A stabilized Lagrange multiplier method for the finite element approximation of contact problems in elastostatics

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Abstract

In this work we consider a stabilized Lagrange (or Kuhn-Tucker) multiplier method in order to approximate the unilateral contact model in linear elastostatics. The particularity of the method is that no discrete inf-sup condition is needed in the convergence analysis. We propose three approximations of the contact conditions well adapted to this method and we study the convergence of the discrete solutions. Several numerical examples in two and three space dimensions illustrate the theoretical results and show the capabilities of the method.

Key words: unilateral contact, finite elements, mixed method, stabilization, a priori error estimate.

Abbreviated title: Stabilized Lagrange multiplier method for contact problems

Mathematics subject classification: 65N30, 74M15

1 Introduction and notation

The numerical implementation of contact and impact problems in solid mechanics generally uses finite element tools (see [22, 24, 29, 38, 39, 48]). An important aspect in these simulations consists of choosing finite element methods which are both easy to implement in practice and accurate from a theoretical point of view. Our aim in this paper is to propose, study and discuss the performances of such a method. In order to focus only on the nonlinearity arising from the unilateral contact problem, we consider in what follows the simplest model: linear elasticity, small strains and no friction.

For this elementary model (or the equivalent Signorini problem) the first convergence analysis with H^1-error estimates on the displacements of a finite element method written as a variational inequality was achieved in [14] and [28] (see also [29]) in the case of linear finite elements. These previous studies were completed in [10] with a wider class of regularity assumptions and [21] with L^2-error estimates. Besides, the mixed methods in which the unknowns are the displacements and the contact pressure (or the equivalent loads at the contact nodes) showed much interest in the numerical implementation. The initial error analysis for a mixed method using continuous linear finite elements or Raviart-Thomas discontinuous elements for the displacement field and discontinuous piecewise polynomial multipliers approximating the pressure on the contact zone was achieved in [15] and [30] (see also [29]). These results were improved and/or generalized in many directions using different kind of multipliers [9, 11, 35], quadratic finite elements [8, 33] or an augmented Lagrangian [16]. In fact, any of the mixed methods cited above need an inf-sup condition (see [3, 12, 13]).

In the present work we consider a mixed finite element method which does not require an inf-sup condition. Such methods which provide stability of the multiplier by adding supplementary

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terms in the weak formulation have been originally introduced and analyzed in [36, 4, 5]. The great advantage of such methods compared to original one in [3] is that the finite element spaces on the primal and dual variables can be chosen independently. Moreover, contrary to penalization techniques, the consistency of the method is preserved. Later, the connection was made in [47] between the stabilized method of Barbosa and Hughes [4, 5] and the former one of Nitsche [43]. The studies in [4, 5] were generalized to a variational inequality framework in [6] (Signorini type problems among others). This method has also been extended to interface problems on nonmatching meshes in [7, 27] and more recently for bilateral (linear) contact problems in [32]. Our aim in this paper is to extend this concept to the unilateral contact problem in elasticity by performing a convergence analysis for various contact conditions and carrying out the corresponding numerical experiments. In addition our convergence analysis generalizes the already known estimates of the nonstabilized case.

Our paper is outlined as follows. In section 2, we introduce the continuous problem modelling the frictionless contact of a linear elastic body with a rigid foundation under the small strains hypothesis. We recall the corresponding variational inequality and the equivalent mixed formulation involving a Lagrange multiplier representing the contact pressure. In section 3, we propose an extension of "Barbosa-Hughes-Nitsche's" concept to the contact problem and we show that the corresponding discrete problem admits a unique solution. Then, we focus on the convergence analysis for a two-dimensional body and for three elementary contact conditions (each of them corresponding to an approximation of the discrete contact condition). We show that any of the approximations are convergent and that the error estimates are optimal if additional regularity assumptions are added. Several numerical experiments are achieved in section 4 dealing with a larger set of methods than in section 3 and also for a three-dimensional body.

Finally, let us introduce some useful notations. In what follows, bold letters like $\mathbf{u}, \mathbf{v}$, indicate vector or tensor valued quantities, while the capital ones (e.g., $\mathbf{V}, \mathbf{K}, \ldots$) represent functional sets involving vector fields. As usual, we denote by $(H^s(.))^d$, $s \in \mathbb{R}, d = 1, 2, 3$ the Sobolev spaces in one, two or three space dimensions (see [1]). The usual norm of $(H^s(D))^d$ (dual norm if $s < 0$) is denoted by $\| \cdot \|_{s,D}$ and we keep the same notation when $d = 1$, $d = 2$ or $d = 3$. The symbol $| \cdot |$ will denote either the length of a line segment or the area of a plane domain.

2 The continuous problem

We consider an elastic body Ω in $\mathbb{R}^2$ where plane small strain assumptions are made. The boundary $\partial\Omega$ of Ω is polygonal and we suppose that $\partial\Omega$ consists in three nonoverlapping parts Γ_D, Γ_N and the contact boundary Γ_C with $\text{meas}(\Gamma_D) > 0$ and $\text{meas}(\Gamma_C) > 0$. The contact boundary is supposed to be a straight line segment. The normal unit outward vector on $\partial\Omega$ is denoted $\mathbf{n} = (n_1, n_2)$ and we choose as unit tangent vector $\mathbf{t} = (-n_2, n_1)$. In its initial stage, the body is in contact on Γ_C with a rigid foundation (the extension to two elastic bodies in contact can be easily made, at least for small strain models) and we suppose that the unknown final contact zone after deformation will be included into Γ_C. The body is clamped on Γ_D for the sake of simplicity. It is subjected to volume forces $\mathbf{f} = (f_1, f_2) \in (L^2(\Omega))^2$ and to surface loads $\mathbf{g} = (g_1, g_2) \in (L^2(\Gamma_N))^2$.

The unilateral contact problem in linear elasticity consists in finding the displacement field

$\mathbf{u} : \Omega \to \mathbb{R}^2$ verifying the equations and conditions (1)–(5):

$$\begin{aligned}
&(1) & \mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f} &= \mathbf{0} && \text{in } \Omega,\\
&(2) & \boldsymbol{\sigma}(\mathbf{u}) &= \mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}) && \text{in } \Omega,\\
&(3) & \mathbf{u} &= \mathbf{0} && \text{on } \Gamma_D,\\
&(4) & \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} &= \mathbf{g} && \text{on } \Gamma_N,
\end{aligned}$$

where $\boldsymbol{\sigma} = (\sigma_{ij}),\ 1 \le i, j \le 2$, stands for the stress tensor field and $\mathbf{div}$ denotes the divergence operator of tensor valued functions. The notation $\boldsymbol{\varepsilon}(\mathbf{v}) = (\boldsymbol{\nabla}\mathbf{v} + \boldsymbol{\nabla}\mathbf{v}^T)/2$ represents the linearized strain tensor field and $\mathbf{A}$ is the fourth order symmetric elasticity tensor having the usual uniform ellipticity and boundedness property. For any displacement field $\mathbf{v}$ and for any density of surface forces $\boldsymbol{\sigma}(\mathbf{v})\mathbf{n}$ defined on $\partial\Omega$ we adopt the following notation

$$\mathbf{v} = v_n\mathbf{n} + v_t\mathbf{t} \quad \text{and} \quad \boldsymbol{\sigma}(\mathbf{v})\mathbf{n} = \sigma_n(\mathbf{v})\mathbf{n} + \sigma_t(\mathbf{v})\mathbf{t}.$$

The conditions describing unilateral contact without friction on Γ_C are:

$$(5) \qquad u_n \le 0, \quad \sigma_n(\mathbf{u}) \le 0, \quad \sigma_n(\mathbf{u})\, u_n = 0, \quad \sigma_t(\mathbf{u}) = 0.$$

The weak variational formulation of (1)–(5) uses the Hilbert spaces

$$\mathbf{V} = \Big\{ \mathbf{v} \in \big(H^1(\Omega)\big)^2 : \quad \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D \Big\}, \qquad W = \Big\{ v_{n|\Gamma_C} : \mathbf{v} \in \mathbf{V} \Big\},$$

and their topological dual spaces $\mathbf{V}'$, W', endowed with their usual norms. Since Γ_C is a straight line segment, we have $H^{1/2}_{00}(\Gamma_C) \subset W \subset H^{1/2}(\Gamma_C)$ which implies $W' \subset H^{-1/2}(\Gamma_C)$. Classically, $H^{1/2}(\Gamma_C)$ is the space of the restrictions on Γ_C of traces on $\partial\Omega$ of functions in $H^1(\Omega)$, and $H^{-1/2}(\Gamma_C)$ is the dual space of $H^{1/2}_{00}(\Gamma_C)$ which is the space of the restrictions on Γ_C of functions in $H^{1/2}(\partial\Omega)$ vanishing outside Γ_C. We refer to [40] and [1] for a detailed presentation of trace operators and/or trace spaces.

We introduce the following convex cone of multipliers on Γ_C:

$$M^- = \Big\{ \mu \in W' : \ \langle \mu, \psi \rangle_{W',W} \ge 0 \quad \text{for all } \psi \in W, \psi \le 0 \ \text{ a.e. on } \Gamma_C \Big\},$$

where the notation $\langle .,. \rangle_{W',W}$ represents the duality pairing between W' and W. Define

$$\begin{aligned}
a(\mathbf{u}, \mathbf{v}) &= \int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v})\, d\Omega, \qquad b(\mu, \mathbf{v}) = \langle \mu, v_n \rangle_{W',W},\\
L(\mathbf{v}) &= \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\Omega + \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{v}\, d\Gamma,
\end{aligned}$$

for any $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{V}$ and μ in W'.

The mixed formulation of the unilateral contact problem without friction (1)–(5) consists then in finding $\mathbf{u} \in \mathbf{V}$ and $\lambda \in M^-$ such that

$$(6) \qquad \begin{cases} a(\mathbf{u}, \mathbf{v}) - b(\lambda, \mathbf{v}) = L(\mathbf{v}), & \forall\, \mathbf{v} \in \mathbf{V},\\[1ex] b(\mu - \lambda, \mathbf{u}) \ge 0, & \forall\, \mu \in M^-. \end{cases}$$

An equivalent formulation of (6) consists in finding $(\mathbf{u}, \lambda) \in \mathbf{V} \times M^-$ satisfying

$$\mathcal{L}(\mathbf{u}, \mu) \le \mathcal{L}(\mathbf{u}, \lambda) \le \mathcal{L}(\mathbf{v}, \lambda), \qquad \forall \mathbf{v} \in \mathbf{V},\ \forall \mu \in M^-,$$

where $\mathcal{L}(\cdot,\cdot)$ is the classical Lagrangian of the system defined as

$$\mathcal{L}(\mathbf{v},\mu) = \frac{1}{2}a(\mathbf{v},\mathbf{v}) - L(\mathbf{v}) - b(\mu,\mathbf{v}). \tag{7}$$

Another classical weak formulation of problem (1)–(5) is a variational inequality: find $\mathbf{u}$ such that

$$\mathbf{u} \in \mathbf{K}, \qquad a(\mathbf{u},\mathbf{v}-\mathbf{u}) \geq L(\mathbf{v}-\mathbf{u}), \qquad \forall \mathbf{v} \in \mathbf{K}, \tag{8}$$

where $\mathbf{K}$ denotes the closed convex cone of admissible displacement fields satisfying the noninterpenetration conditions:

$$\mathbf{K} = \big\{ \mathbf{v} \in \mathbf{V} : \quad v_n \leq 0 \text{ on } \Gamma_C \big\}.$$

The existence and uniqueness of $(\mathbf{u},\lambda)$ solution to (6) has been stated in [29]. Moreover, the first argument $\mathbf{u}$ solution to (6) is also the unique solution of problem (8) and $\lambda = \sigma_n(\mathbf{u})$.

