A posteriori error estimations of residual type for Signorini's problem

Patrick Hild and Serge Nicaise

Université de Franche-Comté,
Laboratoire de Mathématiques de Besançon, UMR CNRS 6623
16 route de Gray, 25030 Besançon, France,
e-mail: hild@math.univ-fcomte.fr
and
Université de Valenciennes et du Hainaut Cambrésis,
MACS, ISTV,
F-59313 - Valenciennes Cedex 9, France,
e-mail: snicaise@univ-valenciennes.fr

Abstract

This paper presents an a posteriori error analysis for the linear finite element approximation of the Signorini problem in two space dimensions. A posteriori estimations of residual type are defined and upper and lower bounds of the discretization error are obtained. We perform several numerical experiments in order to compare the convergence of the terms in the error estimator with the discretization error.

Key words: *finite element method, variational inequalities, a posteriori error estimates, Signorini problem.*
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1 Introduction

The numerical simulation of problems governed by partial differential equations is very often carried out using approximation methods such as finite element methods, finite volume methods, finite differences... An important aspect for the user is to evaluate the discretization errors due to the use of such approximations. This quantification requires the definition of a posteriori error estimators.

In this work we propose and study an a posteriori error estimator of residual type for the linear finite element approximation of the Signorini problem in two space dimensions. Such a problem is represented by a variational inequality so that the definition of an efficient estimator becomes more difficult than in the linear case. Several studies dealing with a posteriori error estimators of residual type for finite element approximations of variational inequalities have already been achieved. Nevertheless it seems that most of the existing work is concerned with the obstacle problem (see e.g., [1, 10, 9] and in particular the recent references [3, 7, 15] in which upper and lower bounds of the error are established in the energy norm and the studies [12, 13] dealing with the maximum norm). Let us recall that there are significant differences in the finite element analysis (a priori and a posteriori) between the obstacle problem and the Signorini problem (since the seventies there exist optimal a priori error estimates with linear finite elements for the obstacle problem whereas such results are not available for the Signorini problem (see [2])). A specificity of the Signorini problem comes from the location of the inequality conditions holding only on a open part Γ_C of the boundary:

$$u \geq 0, \qquad \partial_n u \geq 0, \qquad u\partial_n u = 0 \quad \text{on } \Gamma_C,$$

whereas the inequality conditions hold on the entire domain for the obstacle problem (this is also the case in the elasto-plastic torsion or in the Bingham fluid problems). As far as we know there is no study yielding both upper and lower bounds of the error for the Signorini problem (or the equivalent unilateral contact problem in linear elasticity) written as a variational inequality (or as an equivalent mixed formulation).

An outline of the paper is as follows. Section 2 deals with the continuous setting of the Signorini problem and its piecewise linear conforming finite element approximation in which $u_h \geq 0$ on Γ_C. In section 3 an interpolation operator is introduced which satisfies a stability property from H^1 into L^2, exhibits optimal approximation properties and preserves positivity at the nodes of Γ_C. In section 4 we present a residual error estimator which consists of the standard interior and jump residuals, a term measuring the non-fulfillment of the condition $\partial_n u_h \geq 0$ on Γ_C and a nonstandard term taking into account (roughly speaking) the non-fulfillment of the complementary condition $u_h\, \partial_n u_h = 0$ on Γ_C. A quasi-optimal upper bound of the discretization error is proved in the energy norm. The bound is optimal when $\overline{\Gamma}_C$ is a straight line segment which has no common nodes with the closure of the boundary part submitted to Neumann conditions. Then a lower bound of the error is obtained which becomes optimal when the above-mentioned nonstandard term vanishes. Finally section 5 is concerned with the numerical experiments. We solve numerically three examples of Signorini problems: a first one whose exact solution is not explicitly known, a second one corresponding to a known solution and a third test in which the solution is regular. From the numerical computations we observe that the estimator and the discretization error have the same rate of convergence and therefore the effectivity indexes remain quasi-constant. Therefore we can conclude that our estimator is reliable. Moreover the computations show that the nonstandard term in the estimator is small in comparison with the standard term and that their convergence rates are similar.

