Optimal convergence for discrete variational inequalities modelling Signorini contact in 2D and 3D without additional assumptions on the unknown contact set.

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Abstract

The basic H^1-finite element error estimate of order h with only H^2-regularity on the solution has not been yet established for the simplest 2D Signorini problem approximated by a discrete variational inequality (or the equivalent mixed method) and linear finite elements. To obtain an optimal error bound in this basic case and also when considering more general cases (three-dimensional problem, quadratic finite elements...), additional assumptions on the exact solution (in particular on the unknown contact set, see [5, 20, 35]) had to be used. In this paper we consider finite element approximations of the 2D and 3D Signorini problems with linear and quadratic finite elements. In the analysis, we remove all the additional assumptions and we prove optimal H^1-error estimates with the only standard Sobolev regularity. The main tools are local L^1 and L^2-estimates of the normal constraints and the normal displacements on the candidate contact area and error bounds depending both on the contact and on the non-contact set.

Keywords. Signorini problem, unilateral contact, finite elements, a priori error estimates.

Abbreviated title. Optimal error estimate for Signorini contact

AMS subject classifications. 35J86, 65N30.

1 Introduction and notation

In mechanics of deformable bodies the finite element methods are currently used to approximate Signorini's contact problem or the equivalent scalar valued unilateral problem (see, *e.g.*, [17, 21, 23, 34, 36]). This problem is nonlinear since the displacement field (denoted u) satisfies a nonlinear boundary condition: a component of the solution u is nonpositive (or equivalently nonnegative) on Γ_C which is (a part of) the boundary of the domain Ω (see [31]). The corresponding weak formulation is a variational inequality which admits a unique solution u (see [15]). The finite element approximation u^h solves generally a discrete variational inequality or an equivalent problem with Lagrange multipliers. Note that there also exist other different discretizations such as penalty or Nitsche's methods which cannot be written as variational inequalities with an explicit nonpenetration condition on the displacement. Besides the regularity of the solution u to this kind of problems shows limitations whatever the regularity of the data is (see [2, 22, 26]) and the regularity $H^{5/2}$ can generally not be passed beyond. A consequence is that only finite element methods of order one and of order two are really of interest.

In this paper we consider error analyses involving linear or quadratic finite elements and various discrete nonpenetration conditions in 2D and 3D. The existing $H^1(\Omega)$-error analysis with maximal order of convergence generally uses additional assumptions on the candidate contact

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area which are often technical (when not used, only suboptimal results were available, see, e.g., [19]). So we describe shortly and roughly these additional assumptions:

- linear finite elements in 2D: the analysis in [20] assumes that the exact solution u admits a finite number of points where the transition from contact to noncontact occurs (when u lies in $(H^\tau(\Omega))^2$, $3/2 < \tau \leq 2$),
- quadratic finite elements in 2D: the analysis in [5] does not need any additional assumption when $2 < \tau \leq 5/2$ but in the case $3/2 < \tau \leq 2$ a similar assumption as in the linear 2D case is needed,
- linear finite elements in 3D: the analysis in [20] (when u lies in $(H^\tau(\Omega))^3$, $3/2 < \tau \leq 2$) considers the finite element nodes in the noncontact set of the candidate contact zone whose basis function has a part of its support in the contact set. It assumes, to be brief, that there exists a rectangular contact zone of area αh^2 in a neighborhood of radius βh around these finite element nodes (with α, β fixed),
- quadratic finite elements in 3D: the analysis in [35] when $2 < \tau < 5/2$ uses estimates ("Assumption 4" p. 739) of the L^2-norm of the normal displacement on the tubes of section h centered around the boundary where transition from contact to noncontact occurs.

The results of this paper in Theorems 1, 2, 3 and 4 (see (8),(9),(27),(28)) only use the standard Sobolev regularity. We can summarize them as follows: let $d = 2, 3$ be the space dimension, $k = 1, 2$ the degree of the finite element approximation and h the mesh size. Let u and u^h be the solution of the continuous and the discrete problems. Assume that $u \in (H^\tau(\Omega))^d$ with $3/2 < \tau \leq 3/2 + k/2$. Then

$$\|u - u^h\|_{1,\Omega} \leq Ch^{\tau-1}\|u\|_{\tau,\Omega}.$$

We now specify some notations we shall use. Let ω be a Lebesgue-measurable subset of $\mathbb{R}^d$ with nonempty interior ; the generic point of ω is denoted x. The classical Lebesgue space $L^p(\omega)$ is endowed with the norm

$$\|\psi\|_{L^p(\omega)} = \left(\int_\omega |\psi(x)|^p \, dx\right)^{1/p},$$

when $1 \leq p < \infty$. When $p = \infty$, set

$$\|\psi\|_{L^\infty(\omega)} = \text{ ess sup } \{|\psi(x)| : x \in \omega\}.$$

We will make a constant use of the standard Sobolev space $H^m(\omega)$, $m \in \mathbb{N}$ (we adopt the convention $H^0(\omega) = L^2(\omega)$), provided with the norm

$$\|\psi\|_{m,\omega} = \left(\sum_{0\leq|\alpha|\leq m} \|\partial^\alpha \psi\|^2_{L^2(\omega)}\right)^{1/2},$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi–index in $\mathbb{N}^d$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$ and the symbol ∂^α represents a partial derivative. The fractional Sobolev space $H^\tau(\omega), \tau \in \mathbb{R}_+ \setminus \mathbb{N}$, is defined by the norm (see [1]):

$$\|\psi\|_{\tau,\omega} = \left(\|\psi\|^2_{m,\omega} + \sum_{|\alpha|=m} \int_\omega\int_\omega \frac{(\partial^\alpha\psi(x) - \partial^\alpha\psi(y))^2}{|x-y|^{d+2\nu}} \, dx \, dy\right)^{1/2} = \left(\|\psi\|^2_{m,\omega} + \sum_{|\alpha|=m} |\partial^\alpha\psi|^2_{\nu,\omega}\right)^{1/2},$$

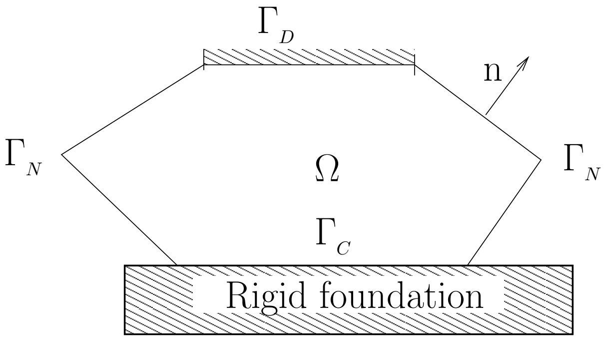

Figure 1: *Elastic body Ω in contact.*

where $\tau = m + \nu, m$ being the integer part of τ and $\nu \in (0, 1)$. As written before, we will often use the seminorm:

$$|\psi|_{\nu,\omega} = \left(\int_\omega \int_\omega \frac{(\psi(x) - \psi(y))^2}{|x - y|^{d+2\nu}} \, dx \, dy \right)^{1/2}.$$

For the sake of simplicity, not to deal with a nonconformity coming from the approximation of the domain, we shall only consider here polygonally shaped domains denoted $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. The boundary $\partial\Omega$ is the union of a finite number of segments (or polygons) $\Gamma_j, 0 \le j \le J$. In such a case, the space $H^\tau(\Omega)$ defined above coincides not only with the set of restrictions to Ω of all functions of $H^\tau(\mathbb{R}^d)$ (see [16]) but also with the Sobolev space defined by Hilbertian interpolation of standard spaces $(H^m(\Omega))_{m \in \mathbb{N}}$ and the norms resulting from the different definitions of $H^\tau(\Omega)$ are equivalent (see [33]). Finally the trace operator $T : \psi \mapsto (\psi_{|\Gamma_j})_{1 \le j \le J}$, maps continuously $H^\tau(\Omega)$ onto $\prod_{j=1}^J H^{\tau-1/2}(\Gamma_j)$ when $\tau > 1/2$ (see, *e.g.*, [25]).

