Residual error estimators for Coulomb friction

Patrick Hild, Vanessa Lleras

Laboratoire de Mathématiques de Besançon, UMR CNRS 6623
Université de Franche-Comté, 16 route de Gray, 25030 Besançon, France.
E-mail: patrick.hild@univ-fcomte.fr, vanessa.lleras@univ-fcomte.fr

July 27, 2009

Abstract
This paper is concerned with residual error estimators for finite element approximations of Coulomb frictional contact problems. A recent uniqueness result by Renard in [72] for the continuous problem allows us to perform an a posteriori error analysis. We propose, study and implement numerically two residual error estimators associated with two finite element discretizations. In both cases the estimators permit to obtain upper and lower bounds of the discretization error.

Key words: Coulomb friction, a posteriori error estimates, residuals.
Abbreviated title: Residual estimators for Coulomb friction
Mathematics subject classification: 65N30, 74M15

1 Introduction and notation

The numerical approximation of frictional contact problems occurring in structural mechanics is generally achieved using the finite element method (see [38, 41, 53, 57, 83]). In order to evaluate and to control the quality of a finite element approximation, a current choice consists in developing adaptive procedures using a posteriori error estimators. The aim of the estimators is to provide the user with global and local quantities which represent in the best way the true error committed by the finite element approximation. Actually there exist various classes of error estimators, anyone showing its specificities and advantages. Some currently used estimators are e.g., those introduced in [7] based on the residual of the equilibrium equations, the estimators linked to the smoothing of finite element stresses (see [85]) and the estimators based on the errors in the constitutive relation, also called ”equilibrated fluxes” (see [56]). A review of different a posteriori error estimators can be found in e.g., [3, 8, 37, 79, 80].

The frictionless unilateral contact problem (or the equivalent scalar valued Signorini problem) shows a nonlinearity on the boundary corresponding to the non-penetration of the materials on the contact area which leads to a variational inequality of the first kind. For this model the residual based method was first considered and studied in [21, 39, 84] using a penalized approach and in [12] by using the error measure technique developed in [9]. More recently the analysis without penalization term was achieved in [46], and in [47] for the corresponding mixed finite element approximation (see also [10]). Besides the study of error in the constitutive relation was performed in [27, 81, 82] for the contact problem and a posteriori estimates for the boundary element method are studied in [63, 64]. More generally, we mention that the analysis of residual error estimators for variational inequalities leads generally to important technical difficulties for any model. Note also that an important work has been devoted to the obstacle
(or obstacle type) problem in which the inequality condition holds on the entire domain (see [1, 4, 15, 16, 17, 22, 35, 49, 52, 55, 59, 68, 69, 77, 78]). Other a posteriori error analyzes involving inequalities linked to plasticity were considered in [20, 70, 71, 75] and the Bingham fluid problem is studied in [76].

When considering friction in addition to the contact model, there are supplementary non-linearities which have to be taken into account. The currently used friction model is the one of Coulomb (although there exist simplified and/or different models: Tresca’s friction, normal compliance, smoothed Coulomb friction... see [53, 74]) whose associated partial differential equation shows numerous mathematical difficulties which remain unsolved. In our work we consider the so-called static friction problem introduced in [30, 31] which roughly speaking corresponds to an incremental problem in the time discretized quasi-static model. For this model, existence of solutions hold when the friction coefficient is small enough, see [32, 33] and the references quoted therein. When the friction coefficient is large, neither existence nor nonexistence result is available. Besides the solutions are generally non unique when the friction coefficient is large enough, see [43, 44]. More recently a first uniqueness result has been obtained in [72] with the assumption that a ”regular” solution exists and that the friction coefficient is sufficiently small. From a numerical point of view it is well known that the finite element problem, associated with the continuous static Coulomb friction model, always admits a solution and that the solution is unique if the friction coefficient is small enough (unfortunately the denomination small depends on the discretization parameter and the bound ensuring uniqueness vanishes as the mesh is refined, see e.g., [41]). Concerning the a posteriori error analysis for the Coulomb model, several studies have been achieved: error in the constitutive relation in [25, 62] as well as an heuristic residual based error estimator for BEM-discretizations in [34]. A simpler model, the so-called Tresca’s friction problem is considered in [13] (see also the study in [14] for a similar problem where residual estimators are analyzed). Note that the latter model is governed by a variational inequality of the second kind (see [6]). Finally an a posteriori error analysis is performed for the friction model with normal compliance in [58].

Our purpose in this paper is to carry out a residual a posteriori error analysis for the Coulomb friction model and to obtain an error estimator with upper and lower bounds involving the discretization error. As far as we know, such a result is not available in the literature.

The paper is organized as follows. In section 2 we introduce the equations modelling the frictional unilateral contact problem between an elastic body and a rigid foundation. We write the problem using a mixed formulation where the unknowns are the displacement field in the body and the frictional contact pressures on the contact area. In the third section, we choose a classical discretization involving continuous finite elements of degree one and continuous piecewise affine multipliers on the contact zone. Section 4 is concerned with the study of the residual estimator which can be seen as the natural one arising from the discrete problem. Thanks to Renard’s uniqueness result we obtain a global upper bound of the error. Then local lower bounds of the error are proved. In section 5 we consider a residual estimator resulting from another discrete model. This second approach has two interesting properties in comparison with the previous one: first it involves less terms coming from the frictional contact and these terms have quite simple expressions. Second, the error analysis we achieve leads to better error bounds. Section 6 is concerned with the numerical experiments and the comparison of both approaches.

Finally we introduce some useful notation and several functional spaces. In what follows, bold letters like $\mathbf{u}, \mathbf{v}$, indicate vector valued quantities, while the capital ones (e.g., $\mathbf{V}, \mathbf{K}, \ldots$) represent functional sets involving vector fields. As usual, we denote by $(L^2(\cdot))^d$ and by $(H^s(\cdot))^d$, $s \geq 0$, $d = 1, 2$ the Lebesgue and Sobolev spaces in one and two space dimensions (see [2]). The usual norm of $(H^s(D))^d$ is denoted by $\| \cdot \|_{s,D}$ and we keep the same notation when $d = 1$ or $d = 2$. For shortness the $(L^2(D))^d$-norm will be denoted by $\| \cdot \|_D$ when $d = 1$ or $d = 2$. In the sequel the symbol $\cdot$ will denote either the Euclidean norm in $\mathbb{R}^2$, or the length of a line segment,
or the area of a plane domain. Finally the notation $a \lesssim b$ means here and below that there exists a positive constant $C$ independent of $a$ and $b$ (and of the meshsize of the triangulation) such that $a \leq Cb$. The notation $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$ hold simultaneously.

## 2 The frictional contact problem in elasticity

We consider the deformation of an elastic body occupying, in the initial unconstrained configuration, a domain $\Omega$ in $\mathbb{R}^2$ where plane strain assumptions are assumed. The Lipschitz boundary $\partial\Omega$ of $\Omega$ consists of $\Gamma_D, \Gamma_N$ and $\Gamma_C$ where the measure of $\Gamma_D$ does not vanish. The body $\Omega$ is clamped on $\Gamma_D$ and subjected to surface traction forces $\mathbf{F}$ on $\Gamma_N$; the body forces are denoted $\mathbf{f}$. In the initial configuration, the part $\Gamma_C$ is a straight line segment considered as the candidate contact surface on a rigid foundation for the sake of simplicity which means that the contact zone cannot enlarge during the deformation process. The contact is assumed to be frictional and the stick, slip and separation zones on $\Gamma_C$ are not known in advance. We denote by $\mu \geq 0$ the given friction coefficient on $\Gamma_C$. The unit outward normal and tangent vectors of $\partial\Omega$ are $\mathbf{n} = (n_1, n_2)$ and $\mathbf{t} = (-n_2, n_1)$ respectively.

The contact problem with Coulomb’s friction law consists of finding the displacement field $\mathbf{u} : \Omega \to \mathbb{R}^2$ satisfying (1)–(6):

\begin{align}
(1) \quad \nabla \mathbf{u} + \mathbf{f} &= 0 \quad \text{in } \Omega, \\
(2) \quad \mathbf{u} &= 0 \quad \text{on } \Gamma_D, \\
(3) \quad \mathbf{u} &= 0 \quad \text{on } \Gamma_D, \\
(4) \quad \mathbf{u} &= 0 \quad \text{on } \Gamma_N.
\end{align}

The notation $\mathbf{u} : \Omega \to \mathcal{S}_2$ represents the stress tensor field lying in $\mathcal{S}_2$, the space of second order symmetric tensors on $\mathbb{R}^2$. The linearized strain tensor field is $\mathbf{\varepsilon}(\mathbf{u}) = (\nabla \mathbf{u} + \nabla^T \mathbf{u})/2$. The variational formulation of problem (1)–(6) in its mixed form consists of finding $(\mathbf{u}, \lambda) = (\mathbf{u}, \lambda_n, \lambda_t) \in V \times M_n \times M_t(\mu \lambda_n) = V \times M(\mu \lambda_n)$ which satisfy (see [48, 72]):

\begin{align}
(5) \quad u_n \leq 0, \quad \sigma_n(\mathbf{u}) \leq 0, \quad \sigma_n(\mathbf{u}) u_n = 0,
\end{align}

and the Coulomb friction law is summarized by the following conditions (see, e.g., [33]):

\begin{align}
(6) \quad \begin{cases}
\mathbf{u}_t = 0 \implies |\sigma_t(\mathbf{u})| \leq \mu |\sigma_n(\mathbf{u})|, \\
\mathbf{u}_t \neq 0 \implies \sigma_t(\mathbf{u}) = -\mu |\sigma_n(\mathbf{u})| \frac{|\mathbf{u}_t|}{|\mathbf{u}_t|}.
\end{cases}
\end{align}

The variational formulation of problem (1)–(6) in its mixed form consists of finding $\mathbf{u} = (\mathbf{u}, \lambda_n, \lambda_t) \in V \times M_n \times M_t(\mu \lambda_n) = V \times M(\mu \lambda_n)$ which satisfy (see [48, 72]):

\begin{align}
(7) \quad \left\{ \begin{array}{ll}
a(\mathbf{u}, \mathbf{v}) + b(\lambda, \mathbf{v}) = L(\mathbf{v}), & \forall \mathbf{v} \in V, \\
b(\nu - \lambda, \mathbf{u}) \leq 0, & \forall \nu = (\nu_n, \nu_t) \in M(\mu \lambda_n),
\end{array} \right.
\end{align}

where

\begin{align}
V = \{ \mathbf{v} \in (H^1(\Omega))^2; \mathbf{v} = 0 \text{ on } \Gamma_D \},
\end{align}

and $M(\mu \lambda_n) = M_n \times M_t(\mu \lambda_n)$ is defined next. We set

\begin{align}
M_n = \{ \nu \in X_n^1; \nu \geq 0 \text{ on } \Gamma_C \}
\end{align}
and, for any $g \in M_n$

$$M_n(g) = \{ \nu \in X'_n : -g \leq \nu \leq g \text{ on } \Gamma_C \}$$

where $X'_n$ (resp. $X'_t$) is the dual space of $X_n$ (resp. $X_t$) with $X_n = \{ v_{n|\Gamma_C} : v \in V \}$ (resp. $X_t = \{ v_{t|\Gamma_C} : v \in V \}$). Note that $H^{1/2}_0(\Gamma_C) \subset X_n \subset H^{1/2}(\Gamma_C)$, $H^{1/2}(\Gamma_C) \subset X_t \subset H^{1/2}(\Gamma_C)$ and that the inequality conditions incorporated in the definitions of $M_n$ and $M_t(g)$ have to be understood in the dual sense.

**Remark 2.1** Note that the previous mixed method is a nonstandard formulation since there is a bootstrap: find $(u, \lambda_n, \lambda_t) \in V \times M_n \times M_t(\mu \lambda_n)$ such that (7) holds. This weak formulation could be written in a different way without the bootstrap and by adding a condition: find $(u, \lambda_n, \lambda_t) \in V \times M_n \times X'_t$ such that $\lambda_t \in M_t(\mu \lambda_n)$ and (7) holds.

In (7), $f \in (L^2(\Omega))^2$, $F \in (L^2(\Gamma_N))^2$ and the standard notations are adopted

$$a(u, v) = \int_{\Omega} (C \varepsilon(u)) : \varepsilon(v) \, d\Omega, \quad L(v) = \int_{\Omega} f \cdot v \, d\Omega + \int_{\Gamma_N} F \cdot v \, d\Gamma,$$

$$b(\nu, v) = \langle \nu_n, v_n \rangle_{X'_n, X_n} + \langle \nu_t, v_t \rangle_{X'_t, X_t}$$

for any $u$ and $v$ in $(H^1(\Omega))^2$ and $\nu = (\nu_n, \nu_t)$ in $X'_n \times X'_t$. In these definitions the notations - and : represent the canonical inner products in $\mathbb{R}^2$ and $S_2$ respectively. It is easy to see that if $(u, \lambda_n, \lambda_t)$ is a solution of (7), then $\lambda_n = -\sigma_n(u)$ and $\lambda_t = -\sigma_t(u)$. The space $X_n$ is equipped with the norm

$$\| w \|_{X_n} = \inf_{v \in V : v_n = w \text{ on } \Gamma_e} \| v \|_{1, \Omega},$$

and a similar expression holds for $\| . \|_{X_t}$. The dual space of $X_n \times X_t$ is endowed with the norm

$$\| \nu \|_{-1/2, \Gamma_C} = \sup_{w \in V \setminus \{0\}} \frac{b(\nu, w)}{\| w \|_{1, \Omega}} \quad \forall \nu = (\nu_n, \nu_t) \in X'_n \times X'_t.$$

To avoid more notation, we will skip over the regularity aspects of the functions defined on $\Gamma_C$ which are beyond the scope of this paper and we write afterwards integral terms instead of duality pairings. Another classical weak formulation of problem (1)–(6) is an inequality problem: find $u$ such that

$$(8) \quad u \in K, \quad a(u, v - u) - \mu \int_{\Gamma_C} \sigma_n(u)(|v_n| - |u_n|) \, d\Gamma \geq L(v - u) \quad \forall v \in K,$$

where $K$ denotes the closed convex cone of admissible displacement fields satisfying the non-penetration conditions:

$$K = \{ v \in V : v_n \leq 0 \text{ on } \Gamma_C \}.$$

When friction is omitted (i.e., $\mu = 0$) then the condition (6) simply reduces to $\sigma_t(u) = 0$ and the frictionless contact problem admits a unique solution according to Stampacchia’s theorem (see e.g., [36, 54]). The existence of a solution to (8) has been first proved for small friction coefficients in [67] (in two space dimensions) and the bounds ensuring existence have been improved and generalized in [51] and [32] (see also [33]). More precisely existence holds if $\mu \leq \sqrt{3 - 4P} / (2 - 2P)$ where $0 \leq P < 1/2$ denotes Poisson’s ratio. Recently some multi-solutions of the problem (1)–(6) are exhibited for triangular or quadrangular domains. These multiple solutions involve either an infinite set of slipping solutions (see [43]) or two isolated (stick and separation) configurations (see [44]) or two isolated (stick and grazing contact) solutions in [45]. Note that these examples of non-uniqueness involve large friction coefficients (i.e., $\mu > \sqrt{1 - P} / P$) and tangential displacements with a constant sign on $\Gamma_C$. Actually, it seems that
no multi-solution has been detected for an arbitrary small friction coefficient in the continuous case, although such a result exists for finite element approximations in [42], but for a variable geometry. The forthcoming partial uniqueness result is obtained in [72]: it defines some cases where it is possible to affirm that a solution to the Coulomb friction problem is in fact the unique solution. More precisely, if a ”regular” solution to the Coulomb friction problem exists (here the denomination ”regular” means, roughly speaking, that the transition is smooth when the slip direction changes) and if the friction coefficient is small enough then this solution is the only one.

