

An error estimate for the Signorini problem with Coulomb friction approximated by finite elements

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Abstract

The present paper is concerned with the unilateral contact model and the Coulomb friction law in linear elastostatics. We consider a mixed formulation in which the unknowns are the displacement field and the normal and tangential constraints on the contact area. The chosen finite element method involves continuous elements of degree one and continuous piecewise affine multipliers on the contact zone. A convenient discrete contact and friction condition is introduced in order to perform a convergence study. We finally obtain a first *a priori* error estimate under the assumptions ensuring the uniqueness of the solution to the continuous problem.

Keywords. unilateral contact, Coulomb friction, uniqueness of solution, finite elements, *a priori* error estimate.

Abbreviated title. Error estimate for discretized Coulomb friction.

AMS subject classifications. 35J85, 65N30, 74M10, 74M15.

Introduction

This study deals with the unilateral contact problem governed by the Coulomb friction law in linear elasticity. We consider a simplified model, the so-called static friction problem which roughly corresponds to an incremental problem in the time discretized quasi-static model and whose solutions are also some particular equilibrium configurations of the dynamic problem.

From a mathematical point of view the early work made on the static problem was accomplished in [15, 17]. These studies concerned the weak formulation of the problem. The first existence results were obtained in [39] for an infinite elastic strip. Thereafter, many existence results followed for general domains, in particular in [18] (see also the references quoted therein). These existence results hold for small friction coefficients and the uniqueness is not discussed. In fact uniqueness does not hold in the general case, at least for large friction coefficients, see [26, 27]. More recently a first uniqueness result has been obtained in [41] with the assumption that a "regular" solution exists and that the friction coefficient is sufficiently small. Besides, the so-called nonlocal Coulomb frictional models mollifying the normal stresses were introduced in [16] and developed in [14, 11, 31]. The smoothing map used in the nonlocal friction model allows to obtain existence results for any friction coefficient. Moreover, uniqueness of a solution can also be established if the friction coefficient is small enough (see [16, 14, 11, 31]). The same type of result (existence for any friction coefficient and uniqueness for small friction coefficients) was obtained in [32, 33] for the normal compliance model, introduced in [40, 37].

From a numerical point of view, the finite element method is commonly used when approximating such frictional contact problems (see, e.g., [31, 24, 21, 35, 43]). It is well known that the finite element problem, associated with the continuous static Coulomb friction model, always

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admits a solution and that the solution is unique if the friction coefficient is small enough (unfortunately the denominator small depends on the discretization parameter and the bound ensuring uniqueness vanishes as the mesh is refined, see e.g., [24]). The former result holds for any reasonable choice of the approximated contact and friction conditions (see [30]). Moreover, a first convergence study of the finite element problem towards the continuous model was accomplished in [22] where convergence is obtained under the assumptions ensuring the existence of a solution in [39] (i.e., small friction coefficient). This result proves the existence of a subsequence of discrete solutions converging towards a solution to the continuous problem. A similar result is obtained in [42] for the quasistatic model.

Our purpose is to carry out a convergence analysis and to obtain an *a priori* error estimate for a finite element discretization of the frictional contact conditions under the assumptions ensuring the uniqueness of a solution to the continuous problem obtained in [41]. As far as we know, this work presents the first error estimate with a convergence rate for this model.

Our paper is outlined as follows. Section 1 is concerned with the setting of the continuous problem, several equivalent weak formulations and a presentation of the tools and techniques leading to the uniqueness result. In section 2 we consider a discretization of the problem with finite elements of degree one and continuous piecewise affine multipliers on the contact zone. We introduce a convenient discrete contact and friction condition which allows us to perform a convergence analysis and to obtain an *a priori* estimate of the discretization error with a quasi-optimal convergence rate of order $h^{1/2}$ in the energy norm under $H^{(3/2)+\varepsilon}$ -regularity assumptions on the displacements.

1 The Signorini problem with Coulomb friction

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a polygonal domain representing the reference configuration of a linearly elastic body whose boundary $\partial\Omega$ consists of three nonoverlapping open parts Γ_N , Γ_D and Γ_C with $\overline{\Gamma_N} \cup \overline{\Gamma_D} \cup \overline{\Gamma_C} = \partial\Omega$. We assume that the measures of Γ_C and Γ_D are positive and, in order to simplify, that Γ_C is a straight line segment when $d = 2$ or a plane surface when $d = 3$. The body is submitted to a Neumann condition on Γ_N with a density of loads $F \in (L^2(\Gamma_N))^d$, a Dirichlet condition on Γ_D (the body is assumed to be clamped on Γ_D to simplify) and to volume loads denoted $f \in (L^2(\Omega))^d$ in Ω . Finally, a unilateral contact condition with static Coulomb friction between the body and a flat rigid foundation holds on Γ_C (see Fig. 1).

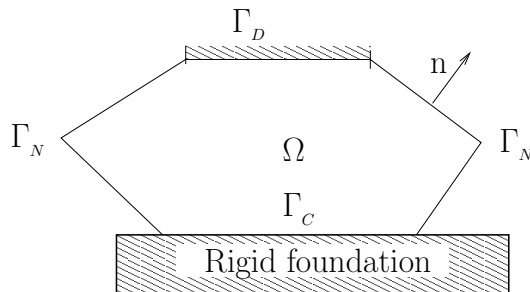


Figure 1: *Elastic body Ω in frictional contact.*

The problem consists in finding the displacement field $u : \overline{\Omega} \rightarrow \mathbb{R}^d$ satisfying:

$$- \operatorname{div} \sigma(u) = f \quad \text{in } \Omega, \quad (1)$$

$$\sigma(u) = \mathcal{A}\varepsilon(u) \quad \text{in } \Omega, \quad (2)$$

$$\sigma(u)\mathbf{n} = F \quad \text{on } \Gamma_N, \quad (3)$$

$$u = 0 \quad \text{on } \Gamma_D, \quad (4)$$

where $\sigma(u)$ represents the stress tensor field, $\varepsilon(u) = (\nabla u + (\nabla u)^T)/2$ denotes the linearized strain tensor field, \mathbf{n} stands for the outward unit normal to Ω on $\partial\Omega$, and \mathcal{A} is the fourth order elastic coefficient tensor which satisfies the usual symmetry and ellipticity conditions and whose components are in $L^\infty(\Omega)$.

On Γ_C , we decompose the displacement and the stress vector fields in normal and tangential components as follows:

$$\begin{aligned} u_N &= u \cdot \mathbf{n}, & u_T &= u - u_N \mathbf{n}, \\ \sigma_N(u) &= (\sigma(u)\mathbf{n}) \cdot \mathbf{n}, & \sigma_T(u) &= \sigma(u)\mathbf{n} - \sigma_N(u)\mathbf{n}. \end{aligned}$$

The unilateral contact condition on Γ_C is expressed by the following complementary condition:

$$u_N \leq 0, \quad \sigma_N(u) \leq 0, \quad u_N \sigma_N(u) = 0, \quad (5)$$

where a vanishing gap between the elastic solid and the rigid foundation has been chosen in the reference configuration.

Denoting by $\mathcal{F} \geq 0$ the given friction coefficient on Γ_C (which is supposed constant for the sake of simplicity), the *static Coulomb friction* condition reads as:

$$\text{if } u_T = 0 \quad \text{then} \quad |\sigma_T(u)| \leq -\mathcal{F}\sigma_N(u), \quad (6)$$

$$\text{if } u_T \neq 0 \quad \text{then} \quad \sigma_T(u) = \mathcal{F}\sigma_N(u) \frac{u_T}{|u_T|}. \quad (7)$$

When $\mathcal{F} = 0$ the friction conditions (6)–(7) merely reduce to $\sigma_T(u) = 0$ on Γ_C .

1.1 Classical weak formulations

This paragraph is devoted to the presentation of different and equivalent weak formulations of the Coulomb friction problem. Let us introduce the following Hilbert spaces:

$$V = \left\{ v \in (H^1(\Omega))^d, v = 0 \text{ on } \Gamma_D \right\},$$

$$X = \left\{ v|_{\Gamma_C} : v \in V \right\} \subset (H^{1/2}(\Gamma_C))^d,$$

$$X_N = \left\{ v_N|_{\Gamma_C} : v \in V \right\}, \quad X_T = \left\{ v_T|_{\Gamma_C} : v \in V \right\},$$

and their topological dual spaces V' , X' , X'_N and X'_T , endowed with their usual norms. Since Γ_C is a straight line segment ($d = 2$) or a plane surface ($d = 3$), we have $H_{00}^{1/2}(\Gamma_C) \subset X_N \subset H^{1/2}(\Gamma_C)$, $(H_{00}^{1/2}(\Gamma_C))^{d-1} \subset X_T \subset (H^{1/2}(\Gamma_C))^{d-1}$ which implies $X'_N \subset H^{-1/2}(\Gamma_C)$ and $X'_T \subset (H^{-1/2}(\Gamma_C))^{d-1}$ where we denote by H^s the standard Sobolev spaces (see [1]). Classically, $H^{1/2}(\Gamma_C)$ is the space of the restrictions on Γ_C of traces on $\partial\Omega$ of functions in $H^1(\Omega)$, and $H^{-1/2}(\Gamma_C)$ is the dual space of $H_{00}^{1/2}(\Gamma_C)$ which is the space of the restrictions on Γ_C of functions in $H^{1/2}(\partial\Omega)$ vanishing outside Γ_C . We refer to [36], [1] and [31] for a detailed presentation of

trace operators and/or trace spaces.