2 Signorini's problem and its finite element discretization

2.1 Setting of the problem

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a polygonal domain representing the reference configuration of a linearly elastic body whose boundary $\partial\Omega$ consists of three nonoverlapping open parts Γ_N, Γ_D and Γ_C with $\overline{\Gamma_N} \cup \overline{\Gamma_D} \cup \overline{\Gamma_C} = \partial\Omega$. We assume that the measures of Γ_C and Γ_D in $\partial\Omega$ are positive and, in order to simplify, that Γ_C is a straight line segment when $d = 2$ or a polygon when $d = 3$. The body is submitted to a Neumann condition on Γ_N with a density of loads $F \in (L^2(\Gamma_N))^d$, a Dirichlet condition on Γ_D (the body is assumed to be clamped on Γ_D to simplify) and to volume loads denoted $f \in (L^2(\Omega))^d$ in Ω. Finally, a (frictionless) unilateral contact condition between the body and a flat rigid foundation holds on Γ_C (see Fig. 1). The problem consists in finding the displacement field $u : \overline{\Omega} \to \mathbb{R}^d$ satisfying (1)–(5):

$$-\operatorname{div} \sigma(u) = f \qquad \text{in } \Omega, \tag{1}$$

$$\sigma(u)\mathrm{n} = F \qquad \text{on } \Gamma_N, \tag{2}$$

$$u = 0 \qquad \text{on } \Gamma_D, \tag{3}$$

where $\sigma(u) = \mathcal{A}\varepsilon(u)$ represents the stress tensor field, $\varepsilon(u) = (\nabla u + (\nabla u)^T)/2$ denotes the linearized strain tensor field, n stands for the outward unit normal to Ω on $\partial\Omega$, and $\mathcal{A}$ is the fourth order elastic coefficient tensor which satisfies the usual symmetry and ellipticity conditions and whose components are in $L^\infty(\Omega)$.

On Γ_C, we decompose the displacement and the stress vector fields in normal and tangential components as follows:

$$u_N = u.\mathrm{n}, \quad u_T = u - u_N \mathrm{n},$$

$$\sigma_N = (\sigma(u)\mathrm{n}).\mathrm{n}, \quad \sigma_T = \sigma(u)\mathrm{n} - \sigma_N \mathrm{n}.$$

The unilateral contact condition on Γ_C is expressed by the following complementarity condition:

$$u_N \leq 0, \ \sigma_N \leq 0, \ u_N \sigma_N = 0, \tag{4}$$

where a vanishing gap between the elastic solid and the rigid foundation has been chosen in the reference configuration. The frictionless condition on Γ_C reads as:

$$\sigma_T = 0. \tag{5}$$

Remark 1 *The contact problem (1)–(5) is the vector valued version of the scalar Signorini problem which consists of finding the field $u : \overline{\Omega} \to \mathbb{R}$ satisfying:*

$$-\Delta u + u = f \ \textit{in } \Omega, \ \ u \leq 0, \frac{\partial u}{\partial \mathrm{n}} \leq 0, u\frac{\partial u}{\partial \mathrm{n}} = 0 \ \textit{on } \partial\Omega.$$

All the results proved in this paper, in particular the error estimates in Theorems 1 to 4, can be straightforwardly extended to the scalar Signorini problem.

Remark 2 *Unilateral contact problems show different kind of regularity limitations caused in particular by the regularity of the data, the mixed boundary conditions (e.g., Neumann-Dirichlet transitions), the corners in polygonal domains, the Signorini condition which generates singularities at contact-noncontact transition points. The first three kind of singularities do not depend on the Signorini conditions (see, e.g., [16, 28]). In the references dealing with singularities of Signorini problems the authors generally study the singularity limitations coming from the Signorini conditions: the work in [26] is restricted to $\mathbb{R}^2$ and considers the Laplace operator on a polygonal domain and allows us to conclude that the solution to the Signorini problem is $H^{5/2-\varepsilon}$ regular in the neighborhood of Γ_C. If Γ_C is not straight, e.g., Γ_C is a union of straight line segments, then additional singularities appear (see section 2.3 in [4] for a study in the two-dimensional case). In the three dimensional case the references [3, 2] prove local $C^{1,1/2}$ regularity results with the Laplace and the Lamé operators respectively in the particular case of an half ball with a flat contact zone Γ_C.*

Let us introduce the following Hilbert space:

$$V = \left\{ v \in (H^1(\Omega))^d : v = 0 \text{ on } \Gamma_D \right\}.$$

The set of admissible displacements satisfying the noninterpenetration conditions on the contact zone is:

$$K = \{ v \in V : v_N = v.\mathrm{n} \leq 0 \quad \text{on } \Gamma_C \}.$$

Let be given the following forms for any u and v in V:

$$a(u, v) = \int_\Omega \mathcal{A}\varepsilon(u) : \varepsilon(v) \, d\Omega, \qquad l(v) = \int_\Omega f.v \, d\Omega + \int_{\Gamma_N} F.v \, d\Gamma,$$

which represent the virtual work of the elastic forces and of the external loads respectively. From the previous assumptions it follows that $a(\cdot,\cdot)$ is a bilinear symmetric V-elliptic and continuous form on $V \times V$ and l is a linear continuous form on V.

The weak formulation of Problem (1)–(5) (written as an inequality), introduced in [15] (see also, *e.g.*, [17, 21]) is:

$$\left\{ \begin{array}{l} \text{Find } u \in K \text{ satisfying:} \\ a(u, v-u) \geq l(v-u), \quad \forall\, v \in K. \end{array} \right. \tag{6}$$

Problem (6) admits a unique solution according to Stampacchia's Theorem.

2.2 Finite element approximation

Let $V_k^h \subset V$ be a family of finite dimensional vector spaces indexed by h coming from a regular family T^h of triangulations or tetrahedralizations of the domain Ω (see [9, 11, 13]). The notation h represents the largest diameter among all elements $T \in T^h$ which are supposed closed. We choose standard continuous and piecewise of degree k functions with $k = 1$ or $k = 2$, *i.e.*:

$$V_k^h = \left\{ v^h \in (C(\overline{\Omega}))^d : v^h|_T \in (P_k(T))^d, \forall T \in T^h, v^h = 0 \text{ on } \Gamma_D \right\},$$

where $P_k(T)$ stands for the space of all polynomials of degree $\leq k$ in the d variables. In the threedimensional case ($d = 3$), we will use inverse inequalities on Γ_C and we suppose that the trace mesh on Γ_C is quasiuniform of characteristic diameter $h_C \leq h$. We next recall some classical nonpenetration conditions when using linear or quadratic finite elements in two and three space dimensions.

2.2.1 The convex cones in the linear case ($k = 1$)

The simplest discrete set of admissible displacements satisfying the nonpenetration conditions on the contact zone is given by:

$$K_1^h = \left\{ v^h \in V_1^h : v_N^h \leq 0 \quad \text{on } \Gamma_C \right\}.$$

We also consider a discrete nonpenetration in average on any contact element $T \cap \Gamma_C$ (segment when $d = 2$ or triangle when $d = 3$):

$$\overline{K_1^h} = \left\{ v^h \in V_1^h, \int_{T \cap \Gamma_C} v_N^h \; d\Gamma \leq 0, \forall T \in T^h \right\}.$$

Note that $K_1^h \subset K$, $\overline{K_1^h} \not\subset K$ and $K_1^h \subset \overline{K_1^h}$.

2.2.2 The convex cones in the quadratic case ($k = 2$)

In what follows we denote by $x_i, 0 \leq i \leq I$ the vertices of the triangulation or tetrahedralization located in $\overline{\Gamma_C}$ and by $m_j, 0 \leq j \leq J$ the midpoints of the contact elements when $d = 2$ (i.e., the midpoints of the segments in $\overline{\Gamma_C}$). When $d = 3$ the m_j ($0 \leq j \leq J$) are the midpoints of the contact element edges (i.e., the midpoints of the edges of the triangles in $\overline{\Gamma_C}$).

The first discrete set of admissible displacements satisfies the nonpenetration conditions at the vertices and the midpoints:

$$K_2^h = \left\{ v^h \in V_2^h, v_N^h(x_i) \leq 0, \forall\, 0 \leq i \leq I, v_N^h(m_j) \leq 0, \forall\, 0 \leq j \leq J \right\},$$

the second one involves an average nonpenetration condition on any contact element (segment in 2D or triangle in 3D):

$$\overline{K_2^h} = \left\{ v^h \in V_2^h, \int_{T\cap\Gamma_C} v_N^h \, d\Gamma \le 0, \forall T \in T^h \right\},$$

and the third one is a combination (specific to the quadratic case) of both previous cases:

$$\widehat{K_2^h} = \left\{ v^h \in V_2^h, \int_{T\cap\Gamma_C} v_N^h \, d\Gamma \le 0, \forall T \in T^h, v_N^h(x_i) \le 0, \forall 0 \le i \le I \right\}.$$

Note that neither of these three convex cones is a subset of K and that $K_2^h \subset \widehat{K_2^h} \subset \overline{K_2^h}$. In the case $d = 3$ we are interested in a fourth convex cone:

$$\widetilde{K_2^h} = \left\{ v^h \in V_2^h, v_N^h(m_j) \le 0, \forall 0 \le j \le J \right\},$$

which satisfies $K_2^h \subset \widetilde{K_2^h} \subset \overline{K_2^h}$ (the similar definition when $d = 2$ does not lead to interesting convergence properties).

Remark 3 *Since we only consider tetrahedralizations of the domain Ω (in the three-dimensional case), the previous inclusions $K_2^h \subset \widehat{K_2^h}$ and $\widetilde{K_2^h} \subset \overline{K_2^h}$ come from the quadrature of order two on the triangle (see, e.g., [11, 13]): $\forall v^h \in V_2^h$*

$$\int_{T\cap\Gamma_C} v_N^h \, d\Gamma = \frac{|T \cap \Gamma_C|}{3} \sum_{j=1}^{3} v_N^h(m_j),$$

where $|T \cap \Gamma_C|$ stands for the surface of $T \cap \Gamma_C$ and m_1, m_2, m_3 represent the three midpoints of the edges. A consequence of the previous quadrature is that the integral on $T \cap \Gamma_C$ of the three basis functions at the vertices vanishes.