We now introduce the space of multipliers $M$ of the functions $\xi$ defined on $\Gamma_C$ such that the following norm $\|\xi\|_M$ is finite:

$$\|\xi\|_M = \sup_{v_t \in X_t \setminus \{0\}} \frac{\|\xi v_t\|_{X_t}}{\|v_t\|_{X_t}}.$$ 

Since $\Gamma_C$ is assumed to be straight, $M$ contains for any $\varepsilon > 0$ the space $H^{1/2+\varepsilon}(\Gamma_C)$ (see [65] for a complete discussion on the theory of multipliers in a pair of Hilbert spaces). The partial uniqueness result is given assuming that $\lambda_t = \mu \lambda_n \xi$, with $\xi \in M$. It is easy to see that it implies $|\xi| \leq 1$ a.e. on the support of $\lambda_n$. More precisely, this implies that $\xi \in \text{Dir}_t(u_t)$ a.e. on the support of $\lambda_n$, where $\text{Dir}_t(.)$ is the subdifferential of the convex map $x_t \mapsto |x_t|$. This means that it is possible to assume that $\xi \in \text{Dir}_t(u_t)$ a.e. on $\Gamma_C$.

**Proposition 2.2** ([72]). Let $(u, \lambda)$ be a solution to Problem (7) such that $\lambda_t = \mu \lambda_n \xi$, with $\xi \in \text{Dir}_t(u_t)$ a.e. on $\Gamma_C$ and $\mu \|\xi\|_M$ is small enough. Then $(u, \lambda)$ is the unique solution to Problem (7).

The case $\xi \equiv 1$ corresponds to an homogeneous sliding direction and the previous result is complementary with the non-uniqueness results obtained in [43, 44, 45]. The multiplier $\xi$ has to vary from $-1$ to $+1$ each time the sign of the tangential displacement changes from negative to positive. The set $M$ does not contain any multiplier having a discontinuity of the first kind. Consequently, in order to satisfy the assumptions of Proposition 2.2, the tangential displacement of the solution $u$ cannot pass from a negative value to a positive value and being zero only at a single point of $\Gamma_C$. For a more precise discussion concerning the assumption $\lambda_t = \mu \lambda_n \xi$, $\xi \in M$, $\xi \in \text{Dir}_t(u_t)$ and the cases where the assumption cannot be fulfilled independently of the regularity of the solution, we refer the reader to [48], Remark 2.

### 3 Mixed finite element approximation

We approximate this problem with a standard finite element method. Namely we fix a regular family of meshes $\mathcal{T}_h$, $h > 0$, [18, 19, 23], made of closed triangles. For $K \in \mathcal{T}_h$, let $h_K$ be the diameter of $K$ and $h = \max_{K \in \mathcal{T}_h} h_K$. The regularity of the mesh implies in particular that for any edge $E$ of $K$ one has $h_E = |E| \sim h_K$. Let us define $E_h$ (resp. $N_h$) as the set of edges (resp. nodes) of the triangulation and set $E_h^{int} = \{ E \in E_h : E \subset \Omega \}$ the set of interior edges of $T_h$ (the edges are supposed to be relatively open). We denote by $E_n = \{ E \in E_h : E \subset \Gamma_N \}$ the set of exterior edges included in the part of the boundary where we impose Neumann conditions, and similarly $E_h^C = \{ E \in E_h : E \subset \Gamma_C \}$. Set $N_h^D = N_h \cap \overline{\Gamma_D}$ (note that the extreme nodes of $\Gamma_D$ belong to $N_h^D$). For an element $K$, we will denote by $E_K$ the set of edges of $K$ and according to the above notation, we set $E_K^{int} = E_K \cap E_h^{int}$, $E_K^N = E_K \cap E_n^{int}$, $E_K^C = E_K \cap E_h^C$. For each interior edge $E$ we fix one of the two normal vectors and we denote it by $n_E$. The jump of some vector valued function $v$ across an edge $E \in E_h^{int}$ at a point $y \in E$ is defined as

$$[v]_E(y) = \lim_{\alpha \to 0^+} v(y + \alpha n_E) - v(y - \alpha n_E), \quad \forall E \in E_h^{int}.$$
Note that the sign of $[\mathbf{v}]_E$ depends on the orientation of $\mathbf{n}_E$. Finally we introduce the patches: denoting by $x$ a node, by $E$ an edge and by $K$ an element, let $\omega_x = \bigcup_{x \in E} K$, $\omega_K = \bigcup_{x \in K} \omega_x$. The finite element space used in $\Omega$ is then defined by

$$V_h = \{ v_h \in (C(\Omega))^2 : \forall K \in T_h, \quad v_h|_K \in (P_1(K))^2, \quad v_h|_{\Gamma_D} = 0 \}.$$  

We recall that the contact area is a straight line segment to simplify. The extension to a contact area which is a broken line can be made without additional technical difficulties (see e.g., [47]).

Proposition 3.2 Using a fixed point argument it can be proven that the problem (9) admits at least a solution (see also [40] for an early convergence result). When friction is absent, an important number of a priori error analyzes have been achieved (see, e.g., [11, 26, 50] and the references therein). Note that even in this simpler case, the proof of an estimate of order $h$ in the $(H^1(\Omega))^2$-norm with only $(H^2(\Omega))^2$ regularity (without any additional assumption) remains an open problem.

Remark 3.1 It is easy to check that the functions in $M_{hn}$ are not necessarily nonnegative. In the same way the functions in $M_{hd}(g)$ do not satisfy $|v_h| \leq g$ everywhere.

The discretized mixed formulation of the frictional contact problem is to find $u_h \in V_h$ and $\lambda_h \in M_h(\mu \lambda_h) = M_{hn} \times M_{hd}(\mu \lambda_h)$ satisfying:

$$\begin{cases} 
  a(u_h, v_h) + b(\lambda_h, v_h) = (L(v_h), \quad \forall v_h \in V_h, \\
  b(\nu_h - \lambda_h, u_h) \leq 0, \quad \forall \nu_h = (\nu_{hn}, \nu_{hd}) \in M_h(\mu \lambda_h). 
\end{cases}$$

(9)

Using a fixed point argument it can be proven that the problem (9) admits at least a solution and that there is a unique solution when $\mu \leq C(h)$ (see [25]). Unfortunately the constant $C(h)$ vanishes when $h$ vanishes ($C(h) \sim h^{1/2}$). The following result proved in [25] gives explicitly the discrete frictional contact conditions.

Proposition 3.2 ([25]) Let $(u_h, \lambda_h)$ be a solution of (9). Suppose that $\dim(W_h) = p$ and let $\psi_{xi}, \ 1 \leq i \leq p$ denote the basis functions of $W_h$ on $\Gamma_C$. The $p$-by-$p$ mass matrix $M = (m_{ij})_{1 \leq i, j \leq p}$ on $\Gamma_C$ is given by $m_{ij} = \int_{\Gamma_C} \psi_x \psi_x$. Let $U_N$ and $U_T$ denote the vectors of components the nodal values of $u_{hn}$ and $u_{hd}$ respectively and let $L_N$ and $L_T$ denote the vectors of components the nodal values of $\lambda_{hn}$ and $\lambda_{hd}$ respectively. Then the discrete frictional contact conditions in (9) are as follows; for any $1 \leq i \leq p$:

$$(ML_N)_i \geq 0, \quad (U_N)_i \leq 0, \quad (ML_N)_i (U_N)_i = 0, \quad |(ML_T)_i| \leq \mu (ML_N)_i, \quad |(ML_T)_i| \leq \mu (ML_N)_i \implies (U_T)_i = 0, \quad (ML_T)_i (U_T)_i \geq 0.$$  

Remark 3.3 The a priori error analysis of (9) remains an open problem although an error estimate is obtained in [48] for a slightly different approximation of the frictional contact conditions (see also [40] for an early convergence result). When friction is absent, an important number of a priori error analyzes have been achieved (see, e.g., [11, 26, 50] and the references therein). Note that even in this simpler case, the proof of an estimate of order $h$ in the $(H^1(\Omega))^2$-norm with only $(H^2(\Omega))^2$ regularity (without any additional assumption) remains an open problem.
We consider the quasi-interpolation operator $\pi_h$: for any $v \in L^1(\Omega)$, we define $\pi_h v$ as the unique element in $V_h = \{v_h \in C(\bar{\Omega}) : \forall K \in T_h, \ v_h|_{\partial K} \in P_1(K), \ v_h|_{\Gamma_D} = 0\}$ such that:

$$\pi_h v = \sum_{x \in N_h \setminus N_h'} \left( \frac{1}{|\omega_x|} \int_{\omega_x} v(y) \, dy \right) \psi_x,$$

where for any $x \in N_h$, $\psi_x$ is the standard basis function in $V_h$ satisfying $\psi_x(x') = \delta_{x,x'}$, for all $x' \in N_h$. Note that we could also consider other quasi-interpolation operators like the ones in \[22\] or \[24\]. The following estimates hold (see, e.g., \[80\]): for any $v \in H^1(\Omega)$ vanishing on $\Gamma_D$, we have $\|v - \pi_h v\|_K \lesssim h_K^2 \|\nabla v\|_{\omega_K}, \forall K \in T_h$, and $\|v - \pi_h v\|_E \lesssim h_E^{1/2} \|\nabla v\|_{\omega_E}, \forall E \in E_h$. Since we deal with vector valued functions we can define a vector valued operator (which we denote again by $\pi_h$) for the sake of simplicity) whose components are defined above. So we get:

**Lemma 3.4** For any $v \in V$ the following estimates hold

(11) $\|v - \pi_h v\|_K \lesssim h_K \|v\|_{1,\omega_K}, \forall K \in T_h,$

(12) $\|v - \pi_h v\|_E \lesssim h_E^{1/2} \|v\|_{1,\omega_E}, \forall E \in E_h.$

## 4 The residual error estimator

### 4.1 Definition of the residual error estimator

The element residual of the equilibrium equation (1) is defined by $\text{div} \sigma(u_h) + f = f$ on $K$. As usual this element residual can be replaced with some simple finite dimensional approximation $f_K \in (P_k(K))^2$ and the difference $f - f_K$ will be treated as data oscillation. A current choice is to take $f_K = \int_K f(x) / |K|$. In the same way $F$ can be approximated by a simple quantity denoted $F_E$ on any $E \in E_h^N$.

**Definition 4.1** The global residual estimator $\eta$ and the local residual error estimators $\eta_K$ are defined by

$$\eta = \left( \sum_{K \in T_h} \eta_K^2 \right)^{1/2},$$

$$\eta_K = \left( \sum_{i=1}^8 \eta_{ik}^2 \right)^{1/2},$$

$$\eta_{1K} = h_K \|f_K\|_{K},$$

$$\eta_{2K} = h_K^{1/2} \left( \sum_{E \in E_h^N} \|J_E \cdot n(u_h)|_{|E}|^2 \right)^{1/2},$$

$$\eta_{3K} = h_K^{1/2} \|\lambda_{hn} + \sigma_n(u_h)|_{K \cap C},$$

$$\eta_{4K} = h_K^{1/2} \|\lambda_{ht} + \sigma_t(u_h)|_{K \cap C},$$

$$\eta_{5K} = \left( \int_{K \cap C} -\lambda_{hn+ut} \right)^{1/2},$$

$$\eta_{6K} = \|\lambda_{hn+ut}|_{K \cap C},$$

$$\eta_{7K} = \left( \int_{K \cap C} (|\lambda_{ht}| \mu \lambda_{hn+ut}) |u_h| + \int_{K \cap C} (\lambda_{ht} u_h)^2 \right)^{1/2},$$

$$\eta_{8K} = \|(|\lambda_{ht}| - \mu \lambda_{hn+ut}) + \|K \cap C,$$
where the notations + and _ denote the positive and negative parts respectively; $J_{E,n}(u_h)$ means the constraint jump of $u_h$ in normal direction, i.e.,

$$J_{E,n}(u_h) = \left\{ \begin{array}{ll}
[\sigma(u_h)n_E]\big|_{E'}, \forall E \in E_h^{\text{int}}, \\
\sigma(u_h)n - F_E, \forall E \in E_h^N.
\end{array} \right.$$

The local and global data oscillation terms are defined by

$$\zeta_K = \left( h_K^2 \sum_{K' \subset \partial K} \|f - f_{K'}\|_K^2 + h_E \sum_{E \in E_h^K} \|F - F_E\|_E^2 \right)^{1/2}, \zeta = \left( \sum_{K \in T_h} \zeta_K^2 \right)^{1/2}.$$

**Remark 4.2** From the previous definition, we see that there are eight contributions for any local estimator $\eta_K$. There are only two classical contributions ($\eta_{1K}$: equilibrium residual and $\eta_{2K}$: interior and Neumann jumps) for all the elements which do not have an edge belonging to $\Gamma_C$. The remaining elements on the contact area have six supplementary terms. The terms $\eta_{3K}$ and $\eta_{4K}$ represent the deviation of the traction from the equilibrium in the mixed finite element approximation, the terms $\eta_{5K}$ and $\eta_{6K}$ (resp. $\eta_{7K}$ and $\eta_{8K}$) represent the nonfulfillment of the unilateral contact conditions (5) (resp. of the friction conditions (6)).

### 4.2 Upper error bound

We now give an upper bound of the discretization error. In the forthcoming theorem we assume that the continuous problem satisfies the uniqueness criterion of [72].

