Abstract

The paper analyses structural models for the evaluation of risky debt following [22] with an approach of optimal stopping problem. Moreover we introduce an investment control parameter and we optimise with respect to the failure threshold and coupon rate. We show that the value of the optimal coupon policy decreases if the strict priority rule is removed.

Keywords: corporate debt, optimal capital structure, default, optimal stopping.

1 Introduction

The analysis of a firm’s financial policy has received much attention since Merton’s paper [25]. Various aspects have been included in order to make the original analysis more realistic: among them, the effects of taxation and the bankruptcy costs as in [22, 23], or the stochasticity of the risk-free interest rate as in [24]. Further the bankruptcy mechanism has been investigated: [22, 23] have developed structural models where the bankruptcy is endogenously triggered by equity-holders in order to maximise the equity value, instead of being exogenously given with relation to the fact that the firm’s value falls below a prescribed level. Recently the management of financial flexibility has been analysed in a dynamic structural model of the firm [15].

We follow [22] assuming that the firm’s asset evolves as a geometric Brownian motion: the activities of the firm are unchanged by the financial structure, according to the Modigliani-Miller theorem [26]. The firm issues debt. The debt is perpetual and pays a constant coupon per instant. Coupon payment determines tax benefits proportionally to the coupon payments. The bankruptcy is determined endogenously by the inability of the firm to raise sufficient equity capital to cover its debt obligations. The strict priority rule holds, therefore if default occurs debt holders receive all assets (except for bankruptcy costs) while stockholders do not receive anything. Thus the optimal control problem is stated with respect to the (coupon policy, default time) pair. In comparison to [22] we
use another method for solving the related optimal stopping problems, recovering his results. The optimal failure threshold is determined by optimising the equity function and then the optimal coupon rate is obtained, which maximises the total value of the firm. Concerning the optimisation with respect to the failure time, it is often supposed to be a constant level hitting time and the idea is then to optimise with respect to the level, see [10, 11, 20, 27, 2]. In the extended setting of [23] it is proved in [5] that the optimal bankruptcy policies are not defined by constant level hitting times. However it does not apply to our case where following [7, 8] the optimal stopping (failure) time is actually a constant level hitting time. The key method is the Laplace transform of this stopping time (cf. [1, 16, 21]) so that the problem is finally turned to optimise the concerned function with respect to the constant level.

The optimal control problem concerns other contexts as well, but with respect to an increasing process $C$ (dividend policy) instead of the stopping time, e.g. Jeanblanc and Shiryaev [18] or Decamps and Villeneuve [4]. Some authors consider the difficult optimal control problem which mixes both points of view. For instance look at the model introduced in [17] or Davis and Zervos [3] where viscosity solutions and variational inequalities are used. Finally Vath et al. [32] are interested in an impulse control model. However, all these are quite different from our topic.

In Section 3 we face the optimisation problem assuming that investments (e.g. in technology, research or development) have effects on the risk-return profile of the firm. Therefore we suppose that the firm’s asset evolves as a geometric Brownian motion, but we introduce an investment control parameter $k$ in the trend and volatility parameters, we then optimise with respect to the failure threshold and coupon rate. The result shows that the failure level decreases as the investment rate is big enough, so the failure time increases with a good investment rate.

Finally in Section 4, the same model from Section 2 is used, but we remove the absolute priority rule, which empirical evidence shows that it rarely holds in financially distressed corporations. More precisely [14, 12] indicated that the absolute priority rule is enforced in only 25% of corporate bankruptcy cases. We compute the optimal coupon policy in the case where, if default occurs, debt holders receive a fraction of the remaining assets and equity holders receive the complement. We obtain that violation of the strict priority rule leads to an increase of the bankruptcy threshold as the value of the optimal coupon policy decreases, thus suggesting that the shareholders’ interest could lead to a too early bankruptcy of the firm.

