EXIT TIME FOR ANCHORED EXPANSION

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Abstract. Let $(X_n)_{n\geq 0}$ be a reversible random walk on a graph G satisfying an anchored isoperimetric inequality. We give upper bounds for exit time (and occupation time in transient case) by X of any set which contains the root. As an application, we consider random environments of $\mathbb{Z}^d$.

Contents

1. Introduction

There exists a lot of connections linking the geometry of a graph and the behaviour of a random walk. One important tool is the isoperimetric profile of the graph. For example, in [5] Coulhon has shown that a control of the isoperimetry of a graph with bounded valence gives an upper bound of the iterated transition probabilities of a simple random walk evolving on this graph. The problem with uniform isoperimetric inequality is its unstability under random perturbations like percolation. Another tool, more robust, has been introduced in the two last decades by Thomassen in[14] and next by Benjamini, Lyons and Schramm in [2]. It is called *anchored or rooted isoperimetric inequality* . Here is the definition. For a graph G, we denote $V(G)$ the set of vertices and $E(G)$ the set of edges.

Definition 1.1. *Let $\mathcal{F}$ a positive increasing function defined on $\mathbb{R}_+$. Let G a graph and $o \in G$. We say that G satisfies an anchored (or rooted) $\mathcal{F}$-isoperimetric* *inequality at o if there exists a constant* $C_{\mathrm{IS}} > 0$ *such that for any connected set* A *which contains o we have:*

$$\frac{|\partial A|}{\mathcal{F}(|A|)} \geq C_{\mathrm{IS}}. \tag{1}$$

∂A *is equal to the set* $\{(x,y) \in E(G);\ x \in A$ *and* $y \notin A\}$ *and* $|B|$ *stands for the cardinal of* B.

When $\mathcal{F} = id$ and G has bounded degree there is an equivalent version of this definition which reads as follows:
G satisfies a strong anchored (or rooted) isoperimetric inequality if

$$\lim_{n\to\infty} \inf\{\frac{|\partial S|}{|S|};\ S \textit{ connected},\ v \in S \textit{ and } |S| \geq n\} := i(G)$$

is strictly positif.
This definition does not depend on the choice of the fixed vertex whereas in the previous definition, the constant C_{IS} depends on the point o. Our object here is to examine what anchored isoperimetric inequality implies for random walk.

1.1. **What we know for anchored expansion.**

The first result known for rooted $\mathcal{F}$-isoperimetric inequality is due to Thomassen. In [14], it is proved that a the simple random walk on a graph G is transient if G satisfies a rooted $\mathcal{F}$-isoperimetric inequality such that $\sum_k \mathcal{F}(k)^{-2} < \infty$. The main step of the proof is to extract a subdivision of the dyadic tree from the initial graph. Then, thanks to hypothesis, it is possible to construct a finite flow on the tree, which proves that the tree is transient.

It was long afterwards that other results did appear for anchored expansion. In 2000 Virag has studied the case of strong anchored isoperimetric inequality. In [15], it is proved that strong anchored isoperimetric inequality on graphs with bounded geometry, implies a positive lim inf speed. Moreover Virag proves that in this case, transitions probability at time n of the random walk are bounded by $e^{-n^{1/3}}$.

Later, still when $\mathcal{F} = id$, Chen and Peres have proved that if G satisfies a strong anchored isoperimetric inequality then so does every infinite cluster of independant percolation with parameter p sufficiently closed to 1. Next, they have shown that strong anchored expansion is preserved under a random stretch if, and only if, the stretching law has an exponential tail. They also proved that for a supercritical Galton Watson tree $\mathbb{T}$ given nonextinction, we have $i(\mathbb{T}) > 0$ a.s.

1.2. **What we don't know. Open questions.**

There is an important collection of conjectures relating to anchored expansion. Here is some of them:
Question 1: does the sub tree of Thomassen satisfy an anchored Isoperimetric inequality ?
Question 2: does a general anchored isoperimetric inequality imply an upper bound of $p_n(x,y)$?

1.3. **Continuous space setting.** The paper is written in the discrete space setting of graphs. The reason is that anchored isoperimetric inequality is a natural tool in random media and is therefore more associated with this setting. In fact the continuous setting (of Riemannian manifolds for instance) works as well, and may be, the proofs are far more readable. As both an introduction to our technicals and an illustration of what the continuous setting results would look like, we begin with a key result written in this setting. Details, especially from potential theory, will only appear later in the paper for graphs.

Let M be a Riemannian manifold with an anchored isoperimetric inequality at root o, that is (1) for finite volume smooth connected domains A containing o. Precisely, $|A| = m(A)$ for the Riemannian volume element m and $|\partial A| = \mu(\partial A)$ for the Riemannian volume element μ on the smooth submanifold ∂A.
Now let fix some A and consider the Brownian motion on M starting at o and killed when hitting ∂A at time τ_A. We denote p_t^A its submarkovian kernel, A_s the level sets of Green function and $u(s)$ their measures.

$$A_s = \left\{ x \in A, G^A(x) = \int_0^\infty p_t^A(x)\ \mathrm{d}t \geq s \right\}, \qquad u(s) = m(A_s).$$

Thanks to harmonic properties of G^A, these level sets are connected and contain the root. Thus, they will also satisfy (1). In the following we use μ for any s and also ν denoting the inward unit normal vector field on ∂A_s. The inward direction is chosen to have G^A increasing.

Theorem 1.2. *The anchored isoperimetric inequality yields a differential inequation*

$$u'(s) \leq -\Big(C_{\mathrm{IS}}\mathcal{F}(u(s))\Big)^2.$$

This naturally leads to upper estimates of $u(s)$ and $\mathbb{E}(\tau_A) = \int_0^\infty u(s)\ \mathrm{d}s$.
For instance if $\mathcal{F}(u) = u^{1-1/d}$, $\mathbb{E}(\tau_A) \leq Cm(A)^{2/d}$,
and if $\mathcal{F}(u) = u$, $\mathbb{E}(\tau_A) \leq C \ln m(A)$.

Proof. Schwarz inequality

$$\Big(C_{\mathrm{IS}}\mathcal{F}(u(s))\Big)^2 \leq \mu(\partial A_s)^2 = \left(\int_{\partial A_s} \mathrm{d}\mu\right)^2 \leq \int_{\partial A_s} \frac{\partial G^A}{\partial \nu}\mathrm{d}\mu \int_{\partial A_s} \frac{\mathrm{d}\mu}{\partial G^A/\partial\nu}$$

involves the flow

$$\int_{\partial A_s} \frac{\partial G^A}{\partial \nu}\mathrm{d}\mu = 1$$

and the derivative of u since whith the co-area formula,

$$u(s) = \int_{G^A \geq s} \mathrm{d}m = \int_s^\infty \left(\int_{G^A = t} \frac{\mathrm{d}\mu}{\partial G^A/\partial\nu} \right) \mathrm{d}t.$$

This yields the differential inequation.