**Theorem 4.3** Let $(u, \lambda)$ be the solution to Problem (7) such that $\lambda_t = \mu \lambda_n \xi$, with $\xi \in M$, $\xi \in \text{Dir}(u)$ a.e. on $\Gamma_C$ and $\mu \|\xi\|_M$ is small enough. Let $(u_h, \lambda_h)$ be a solution to the discrete problem (9). Then

$$\|u - u_h\|_{1,\Omega} + \|\lambda - \lambda_h\|_{-\frac{1}{2},\Gamma_C} \lesssim \eta + \zeta.$$

**Proof:** To simplify the notation we set $e_u = u - u_h$. Let $v_h \in V_h$; from the $V$-ellipticity of $a(\cdot, \cdot)$ and the equilibrium equations in (7) and (9) we obtain:

$$\|e_u\|_{1,\Omega}^2 \lesssim a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h)$$

$$= a(u - u_h, u - v_h) + (u - u_h, v_h - u_h)$$

$$= L(u - v_h) - b(\lambda_h, u - v_h) - a(u_h, u - v_h) + b(\lambda_h - \lambda, v_h - u_h).$$

Integrating by parts on each triangle $K$, using the definition of $J_{E,n}(u_h)$ in (13) and the complementarity conditions $\int_{\Gamma_C} \lambda_n u_n = \int_{\Gamma_C} \lambda h u_h = 0$ gives

$$\|e_u\|_{1,\Omega}^2 \lesssim \int_{\Omega} f \cdot (u - v_h) + \sum_{E \in E_h^N} \int_E (F - F_E) \cdot (u - v_h)$$

$$+ b(\lambda_h, v_h) + b(\lambda, u_h) - \int_{\Gamma_C} \lambda h u_h - \int_{\Gamma_C} \lambda u_t$$

$$- \sum_{E \in E_h^K} \int_E (\sigma(u_h)n) \cdot (u - v_h) - \sum_{E \in E_h^{\text{int}} \cup E_h^N} \int_E J_{E,n}(u_h) \cdot (u - v_h).$$

Splitting up the integrals on $\Gamma_C$ into normal and tangential components gives:

$$\|e_u\|_{1,\Omega}^2 \lesssim \int_{\Gamma_C} \lambda_n u_h + \int_{\Gamma_C} \lambda h u_h + \int_{\Gamma_C} (\lambda_t - \lambda h)(u_h - u_t) + \int_{\Omega} f \cdot (u - v_h)$$

8
\[
- \sum_{E \in E_{n}^{h} \cup E_{n}^{N}} \int_{E} J_{E,n}(u_{h}) \cdot (u - v_{h}) + \sum_{E \in E_{n}^{h}} \int_{E} (F - F_{E}) \cdot (u - v_{h}) \\
+ \sum_{E \in E_{h}^{n}} \int_{E} (\lambda_{n} + \sigma_{n}(u_{h}))(v_{hn} - u_{n}) + \sum_{E \in E_{h}^{N}} \int_{E} (\lambda_{ht} + \sigma_{t}(u_{h}))(v_{ht} - u_{t}) \\
= \int_{\Gamma_{C}} \lambda_{n} u_{hn} + \int_{\Gamma_{C}} \lambda_{hn} u_{n} + \int_{\Gamma_{C}} (\lambda_{n} - \lambda_{ht})(u_{hn} - u_{t}) + I + II + III + IV + V.
\]

We now need to estimate each term of this right-hand side. For that purpose, we take
\[
v_{h} = u_{h} + \pi_{h}(u - u_{h})
\]
where \(\pi_{h}\) is the quasi-interpolation operator defined in Lemma 3.4.

We start with the term \(I\). From the definition of \(v_{h}\) and (11) we get:
\[
\|u - v_{h}\|_{K} = \|e_{u} - \pi_{h} e_{u}\|_{K} \lesssim h_{K} \|e_{u}\|_{1,\omega_{K}}
\]
for any triangle \(K\). This estimate together with Cauchy-Schwarz inequality implies
\[
|I| \lesssim (\eta + \zeta)\|e_{u}\|_{1,\Omega}.
\]

We now consider the interior and Neumann boundary terms in (15): as previously the application of Cauchy-Schwarz’s inequality leads to
\[
|II| \leq \sum_{E \in E_{n}^{h} \cup E_{n}^{N}} \|J_{E,n}(u_{h})\|_{E} \|u - v_{h}\|_{E}.
\]
Therefore using the expression (16) and estimate (12), we obtain
\[
\|u - v_{h}\|_{E} = \|e_{u} - \pi_{h} e_{u}\|_{E} \lesssim h_{E}^{1/2} \|e_{u}\|_{1,\omega_{E}}.
\]
Inserting this estimate in the previous one we deduce that
\[
|II| \lesssim \eta \|e_{u}\|_{1,\Omega}.
\]

Moreover
\[
|III| \lesssim \zeta \|e_{u}\|_{1,\Omega}.
\]

The two remaining terms are handled in a similar way as the previous ones so that
\[
|IV| + |V| \lesssim \eta \|e_{u}\|_{1,\Omega}.
\]
Noting that \(u_{hn} \leq 0\) on \(\Gamma_{C}\), we have
\[
\int_{\Gamma_{C}} \lambda_{n} u_{hn} \leq 0,
\]
and it remains to estimate two terms in (15). Using the discrete complementarity condition \(\int_{\Gamma_{C}} \lambda_{hn} u_{hn} = 0\) implies
\[
\int_{\Gamma_{C}} \lambda_{hn} u_{n} = \int_{\Gamma_{C}} \lambda_{hn}(u_{n} - u_{hn}) = \int_{\Gamma_{C}} (\lambda_{hn}^{+} - \lambda_{hn}^{-})(u_{n} - u_{hn}) \\
\leq - \int_{\Gamma_{C}} \lambda_{hn}^{+} u_{hn} - \int_{\Gamma_{C}} \lambda_{hn}^{-}(u_{n} - u_{hn}) \\
\leq \eta^{2} - \int_{\Gamma_{C}} \lambda_{hn}^{-}(u_{n} - u_{hn}) \\
= \eta^{2} + VI.
\]
The last term in the previous expression is estimated using Cauchy-Schwarz’s and Young’s inequalities:

\[ |VI| \leq \sum_{E \in E_h^n} \| \lambda_{hn} - \lambda \|_E ||u_n - u_{hn}||_E \leq \sum_{E \in E_h^n} \left( \alpha ||u_n - u_{hn}||_E^2 + \frac{1}{4\alpha} \| \lambda_{hn} - \lambda \|_E^2 \right), \]

for any \( \alpha > 0 \). A standard trace theorem implies that

\[
(23) \quad |VI| \leq \alpha \|u_n - u_{hn}||_{H^1}^2 + \frac{1}{4\alpha} \sum_{E \in E_h^n} \| \lambda_{hn} - \lambda \|_E^2 \lesssim \alpha \|e_u\|_{L^2(\Omega)}^2 + \frac{\eta^2}{4\alpha}.
\]

Estimates (22) and (23) give

\[
(24) \quad \int_{\Gamma_C} \lambda_{hn} u_n \lesssim \alpha \|e_u\|_{L^2(\Omega)}^2 + \eta^2 \left(1 + \frac{1}{4\alpha}\right)
\]

for any \( \alpha > 0 \).

We now estimate the term corresponding to the friction:

\[
(25) \quad \int_{\Gamma_C} (\lambda_{ht} - \lambda_t)(u_t - u_{ht}) = \int_{\Gamma_C} (\lambda_{ht} - \mu \lambda_{hn}\xi)(u_t - u_{ht}) + \int_{\Gamma_C} (\mu \lambda_{hn}\xi - \lambda_t)(u_t - u_{ht})
\]

\[
= \int_{\Gamma_C} (\lambda_{ht} - \mu \lambda_{hn}\xi)(u_t - u_{ht}) + \int_{\Gamma_C} \mu (\lambda_{hn} - \lambda_t)\xi(u_t - u_{ht})
\]

where \( \xi \in M, \xi \in \text{Dir}_t(u_t), \lambda_t = \mu \lambda_n \xi \). The second term in (25) is bounded as follows

\[
\left| \int_{\Gamma_C} \mu (\lambda_{hn} - \lambda_t)\xi(u_t - u_{ht}) \right| \lesssim \mu \|\xi\|_M u_t - u_{ht}\|\lambda_n - \lambda_{hn}\|_{L^2(\Gamma_C)}
\]

\[
\lesssim \mu \|\xi\|_M \|u - u_{hn}\|_{L^1(\Omega)} \|\lambda - \lambda_{hn}\|_{L^2(\Gamma_C)}
\]

\[
\lesssim \mu \|\xi\|_M \|u - u_{hn}\|_{L^1(\Omega)} \|u - u_{hn}\|_{L^1(\Omega)} + \eta + \zeta.
\]

In the last inequality, we have used (30). We deduce from Young’s inequality:

\[
(26) \quad \left| \int_{\Gamma_C} \mu (\lambda_{hn} - \lambda_t)\xi(u_t - u_{ht}) \right| \lesssim (1 + \alpha) \mu \|\xi\|_M \|e_u\|_{L^2(\Omega)}^2 + \frac{\mu \|\xi\|_M}{2\alpha} (\eta^2 + \zeta^2)
\]

for any positive \( \alpha \).

Besides, the first term in (25) is handled next:

\[
\int_{\Gamma_C} (\lambda_{ht} - \mu \lambda_{hn}\xi)(u_t - u_{ht})
\]

\[
= \int_{\Gamma_C} \lambda_{ht} u_t - \int_{\Gamma_C} \mu \lambda_{hn} + \xi u_t + \int_{\Gamma_C} \mu \lambda_{hn} + \xi u_{ht} + \int_{\Gamma_C} \mu \lambda_{hn} - \xi (u_t - u_{ht}) - \int_{\Gamma_C} \lambda_{ht} u_{ht}
\]

\[
= \int_{\Gamma_C} (\lambda_{ht} u_t - \mu \lambda_{hn} + |u_t|) + \int_{\Gamma_C} \mu \lambda_{hn} + \xi u_{ht} + \int_{\Gamma_C} \mu \lambda_{hn} - \xi (u_t - u_{ht}) - \int_{\Gamma_C} \lambda_{ht} u_{ht}
\]

\[
\leq \int_{\Gamma_C} (|\lambda_{ht}| - \mu \lambda_{hn} + |u_t|) + \int_{\Gamma_C} (|\lambda_{ht}| - \mu \lambda_{hn} + |u_t|) + \int_{\Gamma_C} (\mu \lambda_{hn} + |u_{ht}|) + \int_{\Gamma_C} \mu \lambda_{hn} - |u_t - u_{ht}|
\]

\[
\leq \int_{\Gamma_C} (|\lambda_{ht}| - \mu \lambda_{hn} + |u_t - u_{ht}| + \int_{\Gamma_C} (|\lambda_{ht}| - \mu \lambda_{hn} + |u_t - u_{ht}|)
\]

\[
+ \int_{\Gamma_C} (\mu \lambda_{hn} + |u_{ht}|) + \int_{\Gamma_C} (\mu \lambda_{hn} - |u_t - u_{ht}|) + \int_{\Gamma_C} (|\lambda_{ht}| |u_{ht}| - \lambda_{ht} u_{ht})
\]
\[
\begin{align*}
\lesssim & \|u - u_h\|_{1,\Omega}(\|(|\lambda_{ht}| - \mu \lambda_{hn+}) + |\Gamma| + \mu \|\lambda_{hn} - \|\Gamma\|)
+ \int_{\Gamma_C} (|\lambda_{ht}| - \mu \lambda_{hn+}) |u_{ht}| - (|\lambda_{ht}| - \mu \lambda_{hn+}) |u_{ht}| + 2 \int_{\Gamma_C} (\lambda_{ht} u_{ht})_-
\lesssim & \|u - u_h\|_{1,\Omega}(\|(|\lambda_{ht}| - \mu \lambda_{hn+}) + |\Gamma| + \mu \|\lambda_{hn} - \|\Gamma\|)
+ \int_{\Gamma_C} (|\lambda_{ht}| - \mu \lambda_{hn+}) |u_{ht}| + \int_{\Gamma_C} (\lambda_{ht} u_{ht})_-.
\end{align*}
\]

From (26) and (27), we obtain for any \(\alpha > 0\):

\[
(28) \quad \int_{\Gamma_C} (\lambda_{ht} - \lambda_t)(u_t - u_{ht}) \lesssim (\alpha + (1 + \alpha)\mu \|\xi\|_M) \|e_u\|_{1,\Omega}^2 + \frac{\mu \|\xi\|_M + 2\alpha + 1 + \mu^2}{2\alpha} (\eta^2 + \zeta^2).
\]

Putting together the estimates (17), (18), (19), (20), (21), (24) and (28) with \(\alpha\) small enough in (15), and using Young's inequality, we deduce that: if \(\mu \|\xi\|_M\) is small enough then

\[
(29) \quad \|u - u_h\|_{1,\Omega} \lesssim \eta + \zeta.
\]

We now search for an upper bound on the discretization error \(\lambda - \lambda_h\) corresponding to the multipliers. Let \(v \in \mathbf{V}\) and \(v_h \in \mathbf{V}_h\). From the equilibrium equations in (7) and (9) we get:

\[
\begin{align*}
b(\lambda - \lambda_h, v) &= b(\lambda, v - v_h) - b(\lambda_h, v - v_h) + b(\lambda - \lambda_h, v_h) \\
&= L(v - v_h) - a(u, v - v_h) - b(\lambda_h, v - v_h) + a(u_h - u, v_h) \\
&= L(v - v_h) - a(u - u_h, v) - a(u_h, v - v_h) - b(\lambda_h, v - v_h).
\end{align*}
\]

An integration by parts on each element \(K\) gives

\[
b(\lambda - \lambda_h, v) = \int_{\Omega} f \cdot (v - v_h) - a(u - u_h, v) - \sum_{E \in E_h^\Omega} \int_{E} J_{E,n}(u_h) \cdot (v - v_h)
- \sum_{E \in E_h^C} \int_{E} (\lambda_{hn} + \sigma_n(u_h))(v_n - v_{hn}) - \sum_{E \in E_h^F} \int_{E} (\lambda_{ht} + \sigma_t(u_h))(v_t - v_{ht})
+ \sum_{E \in E_h^N} \int_{E} (\mathbf{F} - \mathbf{F}_E) \cdot (v - v_h).
\]

Choosing \(v_h = \pi_h v\) where \(\pi_h\) is the quasi-interpolation operator defined in Lemma 3.4 and achieving a similar calculation as in (17), (18), (19) and (20) we deduce that

\[
|b(\lambda - \lambda_h, v)| \lesssim (\|u - u_h\|_{1,\Omega} + \eta + \zeta) \|v\|_{1,\Omega}
\]

for any \(v \in \mathbf{V}\). As a consequence

\[
(30) \quad \|\lambda - \lambda_h\|_{\frac{1}{2} \Gamma_C} \lesssim \|u - u_h\|_{1,\Omega} + \eta + \zeta.
\]

Putting together the two estimates (29) and (30) ends the proof of the theorem. \(\blacksquare\)

### 4.3 Lower error bound

**Theorem 4.4** Let \((u_h, \lambda_h)\) be a solution to the discrete problem (9) and let \(\eta = \eta(u_h, \lambda_h)\) be the corresponding estimator. Let \((u, \lambda)\) be a solution to Problem (7) such that \(\lambda \in (L^2(\Gamma_C))^2\). For all elements \(K\), the following local lower error bounds hold:

\[
\begin{align*}
\eta_{1K} & \lesssim \|u - u_h\|_{1,K} + \zeta_K, \\
\eta_{2K} & \lesssim \|u - u_h\|_{1,\omega_K} + \zeta_K.
\end{align*}
\]

11
For all elements $K$ having an edge in $\Gamma_C$ (i.e., $K \cap \Gamma_C = E$), the following local lower error bounds hold:

\begin{align}
(33) \quad \eta_{jK} & \lesssim h_K^{1/2} \| \lambda - \lambda_h \|_E + \| u - u_h \|_{1,K} + \zeta_K, \quad i = 3, 4, \\
(34) \quad \eta_{jK} & \leq 2(1 + \mu) \left( \| \lambda - \lambda_h \|_E + \| u - u_h \|_{1/2} + \| u - u_h \|_{1,K} + \| \lambda - \lambda_h \|_{1/2} \right), \\
(35) \quad \eta_{jK} & \leq (1 + \mu) \| \lambda - \lambda_h \|_E, \quad l = 6, 8.
\end{align}

**Proof:** We mention that we do not suppose that the solution to the continuous problem is unique. Of course our result holds when $(u, \lambda)$ is the unique solution given by Proposition 2.2. Note also that the solution to the discrete problem is not supposed to be unique.