As usual, we denote by $L^2(.)$ the Lebesgue spaces and by $H^s(.)$, $s \geq 0$, the standard Sobolev spaces. The usual norm and seminorm of $H^s(D)$ are denoted by

$\| \cdot \|_{s,D}$ and $| \cdot |_{s,D}$. For shortness the $L^2(D)$-norm will be denoted by $\| \cdot \|_D$. In the sequel the symbol $| \cdot |$ will denote either the Euclidean norm in $\mathbb{R}^2$, or the length of a line segment, or the area of a plane domain. Finally the notation $a \lesssim b$ means here and below that there exists a positive constant C independent of a and b (and of the meshsize of the triangulation) such that $a \leq C\ b$. The notation $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$ hold simultaneously.

2 Problem set-up and notation

Let Ω be an open subset of $\mathbb{R}^2$, with a polygonal boundary Γ. Let us fix a "partition" of Γ into three open subsets Γ_D, Γ_N and Γ_C, where we will consider Dirichlet, Neumann and Signorini boundary conditions respectively. Since Γ_D, Γ_N and Γ_C are open we then assume that they are disjoint and that $\Gamma := \bar{\Gamma}_D \cup \bar{\Gamma}_N \cup \bar{\Gamma}_C$. We further assume that the measures of Γ_D and Γ_C are positive.

In this paper we consider the standard Signorini problem: for $f \in L^2(\Omega)$ let $u \in H^1(\Omega)$ be the variational solution of

$$(1)\qquad \begin{cases} -\Delta u &= f & \text{in } \Omega, \\ u &= 0 & \text{on } \Gamma_D, \\ \partial_n u &= 0 & \text{on } \Gamma_N, \\ u &\geq 0, \partial_n u \geq 0, u\partial_n u = 0 & \text{on } \Gamma_C, \end{cases}$$

where $\partial_n u$ means the outward normal derivative of u along the boundary. For the sake of simplicity, in our theoretical analysis, we always assume that the Dirichlet and Neumann boundary conditions are homogeneous. Nonhomogeneous boundary conditions can be considered without any significant changes.

To recall the variational formulation of that problem, introduce the closed convex cone $\mathcal{K}$ of $H^1(\Omega)$:

$$\mathcal{K} := \{u \in H^1_D(\Omega) : u \geq 0 \text{ on } \Gamma_C\},$$

where $H^1_D(\Omega)$ is defined as

$$H^1_D(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D\}.$$

Then the variational solution of (1) is the unique solution $u \in \mathcal{K}$ of (see for instance [5])

$$(2)\qquad \int_\Omega \nabla u \cdot \nabla(v-u)\, dx \geq \int_\Omega f(v-u)\, dx, \forall v \in \mathcal{K}.$$

Recall that this problem is equivalent to $u \in \mathcal{K}$ and

$$(3)\qquad \begin{cases} \int_\Omega \nabla u \cdot \nabla v\, dx \geq \int_\Omega f v\, dx, \forall v \in \mathcal{K}, \\ \int_\Omega \nabla u \cdot \nabla u\, dx = \int_\Omega f u\, dx. \end{cases}$$

We approximate this problem by a standard finite element method. Namely we fix a family of meshes $T_h, h > 0$, regular in Ciarlet's sense [5], made of closed triangles. For $K \in T_h$ we recall that h_K is the diameter of K and $h = \max_{K \in T_h} h_K$. The regularity of the mesh implies in particular that for any edge E of K one has $h_E = |E| \sim h_K$.

Let us define E_h (resp. $\mathcal{N}_h$) as the set of edges (resp. nodes) of the triangulation and set $E_h^{int} := \{E \in E_h : E \subset \Omega\}$ the set of interior edges of T_h (the edges are supposed to be relatively open), $E_h^{ext} := E_h \setminus E_h^{int}$. We denote by $E_h^N := \{E \in E_h : E \subset \Gamma_N\}$ (resp. $E_h^D := \{E \in E_h : E \subset \Gamma_D\}$) the set of exterior edges included into the part of the boundary where we impose Neumann (resp. Dirichlet) conditions, and similarly $E_h^C := \{E \in E_h : E \subset \Gamma_C\}$ is the set of exterior edges included into the part of the boundary where we impose Signorini conditions. Set similarly $\mathcal{N}_h^{int} := \mathcal{N}_h \cap \Omega$, $\mathcal{N}_h^N := \mathcal{N}_h \cap \Gamma_N, \mathcal{N}_h^D := \mathcal{N}_h \cap \bar{\Gamma}_D$ (note that the extreme nodes of $\bar{\Gamma}_D$ belong to $\mathcal{N}_h^D$). Let $\mathcal{S}$ denote the set of vertices of Ω and denote by $\mathcal{N}_h^E$ the set of nodes which belong to $\bar{\Gamma}_C \cap \bar{\Gamma}_N$ or to $\Gamma_C \cap \mathcal{S}$. Set finally $\mathcal{N}_h^C := (\mathcal{N}_h \setminus \mathcal{N}_h^E) \cap \Gamma_C$ ($\mathcal{N}_h^C$ contains the nodes in Γ_C which are not vertices of Ω). For an element K, we will denote by E_K the set of edges of K and according to the above notation, we set $E_K^{int} := E_K \cap E_h^{int}$, $E_K^N := E_K \cap E_h^N$, $E_K^C := E_K \cap E_h^C$.