The set of admissible displacements satisfying the noninterpenetration conditions on the contact zone is:

$$K = \{v \in V, v_N \leq 0 \text{ a.e. on } \Gamma_C\}.$$

Let be given the following forms for any u and v in V :

$$a(u, v) = \int_{\Omega} \mathcal{A}\varepsilon(u) : \varepsilon(v) d\Omega,$$

$$l(v) = \int_{\Omega} f \cdot v d\Omega + \int_{\Gamma_N} F \cdot v d\Gamma,$$

which represent the virtual work of the elastic forces and of the external loads respectively. If $\langle \cdot, \cdot \rangle_{X'_N, X_N}$ stands for the duality pairing between X'_N and X_N , then the “virtual work” of the friction forces is given by:

$$j(\mathcal{F}\lambda_N, v_T) = - \langle \mathcal{F}\lambda_N, |v_T| \rangle_{X'_N, X_N}$$

for any $\lambda_N \in X'_N$ and $v_T \in X_T$. From the previous assumptions it follows that:

$$\begin{aligned} a(\cdot, \cdot) &\text{ is a bilinear symmetric } V\text{-elliptic and continuous form on } V \times V : \\ &\exists \alpha > 0, \exists M > 0, a(v, v) \geq \alpha \|v\|_V^2, a(u, v) \leq M \|u\|_V \|v\|_V \quad \forall u, v \in V, \\ l(\cdot) &\text{ linear continuous form on } V, \text{ i.e. } \exists L > 0, |l(v)| \leq L \|v\|_V, \quad \forall v \in V. \end{aligned}$$

Moreover $j(\mathcal{F}\lambda_N, v_T)$ is linear continuous with respect to λ_N and convex lower semi-continuous with regard to v_T if λ_N is a nonpositive element of X'_N (see for instance [2]).

Clearly $a(\cdot, \cdot)$ is an inner product on V and the associated norm:

$$\|v\|_a = (a(v, v))^{1/2},$$

is equivalent to the usual norm of V :

$$\sqrt{\alpha} \|v\|_V \leq \|v\|_a \leq \sqrt{M} \|v\|_V, \quad \forall v \in V.$$

The continuity constant of $l(\cdot)$ can also be given with respect to $\|\cdot\|_a$:

$$\exists L_a > 0, |l(v)| \leq L_a \|v\|_a, \quad \forall v \in V.$$

Constants L and L_a can be chosen such that:

$$\sqrt{\alpha} L_a \leq L \leq \sqrt{M} L_a.$$

The weak formulation of Problem (1)–(7) (written as an inequality), introduced in [15] (see also [17]) is:

$$\begin{cases} \text{Find } u \in K \text{ satisfying:} \\ a(u, v - u) + j(\mathcal{F}\sigma_N(u), v_T) - j(\mathcal{F}\sigma_N(u), u_T) \geq l(v - u), \quad \forall v \in K. \end{cases} \quad (8)$$

Introducing the stresses on the contact boundary as an unknown in the previous formulation one obtains the following equivalent formulation (see [30]):

$$\left\{ \begin{array}{l} \text{Find } u \in V, \lambda_N \in X_N \text{ and } \lambda_T \in X_T \text{ satisfying:} \\ a(u, v) = l(v) + \langle \lambda_N, v_N \rangle_{X'_N, X_N} + \langle \lambda_T, v_T \rangle_{X'_T, X_T}, \quad \forall v \in V, \\ u_N \leq 0, \quad \langle \lambda_N, v_N - u_N \rangle_{X'_N, X_N} \geq 0 \quad \forall v_N \in X_N, v_N \leq 0, \\ \langle \lambda_T, v_T - u_T \rangle_{X'_T, X_T} + j(\mathcal{F}\lambda_N, v_T) - j(\mathcal{F}\lambda_N, u_T) \geq 0, \quad \forall v_T \in X_T. \end{array} \right. \quad (9)$$

Inverting contact and friction relations, one also obtains the classical equivalent hybrid formulation (see [30]):

$$\left\{ \begin{array}{l} \text{Find } u \in V, \lambda_N \in X_N \text{ and } \lambda_T \in X_T \text{ satisfying:} \\ a(u, v) = l(v) + \langle \lambda_N, v_N \rangle_{X'_N, X_N} + \langle \lambda_T, v_T \rangle_{X'_T, X_T}, \quad \forall v \in V, \\ \lambda_N \in \Lambda_N, \quad \langle \mu_N - \lambda_N, u_N \rangle_{X'_N, X_N} \geq 0, \quad \forall \mu_N \in \Lambda_N, \\ \lambda_T \in \Lambda_T(\mathcal{F}\lambda_N), \quad \langle \mu_T - \lambda_T, u_T \rangle_{X'_T, X_T} \geq 0, \quad \forall \mu_T \in \Lambda_T(\mathcal{F}\lambda_N), \end{array} \right. \quad (10)$$

where Λ_N and $\Lambda_T(\mathcal{F}\lambda_N)$ denote the sets of admissible normal and tangential stresses:

$$\Lambda_N = \left\{ \lambda_N \in X'_N : \langle \lambda_N, v_N \rangle_{X'_N, X_N} \geq 0 \quad \forall v_N \leq 0 \right\},$$

$$\Lambda_T(\mathcal{F}\lambda_N) = \left\{ \lambda_T \in X'_T : \langle \lambda_T, v_T \rangle_{X'_T, X_T} + j(\mathcal{F}\lambda_N, v_T) \geq 0, \quad \forall v_T \in X_T \right\}.$$

It is easy to check that the multipliers λ_N and λ_T solving (9) and (10) satisfy $\lambda_N = \sigma_N(u)$ and $\lambda_T = \sigma_T(u)$ at least in a weak sense. The main difficulty in the existence and uniqueness analysis of (8), (9) or (10) comes from the coupling between the friction threshold $\mathcal{F}\sigma_N(u)$ and the contact pressure $\sigma_N(u)$.

Remark 1 *The equivalence between Problems (8) and (9) is easy to obtain here since the assumption $f \in L^2(\Omega)^d$ implies that a generalized Green formula holds (see [31] for instance). The proof can also be made directly as follows. A solution to Problem (9) is obviously a solution to Problem (8). Conversely, if u is solution to Problem (8), then the map $X \ni v \mapsto a(u, \Pi(v)) - l(\Pi(v))$ is linear continuous for any continuous lifting operator $\Pi : X \rightarrow V$. Thus there exists $\lambda \in X'$ such that $\langle \lambda, v \rangle_{X', X} = a(u, \Pi(v)) - l(\Pi(v))$ for all $v \in X$. It is easy to state that in fact $\langle \lambda, v \rangle_{X', X} = a(u, v) - l(v)$ for all $v \in V$ proving $a(u, \Pi(v|_{\Gamma_C}) - v) - l(\Pi(v|_{\Gamma_C}) - v) = 0$ for all $v \in V$. Indeed $\Pi(v|_{\Gamma_C}) - v$ has a vanishing trace on Γ_C and replacing successively $v - u$ by $(\Pi(v|_{\Gamma_C}) - v + u) - u$ and by $(v - \Pi(v|_{\Gamma_C}) + u) - u$ in the inequality of (8) leads to this result. The two inequalities of (9) result then from the replacement of $a(u, v - u) - l(v - u)$ by $\langle \lambda, v - u \rangle_{X', X}$ in the inequality of (8) and separating normal and tangential components (Γ_C is straight here) and remarking that applying a Green formula one has $\sigma_N(u) = \lambda_N$. The equivalence between (9) and (10) is developed in [30] by computing the Fenchel conjugate of $j(\mathcal{F}\sigma_N, \cdot)$ and inverting the normal cone to K .*

1.2 Neumann to Dirichlet operator

We introduce the Neumann to Dirichlet operator on Γ_C and its basic properties. This will allow us to restrict the contact and friction problem on Γ_C and obtain useful estimates.