For $\mathcal{F}(u) = u^{1-1/d}$, computations may be avoided if we compare with the case when A is a ball of radius R in $\mathbb{R}^d$. Then all inequalities are equalities and the result should be that $\mathbb{E}(\tau_A)$ is like R^2. $\square$

1.4. **Results of the paper.**

Let G be a graph and o one particular vertex. Consider a random walk $(X_n)_{n\geq 0}$ on G with transition probability $p(.,.)$ and assume there exists a reversible measure m for X. We use the symmetric kernel $\mu(x,y) := m(x)p(x,y)$ to measure surfaces:

$$\forall A \subset G, \qquad \mu(\partial A) = \sum_{x\in A, y\notin A} \mu(x,y).$$

In this setting the anchored isoperimetric inequality reads:

Definition 1.3. *We say G satisfies the anchored isoperimetric inequality at root o with increasing function $\mathcal{F}$ when for any connected $o \in A \subset G$,*

$$\frac{\mu(\partial A)}{\mathcal{F}(m(A))} \geq C_{IS}. \tag{2}$$

"Connected" means that one can find a discrete path in A between any two points for which $p(x_i, x_{i+1})$ is positive when x_i, x_{i+1} are following points.

No distance will play a role here and the graph is not assumed to be locally finite.

We denote $\mathbb{P}_x$ [resp $\mathbb{E}_x$] the law of the walk starting from point x [resp the expectation], τ_A the exit time and l_A the occupation time (which may be infinite if X is not transient):

$$\tau_A = \inf\{k \geq 0 \ ; \ X_k \notin A\}, \qquad l_A = \mathrm{card}\{k \in \mathbb{N} \ ; \ X_k \in A\}.$$

Theorem 1.4. *If G satisfies (2), then for any subset A we have:*

$$\mathbb{E}_o(\tau_A) \leq 2\int_0^\infty v_+^A(s)\ \mathrm{d}s \tag{3}$$

$$\textit{and} \qquad \mathbb{E}_o(l_A) \leq 2\int_0^\infty v_{+A}(s)\ \mathrm{d}s, \tag{4}$$

where v^A, v are solutions of $\begin{cases} v^A(0) = m(A) \\ (v^A)' = -(C_{IS}\mathcal{F}(v^A))^2, \end{cases}$ *and* $\begin{cases} v(0) = +\infty \\ v' = -(C_{IS}\mathcal{F}(v))^2. \end{cases}$
The truncations in indices mean

$$v_+^A(s) = \begin{cases} 0 \ \textit{if}\ v^A(s) \leq 0 \\ v^A(s)\ \textit{otherwise.} \end{cases} \quad \textit{and}\ v_{+A}(s) = \begin{cases} 0\ \textit{if}\ v(s) \leq 0 \\ m(A)\ \textit{if}\ v(s) \geq m(A) \\ v(s)\ \textit{otherwise.} \end{cases}$$

For comparison when X is transient, note that

$$\int_0^\infty v_+^A(s)\ \mathrm{d}s = \int_{v^{-1}(m(A))}^\infty v_+(s)\ \mathrm{d}s.$$

We consider usual functions $\mathcal{F}$ in Section 3.1.1. It is sometimes useful to precise the values $\mathcal{F}(x) = \mathcal{F}(m(o))$ for $x \leq m(o)$, which is justified in Proposition 2.4.

2. Green functions.

2.1. **Definitions and harmonicity.** The submarkovian kernel of the killed random walk is $p^A(x,y) = \begin{cases} p(x,y) & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$
Although Theorem 1.4 is true for A non connected, we have in this section to assume A is connected. When X is transient, Green function may be defined for the non-killed random walk and we can consider $A = G$ (or the connected component of o if G was not connected, which would have little interest). This leads to the result for l_A in next section.

The discrete Laplacian is

$$\triangle^A f = (Id - P^A) f,$$

where P^A is the operator defined on functions which are zero outside A by

$$\begin{aligned} P^A f(x) &= \mathbb{E}_x(f(X_1)\ 1_{\{x, X_1 \in A\}}) \\ &= \sum_{y \in A} p^A(x,y) f(y). \end{aligned}$$

The Green function is

$$G^A(x,y) = \frac{1}{m(y)} \sum_{k \geq 0} \mathbb{P}_x^A(X_k = y).$$

In particular we denote $G^A(x) = G^A(o,x)$. Note that $G^A(x) = 0$ if $x \notin A$.
Recall that reversibility means $p(x,y)/m(y) = p(y,x)/m(x)$. In other words $p(x,y)/m(y)$ is the precise analog of a density kernel in y starting from x and is symmetric. This explains the factor $1/m(y)$ in the definition of G^A which is symmetric for $x, y \in A$.

Proposition 2.1. $\triangle^A G^A = \frac{\delta_0}{m(0)}$

Proof. For all $x \in A$ we have :

$$\begin{aligned} \triangle^A G^A(x) &= [(Id - P^A)(G^A)](x) \\ &= \frac{1}{m(x)} \sum_{k \geq 0} \mathbb{P}_o^A(X_k = x) - \sum_{k \geq 0} \sum_{y \in A} \frac{p^A(x,y)}{m(y)} \mathbb{P}_o^A(X_k = y) \\ &= \frac{1}{m(x)} \sum_{k \geq 0} \mathbb{P}_o^A(X_k = x) - \sum_{k \geq 0} \sum_{y \in A} \frac{p^A(y,x)}{m(x)} \mathbb{P}_o^A(X_k = y) \\ &= \frac{1}{m(x)} \sum_{k \geq 0} \mathbb{P}_o^A(X_k = x) - \sum_{k \geq 0} \frac{1}{m(x)} \mathbb{P}_o^A(X_{k+1} = x) \\ &= \frac{\mathbb{P}_o^A(X_0 = x)}{m(x)} \\ &= \frac{\delta_0(x)}{m(0)} \end{aligned}$$

And for $x \notin A$, we have $\triangle^A G^A(x) = 0$. □

Corollary 2.2. *G^A is harmonic on $A \smallsetminus o$. As a consequence the level sets $A_s = \{x \in A\ ; G^A(x) \geq s\}$ are connected and contain o. Moreover the inward flow* *through any ∂A_s is 1 or more generally for any $B \subset A$:*

$$\sum_{(x,y)\in\partial B} \mu(x,y)\nabla_{(y,x)}G^A = 1_{\{o\in B\}}. \tag{5}$$

The surface notations are $\partial B = \{(x,y)\ ; x \in B, y \notin B\}$ and $\nabla_{(y,x)}f = f(x) - f(y)$.