2.2.3 The discrete problems

When K^h is one of the (eleven) previous convex cones (five when $d = 2$ and six when $d = 3$), the discrete variational inequality issued from (6) is

$$\left\{ \begin{array}{l} \text{Find } u^h \in K^h \text{ satisfying:} \\ a(u^h, v^h - u^h) \ge l(v^h - u^h), \quad \forall \ v^h \in K^h. \end{array} \right. \tag{7}$$

According to Stampacchia's Theorem, problem (7) admits a unique solution.

3 Error analysis in the two-dimensional case ($d = 2$)

The following two theorems yield optimal convergence rates in the two-dimensional case when considering either linear or quadratic finite elements. Both theorems only use the standard Sobolev regularity assumption of the solution to the continuous problem u.

Theorem 1 *Let $d = 2, k = 1$. Set $K^h = K_1^h$ or $K^h = \overline{K_1^h}$. Let u and u^h be the solutions to Problems (6) and (7) respectively. Assume that $u \in (H^\tau(\Omega))^2$ with $3/2 < \tau \le 2$. Then, there exists a constant $C > 0$ independent of h and u such that*

$$\|u - u^h\|_{1,\Omega} \le Ch^{\tau-1}\|u\|_{\tau,\Omega}. \tag{8}$$

Theorem 2 *Let $d = 2, k = 2$. Set $K^h = K_2^h$ or $K^h = \widehat{K_2^h}$ or $K^h = \overline{K_2^h}$. Let u and u^h be the solutions to Problems (6) and (7) respectively. Assume that $u \in (H^\tau(\Omega))^2$ with $3/2 < \tau \le 5/2$. Then, there exists a constant $C > 0$ independent of h and u such that*

$$\|u - u^h\|_{1,\Omega} \le Ch^{\tau-1}\|u\|_{\tau,\Omega}. \tag{9}$$

Remark 4 *When $k = 2$, note that the optimal error bound for $2 < \tau \le 5/2$ has already been proven in [5], for $K^h = \widehat{K_2^h}$ and $K^h = K_2^h$.*

Proof of Theorem 1 (linear case in 2D). The use of Falk's Lemma (see, *e.g.*, [14, 17, 29]) leads to the following bound:

$$\alpha\|u - u^h\|^2_{1,\Omega} \;\le\; \inf_{v^h \in K^h}\left(\|u - v^h\|^2_{1,\Omega} + \int_{\Gamma_C} \sigma_N(v^h - u)_N \; d\Gamma\right) + \inf_{v \in K}\int_{\Gamma_C} \sigma_N(v - u^h)_N \; d\Gamma, \tag{10}$$

where α is a positive constant which only depends on the continuity and the ellipticity constants of $a(.,.)$. The usual choice for v^h (which we also adopt in this study) is $v^h = \mathcal{I}_1^h u$ where $\mathcal{I}_1^h$ is the Lagrange interpolation operator mapping onto V_1^h. We have $\mathcal{I}_1^h u \in K_1^h \subset \overline{K_1^h}$ and $\|u - \mathcal{I}_1^h u\|_{1,\Omega} \le Ch^{\tau-1}\|u\|_{\tau,\Omega}$ for any $1 < \tau \le 2$.
(**i**): We begin with the case $K^h = K_1^h$ in which the second infimum in (10) disappears since $K_1^h \subset K$. To prove the theorem in this case it remains then to estimate the term

$$\int_{\Gamma_C} \sigma_N(\mathcal{I}_1^h u - u)_N \; d\Gamma,$$

for $u \in (H^\tau(\Omega))^2$, $3/2 < \tau \le 2$. From the trace theorem we deduce that $u_N \in H^{\tau-1/2}(\Gamma_C)$ (hence u_N is continuous), $\sigma_N \in H^{\tau-3/2}(\Gamma_C)$ and $u'_N \in H^{\tau-3/2}(\Gamma_C)$ where u'_N denotes the derivative of u_N along Γ_C. Let $T \in T^h$ with $T \cap \Gamma_C \neq \emptyset$. In the forthcoming proof we will estimate

$$\int_{T\cap\Gamma_C} \sigma_N(\mathcal{I}_1^h u - u)_N \; d\Gamma, \tag{11}$$

and we will denote by h_e the length of the segment $T \cap \Gamma_C$. We define Z_C and Z_{NC} which stand for the contact and the noncontact sets in $T \cap \Gamma_C$ respectively, i.e.:

$$Z_C = \{x \in T \cap \Gamma_C, u_N(x) = 0\},$$

$$Z_{NC} = \{x \in T \cap \Gamma_C, u_N(x) < 0\},$$

and we denote by $|Z_C|$, $|Z_{NC}|$ their measures in $\mathbb{R}$ (so $|Z_C| + |Z_{NC}| = h_e$).

Remark 5 *Since u_N belongs to $H^{\tau-1/2}(\Gamma_C)$ when $3/2 < \tau \le 2$, the Sobolev embeddings ensure that $u_N \in C(\Gamma_C)$ (this remains true in three space dimensions). So Z_C and Z_{NC} are measurable as inverse images of a Borel set by a continuous function.*

If either $|Z_C|$ or $|Z_{NC}|$ equals zero then it is easy to see that the integral term in (11) vanishes. So we suppose that $|Z_C| > 0$ and $|Z_{NC}| > 0$ in the following estimation of (11). We next obtain two estimates of the same error term (11): a first one depending on $|Z_{NC}|$, a second one depending on $|Z_C|$.

Estimate of (11) depending on Z_{NC}. Using Cauchy-Schwarz inequality, estimate (36) in Lemma 2 (see Appendix A) and a standard error estimate on $\mathcal{I}_1^h$ gives

$$\begin{aligned}
\int_{T\cap\Gamma_C} \sigma_N (\mathcal{I}_1^h u - u)_N \, d\Gamma &\leq \|\sigma_N\|_{0,T\cap\Gamma_C} \|(\mathcal{I}_1^h u - u)_N\|_{0,T\cap\Gamma_C} \\
&\leq C \frac{1}{|Z_{NC}|^{1/2}} h_e^{\tau-1} |\sigma_N|_{\tau-3/2,T\cap\Gamma_C} h_e^{\tau-1/2} |u'_N|_{\tau-3/2,T\cap\Gamma_C} \\
&\leq C \frac{h_e^{2\tau-3/2}}{|Z_{NC}|^{1/2}} \left(|\sigma_N|^2_{\tau-3/2,T\cap\Gamma_C} + |u'_N|^2_{\tau-3/2,T\cap\Gamma_C} \right). \qquad (12)
\end{aligned}$$

Estimate of (11) depending on Z_C. This estimate is obtained in a different way. We now use the standard error estimate on $\mathcal{I}_1^h$ (see [11]) and bounds (37), (36) in Lemma 2 (see Appendix A).

$$\begin{aligned}
\int_{T\cap\Gamma_C} \sigma_N (\mathcal{I}_1^h u - u)_N \, d\Gamma &\leq \|\sigma_N\|_{0,T\cap\Gamma_C} \|(\mathcal{I}_1^h u - u)_N\|_{0,T\cap\Gamma_C} \\
&\leq C \|\sigma_N\|_{0,T\cap\Gamma_C} h_e^{1/2} \|u'_N\|_{L^1(T\cap\Gamma_C)} \\
&\leq C \frac{h_e^{2\tau-3/2}}{|Z_C|^{1/2}} \left(|\sigma_N|^2_{\tau-3/2,T\cap\Gamma_C} + |u'_N|^2_{\tau-3/2,T\cap\Gamma_C} \right). \qquad (13)
\end{aligned}$$

We conclude by noting that either $|Z_{NC}|$ or $|Z_C|$ is greater than $h_e/2$ and by choosing the appropriate estimate (12) or (13). So

$$\int_{T\cap\Gamma_C} \sigma_N (\mathcal{I}_1^h u - u)_N \, d\Gamma \leq C h_e^{2(\tau-1)} \left(|\sigma_N|^2_{\tau-3/2,T\cap\Gamma_C} + |u'_N|^2_{\tau-3/2,T\cap\Gamma_C} \right).$$

By summation and using the trace theorem we get

$$\int_{\Gamma_C} \sigma_N (\mathcal{I}_1^h u - u)_N \, d\Gamma \leq C h^{2(\tau-1)} \left(|\sigma_N|^2_{\tau-3/2,\Gamma_C} + |u'_N|^2_{\tau-3/2,\Gamma_C} \right) \leq C h^{2(\tau-1)} \|u\|^2_{\tau,\Omega},$$

from which (8) follows when $K^h = K_1^h$.