Let $\lambda = (\lambda_N, \lambda_T) \in X'$. The solution u to:

$$\begin{cases} \text{Find } u \in V \text{ satisfying:} \\ a(u, v) = l(v) + \langle \lambda, v \rangle_{X', X} \quad \forall v \in V, \end{cases} \quad (11)$$

is unique (see [17]). So it is possible to define the operator

$$\begin{aligned} \mathbb{E} : X' &\longrightarrow X \\ \lambda &\longmapsto u|_{\Gamma_C}. \end{aligned}$$

It is easy to check that the operator \mathbb{E} is affine and continuous. We define the following norms on Γ_C relatively to $a(\cdot, \cdot)$:

$$\begin{aligned} \|v\|_{a, \Gamma_C} &= \inf_{w \in V, w|_{\Gamma_C} = v} \|w\|_a, \\ \|v_N\|_{a, \Gamma_C} &= \inf_{\substack{w \in V, \\ w_N = v_N \text{ on } \Gamma_C}} \|w\|_a = \inf_{\substack{w \in V, \\ w_N = v_N \text{ on } \Gamma_C}} \|w\|_{a, \Gamma_C}, \\ \|v_T\|_{a, \Gamma_C} &= \inf_{\substack{w \in V, \\ w_T = v_T \text{ on } \Gamma_C}} \|w\|_a = \inf_{\substack{w \in V, \\ w_T = v_T \text{ on } \Gamma_C}} \|w\|_{a, \Gamma_C}, \\ \|\lambda\|_{-a, \Gamma_C} &= \sup_{\substack{v \in X \\ v \neq 0}} \frac{\langle \lambda, v \rangle_{X', X}}{\|v\|_{a, \Gamma_C}} = \sup_{\substack{v \in V \\ v \neq 0}} \frac{\langle \lambda, v \rangle_{X', X}}{\|v\|_a}, \\ \|\lambda_N\|_{-a, \Gamma_C} &= \sup_{\substack{v_N \in X_N \\ v_N \neq 0}} \frac{\langle \lambda_N, v_N \rangle_{X'_N, X_N}}{\|v_N\|_{a, \Gamma_C}} = \|(\lambda_N, 0)\|_{-a, \Gamma_C}, \\ \|\lambda_T\|_{-a, \Gamma_C} &= \sup_{\substack{v_T \in X_T \\ v_T \neq 0}} \frac{\langle \lambda_T, v_T \rangle_{X'_T, X_T}}{\|v_T\|_{a, \Gamma_C}} = \|(0, \lambda_T)\|_{-a, \Gamma_C}, \end{aligned}$$

which are equivalent respectively to the norms in X and X'

$$\begin{aligned} \frac{\sqrt{\alpha}}{C_1} \|v\|_X &\leq \|v\|_{a, \Gamma_C} \leq \sqrt{M\gamma} \|v\|_X, \\ \frac{\sqrt{\alpha}}{C_1} \|v_N\|_{X_N} &\leq \|v_N\|_{a, \Gamma_C} \leq \sqrt{M\gamma} \|v_N\|_{X_N}, \\ \frac{\sqrt{\alpha}}{C_1} \|v_T\|_{X_T} &\leq \|v_T\|_{a, \Gamma_C} \leq \sqrt{M\gamma} \|v_T\|_{X_T}, \\ \frac{1}{\sqrt{M\gamma}} \|\lambda\|_{X'} &\leq \|\lambda\|_{-a, \Gamma_C} \leq \frac{C_1}{\sqrt{\alpha}} \|\lambda\|_{X'}. \end{aligned}$$

One has also

$$\|v_N\|_{a, \Gamma_C} \leq \|v\|_{a, \Gamma_C},$$

and

$$\|\lambda_N\|_{-a, \Gamma_C} \leq C_\alpha \|\lambda\|_{-a, \Gamma_C},$$

with a constant $C_\alpha \leq C_1 \gamma \sqrt{M/\alpha}$ (but a better estimate should be possible: following [19], C_α is close to one when the Poisson ratio is close to zero).

With the previous norms, it is possible to state (see [41]) the following equalities, when $u = \mathbb{E}(\lambda)$ and $\bar{u} = \mathbb{E}(\bar{\lambda})$ are the solutions to Problem (11):

$$\|u - \bar{u}\|_a = \|\mathbb{E}(\lambda) - \mathbb{E}(\bar{\lambda})\|_{a, \Gamma_C} = \|\lambda - \bar{\lambda}\|_{-a, \Gamma_C}. \quad (12)$$

1.3 Direct weak inclusion formulation

Let

$$K_N = \{v_N \in X_N : v_N \leq 0 \text{ a.e. on } \Gamma_C\},$$

be the set of admissible normal displacements on Γ_C . The normal cone in X'_N to K_N at $v_N \in X_N$ is defined as:

$$N_{K_N}(v_N) = \left\{ \mu_N \in X'_N : \langle \mu_N, w_N - v_N \rangle_{X'_N, X_N} \leq 0, \forall w_N \in K_N \right\} \text{ if } v_N \in K_N, \quad N_{K_N}(v_N) = \emptyset \text{ if } v_N \notin K_N.$$

The subdifferential of $j(\mathcal{F}\lambda_N, \cdot)$ (i.e., with respect to the second variable) at u_T is given by:

$$\partial_2 j(\mathcal{F}\lambda_N, u_T) = \left\{ \mu_T \in X'_T : j(\mathcal{F}\lambda_N, v_T) \geq j(\mathcal{F}\lambda_N, u_T) + \langle \mu_T, v_T - u_T \rangle_{X'_T, X_T}, \forall v_T \in X_T \right\}.$$

With this notations, Problem (8) can be written:

$$\left\{ \begin{array}{l} \text{Find } u \in V, \lambda_N \in X'_N \text{ and } \lambda_T \in X'_T \text{ satisfying:} \\ (u_N, u_T) = \mathbb{E}(\lambda_N, \lambda_T), \\ -\lambda_N \in N_{K_N}(u_N) \quad \text{in } X'_N, \\ -\lambda_T \in \partial_2 j(\mathcal{F}\lambda_N, u_T) \quad \text{in } X'_T. \end{array} \right. \quad (13)$$

More details resulting from this equivalence can be found in [34].

1.4 A Uniqueness criterion

In [26, 27] some multi-solutions of the problem (1)–(7) are exhibited for triangular or quadrangular domains. These multiple solutions involve either an infinite set of slipping solutions or two isolated (stick and separation) configurations. Note that these examples of non-uniqueness involve large friction coefficients (i.e., $\mathcal{F} > 1$) and tangential displacements with a constant sign on Γ_C . Actually, it seems that no multi-solution has been detected for an arbitrary small friction coefficient in the continuous case, although such a result exists for finite element approximations in [25], but for a variable geometry. The forthcoming partial uniqueness result is obtained in [41]: it defines some cases where it is possible to affirm that a solution to the Coulomb friction problem is in fact the unique solution. More precisely, if a regular solution to the Coulomb friction problem exists (here the denomination regular means, roughly speaking, that the transition is smooth when the slip direction changes) and if the friction coefficient is small enough then this solution is the only one. We recall the main useful tools leading to that result.

Lemma 1 *Let u and \bar{u} be two solutions to Problem (8) and let λ and $\bar{\lambda}$ be the corresponding contact stresses on Γ_C . Then the following estimate holds:*

$$\|u - \bar{u}\|_a^2 = \|\lambda - \bar{\lambda}\|_{-a, \Gamma_C}^2 \leq \langle \zeta - \lambda_T, \bar{u}_T - u_T \rangle_{X'_T, X_T} \quad \forall \zeta \in -\partial_2 j(\mathcal{F}\bar{\lambda}_N, u_T).$$

Proof. From (12), Green formula and (5), we get:

$$\begin{aligned} \|u - \bar{u}\|_a^2 = \|\lambda - \bar{\lambda}\|_{-a, \Gamma_C}^2 &= \langle \bar{\lambda}_N - \lambda_N, \bar{u}_N - u_N \rangle_{X'_N, X_N} + \langle \bar{\lambda}_T - \lambda_T, \bar{u}_T - u_T \rangle_{X'_T, X_T} \\ &\leq \langle \bar{\lambda}_T - \lambda_T, \bar{u}_T - u_T \rangle_{X'_T, X_T}. \end{aligned}$$

Thus

$$\|u - \bar{u}\|_a^2 \leq \langle (\bar{\lambda}_T - \zeta) + (\zeta - \lambda_T), \bar{u}_T - u_T \rangle_{X'_T, X_T} \quad \forall \zeta \in -\partial_2 j(\mathcal{F}\bar{\lambda}_N, u_T).$$

The conclusion follows from (13) and the fact that $-\partial_2 j(\mathcal{F}\bar{\lambda}_N, \cdot)$ is a monotone set-valued mapping. \blacksquare

We now introduce the space of multipliers $M(X_T \rightarrow X_N)$ of the functions $\xi : \Gamma_C \rightarrow \mathbb{R}^d$ satisfying $\xi \cdot n = 0$ *a.e.* on Γ_C and such that the following equivalent norms are finite:

$$\|\xi\|_{M(X_T \rightarrow X_N)} = \sup_{\substack{v_T \in X_T \\ v_T \neq 0}} \frac{\|\xi \cdot v_T\|_{X_N}}{\|v_T\|_{X_T}}, \quad \text{and} \quad \|\xi\|_a = \sup_{\substack{v_T \in X_T \\ v_T \neq 0}} \frac{\|\xi \cdot v_T\|_{a, \Gamma_C}}{\|v_T\|_{a, \Gamma_C}}.$$

Since Γ_C is assumed to be straight, $M(X_T \rightarrow X_N)$ contains for any $\varepsilon > 0$ the space $H^{1/2+\varepsilon}(\Gamma_C)$ when $d = 2$. When $d = 3$, $M(X_T \rightarrow X_N)$ contains $H^1(\Gamma_C) \cap L^\infty(\Gamma_C)$ (see [38] for a complete discussion on the theory of multipliers in a pair of Hilbert spaces).