Proof. For all $x \in A$, Propostion 2.1 may be written

$$\sum_{y\in G} p^A(x,y)(G^A(x) - G^A(y)) = \frac{\delta_0(x)}{m(0)}.$$

Summing over x in B with respect to m we get

$$\sum_{x\in B}\sum_{y\in G} m(x)p^A(x,y)(G^A(x) - G^A(y)) = 1_{\{o\in B\}}.$$

Now the usual integration by parts becomes in this discrete summation a cancellation of terms by symmetry when y also belongs to B. Only (5) remains.

Maximum principle and properties of level sets A_s may be extracted from this result when $o \notin B$. In this case the flow is 0 so there must be an edge x, y with $G^A(y) \geq G^A(x)$. This leads to a contradiction if there was a connected component of A_s not containing o. □

2.2. **Differential inequation.** We use a linearized version of $m(A_s)$, namely

$$u(s) = \sum_{x\in A_s, y\in G} \mu(x,y)\frac{G^A(x) - \max\{s, G^A(y)\}}{G^A(x) - G^A(y)}.$$

For $x \in A_s$ such that $\mu(x,y) > 0 \Rightarrow y \in A_s$, the contribution of x is indeed $m(x)$. Furthermore $u(s) \leq m(A_s)$. The reason for this definition is to have:

Lemma 2.3. *Piecewise linear function u has left derivative*

$$u'(s) = -\sum_{(x,y)\in\partial A_s} \frac{\mu(x,y)}{\nabla_{(y,x)}G^A}.$$

Proof. Variation in s in the definition of $u(s)$ comes from the y's such that $G^A(y) < s$, that is $y \notin A_s$. This is clear but note that it uses $G^A \equiv 0$ outside A and this would not be correct for small values of s and the $\tilde{u}$ at page 8 when occupation time is considered. □

Proposition 2.4. *If G satisfies (2), then:*

$$u' \leq -(C_{\mathrm{IS}}\ \mathcal{F}(u))^2.$$

Proof. Same Schwarz inequality as for Theorem 1.2:

$$\begin{aligned}
\Big(C_{\mathrm{IS}}\mathcal{F}(u(s))\Big)^2 &\leq \Big(C_{\mathrm{IS}}\mathcal{F}(m(A_s))\Big)^2 \\
&\leq \mu(\partial A_s)^2 \\
&\leq \left(\sum_{(x,y)\in A_s} \mu(x,y)\nabla_{(y,x)}G^A\right)\left(\sum_{(x,y)\in\partial A_s} \frac{\mu(x,y)}{\nabla_{(y,x)}G^A}\right) \\
&= -u'(s).
\end{aligned}$$

This is of course correct when $u > 0$, that is when A_s is not empty and contains o. It works therefore with $\mathcal{F}(x) = \mathcal{F}(m(o))$ for $x \le m(o)$. $\square$

3. Applications

3.1. Exit time.

Lemma 3.1. *For any set A we have:*

(i) $\mathbb{E}_o(\tau_A) = \sum_{x\in A} m(x)G^A(x)$,
(ii) $\mathbb{E}_o(l_A) = \sum_{x\in A} m(x)G(x)$ *in the transient case.*

Proof. Given a path $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_n)$ from $\gamma_0 = o$ to A^c, that is only $\gamma_n \notin A$, we denote its probability $\mathbb{P}(\gamma) = p(\gamma_0, \gamma_1) \dots p(\gamma_{n-1}, \gamma_n)$. Its length $l(\gamma) = n = \sum_{x\in A} N_x(\gamma)$ where $N_x(\gamma)$ is the number of indices i such that $\gamma_i = x$. This yields *(i)* since

$$\mathbb{E}_o(\tau_A) = \sum_\gamma l(\gamma)\mathbb{P}(\gamma) \quad \text{and} \quad G^A(x) = \frac{1}{m(x)} \sum_\gamma N_x(\gamma)\mathbb{P}(\gamma).$$

We adapt this argument to prove *(ii)*. We keep $\gamma_n \notin A$ and $\gamma_{n-1} \in A$ but we may have $\gamma_i \notin A$ for $i < n-1$. The probability of the path is not easy to compute but denotes

$$\mathbb{P}(\gamma) = \mathbb{P}_0(\forall i \le n, X_i = \gamma_i \text{ and } \forall i \ge n, X_i \notin A).$$

We also replace the length $l(\gamma)$ by the natural occupation time $N_A(\gamma)$. $\square$

Now we could use $\sum_{x\in A} m(x)G^A(x) = \int_0^\infty m(A_s)\,\mathrm{d}s$. It is a little more intricate since we have control on u which is a linearized version of $m(A_s)$.

Lemma 3.2. *For any set A we have:*

$$\int_0^\infty u(s)\,\mathrm{d}s = \sum_{x\in A, y\in G} \mu(x,y) \min\left\{G^A(x), \frac{G^A(x)+G^A(y)}{2}\right\}.$$

Proof. From the definition of u we just have to compute carefully

$$\int_0^\infty \frac{G^A(x) - \max\{s, G^A(y)\}}{G^A(x) - G^A(y)} 1_{x\in A_s}\,\mathrm{d}s.$$

$\square$

We now have completed the proof of (3) in Theorem 1.4. Factor 2 in the right-hand sides comes from

$$\begin{aligned}\sum_{x\in A} m(x)G^A(x) &= \sum_{x\in A, y\in G} \mu(x,y)G^A(x)\\ &\le 2 \sum_{x\in A, y\in G} \mu(x,y) \min\left\{G^A(x), \frac{G^A(x)+G^A(y)}{2}\right\}.\end{aligned}$$

As far as (3) is concerned, the result first for A connected is clearly sufficient.

To prove (4), we first use the differential inequation with $A = G$, that is we obtain $u(s) \le v(s)$ for

$$u(s) = \sum_{G(x)\ge s, y\in G} \mu(x,y) \frac{G(x) - \max\{s, G(y)\}}{G(x) - G(y)}.$$

Then we argue (here A is not necessarly connected)

$$\mathbb{E}_o(l_A) \leq 2 \sum_{x\in A, y\in G} \mu(x,y) \min\left\{G(x), \frac{G(x)+G(y)}{2}\right\} \leq 2\int_0^\infty \tilde{u}(s)\ \mathrm{d}s,$$

where

$$\tilde{u}(s) = \sum_{x\in A_s, y\in G} \mu(x,y)\frac{G(x)-\max\{s,G(y)\}}{G(x)-G(y)}.$$

It is clear that $\tilde{u}(s) \leq u(s) \leq v(s)$ and $\tilde{u}(s) \leq m(A)$.

