# Non-unique slipping in the Coulomb friction model in two-dimensional linear elasticity

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**Abstract**

This work is concerned with the Coulomb friction model in continuum linear elastostatics. We consider the two-dimensional problem and we recall that an infinity of solutions corresponding to slip may exist when the friction coefficient (or its opposite value) is an eigenvalue of a specific problem. We show that such coefficients exist and we determine them explicitly for a simple class of problems. Finally, we exhibit cases in which the static friction problem admits an infinity of solutions slipping in the same direction.

*Keywords* : Coulomb friction, linear elasticity, multiple solutions of slip type.

## 1. Introduction

The Coulomb friction model [1] is currently chosen when studying contact problems involving friction in solid mechanics. In the simplified case of static linear elasticity this law is generally considered together with the Signorini or unilateral contact model ([2]) which describes the possible separation of the body from the contact surface (or a rigid foundation) and the absence of penetration of the body into the surface.

In the coercive case, it has been proved in [3] (with several generalizations and improvements in [4] and [5]) that the unilateral contact problem with friction admits at least one solution under the condition that the friction coefficient is small enough. Besides, a framework exhibiting non unique solutions for large friction coefficients has been proposed and studied in [6]. In the latter reference examples of nonuniqueness involving two solutions to the same problem are shown when the friction coefficient is large enough: a solution which corresponds to separation of the body from the rigid foundation and a second one representing stick on the contact area.

In this paper we consider a different type of non unique solutions for the static case following the ideas introduced in [7] and developed in [8] where sufficient conditions

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of nonuniqueness are proposed. These sufficient conditions are linked to the existence of eigenvalues for an (a priori) non compact operator. These conditions lead to an infinity of solutions which all remain in slipping contact in the same direction. The main aim of this paper is to show that such eigenvalues exist. This allows us to construct explicitly some examples of non unique solutions with slip for precisely chosen friction coefficients.

The paper is outlined as follows. The frictional contact problem is set in Section 2. We introduce the eigenvalue problem in Section 3 and we recall that an infinity of solutions may exist when the friction coefficient (or its opposite value) is an eigenvalue. In Section 4 we prove that such friction coefficients exist in a simple case involving triangular or quadrilateral bodies and linear displacement fields. Section 5 is concerned with examples in which the static friction problem admits an infinity of solutions with slip in the same direction.

## 2. Setting of the problem

Let a domain $\Omega$ in $\mathbb{R}^2$ be given which represents the initial unconstrained configuration of an elastic body. The boundary of $\Omega$ consists of three non-overlapping parts $\Gamma_D, \Gamma_N$ and $\Gamma_C$ and the measures of $\Gamma_D$ and $\Gamma_C$ are positive. The body $\Omega$ is acted on by surface traction forces $\mathbf{F}$ on $\Gamma_N$, it is submitted to given displacements $\mathbf{U}$ on $\Gamma_D$ and the body forces are denoted by $\boldsymbol{f}$. The part $\Gamma_C$ is considered in the initial unconstrained configuration as the candidate contact surface on a rigid foundation (i.e. the contact zone cannot enlarge during the deformation process). The contact is assumed to be frictional, governed by the Coulomb law with Signorini contact conditions, and the stick, slip and separation zones on $\Gamma_C$ are not known in advance. Let the notation $\mu \geq 0$ stand for the given friction coefficient on $\Gamma_C$ (the case $\mu = 0$ corresponds to absence of friction). The unit outward normal and tangent vectors on the boundary of $\Omega$ are $\boldsymbol{n} = (n_x, n_y)$ and $\boldsymbol{t} = (-n_y, n_x)$ respectively.

The Coulomb friction problem with unilateral contact conditions is to find the displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^2$ satisfying (2.1)–(2.6):

$$\mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} = \mathbf{0} \quad \text{in } \Omega, \tag{2.1}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \mathbf{C}\, \boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{2.2}$$

where the notation $\boldsymbol{\sigma}(\boldsymbol{u})$ represents the stress tensor field lying in the space of second order symmetric tensors on $\mathbb{R}^2$. The linearized strain tensor field is $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{\nabla}^T \boldsymbol{u})/2$ and $\mathbf{C}$ is the fourth order symmetric and elliptic tensor of linear elasticity. Next we define the Dirichlet and Neumann conditions:

$$\boldsymbol{u} = \mathbf{U} \quad \text{on } \Gamma_D, \tag{2.3}$$

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \mathbf{F} \quad \text{on } \Gamma_N. \tag{2.4}$$

The following notation is adopted for any displacement field $\boldsymbol{u}$ and for any density of surface forces $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n}$ defined on the boundary of $\Omega$:

$$\boldsymbol{u} = u_n \boldsymbol{n} + u_t \boldsymbol{t} \quad \text{and} \quad \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \sigma_n(\boldsymbol{u})\boldsymbol{n} + \sigma_t(\boldsymbol{u})\boldsymbol{t}.$$