## 2 Optimal capital structure with tax benefits

Let us consider the following problem: a firm realizes its capital from both debt and equity. The debt is perpetual and pays a constant coupon $C$ per instant. On the failure time $T$, agents who hold debt claims will get the residual value of the firm, while those who hold equity will get nothing. In this section, the strict priority rule holds. The model we consider in this section is the Leland model, but we solve the optimal stopping problems
differently from [22].

We assume that the firm’s activities value is described by process \( V_t = Ve^{X_t} \), where \( X_t \) evolves, under the risk neutral probability measure, as

\[
\begin{align*}
    dX_t := \left( r - \frac{1}{2} \sigma^2 \right) dt + \sigma dW_t, \quad X_0 = 0,
\end{align*}
\]  

(1)

where \( W \) is a standard Brownian motion, \( r \) the constant risk-free rate, \( r \) and \( \sigma > 0 \). When bankruptcy occurs at stopping time \( T \), a fraction \( \alpha \) (0 \( \leq \) \( \alpha \) < 1) of firm value is lost (for instance paid to the ones responsible of the bankruptcy procedures), the debt holders receive the rest, while the stockholders do not receive anything. We suppose that the failure time \( T \) belongs to \( \Delta \), the set of stopping times. Thus, applying contingent claim analysis in a Black-Scholes setting the debt value, for a given stopping (failure) time \( T \), is

\[
    D(V, C, T) = \mathbb{E} \left[ \int_0^T e^{-rs} C ds + (1 - \alpha)e^{-rT} V_T | V_0 = V \right],
\]  

(2)

where the expectation is taken with respect to the risk neutral probability.

Assume that from paying coupons the firm obtains tax deductions, namely \( \tau \), 0 \( \leq \) \( \tau \) < 1, proportionally to the coupon payment. The total value of the (levered) firm includes the firm’s assets value, the tax deduction on debt payments \( C \) minus the value of the bankruptcy costs:

\[
    v(V, C, T) = V + \mathbb{E} \left[ \tau \int_0^T e^{-rs} C ds - \alpha e^{-rT} V_T | V_0 = V \right].
\]  

(3)

Finally the equity value is the total value \( v \) less the value of the debt \( D \):

\[
    E(V, C, T) = V - \mathbb{E} \left[ (1 - \tau) \int_0^T e^{-rs} C ds + e^{-rT} V_T | V_0 = V \right].
\]  

(4)

The aim is to maximise on \( \Delta \), \( T \mapsto E(V, C, T) \). Following [7, 8] and optimal stopping theory (cf. [13] or Theorem 3.3 page 127 in [31]), the failure time, “optimal stopping time”, is a constant level hitting time (cf. [7] or a proof similar to Theorem 2.1 in [8]). Hence default happens at the first time \( T \) when the value \( V \) falls to a constant level \( V_B \).

The value of \( V_B \) is endogenously derived and will be determined with an optimal rule later.

Observe that given (1), it holds that \( \inf \{ t \geq 0 : V_t \leq V_B \} = \inf \{ t \geq 0 : X_t \leq \log \frac{V_B}{V} \} \).

Moreover it holds \( V_T = V_B \), as the process \( V \) is continuous.

Thus, the optimal stopping time problem of equity-holders is turned to maximise \( E(V, C, T) \) by choosing \( V_B \):

\[
    E : V_B \mapsto E(V, C, T) = V - \mathbb{E} \left[ (1 - \tau) \int_0^T e^{-rs} C ds + e^{-rT} V_T | V_0 = V \right],
\]

and to follow optimal strategy yields the properties:

\[
    E(V, C, T) \geq E(V, C, \infty) \quad \text{and} \quad E(V, C, T) \geq 0 \quad \text{for all} \ V \geq V_B.
\]  