3 Discretization with the stabilized Lagrange multiplier method

3.1 Discrete problem

Let $\mathbf{V}^h \subset \mathbf{V}$ be a family of finite dimensional vector spaces (see [19]) indexed by h coming from a family $\mathcal{T}^h$ of triangulations of the domain Ω ($h = \max_{T\in\mathcal{T}^h} h_T$ where h_T is the diameter of T). The family of triangulations is supposed regular, i.e., there exists $\sigma > 0$ such that $\forall T \in \mathcal{T}^h, h_T/\rho_T \leq \sigma$ where ρ_T denotes the radius of the inscribed circle in T. We choose standard continuous and piecewise affine functions, i.e.:

$$\mathbf{V}^h = \left\{ \mathbf{v}^h \in (C(\overline{\Omega}))^2 : \mathbf{v}^h{}_{|T} \in (P_1(T))^2, \forall T \in \mathcal{T}^h, \mathbf{v}^h = \mathbf{0} \text{ on } \Gamma_D \right\}.$$

Next, let be given $\mathbf{x}_0, ..., \mathbf{x}_N$ some distinct points lying in $\overline{\Gamma_C}$ (note that we do not suppose that these nodes coincide with some nodes of the triangulation $\mathcal{T}^h$). These nodes form a monodimensional family of meshes of Γ_C denoted T^H and we set $H = \max_{0\leq i\leq N-1} |\mathbf{x}_{i+1} - \mathbf{x}_i|$. The mesh T^H allows us to define a finite dimensional space W^H approximating W' and a nonempty closed convex set $M^{H-} \subset W^H$ approximating M^-:

$$M^{H-} = \left\{ \mu^H \in W^H : + \text{"nonpositivity condition" on } \Gamma_C \right\}.$$

Remark 3.1 *In our forthcoming study we consider two elementary possible choices of W^H. We can set either*

$$W_0^H = \left\{ \mu^H \in L^2(\Gamma_C) : \mu^H{}_{|(\mathbf{x}_i,\mathbf{x}_{i+1})} \in P_0(\mathbf{x}_i,\mathbf{x}_{i+1}), \forall 0 \leq i \leq N-1 \right\},$$

or

$$W_1^H = \left\{ \mu^H \in C(\Gamma_C) : \mu^H{}_{|(\mathbf{x}_i,\mathbf{x}_{i+1})} \in P_1(\mathbf{x}_i,\mathbf{x}_{i+1}), \forall 0 \leq i \leq N-1 \right\}.$$

From these choices, we provide three elementary definitions of M^{H-}:

$$M_0^{H-} = \left\{ \mu^H \in W_0^H : \mu^H \leq 0 \textit{ on } \Gamma_C \right\}, \tag{9}$$

$$M_1^{H-} = \left\{ \mu^H \in W_1^H : \mu^H \leq 0 \textit{ on } \Gamma_C \right\}, \tag{10}$$

$$M_{1,*}^{H-} = \left\{ \mu^H \in W_1^H : \int_{\Gamma_C} \mu^H \psi^H d\Gamma \geq 0, \forall \psi^H \in M_1^{H-} \right\}. \tag{11}$$

The discrete problem is to find $\mathbf{u}^h \in \mathbf{V}^h$ and $\lambda^H \in M^{H-}$ such that

$$(12) \qquad \left\{ \begin{array}{ll} a(\mathbf{u}^h, \mathbf{v}^h) - b(\lambda^H, \mathbf{v}^h) + \displaystyle\int_{\Gamma_C} \gamma(\lambda^H - \sigma_n(\mathbf{u}^h))\sigma_n(\mathbf{v}^h) d\Gamma = L(\mathbf{v}^h), & \forall\, \mathbf{v}^h \in \mathbf{V}^h, \\ b(\mu^H - \lambda^H, \mathbf{u}^h) + \displaystyle\int_{\Gamma_C} \gamma(\mu^H - \lambda^H)(\lambda^H - \sigma_n(\mathbf{u}^h)) d\Gamma \geq 0, & \forall\, \mu^H \in M^{H-}, \end{array} \right.$$

where γ is defined constant on each element T as $\gamma = \gamma_0 h_T$ where $\gamma_0 > 0$ is independent of h and H. Problem (12) is in fact the optimality system of the Lagrangian

$$\mathcal{L}_\gamma(\mathbf{v}^h, \mu^H) = \frac{1}{2} a(\mathbf{v}^h, \mathbf{v}^h) - L(\mathbf{v}^h) - b(\mu^H, \mathbf{v}^h) - \frac{1}{2} \int_{\Gamma_C} \gamma(\mu^H - \sigma_n(\mathbf{v}^h))^2 d\Gamma.$$

The additional term in this lagrangian compared to the classical one (7) is similar to an augmentation term but, due to its nonpositivity, it corresponds to an augmentation for the multiplier instead of the primal variable. The method is consistent in the sense that λ^H and $\sigma_n(\mathbf{u}^h)$ are both some approximations of $\lambda = \sigma_n(\mathbf{u})$ and the term is vanishing for the solution to the continuous problem. Of course this stabilization term modifies the discrete solution. In fact, it reinforces the correspondence between λ^H and $\sigma_n(\mathbf{u}^h)$.

Note that we can suppose without loss of generality that Γ_C is a straight line segment parallel to the $x-$axis. Let E be an edge of a triangle on Γ_C and let $T \in \mathcal{T}^h$ be the element containing E. Consequently we deduce, for any $\mathbf{v}^h \in \mathbf{V}^h$:

$$\begin{array}{rcl} \|\sigma_n(\mathbf{v}^h)\|_{0,E} & = & \|\sigma_{yy}(\mathbf{v}^h)\|_{0,E} \\ & = & \dfrac{|E|^{1/2}}{|T|^{1/2}} \|\sigma_{yy}(\mathbf{v}^h)\|_{0,T} \\ & \leq & C h_T^{-\frac{1}{2}} \|\sigma_{yy}(\mathbf{v}^h)\|_{0,T} \\ & = & C \left(\dfrac{\gamma}{\gamma_0}\right)^{-\frac{1}{2}} \|\sigma_{yy}(\mathbf{v}^h)\|_{0,T}. \end{array}$$

By summation on all the edges $E \subset \Gamma_C$ we get

$$(13) \qquad \|\gamma^{\frac{1}{2}} \sigma_n(\mathbf{v}^h)\|^2_{0,\Gamma_C} \leq C\gamma_0 \|\sigma_{yy}(\mathbf{v}^h)\|^2_{0,\Omega} \leq C\gamma_0 \|\mathbf{v}^h\|^2_{1,\Omega}.$$

Hence, from Korn inequality and (13), when γ_0 is small enough, there exists $C > 0$ such that for any $\mathbf{v}^h \in \mathbf{V}^h$:

$$a(\mathbf{v}^h, \mathbf{v}^h) - \int_{\Gamma_C} \gamma(\sigma_n(\mathbf{v}^h))^2 d\Gamma \geq C \|\mathbf{v}^h\|^2_{1,\Omega}.$$

The existence of a unique solution to problem (12) when γ_0 is small enough follows from the fact that $\mathbf{V}^h$ and M^{H-} are two nonempty closed convex sets, that $\mathcal{L}_\gamma(.,.)$ is continuous on $\mathbf{V}^h \times W^H$, that $\mathcal{L}_\gamma(\mathbf{v}^h, .)$ (resp. $\mathcal{L}_\gamma(., \mu^H)$) is strictly concave (resp. strictly convex) for any $\mathbf{v}^h \in \mathbf{V}^h$ (resp. for any $\mu^H \in M^{H-}$) and that $\lim_{\mathbf{v}^h \in \mathbf{V}^h, \|\mathbf{v}^h\|_{\mathbf{V}^h} \to \infty} \mathcal{L}_\gamma(\mathbf{v}^h, \mu^H) = +\infty$ for any $\mu^H \in M^{H-}$ (resp. $\lim_{\mu^H \in M^{H-}, \|\mu^H\|_{W^H} \to \infty} \mathcal{L}_\gamma(\mathbf{v}^h, \mu^H) = -\infty$ for any $\mathbf{v}^h \in \mathbf{V}^h$), see [29], pp. 338–339.

Let us define for any $\mathbf{v} \in (H^1(\Omega))^2$ and any $\mu \in L^2(\Gamma_C)$ the following norms:

$$\|\mathbf{v}\| = a(\mathbf{v}, \mathbf{v})^{1/2},$$

$$|||(\mathbf{v}, \mu)||| = \left(\|\mathbf{v}\|^2 + \|\gamma^{1/2} \mu\|^2_{0,\Gamma_C} \right)^{1/2}.$$

Remark 3.2 *In the nonstabilized case (i.e., when $\gamma_0 = 0$), the choice $M^{H-} = M^{H-}_{1,*}$ leads to a conforming (i.e. nonpositive) normal displacement u^h_n when T^H is the mesh induced by $\mathcal{T}^h$ on Γ_C. This property is not preserved in the stabilized case where a slight penetration in the rigid foundation could appear depending on the value of γ_0. Of course, when $M^{H-} = M^{H-}_0$ or $M^{H-} = M^{H-}_1$ a penetration can occur in the stabilized case since the phenomenon already occurs in the nonstabilized case.*

3.2 Convergence analysis

The following proposition yields an abstract error estimate for the stabilized mixed finite element approximation (12) of the unilateral contact problem. Note that our framework extends the one in [6] since it takes also into account the possible nonconformity of the discrete multipliers set (i.e., M^{H-} is not necessarily a subset of M^-).

Proposition 3.3 *Suppose that the solution $(\mathbf{u}, \lambda)$ to Problem (6) is such that $\lambda \in L^2(\Gamma_C)$. Let γ_0 be small enough. Then the solution $(\mathbf{u}^h, \lambda^H)$ to Problem (12) satisfies the following estimate:*

$$
\begin{aligned}
\left|\left|\left|\left(\mathbf{u}-\mathbf{u}^h, \lambda-\lambda^H\right)\right|\right|\right|^2 \leq\ & C\Bigg[\inf_{\mathbf{v}^h\in\mathbf{V}^h}\left(\left|\left|\left|\left(\mathbf{u}-\mathbf{v}^h, \sigma_n(\mathbf{u}-\mathbf{v}^h)\right)\right|\right|\right|^2 + \|\gamma^{-1/2}(u_n - v^h_n)\|^2_{0,\Gamma_C}\right) \\
&+ \inf_{\mu\in M^-}\int_{\Gamma_C}(\mu-\lambda^H)u_n d\Gamma \\
&+ \inf_{\mu^H\in M^{H-}}\int_{\Gamma_C}(\mu^H-\lambda)(u^h_n + \gamma(\lambda^H - \sigma_n(\mathbf{u}^h)))d\Gamma\Bigg],
\end{aligned}
$$

where C is a generic constant independent on h and H but depending on γ_0.