On $\Gamma_C$, the three conditions representing unilateral contact are as follows

$$\begin{cases} u_n \le 0, \\ \sigma_n(\boldsymbol{u}) \le 0, \\ \sigma_n(\boldsymbol{u})\, u_n = 0, \end{cases} \tag{2.5}$$

and the Coulomb friction law on $\Gamma_C$ is summarized by the following conditions:

$$\begin{cases} u_t = 0 \implies |\sigma_t(\boldsymbol{u})| \le \mu|\sigma_n(\boldsymbol{u})|, \\ u_t \neq 0 \implies \sigma_t(\boldsymbol{u}) = -\mu|\sigma_n(\boldsymbol{u})|\dfrac{u_t}{|u_t|}. \end{cases} \tag{2.6}$$

When $\mu = 0$ the friction law in (2.6) simply reduces to the condition $\sigma_t(\boldsymbol{u}) = 0$ and the problem admits a unique solution ([9]). Moreover it is easy to see that the solution $\boldsymbol{u} = \mathbf{0}$ is unique when $\mathbf{U} = \mathbf{F} = \boldsymbol{f} = \mathbf{0}$.

**Remark 2.1** *Note that the true Coulomb friction law involves the tangential contact velocities and not the tangential displacements. However, a problem analogous to the one discussed here is obtained by time discretization of the quasi-static frictional contact evolution problem. In this case* $\boldsymbol{u}$, $\boldsymbol{f}$ *and* $\mathbf{F}$ *stand for* $\boldsymbol{u}((i+1)\Delta t)$, $\boldsymbol{f}((i+1)\Delta t)$ *and* $\mathbf{F}((i+1)\Delta t)$ *respectively and* $u_t$ *has to be replaced by* $u_t((i+1)\Delta t) - u_t(i\Delta t)$, *where* $\Delta t$ *denotes the time step. For simplicity and without any loss of generality only the static case described above will be considered in the following.*

## 3. Sufficient conditions of existence of infinitely many solutions for precise friction coefficients

First we consider a solution $\boldsymbol{u}$ of the unilateral contact problem with friction (2.1)–(2.6) which corresponds to slip in a given direction. Therefore the body comes into contact with the rigid foundation on the entire area $\Gamma_C$ and $\boldsymbol{u}$ solves the following system of equations and conditions:

$$\begin{cases} \mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} = \mathbf{0} \quad \text{in } \Omega, \\ \boldsymbol{\sigma}(\boldsymbol{u}) = \mathbf{C}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \\ \boldsymbol{u} = \mathbf{U} \quad \text{on } \Gamma_D, \\ \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \mathbf{F} \quad \text{on } \Gamma_N, \\ u_n = 0 \quad \text{on } \Gamma_C, \\ \sigma_t(\boldsymbol{u}) = -\mu|\sigma_n(\boldsymbol{u})|\dfrac{u_t}{|u_t|} \quad \text{on } \Gamma_C. \end{cases} \tag{3.1}$$

Moreover we have to suppose that there exist two constants $C > 0$ and $C' > 0$ such that

$$\begin{cases} |u_t| \ge C \quad \text{on } \Gamma_C, \\ \sigma_n(\boldsymbol{u}) \le -C' \quad \text{on } \Gamma_C. \end{cases} \tag{3.2}$$

Next, we consider the following eigenvalue problem which consists of finding $(\lambda, \mathbf{\Phi}) \in \mathbb{C} \times ((H^1(\Omega))^2 - \{\mathbf{0}\})$ such that:

$$\left\{ \begin{aligned} \mathbf{div}\, \boldsymbol{\sigma}(\mathbf{\Phi}) &= \mathbf{0} \quad \text{in } \Omega, \\ \boldsymbol{\sigma}(\mathbf{\Phi}) &= \mathbf{C}\, \boldsymbol{\varepsilon}(\mathbf{\Phi}) \quad \text{in } \Omega, \\ \mathbf{\Phi} &= \mathbf{0} \quad \text{on } \Gamma_D, \\ \boldsymbol{\sigma}(\mathbf{\Phi})\boldsymbol{n} &= \mathbf{0} \quad \text{on } \Gamma_N, \\ \Phi_n &= 0 \quad \text{on } \Gamma_C, \\ \sigma_t(\mathbf{\Phi}) &= \lambda \sigma_n(\mathbf{\Phi}) \quad \text{on } \Gamma_C. \end{aligned} \right. \tag{3.3}$$