The estimates of $\eta_{1K}$ and $\eta_{2K}$ in (31) and (32) are standard (see, e.g., [79]). We now estimate $\eta_{3K}$. Writing $w_E = w_{En} n + w_{Et} t$ on $E \in \mathcal{E}_K^C$ and denoting by $b_E$ the edge bubble function associated with $E$ (i.e., $b_E = 4 \psi_{a_1} \psi_{a_2}$, when $a_1, a_2$ are the two extremities of $E$; we recall that $\psi_x$ is the standard basis function at node $x$ in $V_h$ satisfying $\psi_x(x') = \delta_{x,x'}$ for any node $x'$, see (10)), we choose $w_{En} = (\lambda_h n + \sigma_n(u_h))b_E$ and $w_{Et} = 0$ in the element $K$ containing $E$ (here we make a slight abuse of notation to simplify) and $w_E = 0$ in $\overline{\Gamma} \setminus K$. Therefore

\[
\| \lambda_h + \sigma_n(u_h) \|_E^2 \sim \int_E (\lambda_h + \sigma_n(u_h)) w_{En}
\]

\[
= b(\lambda_h, w_E) + \int_K \sigma(u_h) : \varepsilon(w_E)
\]

\[
= b(\lambda_h, w_E) - \int_K \sigma(u - u_h) : \varepsilon(w_E) + \int_K \sigma(u) : \varepsilon(w_E)
\]

\[
= b(\lambda_h - \lambda, w_E) + L(w_E) - \int_K \sigma(u - u_h) : \varepsilon(w_E)
\]

\[
\lesssim \| \lambda - \lambda_h \|_E \| w_E \|_E + \| f \|_K \| w_E \|_K + \| u - u_h \|_{1,K} \| w_E \|_{1,K}.
\]

An inverse inequality and estimate (31) imply

\[
h_K^{1/2} \| \lambda_h + \sigma_n(u_h) \|_E \lesssim h_K^{1/2} \| \lambda - \lambda_h \|_E + \| u - u_h \|_{1,K} + h_K \| f \|_K
\]

\[
\lesssim h_K^{1/2} \| \lambda - \lambda_h \|_E + \| u - u_h \|_{1,K} + \zeta_K.
\]

This estimate gives the bound of $\eta_{3K}$ in (33). The estimate of $\eta_{4K}$ in (33) is obtained as previously by choosing $w_{En} = 0$ and $w_{Et} = (\lambda_h t + \sigma_t(u_h))b_E$.

We now consider $\eta_{5K}$. If $E \in \mathcal{E}_K^C$, let $F \subset E$ be the part of the edge where $\lambda_{hn} = \lambda_{hn+}$. So

\[
\int_E -\lambda_{hn} u_{hn} = \int_F -\lambda_{hn} u_{hn} = \int_F (\lambda_{hn} - \lambda_n)(u_n - u_{hn}) - \int_F \lambda_{hn} u_n - \int_F \lambda_n u_{hn}
\]

\[
= \int_F (\lambda_{hn} - \lambda_n)(u_n - u_{hn}) - \int_F (\lambda_{hn} - \lambda_n) u_n - \int_F \lambda_n (u_{hn} - u_n)
\]

\[
\leq \| \lambda - \lambda_h \|_E \| u - u_h \|_E + \| \lambda - \lambda_h \|_E \| u_n \|_E + \| u - u_h \|_E \| \lambda \|_E.
\]

The last estimate implies the bound of $\eta_{5K}$ in (34) by taking the square root.

The estimate of $\eta_{6K}$ in (35) is obvious. Since $\lambda_n \geq 0$ we have $0 \leq \lambda_{hn-} \leq |\lambda_n - \lambda_{hn}|$ on $\Gamma_C$. So

\[
\| \lambda_{hn-} \|_E \leq \| \lambda_n - \lambda_{hn} \|_E \leq \| \lambda - \lambda_h \|_E.
\]
Next we estimate $\eta_{\Gamma K}$. If $E \in E^C_K$, let $F \subset E$ be the part of the edge where $- (|\lambda_{ht}| - \mu \lambda_{hn+}) = (|\lambda_{ht}| - \mu \lambda_{hn+})_+$. So

$$\int_E (|\lambda_{ht}| - \mu \lambda_{hn+})_+ u_{ht} + \int_E (\lambda_{ht} u_{ht})_+ = \int_F (|\lambda_{ht}| + \mu \lambda_{hn+})|u_{ht}| + \int_F (\lambda_{ht} u_{ht})_+$$

(36)

$$= \int_F (|\lambda_{ht}| + \mu \lambda_{hn})|u_{ht}| + \int_F (\lambda_{ht} u_{ht})_+ + \int_F \mu \lambda_{hn} - |u_{ht}|.$$

The first term in (36) is estimated as follows using (6):

$$- \int_F (|\lambda_{ht}| - \mu \lambda_{hn})|u_{ht}| = \int_F (|\lambda_{ht}| - |\lambda_t| - \mu(\lambda_{hn} - \lambda_n))(|u_{ht}| - |u_t|)$$

$$- \int_F (|\lambda_{ht}| - |\lambda_t| - \mu(\lambda_{hn} - \lambda_n))|u_t|$$

$$- \int_F (|\lambda_t| - \mu \lambda_{hn})|u_{ht} - u_t|$$

$$\leq (1 + \mu) (\|\lambda - \lambda_h\|_E \|u - u_h\|_E + \|\lambda - \lambda_h\|_E \|u\|_E + \|u - u_h\|_E \|\lambda\|_E).$$

The second term in (36) is estimated by noting that $\lambda_t u_t \geq 0$ on $\Gamma_C$. Hence

$$0 \leq (\lambda_{ht} u_{ht})_- \leq |\lambda_t u_t - \lambda_{ht} u_{ht}|$$

$$= |\lambda_t (u_t - u_{ht}) + (\lambda_t - \lambda_{ht})(u_{ht} - u_t) + (\lambda_t - \lambda_{ht})u_t|.$$

So

$$\int_E (\lambda_{ht} u_{ht})_- \leq \|\lambda - \lambda_h\|_E \|u - u_h\|_E + \|\lambda - \lambda_h\|_E \|u\|_E + \|u - u_h\|_E \|\lambda\|_E.$$

The third term in (36) yields, using the estimate of $\eta_{\Gamma K}$

$$\int_F \mu \lambda_{hn} - |u_{ht}| \leq \int_E \mu \lambda_{hn} - |u_{ht} - u_t| + \int_E \mu \lambda_{hn} - |u_t| \leq \mu \|\lambda - \lambda_h\|_E (\|u - u_h\|_E + \|u\|_E).$$

This proves the bound of $\eta_{\Gamma K}$. Finally we consider the upper bound of $\eta_{\Gamma K}$. We have

$$0 \leq (|\lambda_{ht}| - \mu \lambda_{hn+})_+ = (|\lambda_{ht}| - \mu \lambda_{hn} - \mu \lambda_{hn-})_+ \leq (|\lambda_{ht}| - \mu \lambda_{hn})_+.$$

Since $|\lambda_t| - \mu \lambda_n \leq 0$, we have

$$(|\lambda_{ht}| - \mu \lambda_{hn})_+ \leq \|\lambda_{ht} - |\lambda_t| - \mu \lambda_{hn} + \mu \lambda_n| \leq |\lambda_{ht} - \lambda_t| + \mu |\lambda_{hn} - \lambda_n|$$

Hence

$$\|(|\lambda_{ht}| - \mu \lambda_{hn+})_+ \|_E \leq \|\lambda_t - \lambda_h\|_E + \mu \|\lambda_n - \lambda_{hn}\|_E \leq (1 + \mu) \|\lambda - \lambda_h\|_E.$$

\[\square\]

**Remark 4.5** Assume that $u \in (H^2(\Omega))^2$ (so $\lambda \in (H^1(\Gamma_C))^2$), and that optimal a priori error estimates hold (note that this question is entirely open and that the only aim of the present remark is to try to illustrate our result) and define:

$$\eta_i = \left( \sum_{K \in T_h} \eta^2_{iK} \right)^{1/2}, \quad 1 \leq i \leq 8.$$

Then one would have $\eta_i \lesssim h, 1 \leq i \leq 4; \eta_j \lesssim h^{1/4}, j = 5, 7; \eta_l \lesssim h^{1/2}, l = 6, 8$. So $\eta \lesssim h^{1/4}$. 13
5 A second finite element discretization and the corresponding estimator $\tilde{\eta}$

The aim of this section is to consider a finite element discretization of the frictional contact conditions which allows to obtain a simpler residual error estimator. More precisely a different quadrature formula is used for the frictional contact conditions (see [53] for the early idea).

5.1 Preliminaries

For any $\nu = (\nu_h, \nu_{ht}) \in W_h \times W_h$ and $\nu_h \in V_h$, we define the bilinear form $c(\cdot, \cdot)$ such that

$$c(\nu_h, \nu_h) = \int_{\Gamma_C} (I_h(\nu_h, \nu_h) + I_h(\nu_{ht}, \nu_{ht})) \, d\Gamma$$

where $I_h$ is the classical piecewise affine Lagrange interpolation operator at the nodes of $\Gamma_C$. Let $K_{hn} = \{ \nu_h \in W_h : \nu_h \geq 0 \}$ be the closed convex cone of nonnegative functions in $W_h$. For $g \in K_{hn}$, we set $K_{ht}(g) = \{ \nu_h \in W_h : |\nu_h| \leq g \}$.

Next, we consider the problem of finding $\tilde{u}_h \in V_h$ and $(\tilde{\lambda}_{hn}, \tilde{\lambda}_{ht}) = \tilde{\lambda}_h \in K_h(\mu \tilde{\lambda}_{hn}) = K_{hn} \times K_{ht}(\mu \tilde{\lambda}_{hn})$ satisfying:

$$\begin{align*}
(37) \quad a(\tilde{u}_h, \nu_h) + c(\tilde{\lambda}_h, \nu_h) &= L(\nu_h), \quad \forall \nu_h \in V_h, \\
c(\nu_h - \tilde{\lambda}_h, \tilde{u}_h) &\leq 0, \quad \forall \nu_h = (\nu_{hn}, \nu_{ht}) \in K_h(\mu \tilde{\lambda}_{hn}).
\end{align*}$$

Using the same techniques as in [25] for problem (9), one can prove that the problem (37) admits at least a solution and that there is a unique solution when $\mu \leq C(h)$. The proof of this result can be found in the appendix. Besides one can prove that the pointwise discrete frictional contact conditions incorporated in the inequality of (37) are as follows:

**Proposition 5.1** Let $(\tilde{u}_h, \tilde{\lambda}_h)$ be a solution of (37). Suppose that $\dim(W_h) = p$ and let $\psi_{x_i}, 1 \leq i \leq p$ be the basis functions of $W_h$ on $\Gamma_C$. Let $\tilde{U}_N$ and $\tilde{U}_T$ denote the vectors of components the nodal values of $\tilde{u}_h$ and $\tilde{u}_{ht}$ respectively and let $\tilde{L}_N$ and $\tilde{L}_T$ denote the vectors of components the nodal values of $\tilde{\lambda}_{hn}$ and $\tilde{\lambda}_{ht}$ respectively. Then the discrete frictional contact conditions in (37) are as follows; for any $1 \leq i \leq p$:

$$\begin{align*}
(38) \quad (\tilde{L}_N)_i &\geq 0, \quad (\tilde{U}_N)_i \leq 0, \quad (\tilde{L}_N)_i(\tilde{U}_N)_i = 0, \\
(39) \quad |(\tilde{L}_T)_i| &\leq \mu(\tilde{L}_N)_i, \\
(40) \quad |(\tilde{L}_T)_i| < \mu(\tilde{L}_N)_i \Rightarrow (\tilde{U}_T)_i = 0, \\
(41) \quad (\tilde{L}_T)_i(\tilde{U}_T)_i &\geq 0.
\end{align*}$$

**Proof:** From $\tilde{\lambda}_{hn} \in K_{hn}$, we immediately get (38). Condition

$$\int_{\Gamma_C} I_h((\nu_{hn} - \tilde{\lambda}_{hn})\tilde{u}_h) \, d\Gamma \leq 0, \quad \forall \nu_{hn} \in K_{hn}$$

is equivalent to

$$\begin{align*}
(42) \quad \int_{\Gamma_C} I_h(\nu_{hn}\tilde{u}_h) \, d\Gamma &\leq 0, \quad \forall \nu_{hn} \in K_{hn} \quad \text{and} \quad \int_{\Gamma_C} I_h(\tilde{\lambda}_{hn}\tilde{u}_h) \, d\Gamma = 0.
\end{align*}$$

Choosing in the inequality of (42), $\nu_{hn} = \psi_{x_i}$ and writing $\int_{\Gamma_C} I_h(\psi_{x_i}\tilde{u}_h) = \tilde{u}_h(x_i) \int_{\Gamma_C} \psi_{x_i}$ gives the second inequality in (38). The equality $\int_{\Gamma_C} I_h(\tilde{\lambda}_{hn}\tilde{u}_h) = \sum_{i=1}^{p} \tilde{\lambda}_h(x_i)\tilde{u}_h(x_i) \int_{\Gamma_C} \psi_{x_i} = 0$ implies $(\tilde{L}_N)_i(\tilde{U}_N)_i = 0, 1 \leq i \leq p$. 