(ii): We now consider the case $K^h = \overline{K_1^h}$. As previously we can choose $v^h = \mathcal{I}_1^h u$ since $\mathcal{I}_1^h u \in \overline{K_1^h}$. The first infimum in (10) therefore satisfies the same optimal bound as in the case $K^h = K_1^h$. The second infimum in (10) is handled by choosing $v = 0$. To prove the theorem it remains then to estimate the term

$$-\int_{\Gamma_C} \sigma_N u_N^h \, d\Gamma,$$

where $u^h \in \overline{K_1^h}$ is the discrete solution. We next consider the space X_0^h of the piecewise constant functions on the trace mesh $T^h \cap \Gamma_C$:

$$X_0^h = \left\{ \chi^h \in L^2(\Gamma_C) : \chi^h|_{T\cap\Gamma_C} \in P_0(T\cap\Gamma_C), \forall T \in T^h \right\},$$

and the classical $L^2(\Gamma_C)-$projection operator $\pi_0^h : L^2(\Gamma_C) \to X_0^h$ defined for any $\varphi \in L^2(\Gamma_C)$ by

$$\int_{\Gamma_C} (\varphi - \pi_0^h\varphi)\chi^h \, d\Gamma = 0, \quad \forall \chi^h \in X_0^h.$$

We still denote by h_e the length of the segment $T \cap \Gamma_C$. The operator π_0^h satisfies the following standard estimates for any $0 < r < 1$ and any $\varphi \in H^r(\Gamma_C)$ (see, e.g., [7, 19]):

$$\|\varphi - \pi_0^h\varphi\|_{0,T\cap\Gamma_C} \leq Ch_e^r\, |\varphi|_{r,T\cap\Gamma_C}, \quad \|\varphi - \pi_0^h\varphi\|_{0,\Gamma_C} + h^{-1/2}\|\varphi - \pi_0^h\varphi\|_{1/2,*,\Gamma_C} \leq Ch^r\, |\varphi|_{r,\Gamma_C}, \tag{14}$$

where $\|.\|_{1/2,*,\Gamma_C}$ stands for the dual norm of $\|.\|_{1/2,\Gamma_C}$. When $r = 0$ (resp. $r = 1$) the previous estimates remain true by changing $|\varphi|_{r,.}$ with $\|\varphi\|_{0,.}$ (resp. $\|\varphi'\|_{0,.}$). We have, since $\pi_0^h\sigma_N$ is a nonpositive piecewise constant function on Γ_C:

$$\begin{aligned}
-\int_{\Gamma_C} \sigma_N u_N^h \, d\Gamma &\leq -\int_{\Gamma_C} (\sigma_N - \pi_0^h\sigma_N)u_N^h \, d\Gamma \\
&= -\int_{\Gamma_C} (\sigma_N - \pi_0^h\sigma_N)(u_N^h - u_N) \, d\Gamma - \int_{\Gamma_C} (\sigma_N - \pi_0^h\sigma_N)u_N \, d\Gamma. \qquad (15)
\end{aligned}$$

The first term in (15) is bounded in an optimal way by using (14), the trace theorem and Young's inequality:

$$\begin{aligned}
-\int_{\Gamma_C} (\sigma_N - \pi_0^h\sigma_N)(u_N^h - u_N) \, d\Gamma &\leq \|\sigma_N - \pi_0^h\sigma_N\|_{1/2,*,\Gamma_C}\|u_N^h - u_N\|_{1/2,\Gamma_C} \\
&\leq Ch^{\tau-1}|\sigma_N|_{\tau-3/2,\Gamma_C}\|u^h - u\|_{1,\Omega} \\
&\leq Ch^{2(\tau-1)}|\sigma_N|^2_{\tau-3/2,\Gamma_C} + \frac{\alpha}{2}\|u - u^h\|^2_{1,\Omega}.
\end{aligned}$$

To prove the theorem it remains now to bound the second term in (15). We estimate this term on any element $T \cap \Gamma_C$:

$$-\int_{T\cap\Gamma_C} (\sigma_N - \pi_0^h\sigma_N)u_N \, d\Gamma = \int_{T\cap\Gamma_C} (\sigma_N - \pi_0^h\sigma_N)(\pi_0^h u_N - u_N) \, d\Gamma \tag{16}$$

in two different ways. If either $|Z_C|$ or $|Z_{NC}|$ equals zero then it is easy to see that previous integral term in (16) vanishes. So we suppose that $|Z_C| > 0$ and $|Z_{NC}| > 0$ in the estimation of (16).

Estimate of (16) depending on Z_{NC}. We next use the standard error estimate on π_0^h and bounds (38), (35) in Lemma 2 (see Appendix A).

$$\begin{aligned}
\int_{T\cap\Gamma_C} (\sigma_N - \pi_0^h\sigma_N)(\pi_0^h u_N - u_N) \, d\Gamma &= \int_{T\cap\Gamma_C} \sigma_N(\pi_0^h u_N - u_N) \, d\Gamma \\
&\leq \|\sigma_N\|_{L^1(T\cap\Gamma_C)}\|u_N - \pi_0^h u_N\|_{L^\infty(T\cap\Gamma_C)} \\
&\leq \|\sigma_N\|_{L^1(T\cap\Gamma_C)} h_e^{1/2}\|u_N'\|_{0,T\cap\Gamma_C} \\
&\leq C\frac{h_e^{2\tau-3/2}}{|Z_{NC}|^{1/2}}\left(|\sigma_N|^2_{\tau-3/2,T\cap\Gamma_C} + |u_N'|^2_{\tau-3/2,T\cap\Gamma_C}\right). \qquad (17)
\end{aligned}$$

Estimate of (16) depending on Z_C. Here we use the standard L^2-error estimate on π_0^h in (14) together with bound (38):

$$\begin{aligned}
\int_{T\cap\Gamma_C}(\sigma_N-\pi_0^h\sigma_N)(\pi_0^h u_N-u_N)\,d\Gamma &\leq \|\sigma_N-\pi_0^h\sigma_N\|_{0,T\cap\Gamma_C}\|u_N-\pi_0^h u_N\|_{0,T\cap\Gamma_C}\\
&\leq Ch_e^{\tau-1/2}|\sigma_N|_{\tau-3/2,T\cap\Gamma_C}\|u'_N\|_{0,T\cap\Gamma_C}\\
&\leq C\frac{h_e^{2\tau-3/2}}{|Z_C|^{1/2}}\left(|\sigma_N|^2_{\tau-3/2,T\cap\Gamma_C}+|u'_N|^2_{\tau-3/2,T\cap\Gamma_C}\right). \qquad (18)
\end{aligned}$$

By noting that either $|Z_{NC}|$ or $|Z_C|$ is greater than $h_e/2$, choosing then either estimate (17) or estimate (18), summing over all the contact elements, and then using the trace theorem, we come to the conclusion that:

$$-\int_{\Gamma_C}(\sigma_N-\pi_0^h\sigma_N)u_N\,d\Gamma \leq Ch^{2(\tau-1)}\left(|\sigma_N|^2_{\tau-3/2,\Gamma_C}+|u'_N|^2_{\tau-3/2,\Gamma_C}\right)\leq Ch^{2(\tau-1)}\|u\|^2_{\tau,\Omega}.$$

So (8) holds when $K^h=\overline{K_1^h}$.

■

Proof of Theorem 2 (quadratic case in 2D). The proof is split in two parts, the first one dealing with $3/2<\tau<5/2$ and the second one concerning $\tau=5/2$.
(**i**): $3/2<\tau<5/2$. Since the following inclusions hold,

$$K_2^h\subset\widehat{K_2^h}\subset\overline{K_2^h}, \qquad (19)$$

we only have to prove the optimal approximation error bound when considering K_2^h and the optimal consistency error bound when considering $\overline{K_2^h}$.

- Approximation error (when $K^h=K_2^h$):
  We choose $v^h=\mathcal{I}_2^h u$ where $\mathcal{I}_2^h$ is the Lagrange interpolation operator mapping onto V_2^h. So $v^h\in K_2^h$. Combining the standard error estimate on $\mathcal{I}_2^h$ (see [9, 11, 13]) and Lemma 2 (which holds for $3/2<\tau<5/2$) in the same way as in the linear case (when $K^h=K_1^h$) gives us the optimal approximation bound.

- Consistency error (when $K^h=\overline{K_2^h}$):
  We choose (again) $v=0$. Using (again) the standard error estimate on π_0^h and Lemma 2 in the same way as in the linear case (when $K^h=\overline{K_1^h}$) gives us the optimal bound.

So (9) holds when $3/2<\tau<5/2$ for $K^h=K_2^h$ or $K^h=\widehat{K_2^h}$ or $K^h=\overline{K_2^h}$.