The partial uniqueness result is given assuming that $\lambda_T = \mathcal{F}\lambda_N\xi$, with $\xi \in M(X_T \rightarrow X_N)$. The product $\lambda_N\xi$ has to be understood in the sense that $\langle \lambda_N\xi, v_T \rangle_{X'_T, X_T} = \langle \lambda_N, \xi \cdot v_T \rangle_{X'_N, X_N}$ for all $v_T \in X_T$. It is easy to see that it implies $|\xi| \leq 1$ *a.e.* on the support of λ_N . More precisely, this implies that $\xi \in \text{Dir}_T(u_T)$ *a.e.* on the support of λ_N , where $\text{Dir}_T(\cdot)$ is the subdifferential of the convex map $\mathbb{R}^d \ni x \mapsto |x_T|$. This means that it is possible to assume that $\xi \in \text{Dir}_T(u_T)$ *a.e.* on Γ_C .

Proposition 1 *Let u be a solution to Problem (8) such that $\lambda_T = \mathcal{F}\lambda_N\xi$, with $\xi \in M(X_T \rightarrow X_N)$, $\xi \in \text{Dir}_T(u_T)$ *a.e.* on Γ_C and $\mathcal{F} < (C_\alpha \|\xi\|_a)^{-1}$. Then u is the unique solution to Problem (8).*

Proof. Let \bar{u} be another solution to Problem (8), where $\bar{\lambda}_N$ and $\bar{\lambda}_T$ denote the corresponding contact stresses on Γ_C . According to Lemma 1, we write:

$$\|u - \bar{u}\|_a^2 \leq \langle \zeta - \lambda_T, \bar{u}_T - u_T \rangle_{X'_T, X_T} \quad \forall \zeta \in -\partial_2 j(\mathcal{F}\bar{\lambda}_N, u_T).$$

It is easy to see that a possible choice is $\zeta = \mathcal{F}\bar{\lambda}_N\xi$. Therefore

$$\begin{aligned} \|u - \bar{u}\|_a^2 &\leq \langle \mathcal{F}\xi(\bar{\lambda}_N - \lambda_N), \bar{u}_T - u_T \rangle_{X'_T, X_T} \leq \mathcal{F}\|\lambda_N - \bar{\lambda}_N\|_{-a, \Gamma_C} \|\xi \cdot (u_T - \bar{u}_T)\|_{a, \Gamma_C} \\ &\leq C_\alpha \mathcal{F} \|\xi\|_a \|\lambda - \bar{\lambda}\|_{-a, \Gamma_C} \|u - \bar{u}\|_a \\ &= C_\alpha \mathcal{F} \|\xi\|_a \|u - \bar{u}\|_a^2, \end{aligned}$$

which implies that $\bar{u} = u$ when $\mathcal{F} < (C_\alpha \|\xi\|_a)^{-1}$. ■

In two space dimensions ($d = 2$), the case $\xi \equiv 1$ corresponds to an homogeneous sliding direction and the previous result is complementary to the non-uniqueness results obtained in [26, 27].

As illustrated on Fig. 2 when $d = 2$, the multiplier ξ has to vary from -1 to $+1$ each time the sign of the tangential displacement changes from negative to positive. The set $M(X_T \rightarrow X_N)$ does not contain any multiplier having a singularity of the first kind. Consequently, in order to satisfy the assumptions of Proposition 1, the tangential displacement of the solution u cannot pass from a negative value to a positive value and being zero only at a single point of Γ_C .

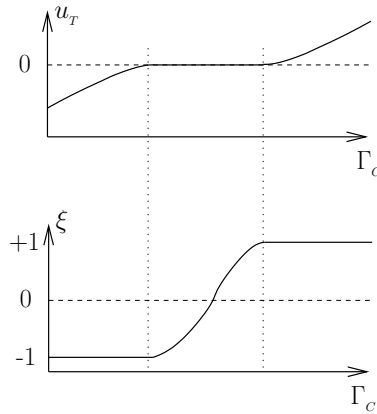


Figure 2: Example of a tangential displacement u_T and a possible corresponding multiplier ξ when $d = 2$.

Remark 2 This remark deals with a more precise discussion concerning the assumption: $\lambda_T = \mathcal{F}\lambda_N\xi$, $\xi \in M(X_T \rightarrow X_N)$, $\xi \in \text{Dir}_T(u_T)$ and the cases where the assumption cannot be fulfilled independently of the regularity of the solution when $d = 2$. On the one hand it is easy to show that the choice of ξ is unique at any point where $\lambda_N \neq 0$ or $u_T \neq 0$. In the first case $\xi = \lambda_T/(\mathcal{F}\lambda_N)$, in the second case $\xi = u_T/|u_T|$ and both expressions coincide when $\lambda_N \neq 0$ and $u_T \neq 0$. On the other hand any $\xi \in [-1, 1]$ can be chosen at the points where $\lambda_N = u_T = 0$. So it remains to determine when ξ lies in $M(X_T \rightarrow X_N)$. If there are no points such that $\lambda_N = u_T = 0$ on Γ_C then the condition $\xi \in M(X_T \rightarrow X_N)$ is linked to the regularity of u (in other words, if u is regular enough then $\xi \in M(X_T \rightarrow X_N)$). If there are some points such that $\lambda_N = u_T = 0$ then it is easy to show that the continuity of ξ can be lost (whatever the regularity of u is) only if some of these points are isolated. A discussion shows then that $\xi \notin M(X_T \rightarrow X_N)$ in three cases. The first one is when u_T passes from a negative to a positive value at such a point (note that this could also occur at a point which is separated from the foundation). The second case corresponds to a stick area surrounding such an isolated point and where the right and left limits of $\lambda_T/(\mathcal{F}\lambda_N)$ differ at this point (where $\lambda_T = \lambda_N = u_T = u_N = 0$). The third case is a combination of both previous cases: a side where $u_T \neq 0$, the other one with $u_T = 0$ and $\lambda_N \neq 0$ and a limit of $\lambda_T/(\mathcal{F}\lambda_N)$ which differs from $u_T/|u_T|$. If the solution is less regular then other cases of nonfulfillment could appear but we think that this assumption (which is needed to obtain the uniqueness of a solution to the continuous problem) takes into account many frictional contact configurations.

2 Finite element approximation

Let $V^h \subset V$ be a family of finite dimensional vector spaces indexed by h coming from a regular family \mathcal{T}^h (see [9]) of triangulations of the domain Ω (h represents the largest diameter among all elements). We choose standard continuous and piecewise affine functions, i.e.:

$$V^h = \left\{ v^h \in (\mathcal{C}(\overline{\Omega}))^d, v^h|_T \in P_1(T), \forall T \in \mathcal{T}^h, v^h = 0 \text{ on } \Gamma_D \right\}. \quad (14)$$

Define

$$\begin{aligned} X_N^h &= \left\{ v_N^h|_{\Gamma_C} : v^h \in V^h \right\}, \\ X_T^h &= \left\{ v_T^h|_{\Gamma_C} : v^h \in V^h \right\}, \\ X^h &= \left\{ v^h|_{\Gamma_C} : v^h \in V^h \right\} = X_N^h \times X_T^h. \end{aligned} \quad (15)$$

Identifying X_N^h and X_T^h with their dual spaces using the L^2 scalar product, we consider that X_N^h and X_T^h are also the finite dimensional approximations of X'_N and X'_T respectively.

