > N_x$, $L(Q)$ (see [6, p. 3052]) holds for all cubes Q of side n with $x \in Q$.

Let $z \in \mathbb{Z}^2$ and let $n > N_z = N_z(\omega)$.

Let Q be a cube of side n which contains z.

Let $x_0 \in \mathcal{C}^{\vee}(Q^+) \cap Q^{\oplus}$ with $Q(x_0, r + k_0)^+ \subset Q^+$ where $C_H n^{\alpha_2} \le r \le n$ and $k_0 = k_0(p, d = 2)$ is the integer chosen in [6, p. 3041].

Let R be such that

$$B_\omega(x_0, (3/2)R \ln R) \subset Q^{\oplus} \qquad \text{and} \tag{5.52}$$

$$(C_H n^{\alpha_2})^{d+2} \le (C_H n^{\alpha_2})^{4(d+2)} < R < R \ln R < n \quad \text{with} \;\; d = 2. \tag{5.53}$$

Then by [6, Theorem 2.18c], $B_\omega(x_0, R \ln R)$ is (C_V, C_P, C_W)- very good with

$$N_{B_\omega(x_0, R \ln R)} \le C_H n^{\alpha_2}$$

with the constants given in [6, section 2].

Then by [6, Theorem 5.11] and (5.53), there is a positive constant C_1, which depends only on the constants C_V, C_P, C_W, such that if $D(x_0, x_1) \le \frac{1}{3} R \ln R$ and if $h : \overline{B(x_1, R)} \to \mathbb{R}$ is non-negative and harmonic in $B(x_1, R)$, then

$$\max_{B(x_1, R/2)} h \le C_1 \min_{B(x_1, R/2)} h. \tag{5.54}$$

Note that since $d = 2$ and $4\alpha_2(d+2) = 4/11 < 1/2$, the conditions (5.53) are verified for $R = 2\sqrt{n}$ when n large enough.

We now apply a standard chaining argument to a well chosen covering by balls (see for instance [37, chapters 3 and 9]). Let $x_0 \in \mathbb{Z}^2$ and consider environments such that $x_0 \in \mathcal{C}_\infty(\omega)$. The main difficulty to carry out the chaining argument is to check that the intersection of "consecutive" balls is not empty. The remainder of the proof is to construct an appropriate covering of $\{x \in \mathcal{C}_\infty; D(x_0, x) = m\}$, for m large enough, with a finite number balls, which does not depend on x_0, m or ω, and such that the Harnack inequality (5.54) holds in each ball. To do so, we need an additional property of the supercritical percolation cluster. We will use the grid constructed below in lemma 5.2.

Let δ_1, δ_2 and δ_3 be three positive real numbers such that

$$2\delta_2 < \delta_1 \quad \text{and} \quad \delta_1 + 2\delta_2 < \delta_3 < \frac{1}{5\mu}\left(\frac{4}{5} - \delta_2\right). \tag{5.55}$$

For instance, choose δ_3 so that $0 < \delta_3 < 4/(50\mu)$, then choose δ_1 so that $0 < 2\delta_1 < \delta_3$ and finally choose δ_2 so that $\delta_2 < \min\{\delta_1/2, 4/(50\mu)\}$.

Let $n > N_{x_0}$.

Furthermore, take n large enough so that there is a Kesten's grid in Q with constant C_K and $R(Q)$ holds (by [6, lemma 2.8]). That is in each vertical and each horizontal strip of width $C_K \ln n$ contains at least $c(p) C_K \ln n$ open disjoint channels. Moreover, since $R(Q)$ holds, $\mathcal{C}^{\vee}(Q) \subset \mathcal{C}^{\vee}(Q^+)$. In particular, $x_0 \in \mathcal{C}^{\vee}(Q^+) \cap Q^{\oplus}$.