(**ii**): $\tau=5/2$. In this case we need to prove (see (9)) that

$$\|u-u^h\|_{1,\Omega}\leq Ch^{3/2}\|u\|_{5/2,\Omega}. \qquad (20)$$

First, we can suppose that the continuous function σ_N (and also u'_N) vanishes somewhere on $T\cap\Gamma_C$ (otherwise the following integral terms in (21) and in (22) equal zero) which leads to the obvious bounds $\|\sigma_N\|_{0,T\cap\Gamma_C}\leq h_e\|\sigma'_N\|_{0,T\cap\Gamma_C}$ and $\|u'_N\|_{0,T\cap\Gamma_C}\leq h_e\|u''_N\|_{0,T\cap\Gamma_C}$. Using (19), it only remains to prove the following optimal bounds:

- Approximation error when $K^h = K_2^h$:
  We choose $v^h = \mathcal{I}_2^h u$. The first term $\|u - \mathcal{I}_2^h u\|_{1,\Omega}$ is optimally bounded using the standard error estimate on $\mathcal{I}_2^h$. It remains to estimate the integral term. Thanks to the standard estimate on $\mathcal{I}_2^h$ and the estimate $\|\sigma_N\|_{0,T\cap\Gamma_C} \leq h_e \|\sigma'_N\|_{0,T\cap\Gamma_C}$, we get

$$\begin{aligned} \int_{T\cap\Gamma_C} \sigma_N (\mathcal{I}_2^h u - u)_N \, d\Gamma &\leq \|\sigma_N\|_{0,T\cap\Gamma_C} \|(\mathcal{I}_2^h u - u)_N\|_{0,T\cap\Gamma_C} \\ &\leq C h_e^3 \|\sigma'_N\|_{0,T\cap\Gamma_C} \|u''_N\|_{0,T\cap\Gamma_C}. \end{aligned} \tag{21}$$

- Consistency error when $K^h = \overline{K_2^h}$:
  From the standard estimate on π_0^h and the bound $\|u'_N\|_{0,T\cap\Gamma_C} \leq h_e \|u''_N\|_{0,T\cap\Gamma_C}$, we get

$$\begin{aligned} \int_{T\cap\Gamma_C} (\sigma_N - \pi_0^h \sigma_N)(\pi_0^h u_N - u_N) \, d\Gamma &\leq \|\sigma_N - \pi_0^h \sigma_N\|_{0,T\cap\Gamma_C} \|u_N - \pi_0^h u_N\|_{0,T\cap\Gamma_C} \\ &\leq C h_e^2 \|\sigma'_N\|_{0,T\cap\Gamma_C} \|u'_N\|_{0,T\cap\Gamma_C} \\ &\leq C h_e^3 \|\sigma'_N\|_{0,T\cap\Gamma_C} \|u''_N\|_{0,T\cap\Gamma_C}. \end{aligned} \tag{22}$$

Summing both previous local quantities (21) and (22) (as in the previous proofs) leads to (20). ■

4 Error analysis in the three-dimensional case ($d = 3$)

Before giving the convergence results, we next explain that the two-dimensional proof could not be extended straightforwardly to the three-dimensional case although the key Lemma 2 holds when $d = 3$. To simplify we first consider the case $d = 3$ in the linear finite element case. In order to extend "straightforwardly" the optimal estimate (8), we would need that all the intermediary estimates used in the proof of Theorem 1 remain true in the three-dimensional case. Since standard estimates on $\mathcal{I}_1^h$ and π_0^h still hold when $d = 3$ (see, e.g., [7] and [9] for the estimates on π_0^h and $\mathcal{I}_1^h$), the last estimates which still need to be satisfied are:

- in the approximation error analysis (when $K^h = K_1^h$ and $K^h = \overline{K_1^h}$):

$$\|u_N - \mathcal{I}_1^h u_N\|_{0,T\cap\Gamma_C} \leq C \|\nabla u_N\|_{L^1(T\cap\Gamma_C)}, \tag{23}$$

- in the consistency error analysis (when $K^h = \overline{K_1^h}$):

$$\|u_N - \pi_0^h u_N\|_{L^\infty(T\cap\Gamma_C)} \leq C \|\nabla u_N\|_{0,T\cap\Gamma_C}. \tag{24}$$

Actually, we cannot prove both estimates (23) and (24). Estimate (24) is the limit case of the Gagliardo-Nirenberg-Sobolev inequality, it does not hold since $H^1(\Gamma_C)$ is not a subset of $C(\Gamma_C)$ when $\Gamma_C \subset \mathbb{R}^2$. The inequality (23) is not satisfied when $d = 3$ due to the fact that the functions of $W^{1,1}(\Gamma_C)$ are not continuous when Γ_C is a subset of $\mathbb{R}^2$. So, we could not extend straightforwardly the optimal error bounds to the three-dimensional case. The next analysis circumvents these both difficulties as follows:

- To circumvent (23), the basic idea is to change the Lagrange interpolation operator with a quasi-interpolation operator adapted to rough functions (see, e.g., [12, 18, 30, 32]) keeping in mind that we also need some positivity preserving properties and more than first order accuracy (which requires, roughly speaking that affine functions are locally reproduced). Since the functions we consider do not necessarily vanish on the boundary of $\Gamma_{_C}$ there is an impossibility result at the extreme points of $\overline{\Gamma_{_C}}$ (see [27]) which forces us to slightly change the convex sets K_1^h and $\overline{K_1^h}$ on the triangles containing an extreme point of $\overline{\Gamma_{_C}}$. We first recall the definition of extreme points, see [27]: $e \in \partial\Gamma_{_C}$ is an extreme point of $\overline{\Gamma_{_C}}$ if there exists an affine function a_e such that

$$a_e(e) = 0 \quad \text{and} \quad a_e(x) > 0, \forall x \in \overline{\Gamma_{_C}},\ x \neq e.$$

In other words, $e \in \partial\Gamma_{_C}$ is an extreme point of $\overline{\Gamma_{_C}}$ if it does not lie in any open segment joining two points of $\overline{\Gamma_{_C}}$. Therefore, a square contains 4 extreme points (the 4 corners), a L-shaped domain contains 5 extreme points (see Figure 2).

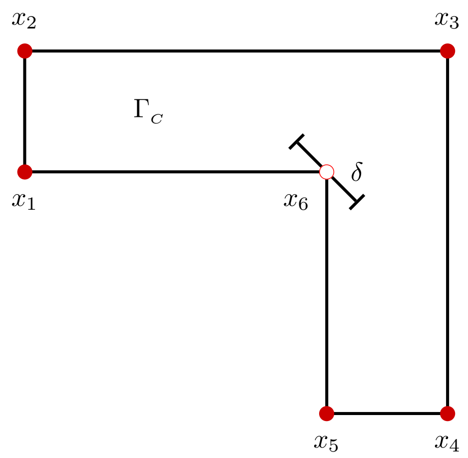

Figure 2: *Example of extreme points in a L-shaped domain, $\Gamma_{_C}$. The vertices x_1 to x_5 are extreme points whereas x_6 is just a boundary point (since we can construct the open segment δ joining two points of $\overline{\Gamma_{_C}}$ and contaning x_6.)*

Let N_e be the set of extreme points of $\overline{\Gamma_{_C}}$. If $e \in N_e$, let $\Delta_e \subset \overline{\Gamma_{_C}}$ be the union of triangles (i.e., the patch) having e as vertex and set $E = \cup_{e \in N_e} \Delta_e$. So we define:

$$K_1^{h,e} \quad = \quad \left\{ v^h \in V_1^h : v_{_N}^h \leq 0 \quad \text{on } \Gamma_{_C} \setminus E, \quad \int_{\Delta_e} v_{_N}^h \, d\Gamma \leq 0, \forall e \in N_e \right\} \tag{25}$$

$$\begin{aligned} \overline{K_1^{h,e}} = \Bigg\{ v^h \in V_1^h, \int_{T \cap \Gamma_{_C}} v_{_N}^h \, d\Gamma \ &\leq \ 0, \forall T \in T^h, T \cap \Gamma_{_C} \subset \Gamma_{_C} \setminus E, \\ \int_{\Delta_e} v_{_N}^h \, d\Gamma \ &\leq \ 0, \forall e \in N_e \Bigg\}. \end{aligned} \tag{26}$$

Note that neither of these two convex cones belongs to K. We have $K_1^{h,e} \subset \overline{K_1^{h,e}}$, $K_1^h \subset K_1^{h,e}$, $\overline{K_1^h} \subset \overline{K_1^{h,e}}$. Moreover if any extreme point of $\overline{\Gamma_{_C}}$ belongs to only one contact triangle, then $\overline{K_1^h} = \overline{K_1^{h,e}}$.

- To circumvent (24), we prove by using Lemma 1, a weaker result than (24) (see (32)) which however allows us to use Lemma 2 to get the same convergence result as in the two-dimensional case.