14
Inequality (39) follows directly from $\tilde{\lambda}_{ht} \in K_{ht}(\mu \tilde{\lambda}_{hn})$. Since
\begin{equation}
\int_{\Gamma_C} I_h((\nu_{ht} - \tilde{\lambda}_{ht})\tilde{u}_{ht}) \ d\Gamma \leq 0, \quad \forall \nu_{ht} \in K_{ht}(\mu \tilde{\lambda}_{hn})
\end{equation}
we choose $\nu_{ht}$ in (43) as follows: $\nu_{ht} = \mu \tilde{\lambda}_{hn}$ at node $x_i$ and $\nu_{ht} = \tilde{\lambda}_{ht}$ at the $p - 1$ other nodes. We obtain
\begin{equation}
\int_{\Gamma_C} I_h((\nu_{ht} - \tilde{\lambda}_{ht})\tilde{u}_{ht}) \ d\Gamma = (\mu \tilde{\lambda}_{hn}(x_i) - \tilde{\lambda}_{ht}(x_i)) \int_{\Gamma_C} \psi_{x_i} \ d\Gamma \leq 0.
\end{equation}
Similarly, take $\nu_{ht} = -\mu \tilde{\lambda}_{hn}$ at node $x_i$ and $\nu_{ht} = \tilde{\lambda}_{ht}$ at the $p - 1$ other nodes. We get
\begin{equation}
\int_{\Gamma_C} I_h((\nu_{ht} - \tilde{\lambda}_{ht})\tilde{u}_{ht}) \ d\Gamma = (-\mu \tilde{\lambda}_{hn}(x_i) - \tilde{\lambda}_{ht}(x_i)) \int_{\Gamma_C} \psi_{x_i} \ d\Gamma \leq 0.
\end{equation}
Putting together estimates (44) and (45) implies (40).
It remains to prove (41). Define $\nu_{ht}$ in (43) as follows: $\nu_{ht} = \frac{1}{2}\tilde{\lambda}_{ht}$ at node $x_i$ and $\nu_{ht} = \tilde{\lambda}_{ht}$ at the $p - 1$ other nodes. Therefore
\begin{equation}
\int_{\Gamma_C} I_h((\nu_{ht} - \tilde{\lambda}_{ht})\tilde{u}_{ht}) \ d\Gamma = -\frac{1}{2}\tilde{\lambda}_{ht}(x_i)\tilde{u}_{ht}(x_i) \int_{\Gamma_C} \psi_{x_i} \ d\Gamma \leq 0.
\end{equation}
Hence inequality (41).

5.2 Definition of the residual error estimator
As for the first discretization the element residual is defined by $\text{div}\sigma(\tilde{u}_h) + f = f$ on $K$. The data $f$ can be replaced by $f_K \in (P_K(K))^2$ and the difference $f - f_K$ will be treated as data oscillation. Similarly $F$ can be approximated by a simpler quantity denoted $F_E$ on any $E \in E_h^N$.

Definition 5.2 The global residual estimator $\tilde{\eta}$ and the local residual error estimators $\tilde{\eta}_K$ are defined by
\[
\tilde{\eta} = \left( \sum_{K \in T_h} \tilde{\eta}_K^2 \right)^{1/2},
\]
\[
\tilde{\eta}_K = \left( \sum_{i=1}^{6} \tilde{\eta}_{ik}^2 \right)^{1/2},
\]
\[
\tilde{\eta}_{1K} = h_K \|f_K\|_{K},
\]
\[
\tilde{\eta}_{2K} = h_K^{1/2} \left( \sum_{E \in E_h^N \cup E_h^N} \|J_{E,n}(\tilde{u}_h)\|_{E}^2 \right)^{1/2},
\]
\[
\tilde{\eta}_{3K} = h_K^{1/2} \|\tilde{\lambda}_{hn} + \sigma_n(\tilde{u}_h)\|_{K \cap \Gamma_C},
\]
\[
\tilde{\eta}_{4K} = h_K^{1/2} \|\tilde{\lambda}_{ht} + \sigma_l(\tilde{u}_h)\|_{K \cap \Gamma_C},
\]
\[
\tilde{\eta}_{5K} = \left( \int_{K \cap \Gamma_C} -\tilde{\lambda}_{hn} \tilde{u}_{hn} \right)^{1/2},
\]
\[
\tilde{\eta}_{6K} = \left( \int_{K \cap \Gamma_C} (\mu \tilde{\lambda}_{hn} |\tilde{u}_{ht}| - \tilde{\lambda}_{ht} \tilde{u}_{ht}) \right)^{1/2},
\]
where we recall that $J_{E,n}(\tilde{u}_h)$ is the constraint jump of $\tilde{u}_h$ in the normal direction defined by (13). As in the previous section, the local and global data oscillation terms $\zeta_K$ and $\zeta$ are defined by (14).
Remark 5.3 From the previous definitions we have \( \tilde{\eta}_{1,K} = \eta_{1,K} \). We mention that there is no term as \( \eta_6 \) and \( \eta_8 \) in \( \tilde{\eta} \) since \( \tilde{\lambda}_{hn} \geq 0 \) and \( |\tilde{\lambda}_{ht}| \leq \mu \tilde{\lambda}_{hn} \).

5.3 Upper error bound

As in the statement of Theorem 4.3 we need to assume that the solution to the continuous problem satisfies the uniqueness criterion of [72] in order to obtain the upper bound of the discretization error.

Theorem 5.4 Let \((u, \lambda)\) be the solution to Problem (7) such that \( \lambda_t = \mu \lambda_n \xi \), with \( \xi \in \text{Dir}_1(u_t) \) a.e. on \( \Gamma_C \) and \( \mu \|\xi\|_M \) is small enough. Let \((\tilde{u}_h, \tilde{\lambda}_h)\) be a solution to the discrete problem (37). Then

\[
\|u - \tilde{u}_h\|_{1,\Omega} + \|\lambda - \tilde{\lambda}_h\|_{-\frac{1}{2},\Gamma_C} \lesssim \tilde{\eta} + \zeta.
\]

Proof: We adopt the following notations for the error term in the displacement: \( \tilde{e}_u = u - \tilde{u}_h \).

As in Theorem 4.3, we obtain for any \( v_h \in V_h \)

\[
\|\tilde{e}_u\|_{1,\Omega}^2 \lesssim \int_\Omega f \cdot (u - v_h) - \sum_{E \in E_n \cup \cup n} \int_E J_{E,n}(\tilde{u}_h) \cdot (u - v_h) + \sum_{E \in E_n} \int_E (\tilde{F} - F_E) \cdot (u - v_h)
\]

\[
+ \sum_{E \in E_h} \int_E (\tilde{\lambda}_hn + \sigma_n(\tilde{u}_h))(v_h - u_n) + \sum_{E \in E_h} \int_E (\tilde{\lambda}_ht + \sigma_t(\tilde{u}_h))(v_t - u_t)
\]

\[
+ \int_{\Gamma_C} (I_h(\tilde{\lambda}_hn(v_h - \tilde{u}_hn)) - \tilde{\lambda}_hn(v_h - \tilde{u}_hn))
\]

\[
+ \int_{\Gamma_C} (I_h(\tilde{\lambda}_ht(v_h - \tilde{u}_ht)) - \tilde{\lambda}_ht(v_h - \tilde{u}_ht))
\]

\[
+ \int_{\Gamma_C} (\tilde{\lambda}_hn - \lambda_n)(u_n - \tilde{u}_hn) + \int_{\Gamma_C} (\tilde{\lambda}_ht - \lambda_t)(u_t - \tilde{u}_ht)
\]

\[(46) \quad = \tilde{I} + \tilde{II} + \tilde{III} + \tilde{IV} + \tilde{V} + \tilde{VI} + \tilde{VII} + \tilde{VIII} + \int_{\Gamma_C} (\tilde{\lambda}_ht - \lambda_t)(u_t - \tilde{u}_ht).\]

As in Theorem 4.3 we take \( v_h \) of the form (16). So

\[(47) \quad |\tilde{I}| + |\tilde{II}| + |\tilde{III}| + |\tilde{IV}| + |\tilde{V}| \lesssim (\tilde{\eta} + \zeta)\|\tilde{e}_u\|_{1,\Omega}.
\]

Now we estimate the two terms in (46) with the interpolation operator using a basic error estimate of numerical integration (trapezoidal formula):

\[
|\tilde{VI}| = \left| \int_{\Gamma_C} \left( I_h(\tilde{\lambda}_hn(\pi_n e_u)n) - \tilde{\lambda}_hn(\pi_n e_u)n) \right) \right|
\]

\[
= \left| \sum_{E \in E_h^C} \int_E \left( I_h(\tilde{\lambda}_hn(\pi_n e_u)n) - \tilde{\lambda}_hn(\pi_n e_u)n) \right) \right|
\]

\[
\lesssim \sum_{E \in E_h^C} h_E^2 |(\tilde{\lambda}_hn(\pi_n e_u)n)'|\]

\[
\lesssim \sum_{E \in E_h^C} h_E^2 |\tilde{\lambda}_hn(\pi_n e_u)n)'|\]

\[
\leq \sum_{E \in E_h^C} h_E^2 \|\tilde{\lambda}_hn\|_E \|(\pi_n e_u)n)'\|_E
\]
\[ \lesssim \sum_{E \in E_h^C} \| \lambda'_{hn} \|_E \| \pi_h \tilde{e}_u \|_{1,K} \]
\[ \lesssim \sum_{E \in E_h^C} \| \lambda'_{hn} \|_E \| \tilde{e}_u \|_{1,\omega_K} \]
\[ = \sum_{E \in E_h^C} \| (\lambda_{hn} + \sigma_n(\tilde{u}_h))' \|_E \| \tilde{e}_u \|_{1,\omega_K} \]
\[ \lesssim \sum_{E \in E_h^C} \| \lambda_{hn} + \sigma_n(\tilde{u}_h) \|_E \| \tilde{e}_u \|_{1,\omega_K} \]
\[ \lesssim \bar{\eta} \| \tilde{e}_u \|_{1,\Omega}, \]

(48)

where \( K \) is the element containing \( E \). Above we have used the Cauchy-Schwarz inequality, the \( H^1 \) stability of \( \pi_h \), proved in Lemma 3.1 of [22] (see also [79]) and the trace inequality on an element (see [79]). In a similar way, we obtain:

\[ \left| V \Pi \right| \lesssim \sum_{E \in E_h^C} h_E^{3/2} \| \lambda_{ht} + \sigma_t(\tilde{u}_h) \|_E \| \tilde{e}_u \|_{1,\omega_K} \leq \bar{\eta} \| \tilde{e}_u \|_{1,\Omega}. \]

(49)

Noting that \( \tilde{u}_{hn} \leq 0 \) and \( \tilde{\lambda}_{hn} \geq 0 \) on \( \Gamma_C \), we have \( \int_{\Gamma_C} \tilde{\lambda}_{hn} u_n \leq 0, \int_{\Gamma_C} \lambda_n u_n = 0 \) and \( \int_{\Gamma_C} \lambda_n \tilde{u}_{hn} \leq 0 \). Consequently, we obtain:

\[ \left| V \Pi \right| \leq \int_{\Gamma_C} -\tilde{\lambda}_{hn} \tilde{u}_{hn} \leq \bar{\eta}^2. \]

(50)

It remains to estimate one term in (46): the one coming from the friction approximation. As in (25) and (26), we obtain

\[ \int_{\Gamma_C} (\lambda_{ht} - \lambda_t)(u_t - \tilde{u}_ht) = \int_{\Gamma_C} (\tilde{\lambda}_{ht} - \mu \tilde{\lambda}_{hn} \xi)(u_t - \tilde{u}_ht) + \int_{\Gamma_C} \mu(\tilde{\lambda}_{hn} - \lambda_n) \xi(u_t - \tilde{u}_ht) \]

where \( \xi \in M, \xi \in \text{Dir}_t(u_t), \lambda_t = \mu \lambda_n \xi, \) and

\[ \left| \int_{\Gamma_C} \mu(\tilde{\lambda}_{hn} - \lambda_n) \xi(u_t - \tilde{u}_ht) \right| \lesssim (1 + \alpha) \mu \| \xi \|_M \| \tilde{e}_u \|_{1,\Omega}^2 + \frac{\mu \| \xi \|_M}{2\alpha} (\bar{\eta}^2 + \zeta^2) \]

(52)

for any positive \( \alpha \). The first term in (51) is handled as follows:

\[ \int_{\Gamma_C} (\lambda_{ht} - \mu \tilde{\lambda}_{hn} \xi)(u_t - \tilde{u}_ht) = \int_{\Gamma_C} \tilde{\lambda}_{ht} u_t - \int_{\Gamma_C} \mu \tilde{\lambda}_{hn} \xi u_t + \int_{\Gamma_C} \mu \tilde{\lambda}_{hn} \xi \tilde{u}_ht - \int_{\Gamma_C} \tilde{\lambda}_{ht} \tilde{u}_ht \]

\[ = \int_{\Gamma_C} (\tilde{\lambda}_{ht} u_t - \mu |u_t|) + \int_{\Gamma_C} (\mu \tilde{\lambda}_{hn} \xi \tilde{u}_ht - \tilde{\lambda}_{ht} \tilde{u}_ht) \]

\[ \leq \int_{\Gamma_C} (|\tilde{\lambda}_{ht}| - |\mu \tilde{\lambda}_{hn}|) |u_t| + \int_{\Gamma_C} (-\tilde{\lambda}_{ht} \tilde{u}_ht + \mu \tilde{\lambda}_{hn} \tilde{u}_ht) \]

\[ \lesssim \bar{\eta}^2. \]

(53)

By (52) and (53), we obtain for any positive \( \alpha \):

\[ \left| \int_{\Gamma_C} (\tilde{\lambda}_{ht} - \lambda_t)(u_t - \tilde{u}_ht) \right| \lesssim (1 + \alpha) \mu \| \xi \|_M \| \tilde{e}_u \|_{1,\Omega}^2 + \frac{\mu \| \xi \|_M + 2\alpha}{2\alpha} (\bar{\eta}^2 + \zeta^2). \]

(54)

Putting together the estimates (47), (48), (49), (50) and (54) in (46) and using Young’s inequality, we come to the conclusion that if \( \mu \| \xi \|_M \) is small enough (see also (29)):

\[ \| u - \tilde{u}_h \|_{1,\Omega} \lesssim \bar{\eta} + \zeta. \]

(55)
Let \( \Gamma \) be either of constant sign on \( E \). The last estimate implies (57) by taking the square root.