Furthermore by (2.6) and Borel-Cantelli, if m is large enough then for all $x, y \in \mathcal{C}_\infty$ such that $|x|_1 \le 3\mu m$, $|y|_1 \le 3\mu m$ and $|x - y|_1 \ge m(\delta_1 - 2\delta_2)/\mu$ we have

$$|x - y|_1 \le D(x, y) \le \mu |x - y|_1.$$

Set $\dfrac{R}{2} = m\delta_3 = \sqrt{n}$.

Furthermore, take m large enough so that (5.52) and (5.53) are verified as well as

$$C_K \ln n < m\delta_2/\mu, \quad 3m\mu < \frac{1}{3} R \ln R \quad \text{and} \quad r < 4m\delta_3.$$

Instead of constructing a finite covering of $\{x \in \mathcal{C}_\infty; D(x_0, x) = m\}$, it is easier to construct a finite covering of the region $\{x \in \mathcal{C}_\infty; \frac{4m}{5\mu} \leq |x - x_0|_1 \leq 2m\}$ which is a larger subset of $\mathbb{Z}^2$.

Let $\mathcal{I} := \{(i; j) \in \mathbb{N}^2; 4/(5\delta_1) \leq i + j \leq 2\mu/\delta_1.\}$ Let M be the cardinal of $\mathcal{I}$.

Let $x_{i,j} = x_0 + (im\delta_1/\mu; jm\delta_1/\mu)$ with $(i; j) \in \mathcal{I}$. Then for each $x_{i,j}$ with $(i; j) \in \mathcal{I}$, there is $\widetilde{x}_{i,j} \in \mathcal{C}_\infty$ such that $|x_{i,j} - \widetilde{x}_{i,j}|_1 \leq m\delta_2/\mu$.

We proceed similarly in the other three quadrants to obtain a set of $4M$ vertices which we denote by $\mathcal{D}$. Note that M does not depend on m.

The finite covering of the region $\frac{4m}{5\mu} \leq |x - x_0|_1 \leq 2m$ is

$$\{B(\widetilde{x}, m\delta_3), \quad \widetilde{x} \in \mathcal{D}\}.$$

Note that each ball contains the center of the four neighbouring balls except those on the boundary of the region. But these are connected to at least one neighbouring ball. Indeed, if $\widetilde{x}, \widetilde{y} \in \mathcal{D}$ are neighbouring centers then by (5.55),

$$D(\widetilde{x}, \widetilde{y}) < \mu|\widetilde{x} - \widetilde{y}|_1 < m(\delta_1 + 2\delta_2) < m\delta_3.$$

If $\widetilde{x} \in \mathcal{D}$ then by (5.55),

$$D(x_0, \widetilde{x}) > \frac{m}{\mu}\left(\frac{4}{5} - \delta_2\right) > 5m\delta_3,$$

$D(x_0, \widetilde{x}) < \mu|x_0 - \widetilde{x}|_1 < 2m\mu$ and $\mu\,(2m + m\delta_2/\mu) < 3m\mu$.

Therefore, $B(x_0, r)$ does not belong to a ball of the covering and u is harmonic in each ball $B(\widetilde{x}, 2m\delta_3)$ with $\widetilde{x} \in \mathcal{D}$. Then the Harnack inequality holds for $R = 2m\delta_3$ since for all $\widetilde{x} \in \mathcal{D}$,

$$D(x_0, \widetilde{x}) < 2m\mu < \frac{1}{3}R\ln R.$$

■

Construction of Kesten's grid

Definition 5.1. *Let* $B_{m,n} = ([0; m] \times [0; n]) \cap \mathbb{Z}^2$.

A horizontal [resp. vertical] channel of $B_{m,n}$ *is a path in* $\mathbb{Z}^2$ $(x_0, x_1, x_2, ..., x_L)$ *such that:*

- $\{x_1, x_2, ..., x_{L-1}\}$ *is contained in the interior of* $B_{m,n}$
- $x_0 \in \{0\} \times [0; n]$ *[resp.* $x_0 \in [0; m] \times \{0\}$*]*
- $x_L \in \{m\} \times [0; n]$ *[resp.* $x_L \in [0; m] \times \{n\}$*]*

We say that two channels are disjoint if they have no vertex in common. Let $N(m, n)$ be the maximal number of disjoint open horizontal channels in $B_{m,n}$.