For an edge E of an element K introduce $n_{K,E} := (n_x, n_y)$ the unit outward normal vector to K along E and the tangent vector $t_{K,E} = n_{K,E}^{\perp} := (-n_y, n_x)$. Furthermore for each edge E we fix one of the two normal vectors and denote it by n_E and set $t_E := n_E^{\perp}$.

The jump of some (scalar or vector valued) function v across an edge E at a point $y \in E$ is defined as

$$[\![v(y)]\!]_E := \begin{cases} \lim_{\alpha \to +0} v(y + \alpha n_E) - v(y - \alpha n_E) & \forall E \in E_h^{int}, \\ v(y) & \forall E \in E_h^{ext}. \end{cases}$$

Note that the sign of $[\![v]\!]_E$ depends on the orientation of n_E. However, terms such as a gradient jump $[\![\nabla v \cdot n_E]\!]_E$ are independent of this orientation.

Finally we will need local subdomains (also called patches). As usual, let ω_K be the union of all elements having a nonempty intersection with K. Similarly for a node x and an edge E, let $\omega_x := \cup_{K : x \in K} K$ and $\omega_E := \cup_{x \in \bar{E}} \omega_x$.

We introduce the spaces

$$V_h := \{v_h \in C(\bar{\Omega}) : v_{h|K} \in \mathbb{P}_1(K), \forall K \in T_h\}, \quad W_h := \{v_h \in V_h : v_h = 0 \text{ on } \Gamma_D\},$$

and we define the closed convex cone

$$\mathcal{K}_h := \mathcal{K} \cap W_h.$$

The finite element approximation u_h of u is the unique solution $u_h \in \mathcal{K}_h$ of

$$\int_\Omega \nabla u_h \cdot \nabla (v_h - u_h)\, dx \geq \int_\Omega f(v_h - u_h)\, dx, \forall v_h \in \mathcal{K}_h. \tag{4}$$

3 The positivity preserving interpolation operator

Inspired from [3] we introduce a new interpolation operator π_h that preserves positivity at the nodes belonging to $\mathcal{N}_h^C$, which satisfies a stability property from H^1 into

L^2 and exhibits optimal approximation properties. For any $v \in H^1(\Omega)$, we define $\pi_h v$ as the unique element in V_h such that:

$$\pi_h v := \sum_{x \in \mathcal{N}_h} \alpha_x(v) \lambda_x,$$

where for any $x \in \mathcal{N}_h$ λ_x is the standard basis function in V_h satisfying $\lambda_x(x') = \delta_{x,x'}$, for all $x' \in \mathcal{N}_h$ and $\alpha_x(v)$ is defined as follows:

$$\begin{aligned}
\alpha_x(v) &:= \frac{1}{|\Delta_x|} \int_{\Delta_x} v(x)\, dx, \forall x \in \mathcal{N}_h^{int} \cup \mathcal{N}_h^N, \\
\alpha_x(v) &:= \frac{1}{|\Gamma_x|} \int_{\Gamma_x} v(x)\, d\sigma(x), \forall x \in \mathcal{N}_h^C \cup \mathcal{N}_h^D.
\end{aligned}$$

The sets Δ_x and Γ_x are fixed in the following way: For $x \in \mathcal{N}_h^{int}$, the set Δ_x is the maximal ball centered at x such that $\Delta_x \subset \omega_x$ (see [3]); for $x \in \mathcal{N}_h^N$, we take the standard patch $\Delta_x := \omega_x$; for $x \in \mathcal{N}_h^C \cup \mathcal{N}_h^D$, take the maximal ball Δ_x centered at x such that $\Delta_x \cap \Omega \subset \omega_x$ and set $\Gamma_x := \Delta_x \cap \Gamma_C$ for $x \in \mathcal{N}_h^C$, $\Gamma_x := \Delta_x \cap \Gamma_D$ for $x \in \mathcal{N}_h^D$.