Proof. One has

$$
\|\gamma^{1/2}(\lambda-\lambda^H)\|^2_{0,\Gamma_C} = \int_{\Gamma_C}\gamma\lambda^2 d\Gamma - 2\int_{\Gamma_C}\gamma\lambda\lambda^H d\Gamma + \int_{\Gamma_C}\gamma(\lambda^H)^2 d\Gamma.
$$

From (6) and (12) one has

$$
\int_{\Gamma_C}\gamma\lambda^2 d\Gamma \leq \int_{\Gamma_C}\gamma\lambda\mu d\Gamma + \int_{\Gamma_C}(\mu-\lambda)u_n d\Gamma - \int_{\Gamma_C}\gamma(\mu-\lambda)\sigma_n(\mathbf{u})d\Gamma, \quad \forall\ \mu\in M^-,
$$

$$
\int_{\Gamma_C}\gamma(\lambda^H)^2 d\Gamma \leq \int_{\Gamma_C}\gamma\lambda^H\mu^H d\Gamma + \int_{\Gamma_C}(\mu^H-\lambda^H)u^h_n d\Gamma - \int_{\Gamma_C}\gamma(\mu^H-\lambda^H)\sigma_n(\mathbf{u}^h)d\Gamma, \quad \forall\ \mu^H\in M^{H-}.
$$

This gives

$$
\begin{aligned}
\|\gamma^{1/2}(\lambda-\lambda^H)\|^2_{0,\Gamma_C} \leq\ & \int_{\Gamma_C}\gamma(\mu-\lambda^H)\lambda d\Gamma + \int_{\Gamma_C}\gamma(\mu^H-\lambda)\lambda^H d\Gamma + \int_{\Gamma_C}(\mu-\lambda)u_n d\Gamma \\
& -\int_{\Gamma_C}\gamma(\mu-\lambda)\sigma_n(\mathbf{u})d\Gamma + \int_{\Gamma_C}(\mu^H-\lambda^H)u^h_n d\Gamma - \int_{\Gamma_C}\gamma(\mu^H-\lambda^H)\sigma_n(\mathbf{u}^h)d\Gamma \\
=\ & \int_{\Gamma_C}(\mu-\lambda^H)u_n d\Gamma + \int_{\Gamma_C}(\mu^H-\lambda)(u^h_n + \gamma(\lambda^H-\sigma_n(\mathbf{u}^h)))d\Gamma \\
& -\int_{\Gamma_C}\gamma(\lambda^H-\lambda)\sigma_n(\mathbf{u}-\mathbf{u}^h)d\Gamma \\
& +\int_{\Gamma_C}(\lambda^H-\lambda)(u_n-u^h_n)d\Gamma, \quad \forall\mu\in M^-, \forall\mu^H\in M^{H-}.
\end{aligned} \tag{14}
$$

According to (12) we have for any $\mathbf{v}^h \in \mathbf{V}^h$:

$$\begin{aligned}
\|\mathbf{u}-\mathbf{u}^h\|^2 &= a(\mathbf{u}-\mathbf{u}^h, \mathbf{u}-\mathbf{u}^h) \\
&= a(\mathbf{u}-\mathbf{u}^h, \mathbf{u}-\mathbf{v}^h) + a(\mathbf{u}-\mathbf{u}^h, \mathbf{v}^h-\mathbf{u}^h) \\
&= a(\mathbf{u}-\mathbf{u}^h, \mathbf{u}-\mathbf{v}^h) + \int_{\Gamma_C} (\lambda-\lambda^H)(v_n^h - u_n^h)d\Gamma \\
&\quad + \int_{\Gamma_C} \gamma(\lambda^H - \sigma_n(\mathbf{u}^h))\sigma_n(\mathbf{v}^h-\mathbf{u}^h)d\Gamma. \qquad (15)
\end{aligned}$$

From the addition of (14) and (15), we deduce

$$\begin{aligned}
\left|\!\left|\!\left|\left(\mathbf{u}-\mathbf{u}^h, \lambda-\lambda^H\right)\right|\!\right|\!\right|^2 \leq\ & a(\mathbf{u}-\mathbf{u}^h, \mathbf{u}-\mathbf{v}^h) + \int_{\Gamma_C} (\lambda-\lambda^H)(v_n^h - u_n)d\Gamma + \int_{\Gamma_C} (\mu-\lambda^H)u_n d\Gamma \\
& + \int_{\Gamma_C} (\mu^H-\lambda)(u_n^h + \gamma(\lambda^H - \sigma_n(\mathbf{u}^h)))d\Gamma \\
& + \int_{\Gamma_C} \gamma(\lambda-\lambda^H)\sigma_n(\mathbf{u}-\mathbf{v}^h)d\Gamma + \int_{\Gamma_C} \gamma(\lambda-\sigma_n(\mathbf{u}^h))\sigma_n(\mathbf{v}^h-\mathbf{u}^h)d\Gamma, \qquad (16)
\end{aligned}$$

for all $\mathbf{v}^h \in \mathbf{V}^h$, $\mu \in M^-$ and $\mu^H \in M^{H-}$. The last term in the previous inequality is estimated as follows by using (13) and recalling that $\lambda = \sigma_n(\mathbf{u})$:

$$\begin{aligned}
& \int_{\Gamma_C} \gamma(\lambda-\sigma_n(\mathbf{u}^h))\sigma_n(\mathbf{v}^h-\mathbf{u}^h)d\Gamma \\
\leq\ & \|\gamma^{1/2}(\sigma_n(\mathbf{u}-\mathbf{u}^h))\|_{0,\Gamma_C}\gamma_0^{1/2}\|h^{1/2}(\sigma_n(\mathbf{v}^h-\mathbf{u}^h))\|_{0,\Gamma_C} \\
\leq\ & C\gamma_0^{1/2}\|\mathbf{v}^h-\mathbf{u}^h\|\left(\|\gamma^{1/2}(\sigma_n(\mathbf{u}-\mathbf{v}^h))\|_{0,\Gamma_C} + \gamma_0^{1/2}\|h^{1/2}(\sigma_n(\mathbf{v}^h-\mathbf{u}^h))\|_{0,\Gamma_C}\right) \\
\leq\ & C\left(\gamma_0\|\mathbf{v}^h-\mathbf{u}^h\|^2 + \|\gamma^{1/2}(\sigma_n(\mathbf{u}-\mathbf{v}^h))\|_{0,\Gamma_C}^2\right) \\
\leq\ & C\left(\gamma_0\|\mathbf{u}-\mathbf{u}^h\|^2 + \gamma_0\|\mathbf{u}-\mathbf{v}^h\|^2 + \|\gamma^{1/2}(\sigma_n(\mathbf{u}-\mathbf{v}^h))\|_{0,\Gamma_C}^2\right). \qquad (17)
\end{aligned}$$

Combining (16) and (17), using Young inequality we come to the conclusion that if γ_0 is sufficiently small:

$$\begin{aligned}
& \left|\!\left|\!\left|\left(\mathbf{u}-\mathbf{u}^h, \lambda-\lambda^H\right)\right|\!\right|\!\right|^2 \\
\leq\ & C\left[\inf_{\mathbf{v}^h\in\mathbf{V}^h}\left(\|\mathbf{u}-\mathbf{v}^h\|^2 + \|\gamma^{1/2}(\sigma_n(\mathbf{u}-\mathbf{v}^h))\|_{0,\Gamma_C}^2 + \|\gamma^{-1/2}(u_n - v_n^h)\|_{0,\Gamma_C}^2\right)\right. \\
& \left. + \inf_{\mu\in M^-}\int_{\Gamma_C}(\mu-\lambda^H)u_n d\Gamma + \inf_{\mu^H\in M^{H-}}\int_{\Gamma_C}(\mu^H-\lambda)(u_n^h + \gamma(\lambda^H-\sigma_n(\mathbf{u}^h)))d\Gamma\right]. \qquad (18)
\end{aligned}$$

Hence the result. □

Remark 3.4 *Of course, if γ_0 is chosen very small, the constant C becomes very large in (18). This means that the best choice for γ_0 is a value slightly smaller than the greater value such that the coercivity is kept. The numerical tests indicate that a large range of values for γ_0 is convenient to preserve a good quality for the numerical solution and a good solvability of the discrete problem (see section 4.2).*

In the following estimates we assume $H^2(\Omega)$-regularity assumptions on the displacements. This regularity assumption is worth some comments in the following remark.

Remark 3.5 *If Γ_D and Γ_N share a common point then the solution $\mathbf{u}$ is expected to contain a singular part so that $\mathbf{u}$ is less regular than $H^2(\Omega)$. For the discussion we refer the reader to e.g., [42, 46]. Since our goal in this paper is to focus on the approximation behavior near Γ_C we can assume that this singular coefficient is zero. A second cause of singularity results from the geometry of the polygonal domains. Again, this cause is not directly connected to the contact problem, and we have simply considered the case where Γ_C is a straight line segment. This case allows to avoid nonconvex domains ([23, 41]). A last cause of nonsmoothness is more fundamental in our problem; this is the Signorini condition. For the analogous problem defined by the Laplace operator and a Signorini type boundary condition, it was proved that the solution is more regular than $H^2(\Omega)$ (see [14, 41]). Even though it has not been established, the singularity due to the transition between contact and noncontact is expected to be in $H^{5/2-\varepsilon}(\Omega)$ for any $\varepsilon > 0$ (this has been proved only in a scalar case in [41]).*

We choose $\mathbf{v}^h = I^h\mathbf{u}$ where I^h stands for the standard Lagrange interpolation operator mapping onto $\mathbf{V}^h$. Obviously we have

$$\|\mathbf{u} - I^h\mathbf{u}\| \leq Ch\|\mathbf{u}\|_{2,\Omega}. \tag{19}$$

Let E be an edge of a triangle on Γ_C and let $T \in \mathcal{T}^h$ be the element containing E.

$$\|\gamma^{-1/2}(u_n - (I^h\mathbf{u})\cdot\mathbf{n})\|_{0,E} \leq Ch_T^{-1/2}h_T^{3/2}\|u_n\|_{3/2,E} \leq Ch\|u_n\|_{3/2,E},$$

(see [19] for instance). By summation on all the edges

$$\|\gamma^{-1/2}(u_n - (I^h\mathbf{u})\cdot\mathbf{n})\|_{0,\Gamma_C} \leq Ch\|u_n\|_{3/2,\Gamma_C} \leq Ch\|\mathbf{u}\|_{2,\Omega}. \tag{20}$$

It remains then to estimate $\|\gamma^{1/2}\sigma_n(\mathbf{u} - I^h\mathbf{u})\|_{0,\Gamma_C}$. Let E be an edge of a triangle $T \in \mathcal{T}^h$. We recall the standard scaled trace inequality (see, e.g., [23, 25, 26, 31])

$$\|v\|_{0,E} \leq C\left(h_T^{-\frac{1}{2}}\|v\|_{0,T} + h_T^{\frac{1}{2}}\|\nabla v\|_{0,T}\right), \quad \forall v \in H^1(T), \tag{21}$$

where C is independent on h and T. Supposing without loss of generality that Γ_C is parallel to the $x-$axis and using the inequality (21) we deduce that (E is an edge on Γ_C of a triangle $T \in \mathcal{T}^h$):

$$\begin{aligned}
\|\sigma_n(\mathbf{u} - I^h\mathbf{u})\|_{0,E} &= \|\sigma_{yy}(\mathbf{u} - I^h\mathbf{u})\|_{0,E} \\
&\leq C\left(h_T^{-\frac{1}{2}}\|\sigma_{yy}(\mathbf{u} - I^h\mathbf{u})\|_{0,T} + h_T^{\frac{1}{2}}\|\nabla\sigma_{yy}(\mathbf{u} - I^h\mathbf{u})\|_{0,T}\right) \\
&= C\left(h_T^{-\frac{1}{2}}\|\sigma_{yy}(\mathbf{u} - I^h\mathbf{u})\|_{0,T} + h_T^{\frac{1}{2}}\|\nabla\sigma_{yy}(\mathbf{u})\|_{0,T}\right) \\
&\leq C\left(h_T^{-\frac{1}{2}}\|\mathbf{u} - I^h\mathbf{u}\|_{1,T} + h_T^{\frac{1}{2}}\|\mathbf{u}\|_{2,T}\right).
\end{aligned}$$

Hence, from [19]:

$$\|h_T^{1/2}\sigma_n(\mathbf{u} - I^h\mathbf{u})\|_{0,E} \leq Ch_T\|\mathbf{u}\|_{2,T},$$

and by summation, the following estimate holds

$$\|\gamma^{1/2}\sigma_n(\mathbf{u}-I^h\mathbf{u})\|_{0,\Gamma_C} \le Ch\|\mathbf{u}\|_{2,\Omega}. \tag{22}$$

Putting together the previous bounds (19),(20) and (22) we deduce that

$$\inf_{\mathbf{v}^h\in\mathbf{V}^h}\left(\left|\left|\left|\left(\mathbf{u}-\mathbf{v}^h,\sigma_n(\mathbf{u}-\mathbf{v}^h)\right)\right|\right|\right|^2 + \|\gamma^{-1/2}(u_n-v_n^h)\|^2_{0,\Gamma_C}\right) \le Ch^2\|\mathbf{u}\|^2_{2,\Omega}. \tag{23}$$

Finally we have to estimate the error terms in Proposition 3.3 coming from the contact approximation:

$$\inf_{\mu^H\in M^{H-}}\int_{\Gamma_C}(\mu^H-\lambda)(u_n^h+\gamma(\lambda^H-\sigma_n(\mathbf{u}^h)))d\Gamma \tag{24}$$

and

$$\inf_{\mu\in M^-}\int_{\Gamma_C}(\mu-\lambda^H)u_n d\Gamma. \tag{25}$$

In order to handle these terms, we need to distinguish the different contact conditions (i.e., we must specify the definition of M^{H-}). We consider hereafter three different discrete contact conditions.