A Kesten's grid in $[-n; n]^2$ is a set of open horizontal and vertical channels of $[-n; n]^2$ such that the horizontal channels are disjoint among themselves and similarly for the vertical channels and moreover, there is at least $c(p)C_K \ln n$ disjoint open channels in each horizontal and in each vertical strip of length n and width $C_K \ln n$ contained in $[-n; n]^2$.

Lemma 5.2. $\mathbb{P}_p$*-almost surely, for* n *large enough, there is a Kesten's grid in* $[-n; n]^2$.

Proof. By [25, Theorem 11.1], for $p > p_c$, there is a positive constant $c(p)$ and some universal constants $0 < c_9, c_{10}, \xi < \infty$, such that

$$\mathbb{P}_p\big(N(m,n) > c(p)n\big) \geq 1 - c_9(m+1)\exp(-c_{10}(p-p_c)^{\xi} n).$$

We apply this result to the number of disjoint open channels in a horizontal strip of length n and width $C_K \ln n$ contained in $[-n;n]^2$. If C_K is large enough so that $c_{10}(p-p_c)^{\xi} C_K > 3$ then

$$\sum_n nc_9(n+1)\exp(-c_{10}(p-p_c)^{\xi} C_K \ln n) < \infty. \tag{5.56}$$

Conclude with Borel-Cantelli lemma. ■

6. An example

In this section, we construct a weighted graph which is not "finitely-partite" and where there is a finite set A for which the harmonic mesure from infinity does not exist.

The first step is to construct a discrete time reversible Markov chain $X = (X_n, n \in \mathbb{N})$ on $\mathbb{N}$. At each step, X jumps at one of its two nearest neighbours except when it is at some vertices $\{x_j; j \geq 1\}$ where X stays at the same vertex with probability $\delta_j/(1+\delta_j)$.

Fix $p > q > 0$ such that $p+q = 1$. The transition probabilities of X at $x \notin \{x_j; j \geq 1\}$ are

$$P_x(X_1 = x-1) = p, \quad P_x(X_1 = x+1) = q, \quad \text{if } \ x \neq 0 \quad \text{and} \ \ P_0(X_1 = 1) = 1.$$

Let $\eta := \inf\{k \geq 1; X_k = X_{k-1}\}$ be the first time that X does not jump.

Claim. It is possible to choose $\{x_j; j \geq 1\}$ and $\{\delta_j; j \geq 1\}$ so that for all $x \in \mathbb{N}$,

$$P_x(\tau_0 < \eta) > 2/3. \tag{6.1}$$

Proof of the claim. Let $(\varepsilon_j; j \geq 1)$ be a decreasing sequence of positive real numbers such that

$$\sum_{j=1}^{\infty} \varepsilon_j < 1/4. \tag{6.2}$$

Set $x_0 = 0$ and $x_1 = 1$. If $x_2, \dots x_j$ are already chosen then δ_j and x_{j+1} will be determined by the following inductive construction.

First choose δ_j such that

$$\delta_j < \frac{\varepsilon_j}{2} P_{x_j}(\tau_{x_{j-1}} < \tau_{x_j}) \tag{6.3}$$

and add a bond from x_j to x_j with a conductance such that

$$P_{x_j}(X_1 = x_j - 1) = p/(1+\delta_j), \quad P_{x_j}(X_1 = x_j + 1) = q/(1+\delta_j) \quad \text{and} \quad P_{x_j}(X_1 = x_j) = \delta_j/(1+\delta_j).$$

then since $p > q$, it is possible to choose x_{j+1} sufficiently large so that

$$P_{x_j}(\tau_{x_{j+1}} < \tau_{x_j}) < \frac{\varepsilon_j}{2} P_{x_j}(\tau_{x_{j-1}} < \tau_{x_j}). \tag{6.4}$$