The finite element discretization of Problem (10) becomes:

$$\left\{ \begin{array}{l} \text{Find } u^h \in V^h, \lambda_N^h \in X_N^h \text{ and } \lambda_T^h \in X_T^h \text{ satisfying:} \\ a(u^h, v^h) = l(v^h) + \int_{\Gamma_C} \lambda_N^h v_N^h d\Gamma + \int_{\Gamma_C} \lambda_T^h v_T^h d\Gamma, \quad \forall v^h \in V^h, \\ \lambda_N^h \in \Lambda_N^h, \quad \int_{\Gamma_C} (\mu_N^h - \lambda_N^h) u_N^h d\Gamma \geq 0, \quad \forall \mu_N^h \in \Lambda_N^h, \\ \lambda_T^h \in \Lambda_T^h(\mathcal{F}\lambda_N^h), \quad \int_{\Gamma_C} (\mu_T^h - \lambda_T^h) u_T^h d\Gamma \geq 0, \quad \forall \mu_T^h \in \Lambda_T^h(\mathcal{F}\lambda_N^h), \end{array} \right. \quad (16)$$

where the approximations of Λ_N and $\Lambda_T(\mathcal{F}\lambda_N)$ have been chosen in the following way:

$$\Lambda_N^h = \Lambda_N \cap X_N^h, \quad (17)$$

$$\Lambda_T^h(\mathcal{F}\lambda_N^h) = \left\{ \lambda_T^h \in X_T^h : \int_{\Gamma_C} \lambda_T^h v_T^h d\Gamma + j(\mathcal{F}\lambda_N^h, v_T^h) \geq 0, \quad \forall v_T^h \in X_T^h \right\}. \quad (18)$$

To simplify our discussion, we assume afterwards that the mesh inherited on the contact zone is quasi-uniform (although there exists some less restrictive assumptions, see e.g. [13]) of size h (to simplify). Another simplification is that we restrict ourselves to the two-dimensional case ($d = 2$) and we assume that the end points of Γ_C do not belong to $\overline{\Gamma_D}$ (in other words $\overline{\Gamma_C} \cap \overline{\Gamma_D} = \emptyset$). More general cases will be discussed in some remarks at the end of the paper.

With this choice of discretisation, the following discrete Babuška-Brezzi inf-sup condition holds (see e.g. [12, 6]):

$$\inf_{\lambda^h \in X^h} \sup_{v^h \in V^h} \frac{\int_{\Gamma_C} \lambda^h v^h d\Gamma}{\|v^h\|_a \|\lambda^h\|_{-a, \Gamma_C}} \geq c_{is} > 0, \quad (19)$$

where $c_{is} \leq 1$ is independent of h . As a consequence, Problem (16) admits a solution for any friction coefficient and the solution is unique for a sufficiently small friction coefficient (where the denomination 'small' may depend on h) (see [30]).

The following lemma shows the relation between the hybrid formulation and the direct formulation of the friction condition in the discrete framework.

Lemma 2 For $\lambda_N^h \in \Lambda_N^h$, a pair $(\lambda_T^h, u_T^h) \in X_T^h \times X_T^h$ satisfies

$$\lambda_T^h \in \Lambda_T^h(\mathcal{F}\lambda_N^h), \quad \int_{\Gamma_C} (\mu_T^h - \lambda_T^h) \cdot u_T^h d\Gamma \geq 0, \quad \forall \mu_T^h \in \Lambda_T^h(\mathcal{F}\lambda_N^h), \quad (20)$$

if and only if the pair satisfies

$$\int_{\Gamma_C} \lambda_T^h \cdot (v_T^h - u_T^h) d\Gamma + j(\mathcal{F}\lambda_N^h, v_T^h) - j(\mathcal{F}\lambda_N^h, u_T^h) \geq 0, \quad \forall v_T^h \in X_T^h. \quad (21)$$

Proof. Let us first assume that (λ_T^h, u_T^h) satisfies (20). For an arbitrary choice $\xi \in \mathcal{F}\lambda_N^h \text{Dir}_T(u_T^h)$ the map $v_T^h \mapsto \int_{\Gamma_C} \xi \cdot v_T^h d\Gamma$ is a linear form on X_T^h and thus by Riesz representation theorem there exists $\mu_T^h \in X_T^h$ such that $\int_{\Gamma_C} \mu_T^h \cdot v_T^h d\Gamma = \int_{\Gamma_C} \xi \cdot v_T^h d\Gamma$ for all $v_T^h \in X_T^h$. This μ_T^h satisfies $\int_{\Gamma_C} \mu_T^h \cdot u_T^h d\Gamma = \int_{\Gamma_C} \mathcal{F}\lambda_N^h |u_T^h| d\Gamma = -j(\mathcal{F}\lambda_N^h, u_T^h)$ and μ_T^h is an element of $\Lambda_T^h(\mathcal{F}\lambda_N^h)$. Now considering this particular μ_T^h in (20) leads to

$$\int_{\Gamma_C} \lambda_T^h \cdot u_T^h d\Gamma + j(\mathcal{F}\lambda_N^h, u_T^h) \leq 0.$$

Together with the fact that λ_T^h is in $\Lambda_T^h(\mathcal{F}\lambda_N^h)$ this leads to the complementarity relation

$$\int_{\Gamma_C} \lambda_T^h \cdot u_T^h d\Gamma + j(\mathcal{F}\lambda_N^h, u_T^h) = 0,$$

which straightforwardly implies (21).

Conversely, let us assume that (λ_T^h, u_T^h) satisfies (21). Then choosing $v_T^h = 0$ in (21) gives $-\int_{\Gamma_C} \lambda_T^h \cdot u_T^h d\Gamma - j(\mathcal{F}\lambda_N^h, u_T^h) \geq 0$ and choosing $v_T^h = 2u_T^h$ gives $\int_{\Gamma_C} \lambda_T^h \cdot u_T^h d\Gamma + j(\mathcal{F}\lambda_N^h, u_T^h) \geq 0$ which implies the complementarity relation

$$\int_{\Gamma_C} \lambda_T^h \cdot u_T^h d\Gamma + j(\mathcal{F}\lambda_N^h, u_T^h) = 0.$$

Taking this into account in (21) leads to

$$\int_{\Gamma_C} \lambda_T^h \cdot v_T^h d\Gamma + j(\mathcal{F}\lambda_N^h, v_T^h) \geq 0, \quad \forall v_T^h \in X_T^h,$$

which implies $\lambda_T^h \in \Lambda_T^h(\mathcal{F}\lambda_N^h)$. Now, from the complementarity relation and for all $\mu_T^h \in \Lambda_T^h(\mathcal{F}\lambda_N^h)$ one has

$$\int_{\Gamma_C} (\mu_T^h - \lambda_T^h) \cdot u_T^h d\Gamma = \int_{\Gamma_C} \mu_T^h \cdot u_T^h d\Gamma + j(\mathcal{F}\lambda_N^h, u_T^h) \geq 0,$$

which implies (20). ■

Remark 3 *The equivalence given by this lemma is a classical result when it deals with the continuous problem. With the particular finite element discretization considered in this section, the result is still valid in the finite dimensional case. One of the reasons is that the space of the multipliers has been chosen in such a way that it can represent the dual space of the discrete trace space X_T^h . But this result does not remain valid when a smaller space for the multipliers is chosen.*

3 The error estimate

Theorem 1 *Let (u, λ) be the solution to Problem (8) (for $d = 2$) such that $\lambda_T = \mathcal{F}\lambda_N\xi$, with $\xi \in M(X_T \rightarrow X_N)$, $\xi \in \text{Dir}_T(u_T)$ a.e. on Γ_C and $\mathcal{F} < c_{is}(C_\alpha\|\xi\|_a)^{-1}$. Assume that $u \in (H^{(3/2)+\varepsilon}(\Omega))^2$ with $\varepsilon > 0$ and let (u^h, λ^h) be a solution to the discrete problem (16). Then, there exists a constant $C > 0$ independent of h and u such that:*

$$\|u - u^h\|_a + \|\lambda - \lambda^h\|_{-a, \Gamma_C} \leq Ch^{1/2}\|u\|_{(H^{(3/2)+\varepsilon}(\Omega))^2}. \quad (22)$$

Proof. Let $v^h \in V^h$. So

$$\begin{aligned} \|u - u^h\|_a^2 &= a(u - u^h, u - u^h) \\ &= a(u - u^h, u - v^h) + a(u, v^h - u^h) - a(u^h, v^h - u^h) \\ &= a(u - u^h, u - v^h) + \int_{\Gamma_C} \lambda_N(v_N^h - u_N^h) d\Gamma + \int_{\Gamma_C} \lambda_T \cdot (v_T^h - u_T^h) d\Gamma \\ &\quad - \int_{\Gamma_C} \lambda_N^h(v_N^h - u_N^h) d\Gamma - \int_{\Gamma_C} \lambda_T^h \cdot (v_T^h - u_T^h) d\Gamma \\ &= a(u - u^h, u - v^h) + \int_{\Gamma_C} (\lambda_N - \lambda_N^h)(v_N^h - u_N^h) d\Gamma + \int_{\Gamma_C} (\lambda_T - \lambda_T^h) \cdot (v_T^h - u_T^h) d\Gamma \\ &\quad + \int_{\Gamma_C} (\lambda_N - \lambda_N^h)(u_N - u_N^h) d\Gamma + \int_{\Gamma_C} (\lambda_T - \lambda_T^h) \cdot (u_T - u_T^h) d\Gamma. \end{aligned}$$