3.1.1. *Examples of $\mathcal{F}$ functions.* If $\mathcal{F}(x) = x^{1-1/d}$ as in $\mathbb{Z}^d$ then Theorem 1.4 gives

$$\begin{aligned} \mathbb{E}(\tau_A) &\leq \frac{d}{C_{\mathrm{IS}}^2} m(A)^{2/d} \\ \text{and } \mathbb{E}(l_A) &\leq \frac{d^2}{C_{\mathrm{IS}}^2(d-2)} m(A)^{2/d} \text{ for } d>2. \end{aligned}$$

Indeed for $d>2$ the Thomassen criterium implies the transience, see below. The computations involve

$$\begin{aligned} v^A(s) &= \left(m(A)^{\frac{2-d}{d}} - C_{\mathrm{IS}}^2 \frac{2-d}{d}s\right)^{\frac{d}{2-d}} \quad \text{for } d\neq 2, \\ v(s) &= \left(C_{\mathrm{IS}}^2\frac{d-2}{d}s\right)^{\frac{-d}{d-2}} \quad \text{for } d>2 \\ \text{and } v^A(s) &= m(A)e^{-C_{\mathrm{IS}}^2 s} \text{ for } d=2. \end{aligned}$$

If $\mathcal{F}(x) = x$ as in a non-amenable graph then Theorem 1.4 gives

$$\begin{aligned} \mathbb{E}(\tau_A) &\leq \frac{1}{C_{\mathrm{IS}}^2}\left(1+2\ln\frac{m(A)}{m(o)}\right) \\ \text{and } \mathbb{E}(l_A) &\leq \frac{1}{C_{\mathrm{IS}}^2}\left(3+2\ln\frac{m(A)}{m(o)}\right). \end{aligned}$$

Here we need the precision $\mathcal{F}(x) = m(o)$ for $x \leq m(o)$ so that

$$\frac{1}{v^A(s)} = \frac{1}{m(A)} + C_{\mathrm{IS}}^2 s$$

does not arise any issue of integration for $s\to\infty$.

We can summarize these computations in:

Proposition 3.3. *Let G a graph satisfying a weighted anchored isoperimetric inequality with function $\mathcal{F}$ and anchored expansion constant C_{IS} (see (2)). Then, there exists constants $c(d)$ and c such that:*

- *if $\mathcal{F}(x) = x^{1-\frac{1}{d}}$ $(d\geq 3)$ we have: $\mathbb{E}_o(l_A) \leq c(d)\ m(A)^{\frac{2}{d}}$,*
- *if $\mathcal{F}(x) = x^{\frac{1}{2}}$ $(d=2)$ we have: $\mathbb{E}_o(\tau_A) \leq c(d)\ m(A)$,*
- *if $\mathcal{F}(x) = x$ we have: $\mathbb{E}_o(\tau_A) \leq \mathbb{E}_o(l_A) \leq c\ \ln(m(A))$.*

Remark 3.4. *These inequalities are sharped. Take the particular case where G satisfies a not ancored isoperimetric inequality.*

3.2. **Transience.** We retrieve Thomassen result's cited in the introduction. Indeed, proposition 2.4 provides a new proof of the transience of the random walk under the sommable assumption on $\mathcal{F}$ without introducing the complex construction of dyadic subtrees of Thomassen. Assume

$$\int_1^{+\infty} \frac{1}{\mathcal{F}(n)^2} < +\infty, \tag{6}$$

for $\mathcal{F} : \mathbb{R}_+ \to \mathbb{R}_+^\star$, not decreasing, with $\mathcal{F}(0) = 0$ and let us prove transience with the help of proposition 2.4.

Let A a connected subset of G containing the origin (A is intended to grow up) and consider random walk killed whenever it leaves A and the associated Green function G_A. Integrating the differential equation (**??**) between time 0 and t gives:

$$\int_{u(t)}^{u(0)} \frac{ds}{\mathcal{F}(s)^2} \geq C_{IS}^2\ t. \tag{7}$$

$\int_1^{u(0)} \frac{ds}{\mathcal{F}(s)^2}$ is bounded by a constant independant of A. Indeed, thanks to hypothesis (6), for all subset A we have: $\int_1^{u(0)} \frac{ds}{\mathcal{F}(s)^2} = \int_1^{m(A)} \frac{1}{\mathcal{F}(s)^2} ds \leq \int_1^{+\infty} \frac{ds}{\mathcal{F}(s)^2} < +\infty$. So for large enough t which depends only on C_{IS} and $\mathcal{F}$, inequality (7) turns into:

$$\int_{u(t)}^{1} \frac{ds}{\mathcal{F}(s)^2} \geq \frac{1}{2} C_{IS}^2\ t.$$

Then, we deduce that:

$$\lim_{t\to+\infty} u(t) = 0 \ \text{ uniformly in } A.$$

In particular, there exists t_0 independant of A such that for all $t \geq t_0$, $u(t) < \inf_G m$. Therefore by definition of u we get that for all set A, $G_A \leq t_0$. Now we can make A growing and finally we deduce that $G < +\infty$ so the walk is transient.

3.3. **Speed.** When $\mathcal{F} = id$, the upper bound of the exit time gives us that the speed of the random walk is strictly positif. We retrieve a weak version of Virag's result. We assume in this subsection that the graph has uniformly localy bounded valency. Let $d(a, b)$ denote the graph distance between point a and b.

Proposition 3.5. *Let G a graph satisfying 1 with $\mathcal{F} = id$ and let $(X_n)_n$ a simple random walk on G. Then we have:*

$$\mathbb{P}(\lim_n \frac{d(o, X_n)}{n} = 0) = 0.$$

Proof. Suppose there exists $\epsilon > 0$ such that $\mathbb{P}(\lim_n \frac{d(o,X_n)}{n} = 0) > \epsilon$. So, we have:

$$\forall \alpha > 0 \quad \mathbb{P}(\exists N_\alpha\ \forall n \geq N_\alpha \quad \frac{d(o, X_n)}{n} \leq \alpha) > \epsilon$$

By considering the event $E_q = \{\exists N_\alpha < q,\ \forall n \geq N_\alpha \quad \frac{d(o,X_n)}{n} \leq \alpha\}$ and by continuity of measure $\mathbb{P}$, we get:

$$\exists N_\alpha \geq 0 \quad \mathbb{P}(\forall n \geq N_\alpha \quad \frac{d(o, X_n)}{n} \leq \alpha) > \frac{\epsilon}{2} \tag{8}$$

Take now $R > 0$, we have:

$$\mathbb{P}(\forall n \in [N_\alpha; \frac{R}{\alpha}] \quad d(o, X_n) < \alpha n) > \frac{\epsilon}{2}$$

On this event we have: $l_{B(o,R)} \geq \frac{R}{\alpha} - N_\alpha$, where l_A is the local time of X in the set A, which is well defined in this case since when $\mathcal{F} = id$ the walk is transient by Thomassen result. Therefore, by using (8), we get

$$\mathbb{E}_o(l_{B(o,R)}) \geq \frac{\epsilon}{2}(\frac{R}{\alpha} - N_\alpha) \tag{9}$$