(5)
From (4) it follows that
\[
E(V, C, T) = V - \frac{(1 - \tau)C}{r} + \left( \frac{(1 - \tau)C}{r} - V_B \right) \mathbb{E} \left[ e^{-rT} | V_0 = V \right],
\] (6)
because \( V_T = V_B \). The first property in (5) is equivalent to
\[
E(V, C, T) - E(V, C, \infty) = \left( \frac{(1 - \tau)C}{r} - V_B \right) \mathbb{E} \left[ e^{-rT} | V_0 = V \right] \geq 0.
\]
In fact this term is the option embodied in equity. Since this is an option to be exercised by the firm, this must have positive value, so it must be \( \frac{(1 - \tau)C}{r} - V_B \geq 0 \). Finally we are lead to the condition:
\[
V_B \leq C \left( 1 - \tau \right) \frac{C}{r}.
\]
As for the second property in (5), observe that if \( V_B \) was chosen by the firm, then the total value of the firm \( v \) would be maximised by setting \( V_B \) as low as possible. Nevertheless, because equity has limited liability, then \( V_B \) cannot be arbitrary small, but \( E(V, C, T) \) must be nonnegative. This limited liability of debt simply means that if the firm declares bankruptcy, it then no longer has to service its debt. A natural constraint on \( V_B \) is \( V_B < V \), indeed, if not, the optimal stopping time would necessarily be \( T = 0 \) and then
\[
E(V, C, T) = V - \frac{(1 - \tau)C}{r} + \left[ \frac{(1 - \tau)C}{r} - V_B \right] = V - V_B < 0.
\]
Finally \( E(V, C, T) \geq 0 \) for all \( V \geq V_B \).

To obtain (6) it remains to compute \( \mathbb{E} \left[ e^{-rT} | V_0 = V \right] \), for which we use the general formula for the Laplace transform of a constant level hitting time by a Brownian motion with drift ([19] pg. 196-197):

**Proposition 2.1** Let \( X_t = \mu t + \sigma W_t \) and \( T_b = \inf \{ s : X_s = b \} \), then for all \( \alpha > 0 \),
\[
\mathbb{E}[e^{-\alpha T_b}] = \exp \left[ \frac{\mu b}{\sigma^2} - \frac{|b|}{\sigma} \sqrt{\frac{\mu^2}{\sigma^2} + 2\alpha} \right].
\]

Thus since \( V_t = V \exp[(r - \frac{1}{2}\sigma^2)t + \sigma W_t] \) we get a simple form of the function \( E \):
\[
E(V, C, V_B) = V - \frac{(1 - \tau)C}{r} + \left[ \frac{(1 - \tau)C}{r} - V_B \right] \left( \frac{V_B}{V} \right)^{2\alpha/\sigma^2},
\]
and the optimal value \( V_B \) is
\[
V_B^*(C) = \frac{(1 - \tau)C}{r + \frac{1}{2}\sigma^2},
\] (7)
which is less than \( \frac{(1 - \tau)C}{r} \).

Now we turn to the optimisation of the total value depending on the threshold \( V_B^* \). We choose \( C \) in order to maximise the application:
\[
C \mapsto v(V, C, V_B^*(C)) = V + \frac{\tau C}{r} - \left( \frac{\tau C}{r} + \alpha V_B^*(C) \right) \left( \frac{V_B^*(C)}{V} \right)^{2\alpha/\sigma^2}
\]
which is a concave function obtaining its maximum at

\[ C^*(V) = V^{r + \frac{1}{2}\sigma^2} \left( \frac{\tau\sigma^2}{2\alpha r(1 - \tau) + \tau(2r + \sigma^2)} \right)^{\sigma^2/2r}. \]  

(8)

Thus we recover the result in [22] (eq. (21)). We easily check the properties (5): on the one hand obviously \( V^*_B(C) < (1 - \tau)\frac{C}{\tau} \), and on the other hand, replacing \( C^*(V) \) in \( V_B^* \), yields

\[ V_B^*(V) = V\left( \frac{\tau\sigma^2}{2\alpha r(1 - \tau) + \tau(2r + \sigma^2)} \right)^{\sigma^2/2r}, \]  

(9)

which actually is less than \( V \), since \( 2\alpha r(1 - \tau) + \tau 2r > 0 \).