The continuous and discrete complementary conditions imply:

$$\int_{\Gamma_C} \lambda_N u_N d\Gamma = \int_{\Gamma_C} \lambda_N^h u_N^h d\Gamma = 0.$$

Hence

$$\begin{aligned} \|u - u^h\|_a^2 &= a(u - u^h, u - v^h) + \int_{\Gamma_C} (\lambda_N - \lambda_N^h)(v_N^h - u_N^h) d\Gamma + \int_{\Gamma_C} (\lambda_T - \lambda_T^h) \cdot (v_T^h - u_T^h) d\Gamma \\ &\quad - \int_{\Gamma_C} \lambda_N u_N^h + \lambda_N^h u_N d\Gamma + \int_{\Gamma_C} (\lambda_T - \lambda_T^h) \cdot (u_T - u_T^h) d\Gamma. \end{aligned}$$

Using the continuity of the bilinear form, we obtain:

$$\begin{aligned} \|u - u^h\|_a^2 &\leq \|u - u^h\|_a \|u - v^h\|_a + \|\lambda - \lambda^h\|_{-a, \Gamma_C} \|u - v^h\|_{a, \Gamma_C} \\ &\quad - \int_{\Gamma_C} \lambda_N u_N^h + \lambda_N^h u_N d\Gamma + \int_{\Gamma_C} (\lambda_T - \lambda_T^h) \cdot (u_T - u_T^h) d\Gamma. \end{aligned} \quad (23)$$

Besides we consider the equilibrium equation. From $V^h \subset V$, we get:

$$a(u, v^h) = l(v^h) + \int_{\Gamma_C} \lambda \cdot v^h d\Gamma \quad \forall v^h \in V^h.$$

Since

$$a(u^h, v^h) = l(v^h) + \int_{\Gamma_C} \lambda^h \cdot v^h d\Gamma \quad \forall v^h \in V^h,$$

we deduce by subtraction that

$$a(u - u^h, v^h) = \int_{\Gamma_C} (\lambda - \lambda^h) \cdot v^h d\Gamma \quad \forall v^h \in V^h.$$

Consequently, for any $v^h \in V^h$ and any $\mu^h \in X^h$:

$$\int_{\Gamma_C} (\lambda^h - \mu^h) \cdot v^h d\Gamma = a(u^h - u, v^h) + \int_{\Gamma_C} (\lambda - \mu^h) \cdot v^h d\Gamma \leq (\|u - u^h\|_a + \|\lambda - \mu^h\|_{-a, \Gamma_C}) \|v^h\|_a.$$

The mesh independent inf-sup condition (19) implies, for any $\mu^h \in X^h$:

$$c_{is} \|\lambda^h - \mu^h\|_{-a, \Gamma_C} \leq \sup_{v^h \in V^h} \frac{\int_{\Gamma_C} (\lambda^h - \mu^h) \cdot v^h d\Gamma}{\|v^h\|_a} \leq \|u - u^h\|_a + \|\lambda - \mu^h\|_{-a, \Gamma_C}.$$

By the triangular inequality we come to the conclusion that

$$\|\lambda - \lambda^h\|_{-a, \Gamma_C} \leq \frac{1}{c_{is}} \|u - u^h\|_a + \left(1 + \frac{1}{c_{is}}\right) \inf_{\mu^h \in X^h} \|\lambda - \mu^h\|_{-a, \Gamma_C}. \quad (24)$$

Keeping in mind that $u \in (H^{(3/2)+\varepsilon}(\Omega))^2$ with $\varepsilon > 0$, so $\lambda \in (L^2(\Gamma_C))^2$ according to the trace theorem, we choose $v^h = I^h u$ where I^h denotes the Lagrange interpolation operator mapping onto V^h and $\mu^h = \pi^h \lambda$ where π^h represents the $(L^2(\Gamma_C))^2$ -projection operator mapping onto X^h . As a consequence (see [7, 9, 13]) if $\varepsilon > 0$ is small enough we have:

$$\inf_{v^h \in V^h} \|u - v^h\|_a \leq Ch^{(1/2)+\varepsilon} \|u\|_{(H^{(3/2)+\varepsilon}(\Omega))^2} \quad (25)$$

and

$$\inf_{\mu^h \in X^h} \|\lambda - \mu^h\|_{-a, \Gamma_C} \leq Ch^{(1/2)+\varepsilon} \|u\|_{(H^{(3/2)+\varepsilon}(\Omega))^2}, \quad (26)$$

where C denotes here and afterwards a positive constant independent of h . We now estimate the terms in (23) coming from the contact approximation. Since $u_N \leq 0$ and $\lambda_N^h \leq 0$ on Γ_C , we deduce that the first term is nonpositive:

$$- \int_{\Gamma_C} \lambda_N^h u_N d\Gamma \leq 0. \quad (27)$$

In order to estimate the second term in (23) coming from the contact approximation we introduce a specific operator. Namely $r^h : L^1(\Gamma_C) \mapsto X_N^h$ is the quasi-interpolation operator defined for any function v in $L^1(\Gamma_C)$ by:

$$r^h v = \sum_{x \in N^h} \alpha_x(v) \psi_x,$$

where N^h represents the set of nodes of $\overline{\Gamma_C}$, ψ_x is the scalar basis function of X_N^h (defined on $\overline{\Gamma_C}$) at node x verifying $\psi_x(x') = \delta_{x,x'}$ for all $x' \in N^h$ and

$$\alpha_x(v) = \left(\int_{\Gamma_C} v \psi_x d\Gamma \right) \left(\int_{\Gamma_C} \psi_x d\Gamma \right)^{-1}.$$

Remark 4 It is straightforward to check that r^h is linear and that it preserves nonpositivity. It is also obvious that $r^h v^h \neq v^h$ when $v^h \in X_N^h$. This operator is different from Clément's one (which consists of making local projections onto P_1 functions, see [10]), from Chen-Nochetto's one (which uses local projections onto P_0 functions, see [8]) and from Ben Belgacem-Renard's one (which consists of making local projections onto the convex cone of nonpositive P_1 functions, see [6]). The main particularity of the operator r^h which directly follows from its definition is that $r^h v^h \leq 0$ when $v^h \in X_N^h$ satisfies only "weak nonpositivity conditions", i.e.,

$$\int_{\Gamma_C} \mu_N^h v^h d\Gamma \geq 0, \quad \forall \mu_N^h \in \Lambda_N^h. \quad (28)$$

This property is not satisfied by the operators in [8] and [10]. Moreover as we see hereafter, the approximation properties of r^h hold for any function without sign condition contrary to the operator in [6].

The approximation properties of r^h are established in [28]. We recall them to render the proof of Theorem 1 self-contained. We first show the L^2 -stability property of r^h .

Lemma 3 There is a positive constant C independent of h such that for any $v \in L^2(\Gamma_C)$ and any $E \in E_C^h$ (E_C^h denotes the set of closed edges lying in $\overline{\Gamma_C}$):

$$\|r^h v\|_{L^2(E)} \leq C \|v\|_{L^2(\gamma_E)},$$

where $\gamma_E = \cup_{\{F \in E_C^h: F \cap E \neq \emptyset\}} F$.

Proof. Let γ_x be the support of the basis function ψ_x in Γ_C . Using the definition of $\alpha_x(v)$, Cauchy-Schwarz inequality, and the uniform regularity of the mesh, we get

$$|\alpha_x(v)| \leq \|v\|_{L^2(\gamma_x)} \|\psi_x\|_{L^2(\gamma_x)} \|\psi_x\|_{L^1(\gamma_x)}^{-1} \leq Ch^{-\frac{1}{2}} \|v\|_{L^2(\gamma_x)}.$$

We obtain by a triangular inequality

$$\|r^h v\|_{L^2(E)} = \left\| \sum_{x \in N^h \cap E} \alpha_x(v) \psi_x \right\|_{L^2(E)} \leq C \|v\|_{L^2(\gamma_E)}.$$

■

The next lemma is concerned with the L^2 -approximation properties of r^h .

Lemma 4 There is a positive constant C independent of h such that for any $v \in H^\eta(\Gamma_C)$, $0 \leq \eta \leq 1$, and any $E \in E_C^h$ (E_C^h denotes the set of closed edges lying in $\overline{\Gamma_C}$):

$$\|v - r^h v\|_{L^2(E)} \leq Ch^\eta \|v\|_{H^\eta(\gamma_E)}, \quad (29)$$

where $\gamma_E = \cup_{\{F \in E_C^h: F \cap E \neq \emptyset\}} F$.