For $x \in \mathcal{N}_h^E$, fix an edge $E_x \in E_h$ containing x and included into $\bar{\Gamma}_C$. Let z_x be the node of E_x different from x and let m_x be the midpoint of E_x. Note that z_x belongs to $\mathcal{N}_h^C$ (the mesh is supposed fine enough). Now for $x \in \mathcal{N}_h^E$, we set

$$\pi_h v(m_x) := \frac{1}{|E_x|} \int_{E_x} v(x)\, d\sigma(x)$$

and we define $\pi_h v$ at x by extrapolation using $\pi_h v(m_x)$ and $\alpha_{z_x}(v)$, namely

$$\alpha_x(v) := 2\pi_h v(m_x) - \alpha_{z_x}(v), \forall x \in \mathcal{N}_h^E.$$

Remark that the mesh regularity assumption implies that the diameters of Δ_x, Γ_x or E_x are equivalent to h_x, h_x being the diameter of ω_x.

It is easy to see that π_h is linear and that $\pi_h(H_D^1(\Omega)) \subset W_h$. Note also that the interpolation operator is of Clément/Chen-Nochetto type in $\Omega \cup \Gamma_N$ and of Scott-Zhang type in $\Gamma \setminus \Gamma_N$, this last modification being made as in [3] to guarantee the following properties:

1) Positivity preserving: if $v \geq 0$ on ω_x ($x \in \mathcal{N}_h \setminus \mathcal{N}_h^E$), then $\pi_h v(x) \geq 0$. Moreover if $v \in \mathcal{K}$, then $\pi_h v(x) \geq 0$, for all $x \in \mathcal{N}_h^C$ (this means that $\pi_h v$ is almost in $\mathcal{K}_h$, the exceptional nodes are the ones in $\mathcal{N}_h^E$ for which the nonnegativeness is not guaranteed).

2) Optimal approximation property: if $v \in \mathbb{P}_1(\omega_x)$, then $\pi_h v(x) = v(x)$, for all $x \in \mathcal{N}_h^{int} \cup \mathcal{N}_h^C$.

Let us now show the stability property from H^1 into L^2 of π_h:

Lemma 3.1 *For all $v \in H^1(\Omega)$ and all $K \in T_h$, $E \in E_h$ such that $K \cap \mathcal{N}_h^E = \emptyset$, $\bar{E} \cap \mathcal{N}_h^E = \emptyset$ one has*

$$\|\pi_h v\|_K \lesssim \|v\|_{\omega_K} + h_K \|\nabla v\|_{\omega_K}, \tag{5}$$

$$\|\pi_h v\|_E \lesssim h_E^{-1/2} \|v\|_{\omega_E} + h_E^{1/2} \|\nabla v\|_{\omega_E}. \tag{6}$$

If there is a node $x \in \mathcal{N}_h^E$ which belongs to K or to $\bar{E}$ then the sets ω_K and ω_E in (5)–(6) have to be replaced by $\cup_{F \in E_h : x \in \bar{F}} \omega_F$.

Proof: For $x \in \mathcal{N}_h^{int} \cup \mathcal{N}_h^N$, by Cauchy-Schwarz's inequality we have

$$|\alpha_x(v)| \leq |\Delta_x|^{-1/2} \|v\|_{\omega_x},$$

and since $|\Delta_x| \sim h_x^2$ we deduce

$$|\alpha_x(v)| \lesssim h_x^{-1} \|v\|_{\omega_x} + \|\nabla v\|_{\omega_x}. \tag{7}$$

Similarly for $x \in \mathcal{N}_h^C \cup \mathcal{N}_h^D$, we have

$$|\alpha_x(v)| \leq |\Gamma_x|^{-1/2} \|v\|_{\Gamma_x},$$

and using the standard trace inequality

$$\|v\|_E \lesssim h_E^{-1/2} \|v\|_K + h_E^{1/2} \|\nabla v\|_K, \forall E \in E_K, \tag{8}$$

we still obtain (7) since $|\Gamma_x| \sim h_x$.