3 Optimal capital structure with investments

The effects of investments, let us say in technology, research or safety, on the firm financial performance has been analysed from the point of view of the firm’s risk-return profile ([6, 30]). These studies indicate that such investments support the firm’s diversification strategy and increase the degree of flexibility; therefore they can result in lower risk for the firm. In this section we build on these results and we consider a model which accounts for an investment decision parameter both in the drift and in the volatility coefficients. We are mainly interested in the impact of the investment on the default time arrival.

The firm’s value evolves as a geometric Brownian motion: its value \( V \) follows under the risk neutral probability the dynamic

\[ dV_t = V_t[r(1 - k)dt + k\sigma dW_t], \]

where \( k \) is the investment control parameter, constant \( 0 < k < 1 \). In [29] a more sophisticated investment-consumption model is considered, however the author does not address a risky debt problem, but he faces to the optimisation of expected utility from the pair (consumption-terminal wealth). Here we take the point of view that a decrease in return depends on “investments” but this can reduce volatility, and finally achieve an advantage when studying default time.

We introduce a process \( X \) such that \( V_t = V e^{X_t} \), so

\[ dX_t = \left[ (1 - k)r - \frac{1}{2}k^2\sigma^2 \right] dt + k\sigma dW_t, \quad X_0 = 0. \]  

(10)

Once again we can follow [7, 8] and optimal stopping theory (cf.[13, 31]), in order to stress that the failure time, “optimal stopping time”, is actually a constant level hitting time as in Section 2. Let \( C \) be the coupon rate. As in Section 2, the debt value, total value of the (levered) firm and equity value are the following:

\[ D(V, C, T, k) = \mathbb{E} \left[ \int_0^T e^{-rs}Cds + (1 - \alpha)e^{-rT}V_T|V_0 = V \right], \]
\[ v(V, C, T, k) = V + \mathbb{E} \left[ \tau \int_0^T e^{-rs} C ds - \alpha e^{-rT} V_T | V_0 = V \right], \]

\[ E(V, C, T, k) = V - (1 - \tau) \frac{C}{r} + \left( (1 - \tau) \frac{C}{r} - V_B \right) \mathbb{E} [e^{-rT} | V_0 = V]. \]

Consider (10) and use Proposition 2.1 to get \( \mathbb{E} [e^{-rT} | V_0 = V] = (\frac{V_B}{V})^{\alpha_k} \), where \( \alpha_k = \frac{(1-k)r}{k^2 \sigma^2} - \frac{1}{2} + \sqrt{\left( \frac{(1-k)r}{k^2 \sigma^2} - \frac{1}{2} \right)^2 + \frac{2r}{k^2 \sigma^2}}. \)

Then the equity value is:

\[ V_B \mapsto E(V, C, V_B, k) = V - (1 - \tau) \frac{C}{r} + \left[ (1 - \tau) \frac{C}{r} - V_B \right] \left( \frac{V_B}{V} \right)^{\alpha_k}, \]

which admits a maximum for \( V_B^*(C, k) = \frac{(1-\tau)C}{r} \frac{\alpha_k}{\alpha_k + 1}. \)

Consider now the total value of the firm

\[ C \mapsto v(V, C, V_B^*(C), k) = V + \frac{\tau C}{r} - \left( \frac{\tau C}{r} + \alpha V_B^*(C) \right) \left( \frac{V_B^*(C)}{V} \right)^{\alpha_k}. \]

As \( \alpha \in ]0, 1[ \), we have:

\[ C^*(V, k) = V \frac{r}{1 - \tau} \frac{\alpha_k + 1}{\alpha_k} \left( 1 + \alpha_k \left( 1 + \alpha \frac{1 - \tau}{\tau} \right) \right)^{-1/\alpha_k}. \]