3.2.1 First contact condition: $M^{H-}=M_0^{H-}$

We first consider the case of nonpositive discontinuous piecewise constant multipliers where M^{H-} is defined by (9). The error estimate is given next.

Theorem 3.6 *Let $(\mathbf{u},\lambda)$ be the solution to Problem (6). Assume that $\mathbf{u}\in(H^2(\Omega))^2$. Let γ_0 be small enough and let $(\mathbf{u}^h,\lambda^H)$ be the solution to the discrete problem (12) where $M^{H-}=M_0^{H-}$. Then, there exists a constant $C>0$ independent of h,H and $\mathbf{u}$ such that:*

$$\left|\left|\left|\left(\mathbf{u}-\mathbf{u}^h,\lambda-\lambda^H\right)\right|\right|\right| \le C(h+H^{3/4})\|\mathbf{u}\|_{2,\Omega}.$$

Proof. Choosing $\mu=0$ in (25) yields:

$$\inf_{\mu\in M^-}\int_{\Gamma_C}(\mu-\lambda^H)u_n d\Gamma \le -\int_{\Gamma_C}\lambda^H u_n d\Gamma \le 0.$$

In (24) we choose $\mu^H=\pi_0^H\lambda$ where π_0^H denotes the $L^2(\Gamma_C)$-projection onto W_0^H. We recall that the operator π_0^H is defined for any $v\in L^2(\Gamma_C)$ by

$$\pi_0^H v\in W_0^H, \quad \int_{\Gamma_C}(v-\pi_0^H v)\mu\, d\Gamma = 0, \quad \forall\mu\in W_0^H,$$

and satisfies the following error estimates for any $0\le r\le 1$:

$$H^{-1/2}\|v-\pi_0^H v\|_{-1/2,\Gamma_C} + \|v-\pi_0^H v\|_{0,\Gamma_C} \le CH^r\|v\|_{r,\Gamma_C}. \tag{26}$$

Obviously $\pi_0^H\lambda\in M_0^{H-}$ and

$$\begin{aligned}\inf_{\mu^H\in M_0^{H-}}\int_{\Gamma_C}(\mu^H-\lambda)(u_n^h+\gamma(\lambda^H-\sigma_n(\mathbf{u}^h)))d\Gamma \;&\le\; \int_{\Gamma_C}(\pi_0^H\lambda-\lambda)u_n^h d\Gamma \\ &+\int_{\Gamma_C}\gamma(\pi_0^H\lambda-\lambda)(\lambda^H-\sigma_n(\mathbf{u}^h))d\Gamma. \end{aligned}\tag{27}$$

The first integral term in (27) is estimated as follows using (26):

$$
\begin{aligned}
\int_{\Gamma_C} (\pi_0^H \lambda - \lambda) u_n^h d\Gamma &= \int_{\Gamma_C} (\pi_0^H \lambda - \lambda)(u_n^h - u_n) d\Gamma + \int_{\Gamma_C} (\pi_0^H \lambda - \lambda) u_n d\Gamma \\
&= \int_{\Gamma_C} (\pi_0^H \lambda - \lambda)(u_n^h - u_n) d\Gamma + \int_{\Gamma_C} (\pi_0^H \lambda - \lambda)(u_n - \pi_0^H u_n) d\Gamma \\
&\le \|\pi_0^H \lambda - \lambda\|_{-1/2,\Gamma_C} \|u_n^h - u_n\|_{1/2,\Gamma_C} + \|\pi_0^H \lambda - \lambda\|_{0,\Gamma_C} \|u_n - \pi_0^H u_n\|_{0,\Gamma_C} \qquad (28) \\
&\le C \left(H \|\lambda\|_{1/2,\Gamma_C} \|\mathbf{u} - \mathbf{u}^h\| + H^{3/2} \|\lambda\|_{1/2,\Gamma_C} \|u_n\|_{1,\Gamma_C} \right) \\
&\le C \left(H \|\mathbf{u}\|_{2,\Omega} \|\mathbf{u} - \mathbf{u}^h\| + H^{3/2} \|\mathbf{u}\|_{2,\Omega}^2 \right).
\end{aligned}
$$

Therefore, for any $\alpha > 0$, we have

$$
\int_{\Gamma_C} (\pi_0^H \lambda - \lambda) u_n^h d\Gamma \le C \left(\alpha \|\mathbf{u} - \mathbf{u}^h\|^2 + \alpha^{-1} H^{3/2} \|\mathbf{u}\|_{2,\Omega}^2 \right).
$$

Now, we consider the second integral term in (27) by using the estimates (26), (13), (22) and the trace inequality $\|\lambda\|_{1/2,\Gamma_C} \le C \|\mathbf{u}\|_{2,\Omega}$:

$$
\begin{aligned}
\int_{\Gamma_C} \gamma (\pi_0^H \lambda - \lambda)(\lambda^H - \sigma_n(\mathbf{u}^h)) d\Gamma &= \int_{\Gamma_C} \gamma (\pi_0^H \lambda - \lambda)(\lambda^H - \lambda) d\Gamma \\
&\quad + \int_{\Gamma_C} \gamma (\pi_0^H \lambda - \lambda) \sigma_n(\mathbf{u} - I^h \mathbf{u}) d\Gamma \\
&\quad + \int_{\Gamma_C} \gamma (\pi_0^H \lambda - \lambda) \sigma_n(I^h \mathbf{u} - \mathbf{u}^h) d\Gamma \\
&\le C \Big(\gamma_0^{1/2} h^{1/2} H^{1/2} \|\mathbf{u}\|_{2,\Omega} \|\gamma^{1/2} (\lambda^H - \lambda)\|_{0,\Gamma_C} \\
&\quad + \gamma_0^{1/2} h^{1/2} H^{1/2} \|\mathbf{u}\|_{2,\Omega} \|\gamma^{1/2} \sigma_n(\mathbf{u} - I^h \mathbf{u})\|_{0,\Gamma_C} \\
&\quad + \gamma_0 h^{1/2} H^{1/2} \|\mathbf{u}\|_{2,\Omega} \|\mathbf{u}^h - I^h \mathbf{u}\| \Big).
\end{aligned}
$$

Since $\|\mathbf{u}^h - I^h \mathbf{u}\| \le \|\mathbf{u} - \mathbf{u}^h\| + Ch \|\mathbf{u}\|_{2,\Omega}$ we deduce, for any small $\alpha > 0$:

$$
\begin{aligned}
&\inf_{\mu^H \in M_0^{H-}} \int_{\Gamma_C} (\mu^H - \lambda)(u_n^h + \gamma(\lambda^H - \sigma_n(\mathbf{u}^h))) d\Gamma \\
&\le C \left(\alpha (\|\mathbf{u} - \mathbf{u}^h\|^2 + \|\gamma^{1/2} (\lambda^H - \lambda)\|_{0,\Gamma_C}^2) + \alpha^{-1} (H^{3/2} + h^2) \|\mathbf{u}\|_{2,\Omega}^2 \right).
\end{aligned}
$$

This last estimate together with (23) and Proposition 3.3 terminates the proof of the theorem. □

Remark 3.7 *1. If an additional regularity assumption is added in the theorem : $\lambda \in H^1(\Gamma_C)$, then it is easy to show that we obtain the following "optimal" error bound: $|||(\mathbf{u} - \mathbf{u}^h, \lambda - \lambda^H)||| \le C(h + H)(\|\mathbf{u}\|_{2,\Omega} + \|\lambda\|_{1,\Gamma_C})$. It suffices then to consider the only suboptimal term (in (28)) and to write $\|\pi_0^H \lambda - \lambda\|_{0,\Gamma_C} \le CH \|\lambda\|_{1,\Gamma_C}$.*

2. An insight into [6] shows that the estimate obtained in a close but scalar framework with discontinuous piecewise constant multipliers (see [6], Theorem 5.1 and section 6) yields a convergence rate of only $h^{1/2}$ with sole $H^2(\Omega)$-regularity assumptions on the displacements. Here we obtain a convergence rate of order $h^{3/4}$ (when $h = H$ in Theorem 3.6).

3.2.2 Second contact condition: $M^{H-} = M_1^{H-}$

Now, we focus on the case of nonpositive continuous piecewise affine multipliers where M^{H-} is given by (10).