With these choices,

$$P_{x_j}(\tau_{x_{j-1}} < \eta) \quad \geq \quad P_{x_j}(\tau_{x_{j-1}} < \tau_{x_j}) + P_{x_j}(\tau_{x_j} < \tau_{x_{j-1}} \wedge \tau_{x_{j+1}} \wedge \eta) P_{x_j}(\tau_{x_{j-1}} < \eta).$$

Hence

$$\begin{aligned} P_{x_j}(\tau_{x_{j-1}} < \eta) &\geq \frac{P_{x_j}(\tau_{x_{j-1}} < \tau_{x_j})}{P_{x_j}(\tau_{x_{j-1}} < \tau_{x_j}) + P_{x_j}(\tau_{x_{j+1}} < \tau_{x_j}) + P_{x_j}(\eta \leq \tau_{x_j})} \\ &\geq 1 - \frac{P_{x_j}(\tau_{x_{j+1}} < \tau_{x_j}) + P_{x_j}(\eta \leq \tau_{x_j})}{P_{x_j}(\tau_{x_{j-1}} < \tau_{x_j})} \\ &\geq 1 - \varepsilon_j \quad \text{by (6.4) and (6.3).} \end{aligned}$$

For all $j \geq 1$, by (6.2), $P_{x_j}(\tau_0 < \eta) > \prod_{\ell=1}^{j} P_{x_\ell}(\tau_{x_{\ell-1}} < \eta) > \prod_{\ell=1}^{j}(1 - \varepsilon_\ell) > 2/3$ and since this probability is monotone decreasing, (6.1) holds for all $x \in \mathbb{N}$. ■

The example is the Markov chain $(Y_n; n \in \mathbb{N})$ on $\mathbb{N} \times \{0, 1\}$ with transition probabilities given by

$$P_{(x,i)}(Y_1 = (x+1, i)) = p, \quad P_{(x,i)}(Y_1 = (x-1, i)) = q, \quad \text{if } x \notin \{x_j; j \geq 1\}, \; i = 0, 1$$

and for $i = 0, 1$, and $j \geq 1$,

$$P_{(x_j,i)}(Y_1 = (x_j + 1, i)) = p/(1 + \delta_j), \quad P_{(x_j,i)}(Y_1 = (x_j - 1, i)) = q/(1 + \delta_j)$$

and

$$P_{(x_j,0)}(Y_1 = (x_j, 1)) = \delta_j/(1 + \delta_j) \quad P_{(x_j,1)}(Y_1 = (x_j, 0)) = \delta_j/(1 + \delta_j).$$

Let $A = \{(0, 0), (0, 1)\}$. Then the first coordinate of (Y_n) conditioned on hitting A has the same distribution as (X_n). It is a special case of Doob's h transform with $h(x) = P_0(\overline{\tau}_0 < \infty) = (q/p)^x$ (see [17] or [28, section 17.6]). Hence, by (6.1), for all $x \in \mathbb{N}$,

$$\overline{H}_A((x, 0), (0, 0)) > 2/3 \quad \text{and} \quad \overline{H}_A((x, 1), (0, 0)) < 1/3$$

and the hitting distribution of A from infinity does not exist.

This example can be slightly modified to provide an example of a recurrent reversible Markov chain with finite sets where the harmonic measure from infinity does not exist.

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Daniel Boivin
Université de Bretagne Occidentale
Laboratoire de Mathématiques CNRS UMR 6205
6 avenue Le Gorgeu, CS93837
F-29238 Brest Cedex 3
France

boivin@univ-brest.fr
http://stockage.univ-brest.fr/ boivin/

Clément Rau
Université Paul Sabatier
Institut de Mathématiques de Toulouse
route de Narbonne
31400 Toulouse
France

rau@math.ups-tlse.fr
http://www.math.univ-toulouse.fr/ rau/