By prop... and since strong anchored isoperimetric inequality implies a subexponential volume growth, there exists $c > 0$ such that:

$$\mathbb{E}_o(l_{B(o,R)}) \leq \ln(|B(o,R)|) \leq cR \tag{10}$$

Choose now α such that $\frac{\epsilon}{2\alpha} > c$. Gathering (9)and (10), we get:

$$\frac{\epsilon}{2}(\frac{R}{\alpha} - N_\alpha) \leq cR$$

Letting R goes to infinity in this last expression, we get a contradiction. □

3.4. **Random environments of $\mathbb{Z}^d$.** We consider discrete time, nearest-neighbor random walk among random (i.i.d) conductances in $\mathbb{Z}^d, d \geq 2$. We assume conductances are bounded from above but we do not require they are bounded from below. After a presentation of random environment in the first subsection, we prove an isoperimetric inequality for big sets in the second part, which enable us to bound the occupation time for sets of volume large enough.

3.4.1. *Random environments, random walks.* Consider the graph $\mathcal{L}^d = (\mathbb{Z}^d, E_d)$ where E_d be the set of non-oriented nearest-neighbor pair. We write $x \sim y$ if $(x,y) \in E_d$. An environment is a function $\omega : E_d \to [0; +\infty[$. Since edges in E_d are not oriented, the edge (x,y) is identified with the edge (y,x) and so it is implicit in the definition that environment are symmetric, ie $\omega(x,y) = \omega(y,x)$ for any pair of neighbors x and y. The value $\omega(x,y)$ is called the conductance of edge (x,y). We always assume that our environment are uniformly bounded from above. Without loss of generality, we may assume that an environment is a function $\omega : E_d \to [0,1]$.

Let $\Omega = [0,1]^{E_d}$ for the set of environments. Let Q be a product probability measure on Ω such that the family $(\omega(e))_{e \in E_d}$ forms independant identically distributed random variables. (Thus, Q is invariant by translation.)

Starting from a point x, a walker or a electric currant can cross only edges with strictly positive conductances. So we call a cluster of the environment ω a connected component of the graph $(\mathbb{Z}^d, \{e \in E_d; \omega(e) > 0\})$. The random variables $(1_{\{\omega(e)>0\}}; e \in E_d)$ are independant Bernoulli variables with parameter $q = Q(\omega(e) > 0)$. By percolation theory, we know that there exists a critical value $p_c = p_c(d) \in]0;1[$ such that for $q < p_c$, Q a.s all cluster of ω are finite and for $q > p_c$, Q a.s there is a unique infinite cluster. We shall assume that the law Q satisfies

$$q = Q(\omega(e) > 0) = 1. \tag{11}$$

X will design the random walk on the graph $\mathcal{L}_d$ starting from the origin with transitions probabilty given by:

$$p^\omega(x,y) = \frac{\omega(x,y)}{\sum_{z \sim x} \omega(x,z)}.$$

We denote by $\mathbb{P}_0^\omega$ the law of X and by $\mathbb{E}_0^\omega$ its expectation. The random walk X admits reversible measures which are proportional to the measure m^ω defined by:

$$m^\omega(x) = \sum_{z\sim x} \omega(x,z).$$

Notice that the usual kernels, μ^ω defined by $\mu^\omega(x,y) = m^\omega(x)p^\omega(x,y)$ are merely equal to $\omega(x,y)$. We keep on using notation μ^ω for convenience.

3.4.2. *(Anchored) isoperimetric inequality.* In order to apply our previous results for the occupation time on a random environment context, we look after a anchored isoperimetric inequality on the graph $\mathcal{L}_d$ with respect to weight ω which enables us to get an estimate of the correct order of the exit time (or occupation time). Differents forms of strong isoperimetric inequality have been raised by many authors (see [8], [12], [11] and [3]) in the percolation context. We use the same kind of idea in random environments but the lack of strictly positive bound from below, enables us to have only a control for big sets. The form which seems to be adapted to our problem and that we are going to prove is the following:

Proposition 3.6. *Let Q a law on environments such that $Q(\omega(e) > 0) = 1$. There exists $\beta_0(Q,d) > 0$ such that Q a.s for all environment ω, there exists $N_0(\omega) \in \mathbb{N}$ such that for all connected sets A which contained 0*

$$(m^\omega(A) \geq N_0(\omega) \Rightarrow \frac{\mu^\omega(\partial_{\mathcal{L}_d} A)}{m^\omega(A)^{1-\frac{1}{d}}} \geq \beta_0,) \tag{12}$$

where $\partial_{\mathcal{L}_d} A = \{(x,y) \in E^d;\ \omega(x,y) > 0,\ x \in A \text{ and } y \notin A\}$.

Proof. Let $A \subset \mathbb{Z}^d$, connected and containing the origin. Let B the infinite connected component of $\mathbb{Z}^d - A$ in graph $\mathcal{L}^d$ and let A' the complementary of B in $\mathbb{Z}^d$, ie: $A' = \mathbb{Z}^d - B$. Then we have:

(i) A' and $\mathbb{Z}^d - A'$ are connected in $\mathcal{L}^d$,
(ii) $A \subset A'$ so $m^\omega(A) \leq m^\omega(A')$,
(iii) $\{e \in \partial_{\mathcal{L}^d} A';\ \omega(e) > 0\} \subset \partial_{\mathcal{L}_d} A$.

Thus, it is sufficient to prove isoperimetric inequality for set A'.

For $n \in \mathbb{N}$, let $E_d(n)$ the set of edges with both end points in $[-n,n]^d$. Consider now the events:

$$\mathcal{A}_n = \{\exists F \subset E_d(n);\ \frac{\sum_{e\in F}\omega(e)}{|F|} \leq \beta,\ F\ \ast\text{-connected and } |F| \geq \ln(n)^{3/2}\},$$

where β is a constant strictly positive which be adjusted later.

Let λ and ξ two constants strictly positives. For a given set of edges $F \subset E_d(n)$, since the random variables $(\omega(e), e \in F)$ are i.i.d and by the Bienaymee Tchebytchef inequality applied to each term $\mathbb{E}_Q(e^{-\lambda\omega(e)})$ for $e \in F$ ($\mathbb{E}_Q$ designs the expectation in regard to law Q.)we have:

$$Q(\frac{\sum_{e\in F}\omega(e)}{|F|} \leq \beta) \leq e^{\lambda\beta|F|}\ [Q(\omega \geq \xi)(e^{-\lambda\xi} - 1) + 1]^{|F|}.$$

Since the number of $\ast$-connected sets of cardinality m included in $E_d(n)$ is bounded by e^{am} (for some constant $a > 0$ which depends only on d), we deduce that:

$$Q(\mathcal{A}_n) \leq \sum_{m \geq \ln(n)^{3/2}} e^{am} e^{\lambda\beta m}\ [Q(\omega \geq \xi)(e^{-\lambda\xi} - 1) + 1]^m.$$

Let $-\gamma = a + \lambda\beta + \ln[Q(\omega > \xi)(e^{-\lambda\xi} - 1) + 1]$.
First, we choose $\xi > 0$ such that $Q(\omega > \xi)$ is closed to 1, which is possible due to assumption (11). Then, we choose $\lambda > 0$ and then $\beta > 0$ small enough such that $\gamma > 0$.