Theorem 3.8 *Let $(\mathbf{u}, \lambda)$ be the solution to Problem (6). Assume that $\mathbf{u} \in (H^2(\Omega))^2$. Let γ_0 be small enough and let $(\mathbf{u}^h, \lambda^H)$ be the solution to the discrete problem (12) where $M^{H-} = M_1^{H-}$. Then, there exists a constant $C > 0$ independent of h, H and $\mathbf{u}$ such that:*

$$\left|\left|\left|\left(\mathbf{u} - \mathbf{u}^h, \lambda - \lambda^H\right)\right|\right|\right| \leq C(h^{1/2} + H^{1/2})\|\mathbf{u}\|_{2,\Omega}. \tag{29}$$

Proof. We choose $\mu = 0$ in (25) which implies

$$\inf_{\mu \in M^-} \int_{\Gamma_C} (\mu - \lambda^H) u_n d\Gamma \leq -\int_{\Gamma_C} \lambda^H u_n d\Gamma \leq 0.$$

In the infimum (24) we choose $\mu^H = 0$. So

$$\begin{aligned}
&\inf_{\mu^H \in M_1^{H-}} \int_{\Gamma_C} (\mu^H - \lambda)(u_n^h + \gamma(\lambda^H - \sigma_n(\mathbf{u}^h)))d\Gamma \\
\leq\; & -\int_{\Gamma_C} \lambda(u_n^h + \gamma(\lambda^H - \sigma_n(\mathbf{u}^h)))d\Gamma \\
=\; & -\int_{\Gamma_C} \lambda\, r^H(u_n^h + \gamma(\lambda^H - \sigma_n(\mathbf{u}^h)))d\Gamma \\
& -\int_{\Gamma_C} \lambda(u_n^h + \gamma(\lambda^H - \sigma_n(\mathbf{u}^h)) - r^H(u_n^h + \gamma(\lambda^H - \sigma_n(\mathbf{u}^h))))d\Gamma \\
\leq\; & -\int_{\Gamma_C} \lambda(u_n^h + \gamma(\lambda^H - \sigma_n(\mathbf{u}^h)) - r^H(u_n^h + \gamma(\lambda^H - \sigma_n(\mathbf{u}^h))))d\Gamma \\
=\; & \int_{\Gamma_C} \lambda(r^H u_n^h - u_n^h)d\Gamma + \int_{\Gamma_C} \lambda(r^H(\gamma(\lambda^H - \sigma_n(\mathbf{u}^h))) - \gamma(\lambda^H - \sigma_n(\mathbf{u}^h)))d\Gamma \qquad (30)
\end{aligned}$$

where $r^H : L^1(\Gamma_C) \mapsto W_1^H$ is the quasi-interpolation operator defined for any function v in $L^1(\Gamma_C)$ by

$$r^H v = \sum_{x \in N^H} \alpha_x(v)\psi_x,$$

where N^H represents the set of nodes $\mathbf{x}_0, ..., \mathbf{x}_N$ in $\overline{\Gamma_C}$, ψ_x is the scalar basis function of W_1^H (defined on $\overline{\Gamma_C}$) at node x verifying $\psi_x(x') = \delta_{x,x'}$ for all $x' \in N^H$ and

$$\alpha_x(v) = \left(\int_{\Gamma_C} v\psi_x \, d\Gamma\right)\left(\int_{\Gamma_C} \psi_x \, d\Gamma\right)^{-1}.$$

Remark 3.9 *It is straightforward to check that r^H is linear and that it preserves nonpositivity. It is also obvious that generally $r^H v^H \neq v^H$ when $v^H \in W_1^H$. This operator is different from Clément's one (which consists of making local projections onto P_1 functions, see [20]), from Chen-Nochetto's one (which uses local projections onto P_0 functions, see [17]) and from Ben Belgacem-Renard's one (which consists of making local projections onto the convex cone of nonpositive P_1 functions, see [11]). The main particularity of the operator r^H which directly follows from its definition is that $r^H v \leq 0$ when v satisfies only "weak nonpositivity conditions", i.e.,*

$$\int_{\Gamma_C} \mu^H v \, d\Gamma \geq 0, \quad \forall \mu^H \in M_1^{H-}.$$

This property is not satisfied by the operators in [17] and [20]. Moreover, as we see hereafter, the approximation properties of r^H hold for any function without sign condition contrary to the operator in [11].

The approximation properties of r^H are proven in [34]. We simply recall hereafter the two main results without proofs. The first result is concerned with L^2-stability property of r^H.

Lemma 3.10 *There is a positive constant C independent of H such that for any $v \in L^2(\Gamma_C)$ and any $E \in T^H$:*

$$\|r^H v\|_{0,E} \;\; \leq \;\; C\|v\|_{0,\gamma_E},$$

where $\gamma_E = \cup_{\{F\in T^H :\, \bar{F}\cap\bar{E}\neq\emptyset\}} \bar{F}$.

Note that the proof of this lemma in [34] uses the assumption that the mesh T^H is quasi-uniform (the quasi uniformity is needed in [34] since we use inverse inequalities). A straightforward calculation shows that the quasi-uniformity assumption is not necessary to obtain L^2-stability. The second result is concerned with the L^2-approximation properties of r^H.

Lemma 3.11 *There is a positive constant C independent of H such that for any $v \in H^\eta(\Gamma_C)$, $0 \leq \eta \leq 1$, and any $E \in T^H$:*

$$\|v - r^H v\|_{0,E} \;\; \leq \;\; CH^\eta\|v\|_{\eta,\gamma_E}, \tag{31}$$

where $\gamma_E = \cup_{\{F\in E_C^H :\, \bar{F}\cap\bar{E}\neq\emptyset\}} \bar{F}$.

Consequently the first integral term in (30) is estimated as follows using (31):

$$\begin{aligned}
\int_{\Gamma_C} \lambda(r^H u_n^h - u_n^h)d\Gamma \;\; &\leq \;\; \int_{\Gamma_C} \lambda(r^H(u_n^h - u_n) - (u_n^h - u_n))d\Gamma + \int_{\Gamma_C} \lambda(r^H u_n - u_n)d\Gamma \\
&\leq \;\; C\left(\|\lambda\|_{0,\Gamma_C} H^{1/2}\|\mathbf{u} - \mathbf{u}^h\| + \|\lambda\|_{0,\Gamma_C} H\|u_n\|_{1,\Gamma_C}\right) \\
&\leq \;\; C\left(H^{1/2}\|\mathbf{u}\|_{2,\Omega}\|\mathbf{u} - \mathbf{u}^h\| + H\|\mathbf{u}\|_{2,\Omega}^2\right).
\end{aligned}$$

Therefore we write for any $\alpha > 0$:

$$\int_{\Gamma_C} \lambda(r^H u_n^h - u_n^h)d\Gamma \leq C\left(\alpha\|\mathbf{u} - \mathbf{u}^h\|^2 + \alpha^{-1}H\|\mathbf{u}\|_{2,\Omega}^2\right).$$

Now, we consider the second integral term in (30):

$$\begin{aligned}
& \int_{\Gamma_C} \lambda(r^H(\gamma(\lambda^H - \sigma_n(\mathbf{u}^h))) - \gamma(\lambda^H - \sigma_n(\mathbf{u}^h)))d\Gamma \\
\leq \;\; & \|\lambda\|_{0,\Gamma_C}\|r^H(\gamma(\lambda^H - \sigma_n(\mathbf{u}^h))) - \gamma(\lambda^H - \sigma_n(\mathbf{u}^h))\|_{0,\Gamma_C} \\
\leq \;\; & C\|\lambda\|_{0,\Gamma_C}\|\gamma(\lambda^H - \sigma_n(\mathbf{u}^h))\|_{0,\Gamma_C} \\
\leq \;\; & C\gamma_0^{1/2}h^{1/2}\|\lambda\|_{0,\Gamma_C}\left\|\gamma^{1/2}\left((\lambda^H - \lambda) + \sigma_n(\mathbf{u} - I^h\mathbf{u}) + \sigma_n(I^h\mathbf{u} - \mathbf{u}^h)\right)\right\|_{0,\Gamma_C} \\
\leq \;\; & C\gamma_0^{1/2}h^{1/2}\|\mathbf{u}\|_{2,\Omega}\left(\|\gamma^{1/2}(\lambda^H - \lambda)\|_{0,\Gamma_C} + h\|\mathbf{u}\|_{2,\Omega} + \gamma_0^{1/2}\|\mathbf{u} - \mathbf{u}^h\|\right).
\end{aligned}$$

As a consequence, we have for any $\alpha > 0$:

$$\begin{aligned}
&\inf_{\mu^H \in M_1^{H-}} \int_{\Gamma_C} (\mu^H - \lambda)(u_n^h + \gamma(\lambda^H - \sigma_n(\mathbf{u}^h)))d\Gamma \\
\leq \;& C\left(\alpha(\|\mathbf{u}-\mathbf{u}^h\|^2 + \|\gamma^{1/2}(\lambda^H - \lambda)\|_{0,\Gamma_C}^2) + \alpha^{-1}(h+H)\|\mathbf{u}\|_{2,\Omega}^2\right).
\end{aligned}$$

The latter bound with (23) and Proposition 3.3 prove of the theorem. □

Remark 3.12 *1. If an additional regularity assumption is added in the theorem: $\lambda \in H^2(\Gamma_C)$ (this assumption is not really relevant: it is only introduced in order to obtain an error estimate of order $h + H$), then it is easy to show that we obtain the following error bound: $\left|\left|\left|(\mathbf{u}-\mathbf{u}^h, \lambda - \lambda^H)\right|\right|\right| \leq C(h+H)(\|\mathbf{u}\|_{2,\Omega} + \|\lambda\|_{2,\Gamma_C})$. It suffices then to choose $\mu^H = I^H\lambda$ in the infimum (30), to write $\|I^H\lambda - \lambda\|_{0,\Gamma_C} \leq CH^2\|\lambda\|_{2,\Gamma_C}$ and to perform some straightforward calculations.*

2. Although the estimate in Theorem 3.8 is only of order $h^{1/2}$ (when $H = h$) we are not able to improve it even in the nonstabilized case where the convergence rate is similar (when the inf-sup condition holds), see [11, 34].

3.2.3 Third contact condition: $M^{H-} = M_{1,*}^{H-}$

This choice corresponds to "weakly nonpositive" continuous piecewise affine multipliers where M^{H-} is given by (11).

Theorem 3.13 *Let $(\mathbf{u}, \lambda)$ be the solution to Problem (6). Assume that $\mathbf{u} \in (H^2(\Omega))^2$. Let γ_0 be small enough and let $(\mathbf{u}^h, \lambda^H)$ be the solution to the discrete problem (12) where $M^{H-} = M_{1,*}^{H-}$. Then, there exists a constant $C > 0$ independent of h, H and $\mathbf{u}$ such that:*

$$\left|\left|\left|\left(\mathbf{u}-\mathbf{u}^h, \lambda - \lambda^H\right)\right|\right|\right| \leq C(h + H^{3/4} + H^{3/2}h^{-1/2})\|\mathbf{u}\|_{2,\Omega}.$$

Proof. Setting $\mu = 0$ in (25), we obtain:

$$\begin{aligned}
\inf_{\mu \in M^-} \int_{\Gamma_C} (\mu - \lambda^H)u_n d\Gamma \;&\leq\; -\int_{\Gamma_C} \lambda^H u_n d\Gamma \\
&=\; \int_{\Gamma_C} \lambda^H (I^H u_n - u_n) d\Gamma - \int_{\Gamma_C} \lambda^H I^H u_n d\Gamma \\
&\leq\; \int_{\Gamma_C} \lambda^H (I^H u_n - u_n) d\Gamma \\
&=\; \int_{\Gamma_C} (\lambda^H - \lambda)(I^H u_n - u_n) d\Gamma + \int_{\Gamma_C} \lambda (I^H u_n - u_n) d\Gamma \\
&\leq\; \|\gamma^{1/2}(\lambda^H - \lambda)\|_{0,\Gamma_C} \|\gamma^{-1/2}(I^H u_n - u_n)\|_{0,\Gamma_C} \\
&\quad + \|\lambda\|_{0,\Gamma_C} \|I^H u_n - u_n\|_{0,\Gamma_C} \\
&\leq\; C\left(H^{3/2}h^{-1/2}\|\mathbf{u}\|_{2,\Omega}\|\gamma^{1/2}(\lambda^H - \lambda)\|_{0,\Gamma_C} + H^{3/2}\|\mathbf{u}\|_{2,\Omega}^2\right),
\end{aligned} \tag{32}$$

where I^H is the Lagrange interpolation operator mapping onto W_1^H. The operator I^H is defined for any $v \in C(\Gamma_C)$ and satisfies the following error estimates for any $1/2 < r \leq 2$:

$$\|v - I^H v\|_{0,\Gamma_C} \leq CH^r\|v\|_{r,\Gamma_C}.$$

In the infimum (24) we choose $\mu^H = \pi_1^H \lambda$ where π_1^H denotes the $L^2(\Gamma_C)$-projection onto W_1^H. The operator π_1^H is defined for any $v \in L^2(\Gamma_C)$ by

$$\pi_1^H v \in W_1^H, \quad \int_{\Gamma_C} (v - \pi_1^H v)\mu \, d\Gamma = 0, \quad \forall \mu \in W_1^H,$$

and satisfies the following error estimates for any $0 \leq r \leq 2$:

$$H^{-1/2}\|v - \pi_1^H v\|_{-1/2,\Gamma_C} + \|v - \pi_1^H v\|_{0,\Gamma_C} \leq CH^r\|v\|_{r,\Gamma_C}. \tag{33}$$