Theorem 3 *Let $d = 3, k = 1$. Set $K^h = K_1^{h,e}$ or $K^h = \overline{K_1^{h,e}}$. Let u and u^h be the solutions to Problems (6) and (7) respectively. Assume that $u \in (H^\tau(\Omega))^3$ with $3/2 < \tau \leq 2$. Then, there exists a constant $C > 0$ independent of h and u such that*

$$\|u - u^h\|_{1,\Omega} \leq Ch^{\tau-1}\|u\|_{\tau,\Omega}. \tag{27}$$

Before proving Theorem 3 we give the convergence result in the quadratic case. As in the linear case, see (25), (26), we can define the modified convex sets $K_2^{h,e}, \widehat{K_2^{h,e}}, \widetilde{K_2^{h,e}}, \overline{K_2^{h,e}}$ by keeping the same definitions as for $K_2^h, \widehat{K_2^h}, \widetilde{K_2^h}, \overline{K_2^h}$ (see subsection 2.2.2) on the triangles of $\Gamma_C \setminus E$ (i.e. except on the patches Δ_e where e an extreme point of $\overline{\Gamma_C}$, see Figure 3). On the patches the nonpenetration condition becomes as in the linear case $\int_{\Delta_e} v_N^h \, d\Gamma \leq 0$.

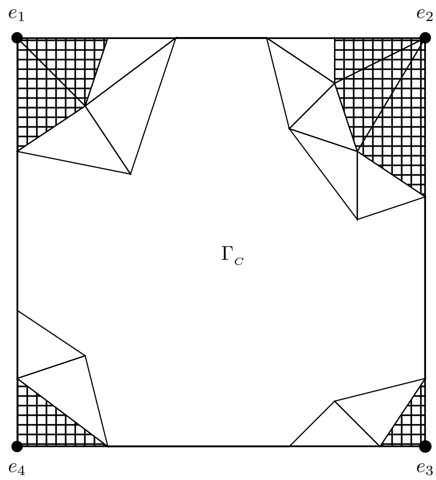

Figure 3: *The four extreme points of the square and the four patches (crosshatched areas) Δ_{e_1} to Δ_{e_4}.*

Theorem 4 *Let $d = 3, k = 2$. Set $K^h = K_2^{h,e}$ or $K^h = \widehat{K_2^{h,e}}$ or $K^h = \widetilde{K_2^{h,e}}$ or $K^h = \overline{K_2^{h,e}}$. Let u and u^h be the solutions to Problems (6) and (7) respectively. Assume that $u \in (H^\tau(\Omega))^3$ with $3/2 < \tau \leq 5/2$. Then, there exists a constant $C > 0$ independent of h and u such that*

$$\|u - u^h\|_{1,\Omega} \leq Ch^{\tau-1}\|u\|_{\tau,\Omega}. \tag{28}$$

Proof of Theorem 3 (linear case in 3D). As previously it suffices to prove the approximation error when $K^h = K_1^{h,e}$ and the consistency error when $K^h = \overline{K_1^{h,e}}$.

- Approximation error (when $K^h = K_1^{h,e}$):
  We choose
$$v^h = \mathcal{I}_1^h u + R_1^h(\mathcal{J}_1^h u_N - \mathcal{I}_1^h u_N)$$

where R_1^h is a discrete extension operator from W_1^h into V_1^h (W_1^h is the normal trace space of V_1^h on Γ_C). Note that the discrete extension operators can be obtained by combining a standard continuous extension operator with a local regularization operator (see, e.g., [30, 8]). The quasi-interpolation operator $\mathcal{J}_1^h : W^{1,1}(\Gamma_C) \mapsto W_1^h$ is defined as follows for a function v: for interior nodes x in Γ_C, we choose the definition of the Chen-Nochetto operator which is positivity preserving and also preserves local affine functions,

$$(\mathcal{J}_1^h v)(x) = \frac{1}{\text{meas}(B)} \int_B v,$$

where B is the largest open ball of center x such that B is contained in the union of the elements containing x (see [10]). Now we consider the nodes on the boundary of $\overline{\Gamma_C}$. For the boundary nodes x in $\overline{\Gamma_C} \cap \overline{\Gamma_D}$, we set $(\mathcal{J}_1^h v)(x) = 0$. For the other boundary nodes x which are not extreme points, we set (see [27]):

$$(\mathcal{J}_1^h v)(x) = \frac{1}{\text{length}(L)} \int_L v,$$

where L is a small line segment of length αh (α is fixed), symmetrically placed around x, and included in $\overline{\Gamma_C}$. Such a definition is both positivity and affine functions preserving. Finally, we have to define $\mathcal{J}_1^h v$ at the remaining extreme nodes. So we consider an extreme node e of $\overline{\Gamma_C}$ and the unions of triangles (i.e., patch) on $\overline{\Gamma_C}$ having e as vertex. We denote this patch by Δ_e. On Δ_e we require that the average of v is preserved: find $\mathcal{J}_1^h v \in P_1(\Delta_e)$ such that

$$\int_{\Delta_e} v - \mathcal{J}_1^h v = 0.$$

It is easy to show that this definition:

- leads to a unique value of $(\mathcal{J}_1^h v)(e)$ (which of course is not necessarily nonpositive),

- preserves locally the affine functions.

From the construction of $\mathcal{J}_1^h$, we deduce that $v^h \in K_1^{h,e}$. Let $v \in W^{1,p}(\Gamma_C)$ with $p \geq 1$, it is easy to prove (using a scaled trace inequality) that for any node x on $\overline{\Gamma_C}$ which is not extreme we have $|(\mathcal{J}_1^h v)(x)| \leq C(h_C^{-2/p}\|v\|_{L^p(\Delta_x)} + h_C^{1-2/p}\|\nabla v\|_{L^p(\Delta_x)})$ where Δ_x is the patch surrounding x. If the node x is extreme we have the same kind of estimate as before where Δ_x has to be changed with the extended patch surrounding Δ_x which we denote again by Δ_x to simplify. So we have on any triangle $T \cap \Gamma_C$, the stability estimate:

$$\|\mathcal{J}_1^h v\|_{0,T\cap\Gamma_C} \leq C(h_C^{1-2/p}\|v\|_{L^p(\Delta_{T\cap\Gamma_C})} + h_C^{2-2/p}\|\nabla v\|_{L^p(\Delta_{T\cap\Gamma_C})})$$

where $\Delta_{T\cap\Gamma_C}$ is the patch surrounding $T \cap \Gamma_C$. Choosing $p = 1$ and using the property that $\mathcal{J}_1^h$ preserves locally the constant functions (note that the triangles containing a node in $\overline{\Gamma_C} \cap \overline{\Gamma_D}$ are handled as in [10]) together with the property $\|v - \pi_0^h v\|_{0,\Delta_{T\cap\Gamma_C}} \leq C\|\nabla v\|_{L^1(\Delta_{T\cap\Gamma_C})}$ (see Corollary 4.2.3 in [37]) we have

$$\|u_N - \mathcal{J}_1^h u_N\|_{0,T\cap\Gamma_C} \leq C\|\nabla u_N\|_{L^1(\Delta_{T\cap\Gamma_C})}, \tag{29}$$

which was the kind of estimate we could not obtain for the Lagrange interpolation operator (see the previous discussion). Besides using the above stability estimate on $\mathcal{J}_1^h$ with $p = 2$ together with the property that $\mathcal{J}_1^h$ preserves locally the affine functions implies that $\mathcal{J}_1^h$

satisfies the same approximation properties as the linear Lagrange interpolation operator. Using the continuity of the extension operator and an inverse inequality gives:

$$\begin{aligned}
\|u - v^h\|_{1,\Omega} &\leq \|u - \mathcal{I}_1^h u\|_{1,\Omega} + C\|\mathcal{J}_1^h u_{_N} - \mathcal{I}_1^h u_{_N}\|_{1/2,\Gamma_C} \\
&\leq \|u - \mathcal{I}_1^h u\|_{1,\Omega} + Ch_C^{-1/2}(\|u_{_N} - \mathcal{I}_1^h u_{_N}\|_{0,\Gamma_C} + \|u_{_N} - \mathcal{J}_1^h u_{_N}\|_{0,\Gamma_C}) \\
&\leq Ch^{\tau-1}\|u\|_{\tau,\Omega} \qquad (30)
\end{aligned}$$

for any $1 < \tau \leq 2$. Combining these estimates with Lemma 2 (and using $\Delta_{T\cap\Gamma_C}$ instead of $T \cap \Gamma_C$ in the Lemma) gives us the optimal approximation bound:

$$\int_{T\cap\Gamma_C} \sigma_{_N}(\mathcal{J}_1^h u - u)_{_N}\ d\Gamma \leq Ch_e^{2(\tau-1)}\left(|\sigma_{_N}|^2_{\tau-3/2,\Delta_{T\cap\Gamma_C}} + |\nabla u_{_N}|^2_{\tau-3/2,\Delta_{T\cap\Gamma_C}}\right).$$

Hence, by summation

$$\int_{\Gamma_C} \sigma_{_N}(v^h - u)_{_N}\ d\Gamma = \int_{\Gamma_C} \sigma_{_N}(\mathcal{J}_1^h u - u)_{_N}\ d\Gamma \leq Ch^{2(\tau-1)}\|u\|^2_{\tau,\Omega}, \qquad (31)$$

which ends the proof of the approximation error.