Proof. When $\eta = 0$ the bound results from the previous lemma. Note that r^h preserves the constant functions on Γ_C . Let be given an arbitrary constant function $c(x) = c$, $\forall x \in \Gamma_C$. From the definition of r^h , we may write, for any $v \in H^\eta(\Gamma_C)$:

$$v - r^h v = v - c - r^h(v - c).$$

Therefore by Lemma 3 we get

$$\|v - r^h v\|_{L^2(E)} \leq C (\|v - c\|_{L^2(E)} + \|v - c\|_{L^2(\gamma_E)}) \leq C \|v - c\|_{L^2(\gamma_E)}, \forall c \in \mathbb{R}. \quad (30)$$

We then choose $c = \int_{\gamma_E} v(x) dx / |\gamma_E|$ in (30) where $|\gamma_E|$ denotes the length of γ_E . Then if $x \in \gamma_E$ and $0 < \eta < 1$ we have

$$\begin{aligned} v(x) - c &= |\gamma_E|^{-1} \int_{\gamma_E} v(x) - v(y) dy \\ &= |\gamma_E|^{-1} \int_{\gamma_E} \frac{v(x) - v(y)}{|x - y|^{\frac{1+2\eta}{2}}} |x - y|^{\frac{1+2\eta}{2}} dy. \end{aligned}$$

Using Cauchy-Schwarz inequality we deduce

$$\begin{aligned} \int_{\gamma_E} (v(x) - c)^2 dx &= |\gamma_E|^{-2} \int_{\gamma_E} \left(\int_{\gamma_E} \frac{v(x) - v(y)}{|x - y|^{\frac{1+2\eta}{2}}} |x - y|^{\frac{1+2\eta}{2}} dy \right)^2 dx \\ &\leq |\gamma_E|^{-2} \int_{\gamma_E} \left(\int_{\gamma_E} \frac{(v(x) - v(y))^2}{|x - y|^{1+2\eta}} dy \int_{\gamma_E} |x - y|^{1+2\eta} dy \right) dx \\ &\leq |\gamma_E|^{2\eta} \int_{\gamma_E} \int_{\gamma_E} \frac{(v(x) - v(y))^2}{|x - y|^{1+2\eta}} dy dx \\ &\leq Ch^{2\eta} \|v\|_{H^\eta(\gamma_E)}^2. \end{aligned}$$

Hence the result.

If $x \in \gamma_E$ and $\eta = 1$ we have

$$v(x) - c = |\gamma_E|^{-1} \int_{\gamma_E} v(x) - v(y) dy = |\gamma_E|^{-1} \int_{\gamma_E} \int_y^x v'(t) dt dy.$$

Hence

$$|v(x) - c| \leq |\gamma_E|^{\frac{1}{2}} \|v'\|_{L^2(\gamma_E)}.$$

The result is then straightforward. ■

End of the proof of Theorem 1. The second term coming from the contact approximation in (23) is handled as follows:

$$- \int_{\Gamma_C} \lambda_N u_N^h d\Gamma = - \int_{\Gamma_C} \lambda_N (u_N^h - r^h u_N^h) d\Gamma - \int_{\Gamma_C} \lambda_N r^h u_N^h d\Gamma.$$

According to (16), u_N^h satisfies a weak nonnegativity condition as in (28). From Remark 4 we deduce that $r^h u_N^h \leq 0$. Hence we have for any small $\varepsilon > 0$:

$$\begin{aligned} - \int_{\Gamma_C} \lambda_N u_N^h d\Gamma &\leq - \int_{\Gamma_C} \lambda_N (u_N^h - r^h u_N^h) d\Gamma \\ &\leq \|\lambda_N\|_{L^2(\Gamma_C)} \|u_N^h - r^h u_N^h\|_{L^2(\Gamma_C)} \\ &\leq \|\lambda_N\|_{L^2(\Gamma_C)} \|(u_N^h - u_N) - r^h (u_N^h - u_N)\|_{L^2(\Gamma_C)} + \|\lambda_N\|_{L^2(\Gamma_C)} \|u_N - r^h u_N\|_{L^2(\Gamma_C)} \\ &\leq Ch^{1/2} \|u\|_{(H^{(3/2)+\varepsilon}(\Omega))^2} \|u_N^h - u_N\|_{H^{1/2}(\Gamma_C)} + Ch \|u\|_{(H^{(3/2)+\varepsilon}(\Omega))^2} \|u_N\|_{H^1(\Gamma_C)} \quad (31) \end{aligned}$$

where the trace theorem $\|\lambda_N\|_{L^2(\Gamma_C)} \leq C\|u\|_{(H^{3/2+\varepsilon}(\Omega))^2}$ (see [20]) and the estimates in Lemma 4 have been used. Putting together estimates (27) and (31) yields for any small $\varepsilon > 0$:

$$-\int_{\Gamma_C} \lambda_N^h u_N + \lambda_N u_N^h d\Gamma \leq Ch^{1/2}\|u\|_{(H^{3/2+\varepsilon}(\Omega))^2} \left(\|u - u^h\|_a + h^{1/2}\|u\|_{(H^{3/2+\varepsilon}(\Omega))^2} \right). \quad (32)$$

We now estimate the terms corresponding to the friction approximation in (23). From the assumptions in the theorem we write:

$$\begin{aligned} \int_{\Gamma_C} (\lambda_T - \lambda_T^h) \cdot (u_T - u_T^h) d\Gamma &= \int_{\Gamma_C} (\lambda_T - \mathcal{F}\lambda_N^h \xi) \cdot (u_T - u_T^h) d\Gamma + \int_{\Gamma_C} (\mathcal{F}\lambda_N^h \xi - \lambda_T^h) \cdot (u_T - u_T^h) d\Gamma \\ &= \int_{\Gamma_C} \mathcal{F}(\lambda_N - \lambda_N^h) \xi \cdot (u_T - u_T^h) d\Gamma + \int_{\Gamma_C} (\mathcal{F}\lambda_N^h \xi - \lambda_T^h) \cdot (u_T - u_T^h) d\Gamma. \end{aligned} \quad (33)$$

The estimate of the first integral term in (33) gives:

$$\int_{\Gamma_C} \mathcal{F}(\lambda_N - \lambda_N^h) \xi \cdot (u_T - u_T^h) d\Gamma \leq C_\alpha \mathcal{F}\|\xi\|_a \|\lambda - \lambda^h\|_{-a, \Gamma_C} \|u - u^h\|_a.$$

The second integral term in (33) is written as follows:

$$\int_{\Gamma_C} (\mathcal{F}\lambda_N^h \xi - \lambda_T^h) \cdot (u_T - u_T^h) d\Gamma = \int_{\Gamma_C} \mathcal{F}\lambda_N^h \xi \cdot (u_T - u_T^h) d\Gamma - \int_{\Gamma_C} \lambda_T^h \cdot u_T d\Gamma + \int_{\Gamma_C} \lambda_T^h \cdot u_T^h d\Gamma.$$

Using the equivalent discrete friction conditions in (21), we obtain for any $v^h \in V^h$:

$$\begin{aligned} \int_{\Gamma_C} (\mathcal{F}\lambda_N^h \xi - \lambda_T^h) \cdot (u_T - u_T^h) d\Gamma &\leq \int_{\Gamma_C} \mathcal{F}\lambda_N^h \xi \cdot (u_T - u_T^h) d\Gamma - \int_{\Gamma_C} \lambda_T^h \cdot u_T d\Gamma + \int_{\Gamma_C} \lambda_T^h \cdot v_T^h d\Gamma \\ &\quad - \int_{\Gamma_C} \mathcal{F}\lambda_N^h |v_T^h| d\Gamma + \int_{\Gamma_C} \mathcal{F}\lambda_N^h |u_T^h| d\Gamma. \end{aligned}$$

Choosing $v_T^h = I^h u_T$ and since $\xi \cdot u_T = |u_T|$, we obtain:

$$\begin{aligned} \int_{\Gamma_C} (\mathcal{F}\lambda_N^h \xi - \lambda_T^h) \cdot (u_T - u_T^h) d\Gamma &\leq \int_{\Gamma_C} \lambda_T^h \cdot (I^h u_T - u_T) d\Gamma + \int_{\Gamma_C} \mathcal{F}\lambda_N^h (|u_T^h| - \xi \cdot u_T^h) d\Gamma \\ &\quad + \int_{\Gamma_C} \mathcal{F}\lambda_N^h (|u_T| - |I^h u_T|) d\Gamma. \end{aligned} \quad (34)$$