As in Theorem 4.4, we only need to estimate \( \tilde{\eta} \).

**Proof:**

(56) \( \| \lambda - \hat{\lambda}_h \|_{\frac{1}{2}, \Gamma_C} \lesssim + \| u - \tilde{u}_h \|_{1, \Omega} + \tilde{\eta} + \zeta \).

Putting together the two estimates (55) and (56) ends the proof of the theorem.

### 5.4 Lower error bound

**Theorem 5.5** Let \((\tilde{u}_h, \tilde{\lambda}_h)\) be a solution to the discrete problem (37) and let \( \tilde{\eta} = \tilde{\eta}(\tilde{u}_h, \tilde{\lambda}_h) \) be the corresponding estimator. Let \((u, \lambda)\) be a solution to Problem (7) such that \( \lambda \in (L^2(\Gamma_C))^2 \). For all elements \( K \), the following local lower error bounds hold:

\[
\tilde{\eta}_{1K} \lesssim \| u - \tilde{u}_h \|_{1,K} + \zeta_K, \\
\tilde{\eta}_{2K} \lesssim \| u - \tilde{u}_h \|_{1,\omega_K} + \zeta_K.
\]

For all elements \( K \) having an edge in \( \Gamma_C \) (i.e., \( K \cap \Gamma_C = E \)), the following local lower error bounds hold:

\[
\tilde{\eta}_{3K} \lesssim h_{E}^{1/2} \| \lambda - \hat{\lambda}_h \|_E + \| u - \tilde{u}_h \|_{1,K} + \zeta_K, \quad i = 3, 4,
\]

(57) \( \tilde{\eta}_{4K} \lesssim \tilde{\eta}_{3K}^{1/2} \| \tilde{u}_h \|_{1,K}^{1/2} \),

(58) \( \tilde{\eta}_{5K} \lesssim (\mu \tilde{\eta}_{3K} + \tilde{\eta}_{4K})^{1/2} \| \tilde{u}_h \|_{1,K}^{1/2} \).

**Proof:** As in Theorem 4.4 we only need to estimate \( \tilde{\eta}_{3K}, \tilde{\eta}_{4K}, \tilde{\eta}_{5K} \) and \( \tilde{\eta}_{6K} \). In addition, the bounds of \( \tilde{\eta}_{3K}, \tilde{\eta}_{4K} \) are obtained as in Theorem 4.4. So we consider \( \tilde{\eta}_{5K} \). If \( E \in E_{K}^C \), one has by the trapezoidal integration formula, an inverse inequality and the scaled trace inequality:

\[
\int_E -\lambda_{hn} \tilde{u}_{hn} = \int_E (I_h(\lambda_{hn} \tilde{u}_{hn}) - \hat{\lambda}_h \tilde{u}_{hn}) \\
\lesssim h_{E}^{1/2} |(\lambda_{hn} \tilde{u}_{hn})''| \\
\lesssim h_{E}^{1/2} |\lambda_{hn}' \tilde{u}_{hn}'| \\
\lesssim h_{E}^{1/2} |\lambda_{hn}'| E \| u_{hn}' \| E \\
= h_{E}^{1/2} |(\lambda_{hn} + \sigma(u_{hn}))''| E \| \tilde{u}_{hn}' \| E \\
\lesssim h_{E}^{1/2} \tilde{\eta}_{3K} \| u_{hn}' \| E \\
\lesssim \tilde{\eta}_{5K} \| \tilde{u}_h \|_{1,K}.
\]

The last estimate implies (57) by taking the square root.

Finally we consider \( \tilde{\eta}_{6K} \). According to Proposition 5.1 we have for any node \( x_i \) in \( \Gamma_C \):

\[
(\mu \lambda_{hn}|\tilde{u}_{hn} - \hat{\lambda}_h \tilde{u}_{hn})(x_i) = ((\mu \lambda_{hn} - |\hat{\lambda}_h|)|\tilde{u}_{hn}|)(x_i) = 0. \quad \text{Let } E \in E_{K}^C; \text{ it is easy to see that } \tilde{u}_{hn} \text{ is either of constant sign on } E \text{ (i.e., nonnegative or nonpositive)} \text{ or } \tilde{u}_{hn}(x_1)\tilde{u}_{hn}(x_2) < 0 \text{ (where } x_1 \text{ and } x_2 \text{ are the extremities of } E) \text{ and } \tilde{u}_{hn} \text{ admits a unique zero denoted } m \text{ in } E.
\]

Let us first consider the second case: we denote \( E_1 = (x_1, m) \) and \( E_2 = (m, x_2) \) and we suppose without loss of generality that \( \tilde{u}_{hn} > 0 \) in \( E_1 \) and \( \tilde{u}_{hn} < 0 \) in \( E_2 \). We denote by \( J_{hn} \) the piecewise affine Lagrange interpolation operator defined in \( E \) at the points \( x_1, m, x_2 \). Since \((\mu \lambda_{hn}|\tilde{u}_{hn} - \hat{\lambda}_h \tilde{u}_{hn})(m) = 0 \) and using the same arguments as for \( \tilde{\eta}_{5K} \), we get:

\[
\int_E (\mu \lambda_{hn}|\tilde{u}_{hn} - \hat{\lambda}_h \tilde{u}_{hn}) = \int_{E_1} (\mu \lambda_{hn}\tilde{u}_{hn} - \hat{\lambda}_h \tilde{u}_{hn}) + \int_{E_2} (-\mu \lambda_{hn}\tilde{u}_{hn} - \hat{\lambda}_h \tilde{u}_{hn}) \\
= \int_{E_1} ((\mu \lambda_{hn}\tilde{u}_{hn} - \hat{\lambda}_h \tilde{u}_{hn}) - J_{hn}(\mu \lambda_{hn}\tilde{u}_{hn} - \hat{\lambda}_h \tilde{u}_{hn}))
\]
\[-\mu \lambda \tilde{u}_{ht} + \lambda h \tilde{u}_{ht} \right) \right) \\
\leq h^3_{E1} (\mu \lambda \tilde{u}_{ht} - \lambda h \tilde{u}_{ht})'' \mid_{E1} + h^3_{E2} (\mu \lambda \tilde{u}_{ht} - \lambda h \tilde{u}_{ht})'' \mid_{E2} \\
\leq h^3_{E1} (\lambda \tilde{u}_{ht})'' \mid_{E1} + h^3_{E2} (\lambda h \tilde{u}_{ht})'' \mid_{E2} \\
\leq h^3_{E1} \lambda_{\tilde{u}_{ht}} u'_{ht} + h^3_{E2} \lambda' h \tilde{u}_{ht} \\
= h^2_{E1} \lambda_{\tilde{u}_{ht}} u'_{ht} + h^2_{E2} \lambda_{\tilde{u}_{ht}} |\tilde{u}_{ht}||E| \\
\leq h^2_{E1} (\lambda_{\tilde{u}_{ht}} + \sigma_n (\tilde{u}_{ht})) |\tilde{u}_{ht}||E| + h^2_{E2} (\lambda_{\tilde{u}_{ht}} + \sigma_t (\tilde{u}_{ht})) |\tilde{u}_{ht}||E| \\
\leq h^2_{E1} (\lambda_{\tilde{u}_{ht}} + \sigma_n (\tilde{u}_{ht})) |\tilde{u}_{ht}||E| + h E |\lambda_{\tilde{u}_{ht}}| |\tilde{u}_{ht}||E| \\
\leq h^{1/2}_{E1} (\mu \tilde{u}_{h3K} + \tilde{h}_{hK}) |\tilde{u}_{ht}||E| \\
\leq (\mu \tilde{u}_{h3K} + \tilde{h}_{hK}) |\tilde{u}_{ht}||E|.
\]

Hence (58) by taking the square root. The first case (\tilde{u}_{ht} is either nonnegative or nonpositive in \(E\)) is straightforward and handled as nonpositive.

\(\square\)

**Remark 5.6** Assume that \(u \in (H^2(\Omega))^2\) (so \(\lambda \in (H^1(\Gamma_C))^2\)), and that optimal a priori error estimates hold (as for the first finite element approximation, this question is entirely open and the only aim of the present remark is to try to illustrate our result). We define:

\[\tilde{\eta}_i = \left( \sum_{K \in \mathcal{T}_h} \tilde{\eta}^2_{K} \right)^{1/2}, \quad 1 \leq i \leq 6.\]

Then it is straightforward to check that \(\tilde{\eta}_i \lesssim h, 1 \leq i \leq 6; \eta_j \lesssim h^{1/2}, j = 5, 6.\) So \(\tilde{\eta} \lesssim h^{1/2}.\)

A deeper insight in the estimates of \(\tilde{\eta}_{hK}\) and \(\tilde{\eta}_{hK}\) (which we prefer to avoid) would show that the estimates in [47], Remark 5.7 could also be applied in our case and this would lead to the estimate: \(\tilde{\eta} \lesssim (\ln(h))^{1/4} h^{3/4}.\)

### 6 Numerical experiments

In this section we achieve the numerical implementation of the residual estimator for both finite element discretizations. The information given by the error estimators is then coupled with a mesh adaptivity procedure. In what follows, we suppose that the bodies are homogeneous isotropic materials so that Hooke’s law (2) becomes:

\[\sigma(u) = \frac{EP}{(1 - 2P)(1 + P)} tr(\varepsilon(u))I + \frac{E}{1 + P} \varepsilon(u)\]

where \(I\) represents the identity matrix, \(tr\) is the trace operator, \(E\) and \(P\) denote Young’s modulus and Poisson’s ratio, respectively with \(E > 0\) and \(0 \leq P < 1/2.\)

Our main aim is to discuss the theoretical results by computing the different contributions of the estimators \(\eta\) and \(\tilde{\eta}\) and their orders of convergence as \(h\) vanishes. In particular we are interested in the following terms (where we adopt the notations of Remarks 4.5 and 5.6):

\[\eta_i = \left( \sum_{K \in \mathcal{T}_h} \eta^2_{i\ell} \right)^{1/2}, 1 \leq i \leq 8, \quad \tilde{\eta}_i = \left( \sum_{K \in \mathcal{T}_h} \tilde{\eta}^2_{i\ell} \right)^{1/2}, 1 \leq i \leq 6.\]

We will also make use of the frictional contact contributions

\[\eta_C = (\sum_{i=3}^{8} \eta^2_i)^{1/2}, \quad \tilde{\eta}_C = (\sum_{i=3}^{6} \tilde{\eta}^2_i)^{1/2}.\]
In the following we denote by $N_C$, the number of elements of the mesh on $\Gamma_C$. In the case of uniform meshes this parameter measures the size of the mesh. Moreover we suppose that the friction coefficient $\mu$ and the meshsize $h$ are such that both discrete problems (9) and (37) admit unique solutions $(u_h, \lambda_h)$ and $(\tilde{u}_h, \tilde{\lambda}_h)$. In such a case it is easy to check that $u_h = \tilde{u}_h$ and that $c(\lambda_h, v_h) = b(\lambda_h, v_h), \forall v_h \in V_h$ which implies that $\eta_2 = \tilde{\eta}_2$.

### 6.1 A first example with slip and separation

We consider the domain $\Omega = [0, 1] \times [0, 1]$ with material characteristics $E = 10^6$ and $P = 0.3$. The body is clamped on $\Gamma_D = \{0 \times [0, 1]\}$, it is initially in contact with $\Gamma_C = \{1 \times [0, 1]\}$ and no force is applied on $\Gamma_N = \{0 \times \{(0) \cup \{1\}\}$. The body $\Omega$ is acted on by a uniform vertical force $f = (0, f_2)$ with $f_2 = -76518$ and the friction coefficient $\mu$ equals 0.2. We use criss-cross meshes (this means that the body is divided into identical squares, each of them being divided into four identical triangles). Figure 1 depicts the initial and deformed configurations with $N_C = 32$. We first observe that all the nodes on $\Gamma_C$ have a negative tangential displacement.

![Figure 1: First example. Initial and deformed configurations with $\mu = 0.2$ and $N_C = 32$.](image)

and that $\Gamma_C$ is divided into two parts: an upper part where the body remains in contact with the axis $x = 1$ (slipping nodes) and the lower part of $\Gamma_C$ where it separates from this axis with a separation point near $(1, 0.65)$, (see Figure 2). In Table 1 we report the convergence rates by averaging the rates between $N_C = 2$ and $N_C = 64$. Note that the convergence rate of the terms: $\eta_1 = \tilde{\eta}_1 = h(\sum_{K \in T_h} \|f_K\|_K^2)^{1/2} \sim h$ is 1.