Now we check the properties (5): on the one hand obviously \( V_B^*(C, k) < (1 - \tau) \frac{C}{r} \) since \( \alpha_k < 1 + \alpha_k \), and on the other hand, replacing \( C^*(V) \) in \( V_B^* \), yields

\[ V_B^*(V) = V \left( 1 + \alpha_k \left( 1 + \alpha \frac{1 - \tau}{\tau} \right) \right)^{-1/\alpha_k}, \quad (11) \]

which actually is less than \( V \) since \( \alpha_k (1 + \alpha \frac{1 - \tau}{\tau}) > 0. \)

At this point it is interesting to compare the above results with those obtained in the no-investment model. The following proposition shows that by choosing a suitable investment level, the firm reaches the failure level later than in the case considered in the previous section.

**Proposition 3.1** Let \( k_1 = \frac{-r + \sqrt{r^2 + (2r \sigma^2)^2}}{2r \sigma^2} \). Then, if \( k > k_1 \), the optimal value, with investment rate \( k \), \( V \left( 1 + \alpha_k \left( 1 + \alpha \frac{1 - \tau}{\tau} \right) \right)^{-1/\alpha_k} \) is less than the optimal value without investment \( V \left( 1 + \frac{2r}{\sigma^2} \left( 1 + \alpha \frac{1 - \tau}{\tau} \right) \right)^{-\sigma^2/2r}. \)

**Proof:**

Note that if \( k \geq k_1 \) then \( \alpha_k \geq \alpha_1 = \frac{2r}{\sigma^2} \). Since the application \( x \mapsto (1 + x\beta)^{-1/x} \) is increasing (here \( \beta = 1 + \alpha \frac{1 - \tau}{\tau} > 1 \)), we only need to compare \( \alpha_k \) and \( \frac{2r}{\sigma^2} \).
This implies that \( 2a \) and \( 2k \) are equivalent to \( k \geq k_2 = \frac{-a + \sqrt{a^2 + 2a(4a + 1)}}{4a + 1} \). Then
\[
\left( \frac{a - k}{k^2} - \frac{1}{2} \right)^2 + \frac{2a}{k^2} \leq \left( 2a - \left( \frac{a - k}{k^2} - \frac{1}{2} \right) \right)^2,
\]
equivalent to \( k \geq k_1 = \frac{-a + \sqrt{a^2 + (2a + 1)^2}}{2a + 1} \), but it can be checked that \( k_2 \leq k_1 \), so the proof is complete.

The firm needs to raise money from issuing the debt, but it does not if the debt holders do not get sufficient coupon gains. Therefore the investment policy needs to take into account the value of the optimal coupon. Thus we turn to coupon holders’ point of view, looking at the application on \([k_1, 1]\), \( k \mapsto C^*(V, \alpha_k) = V \frac{r - \alpha_k}{1 - \tau} \left( 1 + \alpha_k \left( 1 + \alpha \frac{1 - \tau}{1 - \tau} \right)^{-1/\alpha_k} \right) \) and at the comparison with the optimal value without investment \( C^*(V, 2a) \). There is no more monotonicity, so the best way would be to choose in the interval \([k_1, 1]\) the argument maximum of application \( k \mapsto C^*(V, \alpha_k) \) (it exists since the application is continuous on this bounded interval). It depends on parameter \( a \). For instance, if \( a \) is large enough (meaning a large rate \( r \) with respect to volatility), \( C^*(V, \alpha_1) \leq C^*(V, 2a = \alpha_k_1) \), and to invest is not interesting for coupon holders; indeed, in such a case, the return rate \( r \) is great enough to avoid investing. But when \( a \) is small enough, \( C^*(V, \alpha_1) \geq C^*(V, 2a = \alpha_k_1) \); therefore in this case the investment strategy is profitable for coupon holders, too.