It can be checked that the problem (3.3) is to find $(\lambda, \mathbf{\Phi}) \in \mathbb{C} \times ((H^1(\Omega))^2 - \{\mathbf{0}\})$ verifying:

$$T(\mathbf{\Phi}) = \frac{1}{\lambda} \mathbf{\Phi}, \tag{3.4}$$

where the operator $T : \mathbf{V} \mapsto \mathbf{V}$ with $\mathbf{V} = \{\boldsymbol{v} \in (H^1(\Omega))^2 : \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_D, v_n = 0 \text{ on } \Gamma_C, \mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{v}) = \mathbf{0} \text{ in } \Omega, \boldsymbol{\sigma}(\boldsymbol{v})\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma_N\}$, is formally defined as follows: for any $\mathbf{\Phi}$, $T(\mathbf{\Phi})$ is the unique solution of the variational equality

$$\int_\Omega (\mathbf{C}\, \boldsymbol{\varepsilon}(T(\mathbf{\Phi})) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, d\Omega = \int_{\Gamma_C} \sigma_n(\mathbf{\Phi})\, v_t\, d\Gamma, \quad \forall \boldsymbol{v} \in \mathbf{V},$$

where : denotes the canonical inner product in the space of second order tensor fields on $\mathbb{R}^2$.

The following result obtained in [8] gives sufficient conditions for the nonuniqueness of the equilibrium solution $\boldsymbol{u}$ to the problem (2.1)–(2.6) under assumptions which require that the friction coefficient $\mu$ (or $-\mu$) is an eigenvalue of the problem (3.3). Note that the framework in this study deals only with "regular" solutions. Consequently, the normal and tangential stresses we consider on the contact zone are at least defined almost everywhere.

**Proposition 3.1** *Let $\boldsymbol{u}$ be a displacement field satisfying the conditions* (3.1) *and* (3.2)*. Let $(\lambda, \mathbf{\Phi})$ be a solution of the problem* (3.3) *with $\sigma_n(\mathbf{\Phi}) \in L^\infty(\Gamma_C)$ and $\Phi_t \in L^\infty(\Gamma_C)$.*

*If $\lambda = \mu\, sgn(u_t)$ (where $sgn(u_t)$ denotes the sign of $u_t$ on $\Gamma_C$) then there exists $\delta_0 > 0$ such that $\boldsymbol{u} + \delta\mathbf{\Phi}$ is solution of* (2.1)–(2.6) *for any $\delta$ satisfying $|\delta| < \delta_0$.*

**Proof.** We recall the short proof of this result established in [8] to render the paper self-contained. The displacement field $\boldsymbol{u} + \delta\mathbf{\Phi}$ satisfies (2.1)–(2.4) for any $\delta \in \mathbb{R}$. Next we check that $\boldsymbol{u} + \delta\mathbf{\Phi}$ satisfy conditions (2.5)–(2.6) when (3.1)–(3.2) hold, $\lambda = \mu\, sgn(u_t)$ and $|\delta|$ is small enough. On $\Gamma_C$ we have:

$$u_n + \delta\Phi_n = 0 \quad \text{and} \quad \sigma_n(\boldsymbol{u} + \delta\mathbf{\Phi}) \leq -C' + |\delta\sigma_n(\mathbf{\Phi})|$$

and (2.5) holds when $|\delta| \leq C'/\|\sigma_n(\boldsymbol{\Phi})\|_{L^\infty(\Gamma_C)} = \delta_1$. Consider now the conditions (2.6) on $\Gamma_C$. Since $|u_t| \geq C$, we get

$$u_t + \delta\Phi_t \neq 0$$

when $|\delta| < C/\|\Phi_t\|_{L^\infty(\Gamma_C)} = \delta_2$. Assuming that the two previous smallness assumptions on $|\delta|$ hold and since $\lambda = \mu \operatorname{sgn}(u_t)$ we deduce

$$\begin{aligned}
\sigma_t(\boldsymbol{u} + \delta\boldsymbol{\Phi}) = \sigma_t(\boldsymbol{u}) + \sigma_t(\delta\boldsymbol{\Phi}) &= \mu\sigma_n(\boldsymbol{u})\frac{u_t}{|u_t|} + \mu \operatorname{sgn}(u_t)\sigma_n(\delta\boldsymbol{\Phi}) \\
&= \mu\sigma_n(\boldsymbol{u} + \delta\boldsymbol{\Phi})\frac{u_t}{|u_t|} \\
&= -\mu|\sigma_n(\boldsymbol{u} + \delta\boldsymbol{\Phi})|\frac{u_t + \delta\Phi_t}{|u_t + \delta\Phi_t|}.
\end{aligned}$$

Hence $\boldsymbol{u} + \delta\boldsymbol{\Phi}$ solves (2.1)–(2.6) when $|\delta| < \delta_0 = \min(\delta_1, \delta_2)$. □

Let us finally mention that an eigenfunction $\boldsymbol{\Phi}$ solving (3.3) is never a rigid body displacement since $\boldsymbol{\Phi} = \mathbf{0}$ on $\Gamma_D$ which is of positive measure and according to the Korn inequality.