Thus, we get

$$Q(\mathcal{A}_n) \leq e^{-\gamma \ln(n)^{3/2}}.$$

Since this expression in summable in n, by the Borel-Cantelli lemma we deduce that :
for Q a.s ω there exists $n_0(\omega)$ such that for $n \geq n_0(\omega)$, for all $*$-connected set $F \subset E_d(n)$ with $|F| \geq \ln(n)^{3/2}$, we have:

$$\frac{\sum_{e\in F}\omega(e)}{|F|} \geq \beta. \tag{13}$$

Now the set of edges $\partial_{\mathcal{L}_d} A'$ satisfies the three following points :
a. $\partial_{\mathcal{L}_d} A'$ is $*$-connected
b.

$$\begin{aligned} |\partial_{\mathcal{L}_d} A'| &\geq C_d|A'|^{1-1/d} \quad (C_d \text{ is the classical isoperimetric constant in } \mathcal{L}_d.) \\ &\geq C_d|A|^{1-1/d} \\ &\geq \ln(|A|)^{3/2} \quad \text{if } |A| \text{ larger than a certain } n_1. \end{aligned}$$

c. $\partial_{\mathcal{L}_d} A' \subset E_d(|A|)$.

Since $m^\omega(A) \leq 2d|A|$, if we let $N_0(\omega) = 2d\ \max(n_1, n_0^\omega)$ and if $m^\omega(A) \geq N_0(\omega)$, we can apply (13) to $F = \partial_{\mathcal{L}_d} A'$. So we have:

$$\sum_{e\in\partial_{\mathcal{L}_d} A'} \omega(e) \geq \beta|\partial_{\mathcal{L}_d} A'|.$$

Since,

$$|\partial_{\mathcal{L}_d} A'| \geq C_d|A'|^{1-1/d} \geq \frac{C_d}{(2d)^{1-1/d}} m^\omega(A')^{1-1/d} \geq \frac{C_d}{(2d)^{1-1/d}} m^\omega(A)^{1-1/d}$$

and

$$\sum_{e\in\partial_{\mathcal{L}_d} A} \omega(e) \geq \sum_{e\in\partial_{\mathcal{L}_d} A'} \omega(e).$$

We deduce there exists $\beta_0 > 0$ such that:

$$\sum_{e\in\partial_{\mathcal{L}_d} A} \omega(e) \geq \beta_0 m^\omega(A)^{1-1/d}.$$

□

Remark 3.7. *Let ω a fixed environment and $N_0(\omega)$ as in proposition 3.6. Since $Q(\omega > 0) = 1$, there is a finite number of sets B containing 0 and satisfying $m^\omega(B) \leq N_0(\omega)$. Thus for a set A such that $m^\omega(A) \leq N_0(\omega)$, we can have*

$$\mu^\omega(\partial_{\mathcal{L}_d} A) \geq c_\omega := \min\{\sum_{e\in\partial_{\mathcal{L}_d} B} \omega(e);\ 0 \ni B \text{ such that } m^\omega(B) \leq N_0(\omega)\} > 0.$$

This can be re written as well as follow:

$$\mu^\omega(\partial_{\mathcal{L}_d} A) = \sum_{e\in\partial_{\mathcal{L}_d} A} \omega(e) \geq \beta_\omega\ m^\omega(A)^{1-1/d}, \tag{14}$$

with $\beta_\omega = c_\omega / N_0(\omega)^{1-1/d}$*, constant which depends on* ω*.*

Remark 3.8. *In* [10]*, it is proved that we can build environments where the return probability is greater than* $1/n^2$*. By our proposition 3.6, the d-dimensional anchored isoperimetric inequality is satisfied on these environments and so in dimension higher than* 4*, no one can hope to prove that in this case, the return probability is in* $1/n^{d/2}$*.*

3.4.3. *Upper bound for the occupation time.* We apply result of Theorem 1.4 in the particular case of random walk on random environment satisfying asumption (11). Let $B \subset \mathbb{Z}^d$ connected and which contains the origin. We are going to estimate $\mathbb{E}(\tau_B)$ (or $\mathbb{E}(l_B)$ in transient case).

(i) **case** $d \geq 3$

By our isoperimetic inequality (proposition 3.6 and remark 3.7), and by result of Thomassen, we deduce that the walk is transcient. So we can deal with G the whole Green fonction. For $t \geq 0$ we let

$$u(t) = m^\omega(\{x \in B;\ G(0,x) \geq t\}).$$

By proposition 2.4 and thanks to inequality (19), function u satisfies:

$$\begin{cases} u(0) = m^\omega(B) \\ u' \leq -(\beta_0\ u^{1-\frac{1}{d}})^2, \text{ until } u \geq N_0(\omega) \end{cases}$$

Assume volume of B is large enough. In a first time, assume:

$$(H1) \qquad\qquad m^\omega(B) \geq N_0(\omega).$$

Solving this differential equation, we get:

$$u(t) \leq \left[\frac{d-2}{d}\beta_0^2 t + m^\omega(B)^{\frac{2}{d}-1} \right]^{\frac{d}{2-d}} \quad \text{if } t \leq t_0 \tag{15}$$

with

$$t_0 = \frac{1}{\beta_0^2}\frac{d}{d-2}\left(N_0(\omega)^{\frac{2}{d}-1} - m^\omega(B)^{\frac{2}{d}-1}\right)$$

Now Corollary 3.2 gives us the expectation of the occupation time. We have:

$$\mathbb{E}_0(l_B) = \int_0^{+\infty} u(s)\ ds$$

We split into two parts the computation of this integral. First, we have:

$$\int_0^{t_0} u(s)\ ds \leq \frac{d}{2\beta_0^2}\left[m^\omega(B)^{\frac{2}{d}} - N_0(\omega)^{\frac{2}{d}}\right]$$

Secondly we have to deal with the term $\int_{t_0}^{+\infty} u(s)\ ds$

$$\begin{aligned} \int_{t_0}^{+\infty} u(t)\ dt &= \int_{t_0}^{+\infty} m(\{x \in B;\ G(x) \geq t\})\ dt \\ &= \int_{t_0}^{G(0)} m(\{x \in B;\ G(x) \geq t\})\ dt \\ &\leq (G(0) - t_0) m(\{x \in B;\ G(x) \geq t_0\}) \\ &= (G(0) - t_0) N_0(\omega) \\ &\leq G(0) N_0(\omega) \end{aligned}$$

Gathering the two previous computations, we get:

$$\mathbb{E}_0(l_B) \leq \frac{d}{2\beta_0^2}\left[m^\omega(B)^{\frac{2}{d}} - N_0(\omega)^{\frac{2}{d}}\right] + G(0)N_0(\omega).$$

Finally, we have proved that there exists $C > 0$ such that for Q a.s environment ω, there exists $N_\omega \in \mathbb{N}$ such that for any connected subset B which contains the origin with, $m^\omega(B) \geq N_\omega$ then $\mathbb{E}_0(l_B) \leq Cm^\omega(B)^{2/d}$. We can state this result as follow.