- Consistency error (when $K^h = \overline{K_1^{h,e}}$):
We first use the following approximation result proved in [37] which we recall hereafter.

Lemma 1 *Let X_a be a normed linear space with norm $\|.\|_a$ and let $X \subset X_a$ be a Banach space with norm $\|.\|$. Suppose $\|.\| = \|.\|_a + \|.\|_b$ where $\|.\|_b$ is a semi-norm and assume that bounded sets in X are precompact in X_a. Let $Y = X \cap \{x : \|x\|_b = 0\}$. If $L : X \to Y$ is a projection, there is a constant C independent of L such that:*

$$\|x - L(x)\|_a \leq C\|L\|\|x\|_b$$

for all $x \in X$.

Let $\tilde{T}$ be a reference triangle. We set $X_a = L^\infty(\tilde{T})$ and $X = H^{\tau-1/2}(\tilde{T})$. Since $\tau > 3/2$ we have $X \subset X_a$. It is easy to check that $\|\tilde{v}\| = \|\tilde{v}\|_{L^\infty(\tilde{T})} + \|\nabla\tilde{v}\|_{0,\tilde{T}} + |\nabla\tilde{v}|_{\tau-3/2,\tilde{T}}$ is a norm on $H^{\tau-1/2}(\tilde{T})$ since (by using the embedding $X \subset X_a$) it is equivalent to the usual norm $\|\tilde{v}\|_{0,\tilde{T}} + \|\nabla\tilde{v}\|_{0,\tilde{T}} + |\nabla\tilde{v}|_{\tau-3/2,\tilde{T}}$. Moreover it is straightforward that $\|\nabla\tilde{v}\|_{0,\tilde{T}} + |\nabla\tilde{v}|_{\tau-3/2,\tilde{T}}$ is a semi-norm on $H^{\tau-1/2}(\tilde{T})$. Besides the embedding $X \subset X_a$ is compact so that the bounded sets in X are precompact in X_a. Clearly Y is the space of constant functions on $\tilde{T}$. If L stands for the $L^2(\tilde{T})$ projection operator on constant functions on $\tilde{T}$, we get

$$\|\tilde{v} - L\tilde{v}\|_{L^\infty(\tilde{T})} \leq C\|L\|\left(\|\nabla\tilde{v}\|_{0,\tilde{T}} + |\nabla\tilde{v}|_{\tau-3/2,\tilde{T}}\right), \quad \forall\tilde{v} \in H^{\tau-1/2}(\tilde{T}).$$

Now we denote $v(\eta(\tilde{x})) = \tilde{v}(\tilde{x})$ where $\eta : \tilde{T} \to \overline{T \cap \Gamma_C}$ is an affine transformation. By a scaling argument, we obtain $\|v - \pi_0^h v\|_{L^\infty(T\cap\Gamma_C)} = \|\tilde{v} - L\tilde{v}\|_{L^\infty(\tilde{T})}$, $\|\nabla v\|_{0,T\cap\Gamma_C} = \|\nabla\tilde{v}\|_{0,\tilde{T}}$ and $|\nabla v|_{\tau-3/2,T\cap\Gamma_C} = h_e^{3/2-\tau}|\nabla\tilde{v}|_{\tau-3/2,\tilde{T}}$. So

$$\|v - \pi_0^h v\|_{L^\infty(T\cap\Gamma_C)} \leq C\left(\|\nabla v\|_{0,T\cap\Gamma_C} + h_e^{\tau-3/2}|\nabla v|_{\tau-3/2,T\cap\Gamma_C}\right). \qquad (32)$$

The analysis of the consistency error is then the same as in the two-dimensional case by changing X_0^h with the (slightly smaller space) $X_0^{h,e}$ of the piecewise constant functions on the trace mesh and constant on the patches surrounding the extreme points of $\overline{\Gamma_C}$:

$$X_0^{h,e} = \left\{ \chi^h \in L^2(\Gamma_C) \quad : \quad \chi^h|_{T\cap\Gamma_C} \in P_0(T\cap\Gamma_C), \forall T \in T^h, T\cap\Gamma_C \subset \Gamma_C \setminus E, \right.$$
$$\left. \chi^h|_{\Delta_e} \in P_0(\Delta_e), \forall e \in N_e \right\},$$

and by considering the classical $L^2(\Gamma_C)-$projection operator $\pi_0^{h,e} : L^2(\Gamma_C) \to X_0^{h,e}$. The additional term in (32) does not change the estimates (17) and (18) which become respectively

$$\begin{aligned}
& \int_{T\cap\Gamma_C} (\sigma_N - \pi_0^{h,e}\sigma_N)(\pi_0^{h,e}u_N - u_N)\ d\Gamma \\
\leq\ & \|\sigma_N\|_{L^1(T\cap\Gamma_C)} \|u_N - \pi_0^{h,e}u_N\|_{L^\infty(T\cap\Gamma_C)} \\
\leq\ & C\frac{h_e^{2\tau-1}}{|Z_{NC}|^{1/2}} \left(|\sigma_N|^2_{\tau-3/2,T\cap\Gamma_C} + |\nabla u_N|^2_{\tau-3/2,T\cap\Gamma_C} \right) \\
& +C\frac{|Z_C|^{1/2}h_e^{2\tau-2}}{|Z_{NC}|^{1/2}} \left(|\sigma_N|^2_{\tau-3/2,T\cap\Gamma_C} + |\nabla u_N|^2_{\tau-3/2,T\cap\Gamma_C} \right) \\
\leq\ & C\frac{h_e^{2\tau-1}}{|Z_{NC}|^{1/2}} \left(|\sigma_N|^2_{\tau-3/2,T\cap\Gamma_C} + |\nabla u_N|^2_{\tau-3/2,T\cap\Gamma_C} \right)
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
& \int_{T\cap\Gamma_C} (\sigma_N - \pi_0^{h,e}\sigma_N)(\pi_0^{h,e}u_N - u_N)\ d\Gamma \\
\leq\ & \|\sigma_N - \pi_0^{h,e}\sigma_N\|_{0,T\cap\Gamma_C} \|u_N - \pi_0^{h,e}u_N\|_{0,T\cap\Gamma_C} \\
\leq\ & C\frac{h_e^{2\tau-1}}{|Z_C|^{1/2}} \left(|\sigma_N|^2_{\tau-3/2,T\cap\Gamma_C} + |\nabla u_N|^2_{\tau-3/2,T\cap\Gamma_C} \right).
\end{aligned} \tag{34}$$

Hence, since $|Z_{NC}|$ or $|Z_C|$ is greater than Ch_e^2, we choose either (33) or (34) and we come to the conclusion by addition that

$$-\int_{\Gamma_C} (\sigma_N - \pi_0^{h,e}\sigma_N)u_N\ d\Gamma \leq Ch^{2(\tau-1)} \left(|\sigma_N|^2_{\tau-3/2,\Gamma_C} + |\nabla u_N|^2_{\tau-3/2,\Gamma_C} \right) \leq Ch^{2(\tau-1)}\|u\|^2_{\tau,\Omega},$$

which ends the proof of the theorem.

■

Proof of Theorem 4 (quadratic case in 3D). As in the two-dimensional case, the proof is split in two parts, the first one dealing with $3/2 < \tau < 5/2$ and the second one concerning $\tau = 5/2$.

(i): $3/2 < \tau < 5/2$. From the inclusions

$$K_2^{h,e} \subset \widehat{K_2^{h,e}} \subset \overline{K_2^{h,e}} \text{ and } K_2^{h,e} \subset \widetilde{K_2^{h,e}} \subset \overline{K_2^{h,e}},$$

we only have to prove the approximation error bound when considering $K_2^{h,e}$ and the consistency error bound when considering $\overline{K_2^{h,e}}$.

- Approximation error (when $K^h = K_2^{h,e}$):
  We choose
  $$v^h = \mathcal{I}_2^h u + R_2^h(\mathcal{J}_1^h u_{_N} - \mathcal{I}_2^h u_{_N})$$
  where R_2^h is a extension operator from W_2^h into V_2^h (W_2^h is the normal trace space of V_2^h on $\Gamma_{_C}$) and $\mathcal{I}_2^h$ is the Lagrange interpolation operator mapping onto V_2^h. Note that we use again the piecewise affine quasi-interpolation operator $\mathcal{J}_1^h$: this choice is sufficient for our estimates since we only use this operator on $\overline{\Gamma_{_C}}$ where $u_{_N}$ is no more regular than H^2 (since $\tau < 5/2$). Of course $v_{_N}^h$ is piecewise linear and equals $\mathcal{J}_1^h u_{_N}$ on $\Gamma_{_C}$. From the definitions of $K_1^{h,e}$ and $K_2^{h,e}$ it is easy to check that $v^h \in K_2^{h,e}$. Estimate (30) still holds when $3/2 < \tau \leq 5/2$ since $\|u_{_N} - \mathcal{J}_1^h u_{_N}\|_{0,\Gamma_{_C}} \leq C h_C^{\tau-1/2} \|u_{_N}\|_{\tau-1/2,\Gamma_{_C}} \leq C h_C^{\tau-1/2} \|u\|_{\tau,\Omega}$. Estimate (31) is handled exactly as in Theorem 3.