## 4. Examples of eigenvalues and eigenfunctions

The existence of (real) eigenvalues to the problem (3.3) is an open question since the operator $T$ in (3.4) is neither selfadjoint nor compact. Nevertheless we show in this section that the theory can be successfully illustrated in the case when $\Omega$ is a triangle (in which the edges represent $\Gamma_D$, $\Gamma_N$ and $\Gamma_C$) and the displacement field $\boldsymbol{\Phi}$ is linear. The case in which $\Omega$ is a trapezoid leads also to the same kind of results and it is discussed at the end of this section. So we look after pairs $(\lambda, \boldsymbol{\Phi})$ satisfying (3.3).

We consider the triangle $\Omega$ of vertexes $A = (0,0)$, $B = (1,0)$ and $C = (x_c, y_c)$ with $y_c > 0$ and we define $\Gamma_D = ]B, C[$, $\Gamma_N = ]A, C[$, $\Gamma_C = ]A, B[$. The body $\Omega$ lies on a rigid foundation, the half-space delimited by the straight line $(A, B)$ as suggested in Figure 1.

We suppose that the body $\Omega$ is governed by Hooke's law concerning homogeneous isotropic materials so that (2.2) becomes

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \frac{E\nu}{(1-2\nu)(1+\nu)} tr(\boldsymbol{\varepsilon}(\boldsymbol{u}))\mathbf{I} + \frac{E}{1+\nu}\boldsymbol{\varepsilon}(\boldsymbol{u}) \tag{4.1}$$

where $\mathbf{I}$ represents the identity matrix, $tr$ is the trace operator, $E$ and $\nu$ denote Young's modulus and Poisson ratio, respectively with $E > 0$ and $0 \leq \nu < 1/2$. Let $(x = (1,0), y = (0,1))$ stand for the canonical basis of $\mathbb{R}^2$.

We begin with the determination of the eigenfunction $\boldsymbol{\Phi} = (\Phi_x, \Phi_y)$ in (3.3). Since $\boldsymbol{\Phi}$ is linear and $\Phi_y = 0$ on $\Gamma_C \cup \Gamma_D$, we get $\Phi_y = 0$ in $\Omega$. Moreover $\Phi_x = 0$ in $\Gamma_D = ]B, C[$. Hence

$$\Phi_x = y_c x + (1 - x_c)y - y_c, \tag{4.2}$$

$$\Phi_y = 0. \tag{4.3}$$

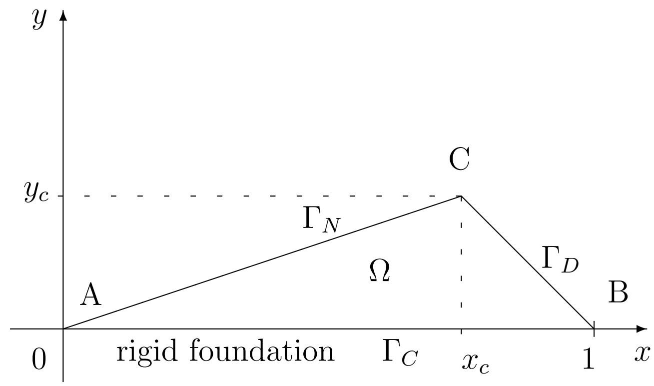


Figure 1: The geometry of the body $\Omega$

Inserting now the expressions (4.2)–(4.3) of $\mathbf{\Phi}$ in the constitutive law (4.1) yields

$$\boldsymbol{\sigma}(\mathbf{\Phi}) = \frac{E}{1+\nu}\begin{pmatrix} \dfrac{y_c(1-\nu)}{1-2\nu} & \dfrac{1-x_c}{2} \\ \dfrac{1-x_c}{2} & \dfrac{\nu\, y_c}{1-2\nu} \end{pmatrix}, \tag{4.4}$$

and $\mathbf{div}\,(\boldsymbol{\sigma}(\mathbf{\Phi})) = \mathbf{0}$. Then we consider the Neumann condition: $\boldsymbol{\sigma}(\mathbf{\Phi})\boldsymbol{n} = \mathbf{0}$ on $\Gamma_N$. Since the unit outward normal vector on $\Gamma_N$ is $\boldsymbol{n} = (-y_c/\sqrt{x_c^2+y_c^2}, x_c/\sqrt{x_c^2+y_c^2})$, the stress vector on $\Gamma_N$ becomes

$$\boldsymbol{\sigma}(\mathbf{\Phi})\boldsymbol{n} = \begin{pmatrix} \dfrac{E(2\nu y_c^2 - 2y_c^2 - x_c^2 + 2x_c^2\nu + x_c - 2x_c\nu)}{2(1-2\nu)(1+\nu)\sqrt{x_c^2+y_c^2}} \\ \dfrac{Ey_c(x_c - 1 + 2\nu)}{2(1-2\nu)(1+\nu)\sqrt{x_c^2+y_c^2}} \end{pmatrix}.$$