Proposition 3.9. *Let $d \geq 3$. There exists constants $C = C(Q,d)$ such that Q a.s for all environment ω:*
for any connected subset B which contains the origin and with volume $m^\omega(B)$ large enough,

$$\mathbb{E}_0(l_B) \leq Cm^\omega(B)^{2/d} \tag{16}$$

(ii) **case** $d = 2$

The same kind of arguments gives the bound in the dimension two replacing the occupation time by the exit time. In recurrence case, for $t \geq 0$ we let

$$u(t) = m^\omega(\{x \in B;\ G^B(0,x) \geq t\}),$$

where G^B is the Green function of the random walk killed outside B. Once again, we use isoperimetric inequality of proposition 3.6 (equation (19)) and moreover we use remark 3.7 (equation (14)). By proposition 2.6, we get:

$$u(t) \leq \begin{cases} m^\omega(B)e^{-\beta_0^2 t} & \text{if } t \leq t_0, \\ N_0(\omega)e^{-\beta_\omega^2(t-t_0)} & \text{if } t > t_0. \end{cases} \tag{17}$$

with $t_0 = \frac{1}{\beta_0^2}\ln\left(\frac{m^\omega(B)}{N_0(\omega)}\right)$. Then,

$$\begin{aligned} \mathbb{E}_0(\tau_B) &= \int_0^{+\infty} u(s)\ ds \\ &\leq m^\omega(B)[\frac{1}{\beta_0^2} + \frac{N_0(\omega)}{m^\omega(B)}(\frac{1}{\beta_\omega^2} - \frac{1}{\beta_0^2})] \end{aligned}$$

Finaly, we get the same proposition.

Proposition 3.10. *Let $d = 2$. There exists constant $C = C(Q,d)$ such that Q a.s for all environment ω:*
for any connected subset B of $\mathcal{C}$ which contains the origin and with $m^\omega(B)$ large enough,

$$\mathbb{E}_0(\tau_B) \leq Cm^\omega(B) \tag{18}$$

3.5. **Percolation of $\mathbb{Z}^d$.** Percolation context can be seen as a particular case of random environments. But assumption (11) does not allow to include percolation case in the previous subsection, although techniques are analogous. We keep same notations as previous subsection.

Pick a number $p \in]0,1[$ and let this time $\omega : E_d \to \{0,1\}$. Q is here the probability measure under which the variable $(\omega(e), e \in E_d)$ are Bernouilli(p) independent variables. Hence each edge e is kept if $\omega(e) = 1$ [resp removed if $\omega(e) = 0$] with probability p [resp $1-p$] in an independant way. If p is larger than some critical value $p_c(d)$, we know by percolation theory that the Q probability that the connected component $\mathcal{C}$ that contains the origine is infinite, is strictly positive.

We denote by $\mathcal{C}^g$ the graph such that $V(\mathcal{C}^g) = \mathcal{C}$ and $E(\mathcal{C}^g) = \{(x,y) \in E_d;\ \omega(x,y) = 1\}$ and $\mathcal{C}_n^g$ the graph such that $V(\mathcal{C}_n^g) = \mathcal{C}_n$ and $E(\mathcal{C}_n^g) = \{(x,y) \in E_d;\ x,y \in \mathcal{C}_n$ and $\omega(x,y) = 1\}$.

From now on and until the end, p would be larger than $p_c(d)$ and we will work on the event $\{\#\mathcal{C} = +\infty\}$.
X will design the simple random walk on the graph $\mathcal{C}^g$ starting from the origin. The random walk X admits reversible measures which are proportional to the measure m (which represents the number of neighbors of x in $\mathcal{C}^g$) such that:

$$m(x) = card\{y;\ \omega(x,y) = 1\}.$$

3.5.1. *(Anchored) isoperimetric inequality.* We have the following proposition which is essentially the same as proposition 3.6, excepted there is the use of renormalization.

Proposition 3.11. *Let $p > p_c(d)$. There exists $\beta_0(p,d) > 0$ such that Q a.s on $\#\mathcal{C} = +\infty$, there exists $N_0(\omega) \in \mathbb{N}$ such that, for all connected sets A of $\mathcal{C}$ which contained 0*

$$(|A| \geq N_0 \Rightarrow \frac{|\partial_{\mathcal{C}^g} A|}{|A|^{1-\frac{1}{d}}} \geq \beta_0,) \tag{19}$$

where $\partial_{\mathcal{C}^g} A = \{(x,y) \in E^d;\ \omega(x,y) = 1\ et\ x \in A\ ;\ y \notin A\}$.

This proposition can be deduced easily from the results of Biskup, Pete or Rau (see respectively proposition 5.1 of [3] or corollary 1.3 of [11] or proposition 1.4 of [12]. For example, here is the isoperimetric profile we can find in Biskup:

Proposition 3.12. *Let $p > p_c(d)$. There exists $\beta_0(p,d) > 0$ and $c_0(p,d) > 0$ such that Q a.s on $\#\mathcal{C} = +\infty$, for n large enough, we have:*

$$\frac{|\partial_{\mathcal{C}^g} A|}{f_{c_0}(|A|)} \geq \beta_0 \quad \textit{for all connected sets } A \textit{ contained in } \mathcal{C}_n, \tag{20}$$

where $f_{c_0}(x) = \begin{cases} 1 & if\ x < (c_0\ \ln(n))^{\frac{d}{d-1}} \\ x^{1-\frac{1}{d}} & if\ x \geq (c_0\ \ln(n))^{\frac{d}{d-1}}, \end{cases}$
and $\partial_{\mathcal{C}^g} A = \{(x,y) \in E^d;\ \omega(x,y) = 1\ et\ x \in A\ ;\ y \notin A\}$.
Inequality (19) is satisfied from a rank which depends on the cluster ω.