Obviously $\pi_1^H\lambda \in M_{1,*}^{H-}$. So

$$\begin{aligned}
&\inf_{\mu^H \in M_{1,*}^{H-}} \int_{\Gamma_C} (\mu^H - \lambda)(u_n^h + \gamma(\lambda^H - \sigma_n(\mathbf{u}^h)))d\Gamma \\
&\leq \int_{\Gamma_C} (\pi_1^H\lambda - \lambda)u_n^h d\Gamma + \int_{\Gamma_C} \gamma(\pi_1^H\lambda - \lambda)(\lambda^H - \sigma_n(\mathbf{u}^h))d\Gamma.
\end{aligned} \tag{34}$$

The first integral term in (34) is estimated as follows using (33):

$$\begin{aligned}
\int_{\Gamma_C} (\pi_1^H\lambda - \lambda)u_n^h d\Gamma &= \int_{\Gamma_C} (\pi_1^H\lambda - \lambda)(u_n^h - u_n)d\Gamma + \int_{\Gamma_C} (\pi_1^H\lambda - \lambda)u_n d\Gamma \\
&= \int_{\Gamma_C} (\pi_1^H\lambda - \lambda)(u_n^h - u_n)d\Gamma + \int_{\Gamma_C} (\pi_1^H\lambda - \lambda)(u_n - \pi_1^H u_n)d\Gamma \\
&\leq \|\pi_1^H\lambda - \lambda\|_{-1/2,\Gamma_C}\|u_n^h - u_n\|_{1/2,\Gamma_C} + \|\pi_1^H\lambda - \lambda\|_{0,\Gamma_C}\|u_n - \pi_1^H u_n\|_{0,\Gamma_C} \\
&\leq C\left(H\|\mathbf{u} - \mathbf{u}^h\|\|\mathbf{u}\|_{2,\Omega} + H^2\|\mathbf{u}\|_{2,\Omega}^2\right).
\end{aligned}$$

Therefore, for any $\alpha > 0$, we have

$$\int_{\Gamma_C} (\pi_1^H\lambda - \lambda)u_n^h d\Gamma \leq C\left(\alpha\|\mathbf{u} - \mathbf{u}^h\|^2 + \alpha^{-1}H^2\|\mathbf{u}\|_{2,\Omega}^2\right).$$

Next, we consider the second integral term in (34) using the bounds in (33), (13), (22) and the trace inequality $\|\lambda\|_{1/2,\Gamma_C} \leq C\|\mathbf{u}\|_{2,\Omega}$:

$$\begin{aligned}
\int_{\Gamma_C} \gamma(\pi_1^H\lambda - \lambda)(\lambda^H - \sigma_n(\mathbf{u}^h))d\Gamma &= \int_{\Gamma_C} \gamma(\pi_1^H\lambda - \lambda)(\lambda^H - \lambda)d\Gamma \\
&\quad + \int_{\Gamma_C} \gamma(\pi_1^H\lambda - \lambda)(\sigma_n(\mathbf{u} - I^h\mathbf{u}))d\Gamma \\
&\quad + \int_{\Gamma_C} \gamma(\pi_1^H\lambda - \lambda)(\sigma_n(I^h\mathbf{u} - \mathbf{u}^h))d\Gamma \\
&\leq C\Big(\gamma_0^{1/2}h^{1/2}H^{1/2}\|\mathbf{u}\|_{2,\Omega}\|\gamma^{1/2}(\lambda^H - \lambda)\|_{0,\Gamma_C} \\
&\quad + \gamma_0^{1/2}h^{1/2}H^{1/2}\|\mathbf{u}\|_{2,\Omega}\|\gamma^{1/2}\sigma_n(\mathbf{u} - I^h\mathbf{u})\|_{0,\Gamma_C} \\
&\quad + \gamma_0 h^{1/2}H^{1/2}\|\mathbf{u}\|_{2,\Omega}\|\mathbf{u}^h - I^h\mathbf{u}\|\Big).
\end{aligned}$$

As a consequence

$$\begin{aligned}
&\inf_{\mu^H \in M_{1,*}^{H-}} \int_{\Gamma_C} (\mu^H - \lambda)(u_n^h + \gamma(\lambda^H - \sigma_n(\mathbf{u}^h)))d\Gamma \\
&\leq C\left(\alpha(\|\mathbf{u} - \mathbf{u}^h\|^2 + \|\gamma^{1/2}(\lambda^H - \lambda)\|_{0,\Gamma_C}^2) + \alpha^{-1}(h^2 + H^2)\|\mathbf{u}\|_{2,\Omega}^2\right).
\end{aligned}$$

The theorem is established by combining Proposition 3.3, (23) and the last estimate. □

Remark 3.14 *1. If an additional regularity assumption is added in the theorem : $u_n \in H^2(\Gamma_C)$ then it is easy to show that we obtain the following error bounds: $\left|\!\left|\!\left|(\mathbf{u}-\mathbf{u}^h, \lambda-\lambda^H)\right|\!\right|\!\right| \leq C(h+H+H^{3/2}h^{-1/2})(\|\mathbf{u}\|_{2,\Omega}+\|u_n\|_{2,\Gamma_C})$. It suffices to observe that the suboptimal estimate in (32) becomes optimal in this case.*

2. When $\gamma = 0$ and T^H is the mesh induced by $\mathcal{T}^h$ on Γ_C then we recover the classical noninterpetration condition for which a convergence rate of only $h^{3/4}$ can be proved with sole $H^2(\Omega)$-regularity assumptions on the displacements (when the inf-sup condition holds). As for the previous contact condition in section 3.2.2, we do not observe a loss of convergence in our study when stabilization is added.

4 Numerical discussion

A matrix formulation of the contact problem (12) can be obtained as follows:

$$(35)\qquad \begin{cases} \text{Find } U \in \mathbb{R}^N \text{ and } L \in \overline{M}^{H-} \text{ such that} \\ (K-C)U - (B-D)^T L = F, \\ (\overline{L}-L)^T((B-D)U + ML) \geq 0, \quad \forall \overline{L} \in \overline{M}^{H-}, \end{cases}$$

where U is the vector of degrees of freedom (d.o.f.) for $\mathbf{u}^h$, L is the vector of d.o.f. for the multiplier λ^H, $\overline{M}^{H-}$ is the set of vectors L such that the corresponding multiplier lies in M^{H-}, K is the classical stiffness matrix coming from the term $a(\mathbf{u}^h, \mathbf{v}^h)$, F is the right hand side corresponding to the Neumann condition and volume forces and B, C, D, M are the matrices corresponding to the terms $b(\lambda^H, \mathbf{v}^h)$, $\int_{\Gamma_C} \gamma\sigma_n(\mathbf{u}^h)\sigma_n(\mathbf{v}^h)\, d\Gamma$, $\int_{\Gamma_C} \gamma\lambda^H\sigma_n(\mathbf{v}^h)\, d\Gamma$, $\int_{\Gamma_C} \gamma\lambda^H\mu^H\, d\Gamma$ respectively.

We present some numerical tests in a slightly larger framework than the theoretical results of section 3. Some of the experiments are achieved with quadratic (P_2) finite element methods and not only in two dimensions but also in three dimensions. The space of multipliers is always the one such that nonpositivity occurs at the finite element nodes (for P_0 and P_1 multipliers, this corresponds to M_0^{H-} and M_1^{H-}). The mesh on the boundary for the multiplier is taken to be the one induced by the mesh of the whole body (in that case, $T^H = \mathcal{T}^h{}_{|\Gamma_C}$ and $H \leq h$).

A Hertz contact problem between a disc (plane strain approximation of a cylinder) or a sphere and a rigid plane is considered. In both cases, an imaginary elastic body of Lamé coefficients $\lambda = 10MPa, \mu = 5MPa$ (Young modulus: $26.66MPa$, Poisson ratio: 0.33) is submitted to its own weight with a gravity constant $g = 9.81m/s^2$ and a density $\rho = 6000kg/m^3$. The disc and the sphere are both of radius $20mm$. Curved meshes are used to discretize the disc and the sphere. The initial gap between the body and the rigid foundation is not vanishing in that case.

The (potential) contact boundary Γ_C is the part of the boundary in which the unit outward normal has a negative vertical component (lower part of the boundary). An homogeneous Neumann condition is applied on the rest of the boundary (upper part). To avoid multi-solutions, global linearized rotations and horizontal rigid motions are prescribed.

4.1 Numerical solution

There are various techniques to solve numerically Problem (35). Some of them are described and compared in [37]. Here we use the semi-smooth Newton method introduced for contact and friction problems in [2]. The key point of this method is to express the inequality of (35) as an equivalent projection:

$$(36)\qquad L = P_{\overline{M}^{H-}}(L - r((B-D)U + ML)).$$

This transforms the inequality into a nonlinear equation. The Newton algorithm is then applied on this problem. The terminology semi-smooth comes from the fact that projections are only piecewise differentiable. It is proven in [18] that this semi-smooth Newton converges when the starting point is not too far from the solution. Practically, it is one of the most robust algorithms to solve contact problems with or without friction.

In order to write a Newton step, one has to compute the derivative of the projection (36). An analytical expression can only be obtained when the projection itself is simple to express. This is the case for instance when the set M^{H-} is chosen to be the set of multipliers having nonpositive values on each finite element node of the contact boundary (such as M_0^{H-} or M_1^{H-}). In this case, the projection can be expressed componentwise (see [37]).

Remark 4.1 *As described also in [37], a second case where the derivative of the projection is easy to express occurs when M^{H-} is the set of multipliers where the corresponding equivalent nodal forces on the contact boundary are nonpositive (such as M_{1*}^{H-} for the nonstabilized case, this corresponds to a nonpositive normal displacement on these finite element nodes). Since the multiplier is present in the stabilization term, the projection is no longer easy to express. Consequently, when the stabilization is used, nonpositive multipliers are the best choice.*

4.2 Numerical experiments in two dimensions

On Fig. 1 we show some of the meshes used for the computation on the disc. Theses meshes and all the computations have been obtained with GETFEM++, the C++ finite element library developed by our team (see [44]). The current choice is to take curved elements on the boundary with a quadratic geometric transformation between the reference element and the real ones. Theoretically, in the linear case, this is sufficient to recover the optimality for P_2 finite elements (isoparametric elements). This is not strictly necessary for P_1 finite elements but, for the sake of simplicity, the same meshes have been used for P_1 and P_2 finite elements.