<table>
<thead>
<tr>
<th>Errors</th>
<th>$N_C = 1$</th>
<th>$N_C = 2$</th>
<th>$N_C = 4$</th>
<th>$N_C = 8$</th>
<th>$N_C = 16$</th>
<th>$N_C = 32$</th>
<th>$N_C = 64$</th>
<th>Convergence rates</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta_2 = \tilde{\eta}_2$</td>
<td>71943</td>
<td>89950</td>
<td>72476</td>
<td>48412</td>
<td>29533</td>
<td>17687</td>
<td>12504</td>
<td>0.57</td>
</tr>
<tr>
<td>$\eta_3$</td>
<td>32980</td>
<td>21134</td>
<td>6826.6</td>
<td>2366.7</td>
<td>960.54</td>
<td>565.63</td>
<td>322.29</td>
<td>1.21</td>
</tr>
<tr>
<td>$\tilde{\eta}_3$</td>
<td>11092</td>
<td>8681.4</td>
<td>4165.7</td>
<td>1868.4</td>
<td>778.11</td>
<td>391.41</td>
<td>223.87</td>
<td>1.06</td>
</tr>
<tr>
<td>$\eta_4$</td>
<td>30028</td>
<td>15319</td>
<td>6299.3</td>
<td>2594.8</td>
<td>1012.5</td>
<td>457.45</td>
<td>244.01</td>
<td>1.19</td>
</tr>
<tr>
<td>$\tilde{\eta}_4$</td>
<td>29379</td>
<td>14325</td>
<td>6079.3</td>
<td>2542.3</td>
<td>997.58</td>
<td>448.78</td>
<td>239.20</td>
<td>1.18</td>
</tr>
<tr>
<td>$\eta_5$</td>
<td>13.674</td>
<td>8.1415</td>
<td>3.0994</td>
<td>1.5381</td>
<td>0.50073</td>
<td>0.21377</td>
<td>0.063429</td>
<td>1.56</td>
</tr>
<tr>
<td>$\tilde{\eta}_5$</td>
<td>14.503</td>
<td>3.9747</td>
<td>3.1219</td>
<td>0.77988</td>
<td>0.42660</td>
<td>0.13956</td>
<td>0.039897</td>
<td>1.33</td>
</tr>
<tr>
<td>$\eta_6$</td>
<td>12680</td>
<td>11242</td>
<td>1599.8</td>
<td>1945.9</td>
<td>416.70</td>
<td>385.61</td>
<td>79.730</td>
<td>1.43</td>
</tr>
<tr>
<td>$\tilde{\eta}_6$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>$\eta_7$</td>
<td>12.121</td>
<td>13.619</td>
<td>4.8373</td>
<td>4.6372</td>
<td>1.8290</td>
<td>1.4717</td>
<td>0.54418</td>
<td>0.93</td>
</tr>
<tr>
<td>$\tilde{\eta}_7$</td>
<td>2535.9</td>
<td>2248.4</td>
<td>319.95</td>
<td>389.18</td>
<td>83.339</td>
<td>77.122</td>
<td>15.946</td>
<td>1.43</td>
</tr>
</tbody>
</table>

Table 1: Contributions in $\eta$ and $\tilde{\eta}$ for the first example.
From the computations we see that all the terms $\eta_i$ and $\tilde{\eta}_i$ converge towards zero as $h$ vanishes and that $\eta_2 = \tilde{\eta}_2$ is obviously the term converging the slowest towards zero. The main part of the error in $\eta$ and $\tilde{\eta}$ is located near the singular points $(0,0)$ and $(0,1)$. The error terms for which no optimal error analysis is available (i.e., $\eta_5, \eta_6, \eta_7, \eta_8, \tilde{\eta}_5, \tilde{\eta}_6$) vanish faster than all the others except $\eta_7$ which has a slower convergence rate. Note that $\tilde{\eta}_6 = 0$ since $u_{ht} < 0$ and $\mu\tilde{\lambda}_{hn} = -\tilde{\lambda}_{ht}$ on $\Gamma_C$. We note also that the error $\tilde{\eta}_5$ is located on one element near the separation point whereas $\eta_5, \eta_6, \eta_7, \eta_8$ are located on $\Gamma_C$, especially in the separation area.

Next we couple the error estimator with a mesh adaptivity procedure. The aim of adaptive procedures is to offer the user a level of accuracy denoted $\eta_0$ with a minimal computational cost. We use the $h$-version in which the size and the topology of the elements are modified but the same kind of basis functions for the different meshes are retained. A mesh $T^*$ is said to be optimal with respect to a measure of the error $\eta^*$ if (see [61]):

$$\begin{cases}
\eta^* = \eta_0 \\
N \text{ minimal (} N \text{ number of unknowns (or degrees of freedom) when using } T^*)
\end{cases}$$

To solve problem (59), the following procedure is applied:

1. an initial analysis is performed on a relatively uniform and coarse mesh $T$,
2. the corresponding global error $\eta$ (resp. $\tilde{\eta}$) and the local contributions $\eta_K$ (resp. $\tilde{\eta}_K$) are computed,
3. the characteristics of the optimal mesh $T^*$ are determined in order to minimize the computational costs in respect of the global error,
4. a second finite element analysis is performed on the mesh $T^*$.

The optimal mesh $T^*$ is determined by the computation of a size modification coefficient $r_K$ on each element $K$ of the mesh $T$: $r_K = h^*_{K}/h_K$, where $h^*_{K}$ represents the size that must be imposed to the elements of $T^*$ in the region of $K$ in order to ensure optimality. The computation of the coefficients $r_K$ uses the rate of convergence of the error which depends on the used element but also on the regularity of the solution [28]. So, to compute the coefficients $r_K$, we use a technique detailed in [29] that automatically takes into account the steep gradient regions. The mesh $T^*$ is generated by an automatic mesher able to respect accurately a map of sizes. If the user wishes
more accuracy, then the procedure is repeated as far as a precision close to \( \eta_0 \) is reached (see [28]).

Applying this procedure to the example, we obtain a family of adapted meshes which are refined near the singularities \((0, 0)\) and \((0, 1)\) (see Figure 3). We also observe that the difference between the values of \( \eta \) and \( \tilde{\eta} \) is not significant when refining and we note that the contact contributions \( \eta_C \) (resp. \( \tilde{\eta}_C \)) are dominated by \( \eta_3, \eta_4 \) (resp. \( \tilde{\eta}_3, \tilde{\eta}_4 \)), the other terms being small (this observation also holds for examples 2 and 3 considered hereafter). Denoting by \( N \) the number of unknowns, we observe that the estimators \( \eta \) and \( \tilde{\eta} \), computed on adaptively generated meshes, behave like \( N^{-0.5} \) and that the contact contributions behave approximately like \( N^{-0.8} \). Figure 3 depicts \( \tilde{\eta} \) and \( \tilde{\eta}_C \) as functions of \( N \).

Figure 3: First example. Left: Adapted mesh. Right: Convergence of the error estimator \( \tilde{\eta} \) and its frictional contact contribution \( \tilde{\eta}_C \) with adaptive refinement.

### 6.2 A second example with stick, slip and separation

Next we study an example where none of the terms \( \eta_i \) and \( \tilde{\eta}_i \) vanish \((i \geq 2)\), where the three different zones characterizing friction (stick, slip, separation) exist and with softer corner singularities than in the previous example. We consider the geometry \( \hat{\Omega} = [0, 2] \times [0, 1] \) and we adopt symmetry conditions (i.e., \( u_n = 0, \sigma_t(u) = 0 \)) on \( \Gamma_S = \{1\} \times [0, 1] \). We achieve the computations on the square \( \Omega = [0, 1] \times [0, 1] \). We set \( \Gamma_C = [0, 1] \times \{0\} \) and \( \Gamma_N = ([0, 1] \times \{1\}) \cup (\{0\} \times [0, 1]) \). A Poisson ratio of \( P = 0.2 \), a Young modulus of \( E = 10^4 \) and a friction coefficient \( \mu = 0.5 \) are chosen. A density of surface forces \( F \) of magnitude 1 oriented inwards \( \Omega \) is applied on \( \{0\} \times [0.5, 1] \) and \( [0.5, 1] \times \{1\} \). Such a configuration corresponds to a \( K \)-elliptic case (see [41], Theorem 6.3).

Figure 4 depicts the initial and deformed configurations of the body. Here again \( \Gamma_C \) shows a separation and a contact part with a transition point near \((0.26, 0)\). In addition the contact part is divided into a slip part (on its left) and a stick part (on its right) with a transition point from slip to stick near \((0.47, 0)\), (see Figures 4 and 5).

It is easy to check that the symmetry conditions on \( \Gamma_C \) lead to supplementary error terms similar to the ones in \( \eta_1 \) and \( \tilde{\eta}_1 \) and we add these terms to \( \eta_2 = \tilde{\eta}_2 \). Moreover we have \( \eta_1 = \tilde{\eta}_1 = 0 \). The results concerning \( \eta \) and \( \tilde{\eta} \) are reported in Table 2 where the convergence rates are averaged between \( N_C = 2 \) and \( N_C = 128 \).
Figure 4: Second example. Initial and deformed configurations with $\mu = 0.5$ and $N_C = 32$ (deformation is amplified by a factor 2000).

Figure 5: Second example. Left: normal and tangential displacements ($\tilde{u}_{hn}, \tilde{u}_{ht}$) on $\Gamma_C$. Right: normal and tangential multipliers ($\tilde{\lambda}_{hn}, -\tilde{\lambda}_{ht}$) on $\Gamma_C$.

<table>
<thead>
<tr>
<th>Errors $\times 10^5$</th>
<th>$N_C = 2$</th>
<th>$N_C = 4$</th>
<th>$N_C = 8$</th>
<th>$N_C = 16$</th>
<th>$N_C = 32$</th>
<th>$N_C = 64$</th>
<th>$N_C = 128$</th>
<th>Convergence rates</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta_2 = \eta_2$</td>
<td>87774</td>
<td>53444</td>
<td>32022</td>
<td>18577</td>
<td>10449</td>
<td>574085</td>
<td>310985</td>
<td>0.80</td>
</tr>
<tr>
<td>$\eta_3$</td>
<td>16925</td>
<td>5164.72</td>
<td>2111.32</td>
<td>857.613</td>
<td>359.365</td>
<td>113.333</td>
<td>43.3032</td>
<td>1.44</td>
</tr>
<tr>
<td>$\eta_4$</td>
<td>10176</td>
<td>4448.27</td>
<td>1665.89</td>
<td>642.493</td>
<td>256.814</td>
<td>93.1164</td>
<td>35.8664</td>
<td>1.36</td>
</tr>
<tr>
<td>$\eta_5$</td>
<td>17166</td>
<td>7553.27</td>
<td>3860.54</td>
<td>1818.65</td>
<td>848.092</td>
<td>388.834</td>
<td>184.881</td>
<td>1.09</td>
</tr>
<tr>
<td>$\eta_6$</td>
<td>9237.09</td>
<td>5292.69</td>
<td>2825.89</td>
<td>1376.62</td>
<td>631.021</td>
<td>278.175</td>
<td>127.115</td>
<td>1.03</td>
</tr>
<tr>
<td>$\eta_7$</td>
<td>39.1890</td>
<td>6.22418</td>
<td>3.65335</td>
<td>3.48613</td>
<td>2.18880</td>
<td>0.87304</td>
<td>0.113440</td>
<td>1.41</td>
</tr>
<tr>
<td>$\eta_8$</td>
<td>52.3904</td>
<td>24.4419</td>
<td>9.21759</td>
<td>2.94782</td>
<td>0.220389</td>
<td>0.544605</td>
<td>0.197534</td>
<td>1.34</td>
</tr>
<tr>
<td>$\eta_9$</td>
<td>8624.25</td>
<td>1500.79</td>
<td>228.240</td>
<td>505.892</td>
<td>647.810</td>
<td>526.079</td>
<td>807.287</td>
<td>1.68</td>
</tr>
<tr>
<td>$\eta_0$</td>
<td>34.2719</td>
<td>16.9743</td>
<td>6.48435</td>
<td>1.95881</td>
<td>0.607762</td>
<td>0.212090</td>
<td>0.076817</td>
<td>1.47</td>
</tr>
<tr>
<td>$\eta_7$</td>
<td>33.4342</td>
<td>19.1663</td>
<td>9.98284</td>
<td>7.51870</td>
<td>6.30249</td>
<td>3.09423</td>
<td>0.494072</td>
<td>1.01</td>
</tr>
<tr>
<td>$\eta_8$</td>
<td>4157.91</td>
<td>780.210</td>
<td>509.431</td>
<td>501.907</td>
<td>323.932</td>
<td>113.182</td>
<td>9.04330</td>
<td>1.47</td>
</tr>
</tbody>
</table>

Table 2: Contributions in $\eta$ and $\tilde{\eta}$ for the second example.

We observe that the errors $\eta$ and $\tilde{\eta}$ are mainly located near the singularities (0, 0.5) and (0.5, 1) and also near the transition point between contact and separation. The error near the transition point between stick and slip is much smaller. As in the previous example, $\eta_2 = \tilde{\eta}_2$
is the main term in the estimator with the lowest (but greater than in the previous example) convergence rate and the error terms for which no optimal convergence result is available (i.e., \(\eta_5, \eta_6, \eta_7, \eta_8, \tilde{\eta}_5, \tilde{\eta}_6\)) vanish with an higher rate than theoretically expected. The particularity in this example is that many terms (in particular \(\eta_6\)) converge towards 0 with a nonuniform convergence rate.

We then apply the adaptive procedure described before and we depict the initial mesh and two refined meshes in Figure 6. As previously the error decay using refinement behaves like \(N^{-0.5}\) and it is a bit faster than the error decay using refined meshes (near \(N^{-0.45}\), see Figure 7). Figure 7 also shows the convergence of the contact contribution \(\tilde{\eta}_C\) and we observe that \(\tilde{\eta}_C/\tilde{\eta} \sim N^{-0.2}\) which therefore vanishes when \(N \to \infty\). The results are similar when considering \(\eta\) instead of \(\tilde{\eta}\).

![Figure 6: Second example. Initial (left) and refined meshes in adaptive procedure](image)

![Figure 7: Second example. Left: Convergence of the error estimator \(\tilde{\eta}\) with uniform and adaptive refinement. Right: Convergence of the error estimator \(\tilde{\eta}\) and its frictional contact contribution \(\tilde{\eta}_C\) with adaptive refinement.](image)

### 6.3 Third example: a case with small friction, comparison with an example in the literature

Finally we consider an example from the literature (see [81], “square on a plane”) which is somewhat more regular than the previous ones. Namely we consider the geometry \(\hat{\Omega} = [0,1] \times [0,1]\) with symmetry conditions on \(\Gamma_S = \{0.5\} \times [0,1]\) and we compute the solutions on \(\Omega = [0,0.5] \times [0,1]\). We set \(\Gamma_C = [0,0.5] \times \{0\}\), \(\Gamma_N = ([0,0.5] \times \{1\}) \cup ([0] \times [0,1])\), \(P = 0.3\) and \(E = 10^4\). A density of inward oriented surface forces \(F(x,y) = -x^2(1-x)^2\) (resp.
\(F(x, y) = 2y^2(1 - y)^2\) is applied on \([0, 0.5] \times \{1\}\) (resp. \(\{0\} \times [0, 1]\)). We choose a small friction coefficient \(\mu = 0.1\) keeping in mind that the numerical example in [81] is frictionless. Figure 8 depicts the initial and deformed configurations of the body (with \(N_C = 64\)). The boundary part \(\Gamma_C\) shows a transition point between contact and separation near \((0.08, 0)\). Due to the (small) friction we observe that (only) the last contact element near \((0.5, 0)\) is stuck on the foundation. Figure 9 shows the surface displacements and tractions on \(\Gamma_C\). The adaptive procedure is summarized in Figures 10 and 11. The initial mesh and two refined meshes are shown in Figure 10: the refined meshes are more uniform than in the previous examples and contain more small elements near the boundary (except where symmetry holds). Note that the error decay is optimal (like \(N^{-0.5}\)) when uniform meshes are used and that the frictional contact contribution in the error estimator behaves approximately like \(N^{-0.85}\), see Figure 11. These results obtained for a small friction coefficient show many similarities with the ones obtained in [81] without friction.
Figure 10: Third example. Initial (left) and refined meshes in adaptive procedure.