Let us prove quickly that proposition 3.12 implies 3.11. Let β_0, c_0 as in propostion 3.12 and let $n_0(\omega)$ the rank from which inequality (20) is true. Take $N_0 = n_0$, et let $A \subset \mathcal{C}$ which contains the origin and with $|A| \geq N_0$. We notice that $|A| \subset \mathcal{C}_{|A|}$ and since $|A| \geq n_0$ and $|A| \geq (c_0 \ln(|A|))^{\frac{d}{d-1}}$ (for $|A|$ large), we can apply inequality (20) of proposition 3.12 and so inequality (19) is proved.

Let us raise two points concerning this isoperimetric inequality. First, one can think that we can hope to get a "better " isoperimetric inequality for set which contains the origin than for any sets. In fact, a precise rereading of these proofs shows us that additional assumption of the fact that the origin is contained in the set A, does not simplify the proof. Secondly, notice that, thanks to proposition 5.1 in [3], anchored isoperimetric inequality with any root in $\mathcal{C}_n$, holds with the same isoperimetric constant (in [3] there is no root). So proposition 3.11 is still true with the same constant β_0 for any roots. Thus, the estimates we get in the next subsection for the occupation time, are satisfied for all connected sets contained in $\mathcal{C}$ with the same constant(see remark 3.14). Finally, we choose this form because for big sets we retrieve the isoperimetry of $\mathbb{Z}^d$. This suggests that we can hope to have a good control of the occupation time of big set B, since our method is based in a study of a serie of decreasing sets from B.

3.5.2. *Upper bound for the exit or occupation time in percolation cluster.* Let $B \subset \mathcal{C}$ connected and which contains the origin. Since m is bounded between 1 and $2d$, we deduce that inequality (19) is still true by counting boundary and volume respectively by measure m and μ ($\mu = 1$ here). This can affect only constants β_0 and N_0. That is: there exists $\beta > 0$ and $N > 0$ such that Q a.s on $\{\#\mathcal{C} = +\infty\}$,

$$\begin{cases} \frac{\mu(\partial_{\mathcal{C}^g} A)}{m(A)^{1-\frac{1}{d}}} \geq \beta & \text{for all connected sets } A \text{ which contained } 0 \text{ with } m(A) \geq N, \\ \text{and} & \\ \mu(\partial_{\mathcal{C}^g} A) \geq \beta & \text{for all connected sets } A \text{ which contained } 0 \text{ with } m(A) \leq N. \end{cases}$$

Now, by the same way as for random environments, computation of $u(t) = m(\{x \in B;\ G^B(0,x) \geq t\})$ due to proposition 2.4 and computation of integral of u on $\mathbb{R}_+$ gives the following proposition:

Proposition 3.13. *Let $p > p_c(d)$ and $d \geq 2$. There exist constants $C = C(p,d)$ such that Q a.s on the event $\{\#\mathcal{C} = +\infty\}$:*
for any connected subset B of $\mathcal{C}$ which contains the origin and with volume large enough,

$$\begin{cases} \mathbb{E}_0(l_B) \leq C|B|^{2/d} & \text{if } d \geq 3, \\ \mathbb{E}_0(\tau_B) \leq C|B| & \text{if } d = 2. \end{cases} \tag{21}$$

3.5.3. *Some remarks.*

Remark 3.14. *Since proposition 3.12 is true for all connected set A contained in $\mathcal{C}_n$, we can replace the root by any point of $\mathcal{C}_n$ with keeping the same constants. So we can state an improvement of propositions 3.9 and 3.10 as follow:*
Let $p > p_c(d)$ and $d \geq 2$. Let $x_0 \in \mathcal{C}$. There exists constant $C = C(p,d)$ such that Q a.s on the event $\{\#\mathcal{C} = +\infty\}$, for large enough connected set B which contains x_0, one have:

$$\mathbb{E}_{x_0}(l_B) \leq C|B|^{2/d} \tag{22}$$

Constant C is the same as in proposition 3.9, whereas the size of B from which (22) holds depends on x_0 and ω.

Remark 3.15. *We retrieve a consequence of results of Barlow. Indeed, in* [1] *it is proved that:*

Theorem 3.16. *There exists Ω_1 with $Q(\Omega_1) = 1$ and random variables $Sx; x \in \mathbb{Z}^d$ such that for each $x \in \mathcal{C}$ and for all $\omega \in \Omega_1$, $Sx(\omega) < \infty$ and there exists constants $ci = c_i(d;p) > 0$ such that for all $x, y \in \mathcal{C}$ and $t \geq 1$ with*

$$k \geq S_x(\omega) \vee |x-y|_1$$

the transition density $\mathbb{P}_x(X_k = y)$ of X satisfies:

$$\nu(y) c_1 k^{-d/2} e^{-c_2 \frac{|x-y|_1^2}{k}} \leq \mathbb{P}_x(X_k = y) \leq \nu(y) c_3 k^{-d/2} e^{\frac{-c_4 |x-y|_1^2}{k}}.$$

Let $x_0 \in \mathcal{C}$ and let $B \subset \mathcal{C}$ connected which contains the point x_0. First, for all $k \geq 0$ we can write:

$$\mathbb{P}_{x_0}(\tau_B > k) \leq \sum_{y \in B} \mathbb{P}_{x_0}(X_k = y)$$

. With the help of the previous Theorem (we keep the same notation), there exists a constant $c > 0$ such that Q a.s for all $y \in B$ and for all $k \geq S_{x_0}(\omega) \vee |x_0 - y|_1$, we have :

$$\mathbb{P}_{x_0}(X_k = y) \leq c\nu(y)k^{-d/2}.$$

Hence,

$$\mathbb{P}_{x_0}(\tau_B > k) \leq c\nu(B)k^{-d/2}.$$

Let $k_0 = (2c.\nu(B))^{2/d}$ and fix the environement ω. For B large enough, the condition $k_0 \geq S_{x_0}(\omega) \vee |x_0 - y|_1$ is satisfied (once again the size from which the condition is satisfied depends on the point x_0 and ω).

Then, for all x_0 and for B large enough, connected and which contains x_0, we have :

$$\mathbb{P}_{x_0}(\tau_B > k_0) \leq 1/2.$$

So, for all $i \geq 0$,

$$\mathbb{P}_{x_0}(\tau_B > ik_0) \leq (1/2)^i.$$

And then,

$$\begin{aligned}
\mathbb{E}_{x_0}(\tau_B) &\leq \sum_{i \geq 0}(i+1)k_0 \; \mathbb{P}_{x_0}(\tau_B \in [ik_0; (i+1)k_0[) \\
&\leq \sum_{i \geq 0}(i+1)k_0 \; \mathbb{P}_{x_0}(\tau_B > (i+1)k_0) \\
&\leq c'k_0 \\
&\leq c''\nu(B)^{2/d}.
\end{aligned}$$

Finally, we well retrieve inequality (22) of remark 3.14.

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