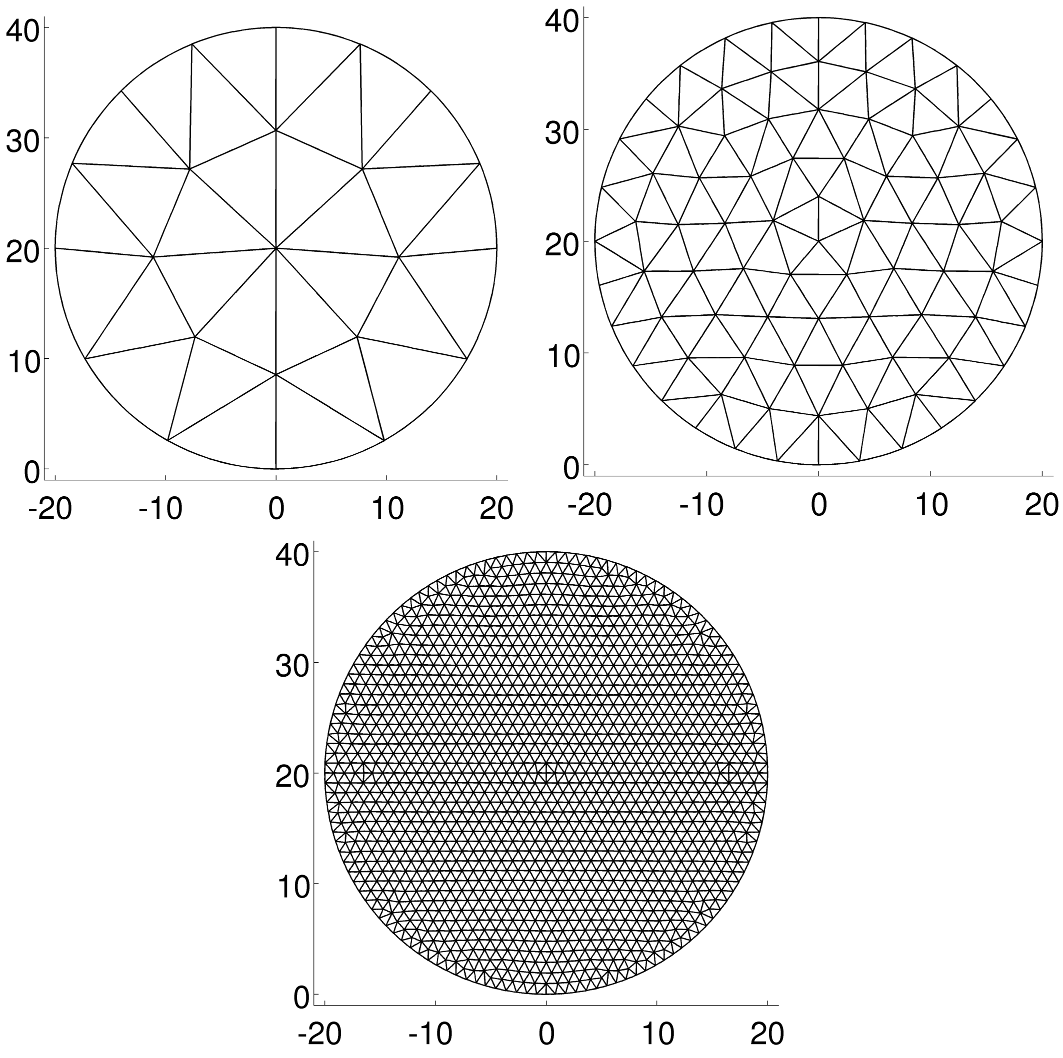

Figure 1: Examples of curved meshes used for the disc.

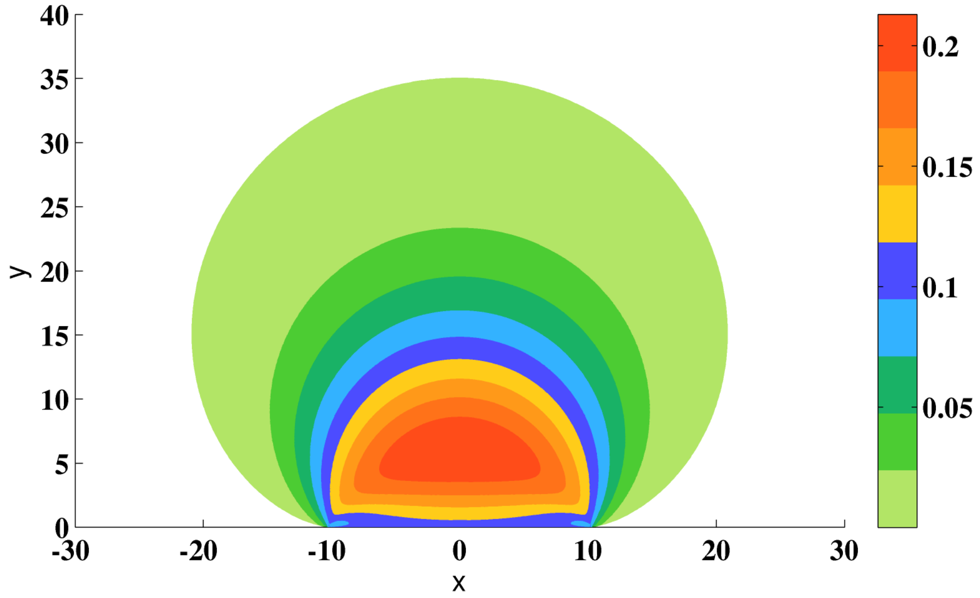

Figure 2: Deformation of the disc on the finest mesh with Von Mises stresses (in MPa) contour plot.

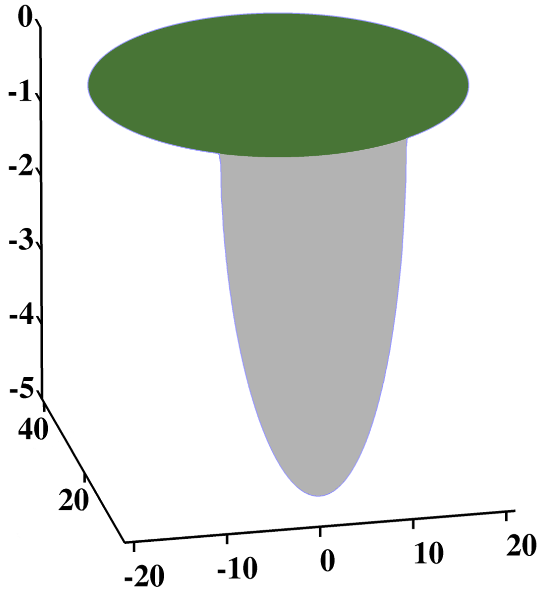

Figure 3: Contact pressure (in MPa) on the contact zone on the finest mesh.

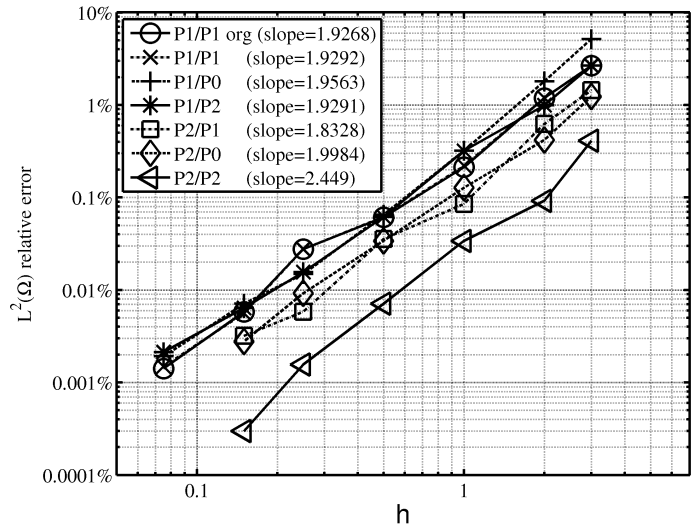

Figure 4: $L^2(\Omega)$-error convergence curves for the disc with $\gamma_0 = 10^{-3}$.

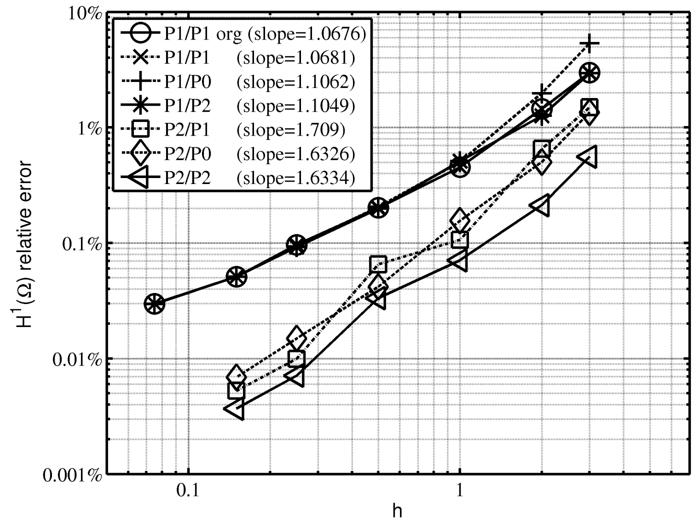

Figure 5: $H^1(\Omega)$-error convergence curves for the disc with $\gamma_0 = 10^{-3}$.

The Hertz contact theory gives only an approximate solution when the radius of the real contact zone is small compared to the radius of the body (in particular it does not take into account how the load is applied). We compare here the discrete solution with a reference solution computed on a very fine mesh. This reference solution is computed using a nodal contact condition with a P_2 finite element method (no explicit multipliers, the normal displacement is nonpositive on each finite element node on the contact boundary). The deformation obtained for this reference solution is represented on Fig. 2. The contact pressure is drawn on Fig. 3.

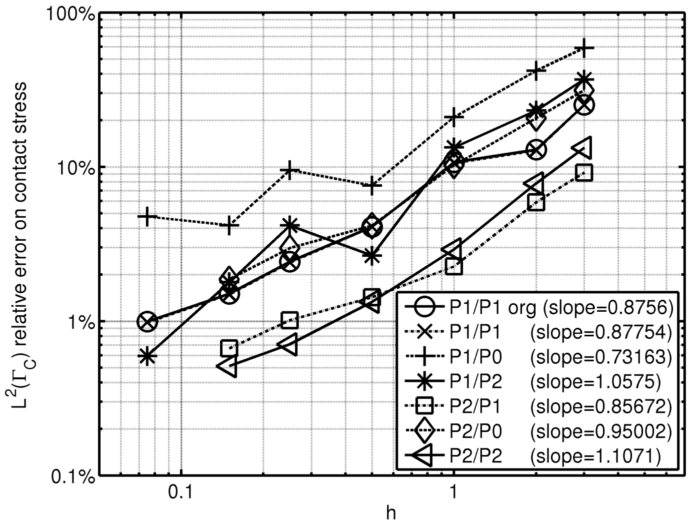

Figure 6: $L^2(\Gamma_C)$-error convergence curves on the contact stress for the disc with $\gamma_0 = 10^{-3}$.

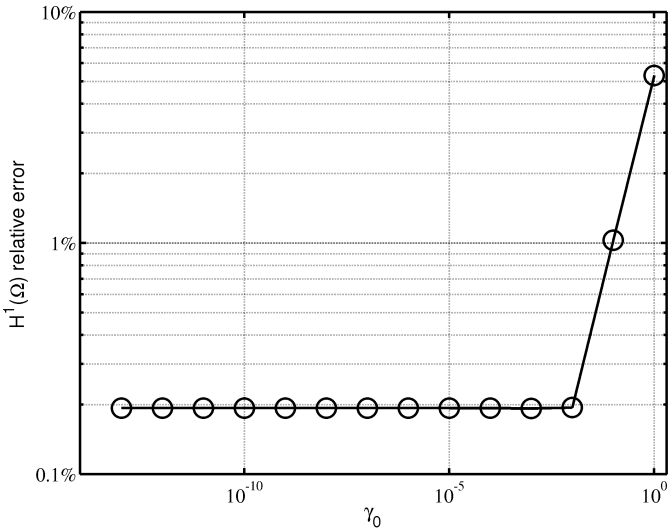

Figure 7: $H^1(\Omega)$-error for different values of γ_0 for P_1/P_2 experiment and $h = 2mm$.