The estimate of the first term in (34) is achieved as follows by using the error estimates in [9]:

$$\begin{aligned} \int_{\Gamma_C} \lambda_T^h \cdot (I^h u_T - u_T) d\Gamma &= \int_{\Gamma_C} (\lambda_T^h - \lambda_T) \cdot (I^h u_T - u_T) d\Gamma + \int_{\Gamma_C} \lambda_T \cdot (I^h u_T - u_T) d\Gamma \\ &\leq Ch^{1/2}\|u\|_{(H^{3/2}(\Omega))^2} \left(\|\lambda - \lambda^h\|_{-a, \Gamma_C} + h^{1/2}\|u\|_{(H^{3/2+\varepsilon}(\Omega))^2} \right). \end{aligned}$$

The estimate of the second term in (34) uses the fact that $\lambda_N^h \leq 0$ and $|u_T^h| - \xi \cdot u_T^h \geq 0$ so that:

$$\int_{\Gamma_C} \mathcal{F}\lambda_N^h (|u_T^h| - \xi \cdot u_T^h) d\Gamma \leq 0.$$

Finally, the third term in (34) yields:

$$\begin{aligned}
\int_{\Gamma_C} \mathcal{F} \lambda_N^h (|u_T| - |I^h u_T|) d\Gamma &\leq \mathcal{F} \|\lambda_N^h\|_{L^2(\Gamma_C)} \| |u_T| - |I^h u_T| \|_{L^2(\Gamma_C)} \\
&\leq \mathcal{F} \|\lambda_N^h\|_{L^2(\Gamma_C)} \|u_T - I^h u_T\|_{L^2(\Gamma_C)} \\
&\leq C \mathcal{F} \|\lambda_N^h\|_{L^2(\Gamma_C)} h \|u\|_{(H^{3/2}(\Omega))^2}.
\end{aligned}$$

Besides, using the (global) $L^2(\Gamma_C)$ -projection operator π_N^h onto X_N^h (the notation π^h stands for the $(L^2(\Gamma_C))^2$ -projection operator onto X^h) and an inverse inequality (see, e.g., [7, 9, 13]) we write:

$$\begin{aligned}
\|\lambda_N^h\|_{L^2(\Gamma_C)} &\leq \|\lambda_N^h - \pi_N^h \lambda_N\|_{L^2(\Gamma_C)} + \|\pi_N^h \lambda_N - \lambda_N\|_{L^2(\Gamma_C)} + \|\lambda_N\|_{L^2(\Gamma_C)} \\
&\leq C \left(h^{-1/2} \|\lambda^h - \pi^h \lambda\|_{-a, \Gamma_C} + \|\lambda_N\|_{L^2(\Gamma_C)} \right) \\
&\leq C \left(h^{-1/2} \|\lambda - \lambda^h\|_{-a, \Gamma_C} + \|u\|_{(H^{3/2}(\Omega))^2} \right).
\end{aligned}$$

Therefore:

$$\int_{\Gamma_C} \mathcal{F} \lambda_N^h (|u_T| - |I^h u_T|) d\Gamma \leq C \mathcal{F} h^{1/2} \|u\|_{(H^{3/2}(\Omega))^2} \left(\|\lambda - \lambda^h\|_{-a, \Gamma_C} + h^{1/2} \|u\|_{(H^{3/2}(\Omega))^2} \right).$$

We come to the conclusion that the term dealing with the friction approximation in (23) is bounded as follows:

$$\begin{aligned}
\int_{\Gamma_C} (\lambda_T - \lambda_T^h) \cdot (u_T - u_T^h) d\Gamma &\leq C_\alpha \mathcal{F} \|\xi\|_a \|\lambda - \lambda^h\|_{-a, \Gamma_C} \|u - u^h\|_a + C(1 + \mathcal{F})h \|u\|_{(H^{3/2}(\Omega))^2}^2 \\
&\quad + C(1 + \mathcal{F})h^{1/2} \|u\|_{(H^{3/2}(\Omega))^2} \|\lambda - \lambda^h\|_{-a, \Gamma_C}. \tag{35}
\end{aligned}$$

Finally, the result is obtained by using (25)–(26) and putting together (23), (24), (32) and (35). \blacksquare

Remark 5 *The quasi-optimal rate of convergence of order 1/2 in the theorem does not depend on $\varepsilon > 0$. Actually we are not able to obtain a better convergence rate even if ε increases. A bit more regularity than $H^{3/2}$ is needed to apply the trace theorem, when writing $\|\lambda_N\|_{L^2(\Gamma_C)} \leq C \|u\|_{(H^{3/2}(\Omega))^2}$ (see [20]). The choice of the regularity assumptions $u \in (H^{(3/2)+\varepsilon}(\Omega))^2$ in the numerical analysis of contact problems is discussed in [4], Remark 2.4 (i) and [5], Remark 4.4. If u is less regular than $H^{3/2}$ then the normal and tangential constraints cannot be expressed pointwise and the frictional contact conditions cannot be simply written as in (5)–(7). In the frictionless case, when $u \in (H^\nu(\Omega))^2$ with $1 < \nu < 3/2$ the error analysis of a finite element approximation is achieved in [4]. Actually we are not able to extend these results to the frictional case.*

Remark 6 *If one (or both) end points of $\overline{\Gamma_C} = [x_0, x_n]$ is subjected to Dirichlet conditions then the previous study can be extended with some modifications. Suppose for instance that $\overline{\Gamma_C} \cap \overline{\Gamma_D} = \{x_0\}$ and that the definition of V^h in (14) remains unchanged. If we still keep the same definition of X^h as in (15) then the estimate (26) does not hold in the general case. So we use a mortar approach introduced in [7]: denoting by $x_i, 0 \leq i \leq n$ the nodes on $\overline{\Gamma_C}$, we set:*

$$X_N^h = \left\{ \mu_N^h \in \mathcal{C}(\overline{\Gamma_C}), \mu_N^h|_{[x_i, x_{i+1}]} \in P_1([x_i, x_{i+1}]), \forall 1 \leq i \leq n-1, \mu_N^h|_{[x_0, x_1]} \in P_0([x_0, x_1]) \right\}.$$

The particularity of this space is that the functions are constant on the extreme segment $[x_0, x_1]$. We choose the same kind of approximation for X_T^h and we set $X^h = X_N^h \times X_T^h$. In this case the discrete Babuška-Brezzi inf-sup condition (19) still holds (see [3]). Moreover, estimate (26) remains valid (see [7], Lemma 4.1). Keeping the same definitions of Λ_N^h and $\Lambda_T^h(\mathcal{F}\lambda_N^h)$ as in (17) and (18), we note that the equivalence in Lemma 2 still holds in this case since the dimensions of the multiplier and tangential displacement spaces are the same (see also Remark 1) and the inf-sup condition is satisfied. According to [30], Problem (16) admits a solution for any friction coefficient and the solution is unique for a sufficiently small friction coefficient. The following result is then obtained: if (u^h, λ^h) is a solution to (16) then estimate (22) is recovered.

Remark 7 In the three-dimensional case, the convergence result should hold (at least when $\overline{\Gamma_C} \cap \overline{\Gamma_D} = \emptyset$) and the main task would be to generalize the estimate (29).

Remark 8 If $\mathcal{F} = 0$ then the continuous problem admits a unique solution. Choosing then the same approximation method as in (16) (therefore $\lambda_T^h = 0$ and the discrete solution is unique) and accomplishing the convergence analysis has led to an upper bound of the error of order $h^{1/2}$ under H^2 -regularity hypotheses (see [6]). The estimate obtained in the present paper improves the bound in [6] since we obtain the same convergence rate with less regularity assumptions ($H^{(3/2)+\varepsilon}$ with ε arbitrary small instead of H^2). Moreover we observe that there is no loss of convergence when the friction terms are added. Nevertheless we mention that there exist in the frictionless case a standard finite element approximation which leads to an upper bound of the error of order $h^{3/4}$ under H^2 -regularity hypotheses (see [24, 23]) and of order h with some additional assumptions concerning the finiteness of transition points between contact and separation (see [29]). Actually we are not able to extend these results to the frictional case.

Remark 9 Note that we do not prove that the solution to the discrete problem is unique under the assumptions of Theorem 1. This seems to be an open question which is actually under investigation. Note also that this possible loss of uniqueness would not be embarrassing in the a priori error analysis of Theorem 1. As a matter of fact, even if there are multiple solutions to the discrete problem, any solution would converge towards the unique solution of the continuous model. Besides the bound ensuring uniqueness in Proposition 1 is $\mathcal{F} < (C_\alpha \|\xi\|_a)^{-1}$ and we establish the error estimate only for $\mathcal{F} < c_{is}(C_\alpha \|\xi\|_a)^{-1}$. It should be interesting to see whether or not it is possible to prove an error estimate for all the uniqueness cases of Proposition 1.

Conclusion

This work is a contribution to the numerical analysis of the unilateral contact problem governed by Coulomb's law of friction in elastostatics. As far as we know this study establishes a first error estimate with a convergence rate for this model. From the previous remarks we can reasonably conclude that the present convergence analysis could be generalized in many directions.

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