Figure 11: Third example. Left: Convergence of the error estimator $\tilde{\eta}$ with uniform and adaptive refinement. Right: Convergence of the error estimator $\tilde{\eta}$ and its frictional contact contribution $\tilde{\eta}_C$ with adaptive refinement.

7 Conclusion and perspectives

In this paper we propose, analyze and implement two residual error estimators $\eta$ and $\tilde{\eta}$ corresponding to two finite element discretizations of the static Coulomb friction problem by using the partial uniqueness result obtained in [72]. To our knowledge our study yields the first results (for the Coulomb friction problem) involving residual estimators with both upper and lower bounds of the discretization error. From the definitions and the theoretical estimates we observe that $\tilde{\eta}$ is simpler to define and it yields better bounds. From the numerical experiments, we observe that all the terms in $\eta$ and $\tilde{\eta}$ for which no optimal theoretical results can be provided behave better than theoretically expected and that both approaches are worth to be considered.

Another line of research could consist in obtaining a uniqueness result for the quasi-static problem by adapting the techniques in [72] and then to perform an a posteriori analysis (note that the existence results obtained in [5, 73] for the quasi-static problem are of the same type than the ones for the static problem).

Another (difficult) study consists to extend the estimators obtained in this paper to the so-called XFEM method for crack problems (see [66]) where frictional contact occurs on the crack lips and where the mesh of the body does not coincide with the crack. This study is actually under investigation in [60].
Appendix

Proposition 7.1 For any positive $\mu$, Problem (37) admits at least a solution.

Proof: Let $\mu > 0$ be given. We introduce the problem of friction $P(g_{hn})$ with a given threshold $\mu g_{hn}$ and $g_{hn} \in K_{hn}$. It consists of finding $u_h \in V_h$ and $(\lambda_{hn}, \lambda_{ht}) = \lambda_h \in K_h(\mu g_{hn}) = K_{hn} \times K_{ht}(\mu g_{hn})$ satisfying:

\[
P(g_{hn}) \begin{cases} a(u_h, v_h) + c(\lambda_h, v_h) = L(v_h), & \forall v_h \in V_h, \\ c(\nu_h - \lambda_h, u_h) \leq 0, & \forall \nu_h = (v_{hn}, v_{ht}) \in K_h(\mu g_{hn}). \end{cases}
\]

Problem (60) is equivalent of finding a saddle-point $(u_h, \lambda_{hn}, \lambda_{ht}) = (u_h, \lambda_h) \in V_h \times K_h(\mu g_{hn})$ verifying

\[
\mathcal{L}(u_h, \nu_h) \leq \mathcal{L}(u_h, \lambda_h) \leq \mathcal{L}(v_h, \lambda_h), \quad \forall v_h \in V_h, \forall \nu_h \in K_h(\mu g_{hn}),
\]

where

\[
\mathcal{L}(v_h, \nu_h) = \frac{1}{2}a(v_h, v_h) + \int_{\Gamma_C} I_h(v_{hn}v_{hn}) \ d\Gamma + \int_{\Gamma_C} I_h(v_{ht}v_{ht}) \ d\Gamma - L(v_h).
\]

By using standard arguments on saddle-point problems as in [41] (Theorem 3.9, p.339), we deduce that there exists such a saddle-point. The strict convexity of $a(\cdot, \cdot)$ implies that the first argument $u_h$ is unique. Suppose that the second argument is not unique: then the equality in (60) implies

\[
c(\lambda_{hn}^1 - \lambda_{hn}^2, v_h) = 0, \quad \forall v_h \in V_h.
\]

The definition of $W_h$ allows us to choose $v_h = \lambda_{hn}^1 - \lambda_{hn}^2$ on $\Gamma_C$. From the definition of $c(\cdot, \cdot)$ we come to the conclusion that $\lambda_{hn}^1 - \lambda_{hn}^2 = 0$. Consequently, the second argument $\lambda_h$ is unique and (60) admits a unique solution. The next lemma is a straightforward consequence of the definition of problems (37) and (60).

Lemma 7.2 The solutions of Coulomb’s discrete frictional contact problem (37) are the solutions of $P(\lambda_{hn})$ where $\lambda_{hn}$ is a fixed point of $\Phi_h$. The functional $\Phi_h$ is defined as follows:

\[
\Phi_h : \begin{array}{rcl} K_{hn} & \longrightarrow & K_{hn} \\ g_{hn} & \longrightarrow & \lambda_{hn}, \end{array}
\]

where $(u_h, \lambda_h)$ is the solution of $P(g_{hn})$.

To establish existence of a fixed point of $\Phi_h$, we use Brouwer’s fixed point theorem. First we prove that the mapping $\Phi_h$ is continuous. Set $\tilde{V}_h = \{ v_h \in V_h : v_{ht} = 0 \ \text{on} \ \Gamma_C \}$. From the definition of $W_h$, it is easy to check that the definition of $\| \cdot \|_{-\frac{1}{2}, h}$ given by

\[
\| \nu \|_{-\frac{1}{2}, h} = \sup_{v_h \in \tilde{V}_h} \frac{\int_{\Gamma_C} I_h(\nu v_{hn}) \ d\Gamma}{\| v_h \|_{1, \Omega}},
\]

is a norm on $W_h$. Let $(u_h, \lambda_{hn}, \lambda_{ht})$ and $(\bar{u}_h, \bar{\lambda}_{hn}, \bar{\lambda}_{ht})$ be the solutions of $P(g_{hn})$ and $P(\bar{g}_{hn})$ respectively. On the one hand, we get

\[
a(u_h, v_h) + \int_{\Gamma_C} I_h(\lambda_{hn}v_{hn}) \ d\Gamma = L(v_h), \quad \forall v_h \in \tilde{V}_h,
\]

\[
a(\bar{u}_h, v_h) + \int_{\Gamma_C} I_h(\bar{\lambda}_{hn}v_{hn}) \ d\Gamma = L(v_h), \quad \forall v_h \in \tilde{V}_h.
\]

27
Subtracting the previous equalities and using the continuity of the bilinear form $a(.,.)$ gives
\[
\int_{\Gamma} I_h((\lambda_{hn} - \overline{\lambda}_{hn})v_{hn}) \, d\Gamma = a(\overline{u}_h - u_h, v_h) \lesssim \|u_h - \overline{u}_h\|_{1,\Omega}\|v_h\|_{1,\Omega} \quad \forall v_h \in \tilde{V}_h.
\]

Hence, we get a first estimate
\[
(61) \quad \|\lambda_{hn} - \overline{\lambda}_{hn}\|_{\frac{1}{2},\Gamma} \lesssim \|u_h - \overline{u}_h\|_{1,\Omega}.
\]

On the other hand, we have from (37)
\[
(62) \quad a(u_h, v_h) + \int_{\Gamma} I_h(\lambda_{hn}v_{hn}) \, d\Gamma + \int_{\Gamma} I_h(\lambda_{ht}v_{ht}) \, d\Gamma = L(v_h), \quad \forall v_h \in V_h,
\]
\[
(63) \quad a(\overline{u}_h, v_h) + \int_{\Gamma} I_h(\overline{\lambda}_{hn}v_{hn}) \, d\Gamma + \int_{\Gamma} I_h(\overline{\lambda}_{ht}v_{ht}) \, d\Gamma = L(v_h), \quad \forall v_h \in V_h.
\]

Choosing $v_h = u_h - \overline{u}_h$ in (62) and $v_h = \overline{u}_h - u_h$ in (63) implies by addition:
\[
(64) \quad a(u_h - \overline{u}_h, u_h - \overline{u}_h) = \int_{\Gamma} I_h((\lambda_{hn} - \overline{\lambda}_{hn})(u_{hn} - \overline{u}_{hn})) \, d\Gamma + \int_{\Gamma} I_h((\lambda_{ht} - \overline{\lambda}_{ht})(u_{ht} - \overline{u}_{ht})) \, d\Gamma.
\]

Let us notice that the inequality in (60) is obviously equivalent to the two following conditions:
\[
(65) \quad \int_{\Gamma} I_h((\nu_{hn} - \lambda_{hn})u_{hn}) \, d\Gamma \leq 0, \quad \forall \nu_{hn} \in K_{hn},
\]
\[
(66) \quad \int_{\Gamma} I_h((\nu_{ht} - \lambda_{ht})u_{ht}) \, d\Gamma \leq 0, \quad \forall \nu_{ht} \in K_{ht}(\mu_{ghn}).
\]

According to the definition of $K_{hn}$, we can choose $\nu_{hn} = 0$ and $\nu_{hn} = 2\lambda_{hn}$ in (65) which gives
\[
\int_{\Gamma} I_h(\lambda_{hn}u_{hn}) \, d\Gamma = 0 \quad \text{and} \quad \int_{\Gamma} I_h(\nu_{hn}u_{hn}) \, d\Gamma \leq 0, \quad \forall \nu_{hn} \in K_{hn},
\]
from which we deduce that
\[
\int_{\Gamma} I_h((\lambda_{hn} - \overline{\lambda}_{hn})(u_{hn} - \overline{u}_{hn})) \, d\Gamma \leq 0.
\]

Hence (64) becomes
\[
(67) \quad \|u_h - \overline{u}_h\|^2_{1,\Omega} \lesssim \int_{\Gamma} I_h((\lambda_{ht} - \overline{\lambda}_{ht})(u_{ht} - \overline{u}_{ht})) \, d\Gamma.
\]

From the definition of $K_{ht}(\mu_{ghn})$, we get
\[
\int_{\Gamma} I_h(\lambda_{ht}\overline{u}_{ht}) \, d\Gamma \leq \int_{\Gamma} I_h(|\lambda_{ht}|\,|\overline{u}_{ht}|) \, d\Gamma \leq \int_{\Gamma} I_h(\mu_{ghn}|\overline{u}_{ht}|) \, d\Gamma.
\]

A similar expression can be obtained when integrating the term $I_h(\overline{\lambda}_{ht}u_{ht})$. Besides from (66),
\[
-\int_{\Gamma} I_h(\lambda_{ht}u_{ht}) \, d\Gamma \leq -\int_{\Gamma} I_h(\nu_{ht}u_{ht}) \, d\Gamma
\]
\[
= -\sum_{i=1}^{p} \nu_{ht}(\mathbf{x}_i)u_{ht}(\mathbf{x}_i)\int_{\Gamma} \psi_{\mathbf{x}_i} \, d\Gamma, \quad \forall \nu_{ht} \text{ such that } |\nu_{ht}| \leq \mu_{ghn}.
\]

28
If \( u_{ht}(x_i) \geq 0 \), we choose \( \nu_{ht}(x_i) = \mu g_{hn}(x_i) \) and if \( u_{ht}(x_i) \leq 0 \), we choose \( \nu_{ht}(x_i) = -\mu g_{hn}(x_i) \). This yields the following bound:

\[
- \int_{\Gamma_c} I_h(\lambda_{ht} u_{ht}) \, d\Gamma \leq -\mu \sum_{i=1}^{p} g_{hn}(x_i) |u_{ht}(x_i)| \int_{\Gamma_c} \psi_{x_i} \, d\Gamma = - \int_{\Gamma_c} I_h(\mu g_{hn} |u_{ht}|) \, d\Gamma.
\]

A similar expression can be obtained when integrating the term \( I_h(\lambda_{ht} \bar{u}_{ht}) \). Finally, (67) becomes

\[
\|u_h - \bar{u}_{ht}\|_{1,\Omega}^2 \lesssim \mu \int_{\Gamma_c} I_h((g_{hn} - \bar{g}_{hn})(|\bar{u}_{ht}| - |u_{ht}|)) \, d\Gamma \leq \mu \int_{\Gamma_c} I_h(|g_{hn} - \bar{g}_{hn}|) \, d\Gamma \lesssim \mu C(h) \|g_{hn} - \bar{g}_{hn}\|_{1,\Omega}^2
\]

where the equivalence of norms in finite dimensional spaces have been used as well as the trace theorem. Combining (68) and (61) implies that there exists a constant \( C(h) \) such that

\[
\|\lambda_{hn} - \bar{\lambda}_{hn}\|_{1,\Omega} \lesssim \mu C(h) \|g_{hn} - \bar{g}_{hn}\|_{1,\Omega}^2.
\]

Hence \( \Phi_h \) is continuous.

Let \( (u_h, \lambda_{hn}, \lambda_{ht}) \) be the solution of \( P(g_{hn}) \). Taking \( v_h = u_h \) in (60) gives

\[
a(u_h, u_h) + \int_{\Gamma_C} I_h(\lambda_{hn} u_{hn}) \, d\Gamma + \int_{\Gamma_C} I_h(\lambda_{ht} u_{ht}) \, d\Gamma = L(u_h).
\]

According to

\[
\int_{\Gamma_C} I_h(\lambda_{hn} u_{hn}) \, d\Gamma = 0 \quad \text{and} \quad \int_{\Gamma_C} I_h(\lambda_{ht} u_{ht}) \, d\Gamma \geq 0,
\]

we deduce from (70), the \( V \)-ellipticity of \( a(.,.) \) and the continuity of \( L(.) \):

\[
\|u_h\|_{1,\Omega}^2 \lesssim a(u_h, u_h) \leq L(u_h) \lesssim \|u_h\|_{1,\Omega}.
\]

So, we deduce that \( \|u_h\|_{1,\Omega} \) is bounded. In other respects

\[
a(u_h, v_h) + \int_{\Gamma_C} I_h(\lambda_{hn} v_{hn}) \, d\Gamma = L(v_h), \quad \forall v_h \in \bar{V}_h,
\]

leads to

\[
\int_{\Gamma_C} I_h(\lambda_{hn} v_{hn}) \, d\Gamma \lesssim \|u_h\|_{1,\Omega} \|v_h\|_{1,\Omega} + \|v_h\|_{1,\Omega}, \quad \forall v_h \in \bar{V}_h.
\]

Therefore \( \|\Phi_h(g_{hn})\|_{1,\Omega} \lesssim \|\lambda_{hn}\|_{1,\Omega} \leq 1 \), for all \( g_{hn} \in M_{hn} \). This boundedness of \( \Phi_h \) together with the continuity of \( \Phi_h \) proves that there exists at least a solution of Coulomb’s discrete frictional contact problem according to Brouwer’s fixed point theorem.

**Remark 7.3** From (69), we obtain a mesh size dependent uniqueness result when \( \mu \, C(h) < 1 \). This means that uniqueness holds when \( \mu \) is small enough where the denomination “small” depends on the discretization parameter. A more detailed study would show that this uniqueness criterion disappears when \( h \) vanishes (i.e., \( \lim_{h \to 0} C(h) = +\infty \)).

This work is supported by “l’Agence Nationale de la Recherche”, project ANR-05-JCJC-0182-01.
References