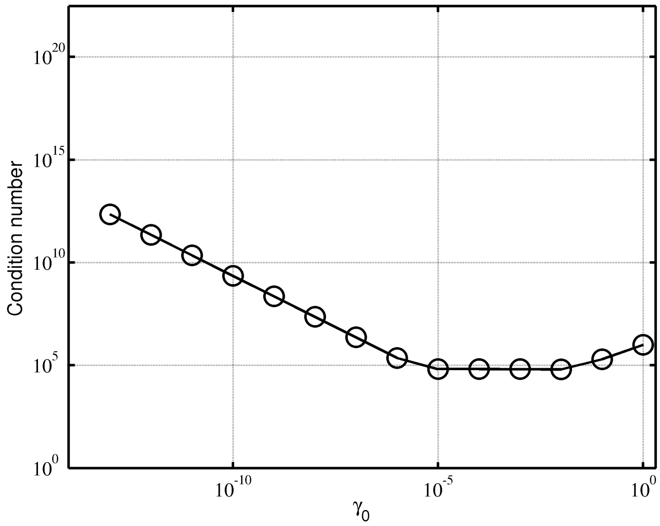

Figure 8: Sensitivity of the condition number of the Newton tangent linear system for different values of γ_0 for the P_1/P_2 experiment and $h = 2mm$.

The error curves between the discrete solution and the reference solution are given on Fig. 4 for the $L^2(\Omega)$-norm on the displacements, on Fig. 5 for the $H^1(\Omega)$-norm on the displacements and on Fig. 6 for the $L^2(\Gamma_C)$-norm on the contact stresses. The experiments are performed for six different stabilized situations and compared with a case without stabilization (denoted P1/P1 org). The notation Pi/Pj means that the displacement is approximated with a Pi finite element method and the multiplier with a Pj one. One can observe on Figs. 4, 5 and 6, that there is little difference between the P1/Pi methods with or without stabilization. The convergence rates are slightly over-optimal in all cases, a phenomenon which we cannot really explain (this is perhaps due to the quadratic geometric transformation of the elements which are more deformed for coarse meshes). It seems that for fine meshes the rate of convergence becomes closer to the theoretical ones. The $L^2(\Omega)$-norm error curves suggest that a kind of Aubin-Nitsche lemma might be established for the contact problem. It seems that there is a gain of at least half an order compared to the $H^1(\Omega)$-norm error curves. Of course, the particularity of the stabilized version is that the P1/P0 and P1/P2 version work without any problem. In the two-dimensional case, the P1/P0 version usually works without stabilization even though the inf-sup condition is not satisfied, but the P1/P2 version cannot work without stabilization.

The P2/Pi versions have a better convergence rate, especially in the $H^1(\Omega)$-norm. A slope of 2 is not attained due to the regularity of the solution. As mentioned in Remark 3.5 the singularity due to the transition between contact and noncontact is expected to be in $H^{5/2-\varepsilon}(\Omega)$ for any $\varepsilon > 0$. Theoretically, this limits the convergence rate to 3/2 in $H^1(\Omega)$-norm.

The use of the stabilization leads to the problem of choosing the stabilization parameter γ_0. In fact, Fig. 7 and Fig. 8 show that this choice can be made in a very large range of values without affecting the quality of the solution. In Fig. 7, the $H^1(\Omega)$-norm is plotted as a value of γ_0 with a fixed mesh in the P1/P2 case. We observe that the only criterion is that γ_0 has to be chosen sufficiently small. On Fig. 8 the condition number of the tangent system of an iteration in the Newton algorithm plotted for the same experiment. For very small values of γ_0 the condition number becomes very large. The reason is that the inf-sup condition is not satisfied in that case and when γ_0 vanishes, the system tends to the nonstabilized system which is singular. As a consequence, a choice of a too small γ_0 can significantly slow down the convergence of the numerical algorithm used to solve the problem.

4.3 Numerical experiments in three dimensions

Fig. 9 shows the deformation of a sphere in contact with a rigid plane (the plane is not represented). The first drawing is the deformation with one of the meshes used in the convergence test with a P_1 finite element method. The second picture represents the solution for the finest mesh used to compute the reference solution with isoparametric P_2 finite elements (about 474000 d.o.f.); and the third drawing depicts a slice of the reference solution with the Von-Mises stresses inside the sphere.

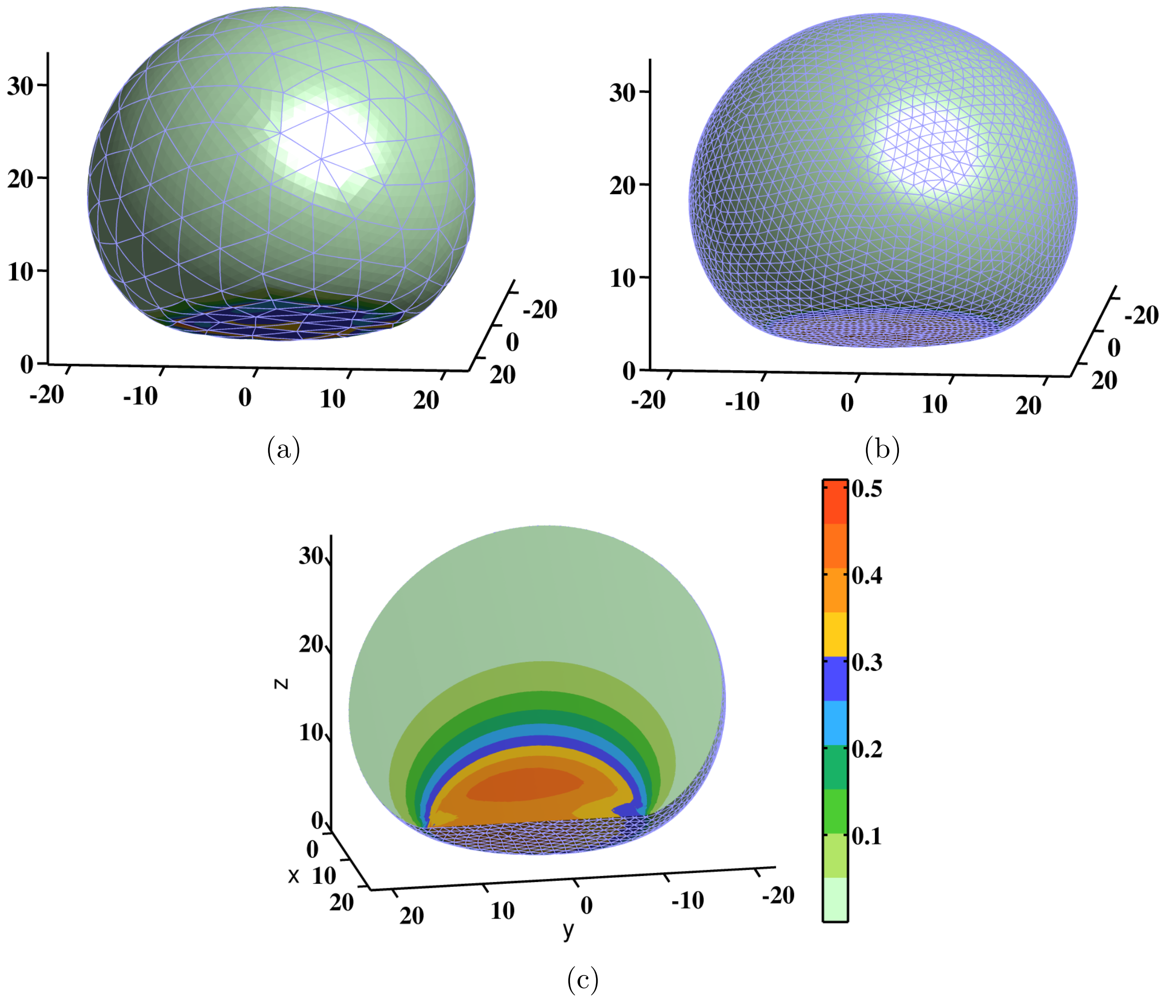

Figure 9: Deformation of the sphere with one of the test meshes and a P_1 f.e.m. (a), for the reference solution with a P_2 f.e.m. (b), for a slice with Von Mises stresses on the reference solution (c).

The convergence curves for P_1/P_0 and P_1/P_1 finite elements with stabilization are shown on Fig 10 for the $L^2(\Omega)$-norm and on Fig 11 for the $H^1(\Omega)$-norm. Here again a slight super-convergence is observed which could result from the used quadratic geometric transformation.

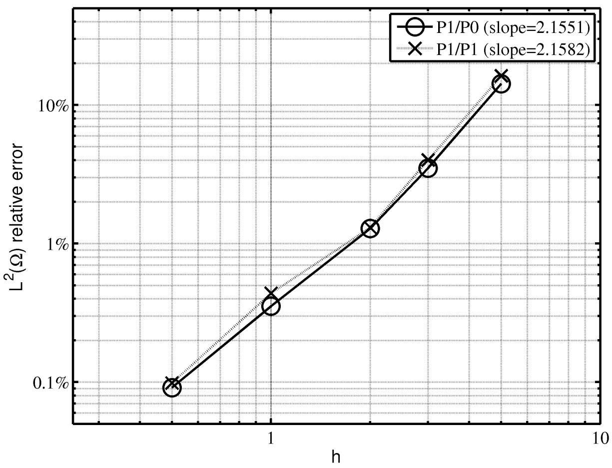

Figure 10: $L^2(\Omega)$-error convergence curves for the sphere with $\gamma_0 = 10^{-3}$.

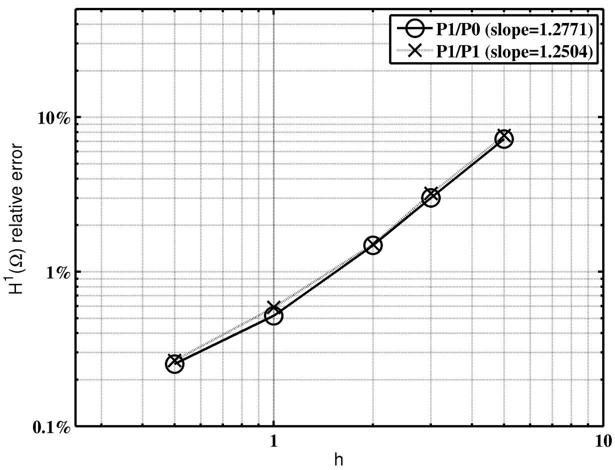

Figure 11: $H^1(\Omega)$-error convergence curves for the sphere with $\gamma_0 = 10^{-3}$.

5 Conclusion

We perform an adaptation of the Barbosa-Hughes stabilization technique to the nonlinear small strain elastostatics problem with unilateral contact. The characteristic of this method is to circumvent the Babuška-Brezzi inf-sup condition. For this model whose particularity is the presence of inequality conditions on the boundary, we achieve an error analysis of the stabilized finite element method when different classical approximations of the contact conditions are considered. For any contact condition we prove the same convergence rates as the existing ones in the nonstabilized case when the inf-sup condition holds.

The advantages of the stabilization method can be exploited whenever the Babuška-Brezzi inf-sup condition is difficult or impossible to obtain. This is the case when an enrichment of the finite element space approximating the contact pressure is made (for instance to take into account a singular behavior) or more simply when one intends to use the popular P_1/P_0 method (see the first contact condition in section 3.2.1).

Note that our convergence results apply straightforwardly to the simpler (scalar) Signorini problem. These techniques could also be directly adapted to the Coulomb friction model, even if in this case, the theoretical error estimates are far more difficult to obtain. Nevertheless, it should be possible to obtain some results in the framework considered in [34, 45]. Finally, the three-dimensional tests suggest also that some results could be generalized to this case.

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