Keeping in mind that $0 \le \nu < 1/2$ and $y_c > 0$, the Neumann condition is equivalent to the two following equalities (4.5) and (4.6):

$$\nu = \frac{1-x_c}{2}, \tag{4.5}$$

$$y_c = x_c\sqrt{\frac{1-x_c}{1+x_c}}. \tag{4.6}$$

Hence

$$x_c \in ]0,1[, \quad y_c = x_c\sqrt{\frac{1-x_c}{1+x_c}}. \tag{4.7}$$

The admissible line $\gamma$ in which are located the pairs $(x_c, y_c)$ satisfying (4.7) is depicted in Figure 2.

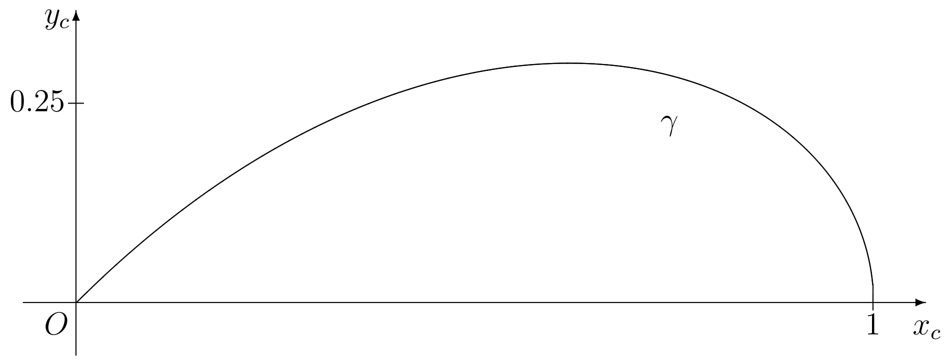


Figure 2: The admissible line $\gamma$ for point $C = (x_c, y_c)$.

In this case the normal and tangential constraints on $\Gamma_C$ given by (4.4) (with $\boldsymbol{n} = (0, -1)$ and $\boldsymbol{t} = (1, 0)$) become

$$\sigma_n(\boldsymbol{\Phi}) = \frac{E(1-x_c)}{3-x_c}\sqrt{\frac{1-x_c}{1+x_c}}$$

$$\sigma_t(\boldsymbol{\Phi}) = \frac{E(1-x_c)}{x_c-3}$$

so that

$$\frac{\sigma_t(\boldsymbol{\Phi})}{\sigma_n(\boldsymbol{\Phi})} = -\sqrt{\frac{1+x_c}{1-x_c}} = -\sqrt{\frac{1-\nu}{\nu}}.$$

The discussion is summarized in the following proposition.

**Proposition 4.1** *Let be given the triangle $\Omega$ of vertexes $A = (0,0)$, $B = (1,0)$ and $C = (x_c, x_c\sqrt{(1-x_c)/(1+x_c)})$ with $x_c \in ]0,1[$. Set $\Gamma_D = ]B, C[$, $\Gamma_N = ]A, C[$, $\Gamma_C = ]A, B[$. Let $E > 0$ and $\nu = (1-x_c)/2$.*

*Then the problem (3.3) admits at least an eigenpair $(\lambda, \boldsymbol{\Phi})$ with $\lambda < 0$ and $\boldsymbol{\Phi}$ linear. More precisely:*

$$\lambda = -\sqrt{\frac{1+x_c}{1-x_c}}$$

*and*

$$\boldsymbol{\Phi} = \left( x_c\sqrt{\frac{1-x_c}{1+x_c}}(x-1) + (1-x_c)y, 0 \right). \tag{4.8}$$

**Remark 4.2** *If instead a triangle we consider a trapezoid of vertexes $A = (0,0)$, $B' = (\theta, 0)$, $C = (x_c, y_c)$ and $D = (x_c + \theta(1-x_c), (1-\theta)y_c)$ with $y_c > 0$ and $0 < \theta < 1$ the discussion is the same as previously. In fact it suffices to define $\Gamma_D = ]C, D[$, $\Gamma_N = ]A, C[ \cup ]B', D[$, $\Gamma_C = ]A, B'[$ and to observe that the lines $AC$ and $B'D$ are parallel (see Figure 3). The result of the proposition is still valid in this case.*