- The consistency error (when $K^h = \overline{K_2^{h,e}}$) is estimated as in Theorem 3.

(**ii**): $\tau = 5/2$. Obviously $\sigma_{_N}$ and $\nabla u_{_N}$ are not continuous on $\Gamma_{_C}$ contrary to the two-dimensional case. A deeper insight into the proofs of the previous theorems shows us that the only result which is missing to complete the proof is an extension of Lemma 2 when $d = 3$ and $\tau = 5/2$. This is the aim of Lemma 3 which allows us to end the proof.

■

Appendix A. Some local L^1 and L^2-estimates for $\sigma_{_N}$ and $\nabla u_{_N}$

Lemma 2 *Let $d = 2$ or $d = 3$. Set $3/2 < \tau < 5/2$. Let h_e be the diameter of the $d-1$ dimensional trace element $T \cap \Gamma_{_C}$ and $|Z_{_C}|, |Z_{_{NC}}|$ stand for the measures in $\mathbb{R}^{d-1}$ of the contact and noncontact sets $Z_{_C}, Z_{_{NC}}$ in $T \cap \Gamma_{_C}$ respectively. Assume that $|Z_{_C}| > 0$ and $|Z_{_{NC}}| > 0$. The following L^1 and L^2-estimates hold for $\sigma_{_N}$ and $\nabla u_{_N}$:*

$$\|\sigma_{_N}\|_{L^1(T\cap\Gamma_{_C})} \leq \frac{|Z_{_C}|^{1/2}}{|Z_{_{NC}}|^{1/2}} h_e^{\tau-2+d/2} |\sigma_{_N}|_{\tau-3/2,T\cap\Gamma_{_C}}, \tag{35}$$

$$\|\sigma_{_N}\|_{0,T\cap\Gamma_{_C}} \leq \frac{1}{|Z_{_{NC}}|^{1/2}} h_e^{\tau-2+d/2} |\sigma_{_N}|_{\tau-3/2,T\cap\Gamma_{_C}}, \tag{36}$$

$$\|\nabla u_{_N}\|_{L^1(T\cap\Gamma_{_C})} \leq \frac{|Z_{_{NC}}|^{1/2}}{|Z_{_C}|^{1/2}} h_e^{\tau-2+d/2} |\nabla u_{_N}|_{\tau-3/2,T\cap\Gamma_{_C}}, \tag{37}$$

$$\|\nabla u_{_N}\|_{0,T\cap\Gamma_{_C}} \leq \frac{1}{|Z_{_C}|^{1/2}} h_e^{\tau-2+d/2} |\nabla u_{_N}|_{\tau-3/2,T\cap\Gamma_{_C}}. \tag{38}$$

Proof. We begin with estimate (36). Since u is solution of (6), the unilateral contact conditions in (4) hold in the weak sense so we deduce that $\sigma_N = 0$ a.e. on Z_{NC}. Therefore

$$\begin{aligned}
\|\sigma_N\|^2_{0,T\cap\Gamma_C} &= \int_{Z_C} \sigma_N(s)^2 \, ds \\
&= |Z_{NC}|^{-1} \int_{Z_C} \int_{Z_{NC}} (\sigma_N(s) - \sigma_N(t))^2 \, dt \, ds \\
&= |Z_{NC}|^{-1} \int_{Z_C} \int_{Z_{NC}} \frac{(\sigma_N(s) - \sigma_N(t))^2}{|s-t|^{d-1+2\nu}} |s-t|^{d-1+2\nu} \, dt \, ds \\
&\le |Z_{NC}|^{-1} \sup_{Z_C \times Z_{NC}} (|s-t|^{d-1+2\nu}) \int_{Z_C} \int_{Z_{NC}} \frac{(\sigma_N(s) - \sigma_N(t))^2}{|s-t|^{d-1+2\nu}} \, dt \, ds \\
&\le |Z_{NC}|^{-1} h_e^{d-1+2\nu} |\sigma_N|^2_{\nu,T\cap\Gamma_C} \\
&= |Z_{NC}|^{-1} h_e^{2(\tau-2+d/2)} |\sigma_N|^2_{\tau-3/2,T\cap\Gamma_C}
\end{aligned}$$

which proves (36). We then obtain (35) by using Cauchy-Schwarz inequality.

Both remaining estimates (37) and (38) deal with ∇u_N (i.e., u'_N when $d = 2$). We first use a non trivial result (see, e.g., [24]) which claims that if v lies in $H^1(\omega)$ then $\nabla v = 0$ a.e. on any "level set" whatever the space dimension is. When v is continuous which is the case in the present study since $v = u_N$ is continuous on $\omega = \Gamma_C$, then the level set can be understood in the classical sense (otherwise a convenient definition should be used). Since Z_C is the set of level 0, we have $\nabla u_N = 0$ a.e. on Z_C. So both estimates (37) and (38) are proved exactly as (35) and (36) by changing σ_N with u'_N (resp. both partial derivatives of u_N) when $d = 2$ (resp. when $d = 3$) and inverting Z_C and Z_{NC}.

■

Lemma 3 *Let $d = 3$ and $\tau = 5/2$. Let h_e be the diameter of the two-dimensional triangle $T \cap \Gamma_C$ and $|Z_C|, |Z_{NC}|$ stand for the measures in $\mathbb{R}^2$ of the contact and noncontact sets Z_C, Z_{NC} in $T \cap \Gamma_C$ respectively. Assume that $|Z_C| > 0$ and $|Z_{NC}| > 0$. The following L^1 and L^2-estimates hold for σ_N and ∇u_N:*

$$\begin{aligned}
\|\sigma_N\|_{L^1(T\cap\Gamma_C)} &\le \sqrt{2}\frac{|Z_C|^{1/2}}{|Z_{NC}|^{1/2}} h_e^2 \|\nabla \sigma_N\|_{0,T\cap\Gamma_C}, \\
\|\sigma_N\|_{0,T\cap\Gamma_C} &\le \frac{\sqrt{2}}{|Z_{NC}|^{1/2}} h_e^2 \|\nabla \sigma_N\|_{0,T\cap\Gamma_C}, \\
\|\nabla u_N\|_{L^1(T\cap\Gamma_C)} &\le \sqrt{2}\frac{|Z_{NC}|^{1/2}}{|Z_C|^{1/2}} h_e^2 \|H u_N\|_{0,T\cap\Gamma_C}, \\
\|\nabla u_N\|_{0,T\cap\Gamma_C} &\le \frac{\sqrt{2}}{|Z_C|^{1/2}} h_e^2 \|H u_N\|_{0,T\cap\Gamma_C},
\end{aligned} \tag{39}$$

where H stands for the Hessian matrix.

Proof. The proof is slightly different from the previous lemma and it follows the same steps as the original proof by H. Poincaré of the famous Poincaré-Wirtinger inequality. We begin with estimate (39). Since $\sigma_N = 0$ a.e. on Z_{NC} we write:

$$
\begin{aligned}
\|\sigma_N\|^2_{0,T\cap\Gamma_C} &= |Z_{NC}|^{-1}\int_{Z_C}\int_{Z_{NC}} (\sigma_N(s)-\sigma_N(t))^2\ dt\ ds \\
&= |Z_{NC}|^{-1}\int_{Z_C}\int_{Z_{NC}} \left(\int_0^1 (s-t)\cdot\nabla\sigma_N(us+(1-u)t)\ du\right)^2\ dt\ ds \\
&\leq |Z_{NC}|^{-1}h_e^2\int_{T\cap\Gamma_C}\int_{T\cap\Gamma_C}\int_0^1 |\nabla\sigma_N(us+(1-u)t)|^2\ du\ ds\ dt \\
&= 2|Z_{NC}|^{-1}h_e^2\int_{T\cap\Gamma_C}\int_{T\cap\Gamma_C}\int_{1/2}^1 |\nabla\sigma_N(us+(1-u)t)|^2\ du\ ds\ dt,
\end{aligned}
$$

where the symmetry between t and s has been used in the last identity. Setting $v = us+(1-u)t$, we get

$$
\begin{aligned}
\|\sigma_N\|^2_{0,T\cap\Gamma_C} &\leq 2|Z_{NC}|^{-1}h_e^2\int_{T\cap\Gamma_C}\int_{1/2}^1 \frac{1}{u^2}\int_{T\cap\Gamma_C} |\nabla\sigma_N(v)|^2\ dv\ du\ dt \\
&\leq 2|Z_{NC}|^{-1}h_e^4\|\nabla\sigma_N\|^2_{0,T\cap\Gamma_C},
\end{aligned}
$$

which proves (39). The three remaining estimates are then proved as in the previous lemma. ■

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