## 4 Optimality from shareholders’ point of view

Finally, we consider the model in Section 2, the firm value \( V_t = Ve^{X_t} \), where \( X_t \) evolves following Equation (1). But, previously we assumed that the absolute priority rule is in force, that is debt holders receive all assets (except bankruptcy costs) if default occurs, and stockholders do not receive anything. We now suppose that debt holders receive a fraction \((1-b)\) (with \(0 < b(1-\alpha) < 1\)) of the remaining assets \((1-\alpha)V_B\) and stockholders \(b(1-\alpha)V_B\). We get:

\[
D(V, C, T) = \mathbb{E} \left[ \int_0^T e^{-rs}C ds + (1-b)(1-\alpha)e^{-rT}V_T \right],
\]

\[
E(V, C, T) = V - \mathbb{E} \left[ \int_0^T (1-\tau)e^{-rs}C ds + (1-b(1-\alpha))e^{-rT}V_T \right],
\]

\[
v(V, C, T) = V + \mathbb{E} \left[ \tau \int_0^T e^{-rs}C ds - \alpha e^{-rT}V_T \right].
\]
The equity value with respect to $V_B$ is:

$$E(V, C, V_B) = V - \frac{(1-\tau)C}{r} + \left[\frac{(1-\tau)C}{r} - (1-b(1-\alpha))V_B\right] \left(\frac{V_B}{V}\right)^{2r/\sigma^2},$$

then the optimal value $V_B$ is

$$V_B^*(C) = (1-\tau)\frac{C}{(r+\frac{1}{2}\sigma^2)(1-b(1-\alpha))}.$$ (12)

We now turn to the optimisation of the total value depending on the threshold $V_B^*(C)$:

$$C \mapsto v(V, C, V_B^*(C)) = V + \frac{\tau C}{r} - \left(\frac{\tau C}{r} + \alpha V_B^*(C)\right) \left(\frac{V_B^*(C)}{V}\right)^{2r/\sigma^2},$$

which is a concave function obtaining its maximum at

$$C^*(V) = V\frac{(r+\frac{1}{2}\sigma^2)(1-b(1-\alpha))}{1-\tau} \left(\frac{\tau \sigma^2}{\frac{2r(1-\tau)}{1-b(1-\alpha)} + \tau(2r + \sigma^2)}\right)^{\sigma^2/2r}.\quad (13)$$

Obviously, we recover Leland’s result [22] (section C, page 1242). But our purpose is to stress the negative influence of parameter $b > 0$ on the firm’s value: a direct comparison of (7) and (8) with (12) and (13) shows that, when $0 < b < 1$, violation of the strict priority rule leads to an increase of the failure threshold, while the optimal coupon policy decreases.

Thus the shareholders’ interest contradicts the firm’s interest; their power leads to a too early failure of the firm. For instance, it could be leading to unemployment. A suggestion could be, instead of always optimising the shareholders’ wealth, to build other financial strategies to avoid a total economic disaster, i.e. the bankruptcy of the firm, which can arise as the consequence of shareholders’ optimal strategies.

On this topic, we can refer to Peyrelevade [28] page 87,

*D’autres voies peuvent être envisagées: encourager par la fiscalité le réinvestissement des bénéfices plutôt que la distribution et rendre plus coûteux, voire interdire les rachats d’actions; autoriser des dividendes beaucoup plus élevés pour les titres détenus depuis plus longtemps, de façon à stabiliser les actionnariats; à nouveau grâce à la fiscalité sur les plus-values, favoriser les détentions longues et décourager les allers-retours...*

Along these lines we report an article appeared in Le Monde [9]: Cécile Ducourtieux concludes as following:

*Les financiers renvoient dos à dos versement de dividendes et investissement. Selon eux, en faisant la part trop belle aux actionnaires, par manque de confiance en l’avenir, d’imagination ou d’opportunités d’investissement, les entreprises pourraient compromettre leur valeur à long terme. “C’est moins risqué de rendre de l’argent aux actionnaires que de l’investir. Mais les niveaux actuels d’investissement, plutôt bas, auront un impact sur les niveaux futurs de profits”, analyse Thomas Aubrey, un dirigeant de la société de conseil britannique*
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References