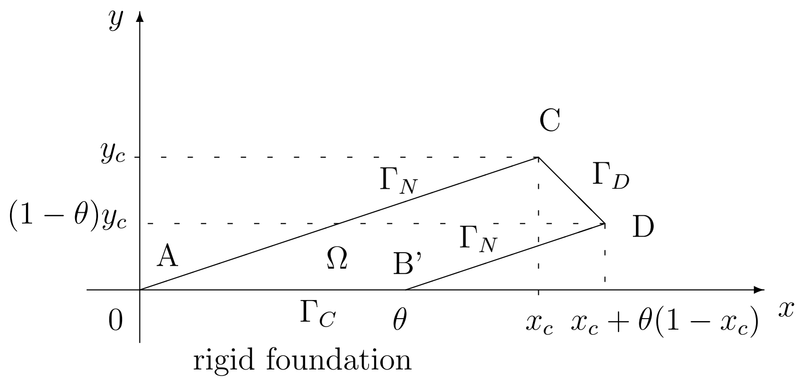


Figure 3: Case where $\Omega$ is a trapezoid

**Remark 4.3** *The eigenvalue $\lambda$ in the proposition varies between $-1$ and $-\infty$ when $x_c$ describes the interval $]0,1[$ (or equivalently when $\nu$ varies between $1/2$ and $0$). According to Proposition 3.1 such values of $\lambda$ correspond to friction coefficients $\mu$ in $]1,\infty[$.*

## 5. Examples of infinitely many solutions with slip

Afterwards we look after fields $\boldsymbol{u}$ and $\boldsymbol{\Phi}$ satisfying the assumptions of Proposition 3.1 in order to exhibit some examples of non-unique solutions to the frictional contact problem with slip. So we consider the triangular or quadrangular geometries of the previous section (see Figures 1 and 3).

Having at our disposal from Proposition 4.1 an eigenpair $(\lambda, \boldsymbol{\Phi})$ satisfying (3.3), we now focus on the field $\boldsymbol{u} = (u_x, u_y)$ solving problem (3.1)–(3.2). To simplify our discussion, we search a linear displacement field $\boldsymbol{u}$. Since $u_y = 0$ on $\Gamma_C$, it can be written

$$u_x = ax + by + c, \tag{5.1}$$

$$u_y = dy, \tag{5.2}$$

with $a$, $b$, $c$, $d$ in $\mathbb{R}$. According to Propositions 3.1 and 4.1 the friction coefficient is necessarily

$$\mu = \sqrt{\frac{1+x_c}{1-x_c}}$$

and $u_t < 0$ on $\Gamma_C$ so that $c < 0$ and $a + c < 0$.

Inserting the expression of $\boldsymbol{u}$ in the constitutive law (4.1) and since $\nu = (1-x_c)/2$, we get (with $\boldsymbol{n} = (0,-1)$ and $\boldsymbol{t} = (1,0)$)

$$\sigma_n(\boldsymbol{u}) = \frac{E((1-x_c)a + (1+x_c)d)}{x_c(3-x_c)}, \quad \sigma_t(\boldsymbol{u}) = \frac{Eb}{x_c - 3},$$

hence $(1-x_c)a + (1+x_c)d < 0$ and $b < 0$.

Writing $\sigma_t(\boldsymbol{u}) = -\mu\sigma_n(\boldsymbol{u})$ on $\Gamma_C$ yields the conditions:

$$b = \sqrt{\frac{1+x_c}{1-x_c}}\,\frac{(1-x_c)a + (1+x_c)d}{x_c}.$$

The displacement field $\mathbf{U} = (U_x, U_y)$ on $\Gamma_D$ becomes

$$U_x = ax + \left(\sqrt{\frac{1+x_c}{1-x_c}}\,\frac{(1-x_c)a + (1+x_c)d}{x_c}\right) y + c, \tag{5.3}$$

$$U_y = dy. \tag{5.4}$$

In the case where $\Omega$ is a triangle, the densities of surface forces $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \mathbf{F} = (F_x, F_y)$ on the boundary part $\Gamma_N$ with $\boldsymbol{n} = (-\sqrt{(1-x_c)/2}, \sqrt{(1+x_c)/2})$ are then

$$F_x = \frac{2\sqrt{2}Ed}{\sqrt{1-x_c}(3-x_c)}, \tag{5.5}$$

$$F_y = 0. \tag{5.6}$$

In the case where $\Omega$ is a trapezoid, it suffices to consider also the second part of $\Gamma_N$ which is precisely the straight line segment $B'D$ depicted in Figure 3 with $\boldsymbol{n} = (\sqrt{(1-x_c)/2}, -\sqrt{(1+x_c)/2})$. Obviously $\mathbf{div}\,(\boldsymbol{\sigma}(\boldsymbol{u})) = \mathbf{0}$. Combining (5.1)–(5.2) with (4.8) yields the expression of $\boldsymbol{u} + \delta\boldsymbol{\Phi} = ((u+\delta\Phi)_x, (u+\delta\Phi)_y)$:

$$(u+\delta\Phi)_x = \left(a + \delta x_c\sqrt{\frac{1-x_c}{1+x_c}}\right)x + \left(\sqrt{\frac{1+x_c}{1-x_c}}\,\frac{(1-x_c)a + (1+x_c)d}{x_c} + \delta(1-x_c)\right)y$$
$$+c - \delta x_c\sqrt{\frac{1-x_c}{1+x_c}}, \tag{5.7}$$

$$(u+\delta\Phi)_y = dy, \tag{5.8}$$

with

$$c < 0,\ a + c < 0,\ (1-x_c)a + (1+x_c)d < 0. \tag{5.9}$$

Next we determine the number $\delta_0$ introduced in Proposition 3.1. From the proof of Proposition 3.1, we deduce that

$$\delta_0 = \min\left(\frac{\inf_{\Gamma_C}|\sigma_n(\boldsymbol{u})|}{\sup_{\Gamma_C}|\sigma_n(\boldsymbol{\Phi})|}, \frac{\inf_{\Gamma_C}|u_t|}{\sup_{\Gamma_C}|\Phi_t|}\right)$$
$$= \min\left(\frac{|(1-x_c)a + (1+x_c)d|\sqrt{1+x_c}}{x_c(1-x_c)\sqrt{1-x_c}}, \frac{\min(|c|, |a+c|)\sqrt{1+x_c}}{x_c\sqrt{1-x_c}}\right). \tag{5.10}$$

We finally remark that the displacement field $\boldsymbol{u} + \delta\boldsymbol{\Phi}$ moves points $A$, $B$ and $C$ to the new positions given by $\overline{A}$, $\overline{B}$ and $\overline{C}$ respectively.

$$\overline{A} = \left(c - \delta x_c\sqrt{\frac{1-x_c}{1+x_c}}, 0\right),$$

$$\overline{B} = (1 + a + c, 0),$$

$$\overline{C} = \left( (d+1)x_c + a + c + d, (1+d)x_c\sqrt{\frac{1-x_c}{1+x_c}} \right).$$

The latter discussion and the statement of Proposition 3.1 prove the existence of an infinity of solutions to the Coulomb friction problem (2.1)–(2.6) in the cases which we describe hereafter. The result is given in the case of a triangle but the same result holds also when $\Omega$ is a trapezoid according to Remark 4.2.

**Proposition 5.1** *Let be given the triangle $\Omega$ of vertexes $A = (0, 0)$, $B = (1, 0)$ and $C = (x_c, x_c\sqrt{(1 - x_c)/(1 + x_c)})$ with $x_c \in ]0, 1[$. Set $\Gamma_D = ]B, C[$, $\Gamma_N = ]A, C[$, $\Gamma_C = ]A, B[$. Let $E > 0$, $\nu = (1 - x_c)/2$, $\mathbf{U}$ and $\mathbf{F}$ given by* (5.3)–(5.4) *and* (5.5)–(5.6)*. Assume that $\boldsymbol{f} = \mathbf{0}$ and that* (5.9) *holds.*

*If the friction coefficient $\mu$ is such that $\mu = \sqrt{(1 + x_c)/(1 - x_c)}$ then the problem* (2.1)–(2.6) *admits an infinity of solutions. In particular if $\delta_0$ is given by* (5.10) *then any displacement field given by* (5.7)–(5.8) *with $|\delta| < \delta_0$ solves the problem* (2.1)–(2.6)*.*

An immediate consequence of the previous study is the following result.

**Corollary 5.2** *The Coulomb friction problem* (2.1)–(2.6) *may admit an infinity of solutions with slip when $\mu > 1$.*

Finally we illustrate the Proposition 5.1 with a simple "numerical" example and an illustration.

**Example 5.3** *Set $x_c = 3/5, E = 1, a = -1/2, c = -1/2, d = 0$. According to Proposition 5.1, we obtain $C = (3/5, 3/10), \nu = 1/5$, $\boldsymbol{f} = \mathbf{0}$ in $\Omega$, $\mathbf{F} = \mathbf{0}$ on $\Gamma_N$, $\mathbf{U} = (-1, 0)$ on $\Gamma_D$. If the friction coefficient $\mu$ is such that $\mu = 2$ then any displacement field $\boldsymbol{u}_\delta$ defined by*

$$(u_\delta)_x = \Big( -\frac{1}{2} + \frac{3}{10}\delta \Big)x + \Big( -\frac{2}{3} + \frac{2}{5}\delta \Big)y - \frac{1}{2} - \frac{3}{10}\delta,$$

$$(u_\delta)_y = 0,$$

*with $|\delta| < \delta_0 = 5/3$ solves the problem* (2.1)–(2.6)*. Figure 4 depicts the initial configuration and three of the many possible deformed configurations characterized by the position of the point $\overline{A}$. We also remark that the fields $\boldsymbol{u}_{\delta_0}$ (rigid body displacement) and $\boldsymbol{u}_{-\delta_0}$ (solution in which point A does not move) satisfy also* (2.1)–(2.6) *in this example. Finally we mention that this example involves important strains (although the small strain hypothesis has been adopted). Of course this is in order to have a better graphical representation and it can be avoided.*

**Remark 5.4** *This remark deals with some links and differences between the nonuniqueness examples for the continuous problem* (2.1)–(2.6) *obtained in this paper and the already known nonuniqueness examples in the finite dimensional context. In two space*

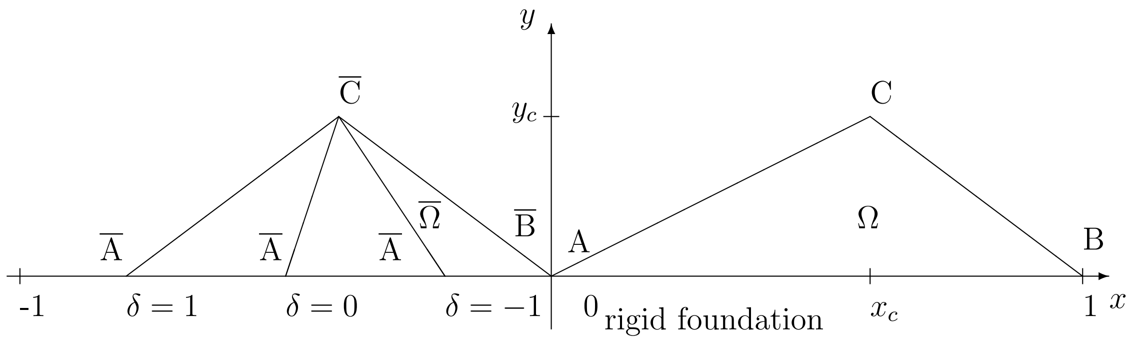


Figure 4: Initial configuration $\Omega$ and deformed configurations $\overline{\Omega}$

*dimensions, a simple example using springs obtained in* [10] *shows that non unique solutions exist when the friction coefficient exceeds a critical value and for certain directions of loading. In the more realistic three dimensional case, nonuniqueness is also exhibited and discussed together with a stability analysis in* [11]. *In* [12] *a study similar to that in* [10] *is performed using one linear finite element (which corresponds to a triangle as in this paper with* $x_c = 0$ *and* $y_c = 1$*) instead of springs: the one element model admits a unique solution when the friction coefficient is below a critical value, it admits an infinity of solutions (with slip in the same direction as in this paper) for a specific friction coefficient and for a precise direction of loading (but not for a specific material linked to a precise geometry as required in this paper). Finally the elementary finite element model shows one, two or three solutions for large friction coefficients depending on the directions of loading. Besides it can be checked that all the nonuniqueness examples in this simple discrete case are not nonuniqueness examples for the continuous problem* (2.1)–(2.6). *In fact the difference between two solutions does not satisfy zero Neumann conditions on* $\Gamma_N$.

*It can be observed that the nonuniqueness examples of* (2.1)–(2.6) *obtained in this paper are linear. Therefore these fields are also solutions of a certain finite element approximation. Of course we focus in this paper only on a specific "class" of nonuniqueness examples in continuum mechanics and there certainly exist several other cases of nonuniqueness (examples involving stick and separation are studied in* [6]*). Finally there exist to our knowledge in the simple static case very few links between the continuous model* (2.1)–(2.6) *and its finite dimensional approximation: some convergence results for small friction coefficients are established in* [13].

## 6. Conclusion

This work is a contribution to the understanding of a phenomenon in the Coulomb friction model in continuum linear elasticity. We prove the existence of an infinity of solutions slipping in the same direction for precise friction coefficients. We recall that these solutions are of the following type: $\boldsymbol{u} + \delta\boldsymbol{\Phi}$ for any small $|\delta|$. The notation $\boldsymbol{u}$

denotes a solution which slips in a given direction and $\mathbf{\Phi}$ is an eigenfunction whose associated eigenvalue is precisely the friction coefficient or its opposite value.

Nevertheless several questions linked to this study remain open. Among them, the existence of an example with such multiple solutions for arbitrary small friction coefficients seems to be an open problem. Another question concerns the size and the type (finite, countable or not, continuous) of the set of friction coefficients for which the phenomenon occurs.

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