Finally for $x \in \mathcal{N}_h^E$ one has

$$\begin{aligned} |\alpha_x(v)| &\leq 2|\pi_h v(m_x)| + |\alpha_{z_x}(v)| \\ &\lesssim h_x^{-1}(\|v\|_{\omega_{z_x}} + \|v\|_{\omega_x}) + (\|\nabla v\|_{\omega_{z_x}} + \|\nabla v\|_{\omega_x}) \\ &\lesssim h_x^{-1} \|v\|_{\omega_{E_x}} + \|\nabla v\|_{\omega_{E_x}} \end{aligned}$$

by the standard trace inequality (8) and the estimate (7) already obtained for $|\alpha_{z_x}(v)|$.

The conclusion follows from the fact that $\|\lambda_x\|_K \lesssim h_K$, for any node x of K and $\|\lambda_x\|_E \lesssim h_E^{1/2}$, for any node x of $\bar{E}$. ∎

Lemma 3.2 *For any $v \in H^1(\Omega)$ we have*

$$\|v - \pi_h v\|_K \lesssim h_K \|\nabla v\|_{\omega_K}, \forall K \in T_h \text{ such that } K \cap \mathcal{N}_h^E = \emptyset, \tag{9}$$

$$\|v - \pi_h v\|_E \lesssim h_E^{1/2} \|\nabla v\|_{\omega_E}, \forall E \in E_h \text{ such that } \bar{E} \cap \mathcal{N}_h^E = \emptyset. \tag{10}$$

If there is a node $x \in \mathcal{N}_h^E$ which belongs to K or to $\bar{E}$ then the sets ω_K and ω_E in (9)–(10) have to be replaced by $\cup_{F \in E_h : x \in \bar{F}} \omega_F$.

Proof: The proof is relatively standard and is based on the above stability properties and the fact that π_h preserves the constant functions (see [6, 14], or [3, Coro 3.1]). Let us give the proof for the sake of completeness. Let be given an arbitrary constant function $c(x) = c, x \in \bar{\Omega}$, then from the definition of π_h, we may write for any $v \in H^1(\Omega)$:

$$v - \pi_h v = v - c - \pi_h(v - c).$$

Therefore by Lemma 3.1 we get

$$\|v - \pi_h v\|_K \lesssim \|v - c\|_{\omega_K} + h_K \|\nabla v\|_{\omega_K}, \forall c \in \mathbb{R}, \tag{11}$$

$$\|v - \pi_h v\|_E \lesssim h_E^{-1/2} \|v - c\|_{\omega_E} + h_E^{1/2} \|\nabla v\|_{\omega_E}, \forall c \in \mathbb{R}. \tag{12}$$

The conclusion follows from a standard interpolation property choosing in (11), (12) $c := \int_X v(x)\, dx / |X|$ where $X := \omega_K$, $X := \omega_E$ or $X := \cup_{F \in E_h : x \in \bar{F}} \omega_F$ (see [5]). ∎

We will now exploit the maximal approximation property to deduce the next result:

Lemma 3.3 *For any $v_h \in W_h$ we have*

$$|v_h(x) - \pi_h v_h(x)| \lesssim h_x^{1/2} \sum_{E \in E_h^{int} \cup E_h^N : x \in \bar{E}} \|J_{E,n}(v_h)\|_E, \forall x \in \mathcal{N}_h^{int} \cup \mathcal{N}_h^C, \tag{13}$$

$$|v_h(x) - \pi_h v_h(x)| \lesssim h_x^{1/2} \sum_{E \in E_h^{int} : z_x \in \bar{E}} \|J_{E,n}(v_h)\|_E, \forall x \in \mathcal{N}_h^E, \tag{14}$$

where $J_{E,n}(v_h)$ means the gradient jump of v_h in the normal direction, i.e.,

$$J_{E,n}(v_h) \quad := \quad \begin{cases} [\![\frac{\partial v_h}{\partial n_E}]\!]_E, \forall E \in E_h^{int}, \\ \frac{\partial v_h}{\partial n_E}, \forall E \in E_h^N. \end{cases}$$

Proof: If $x \in \mathcal{N}_h^{int}$ the estimate (13) is proved in Lemma 3.3 of [3], we adapt this proof for $x \in \mathcal{N}_h^C \cup \mathcal{N}_h^E$.

If $x \in \mathcal{N}_h^C$, denote by $E_j, j = 1, 2$ the two edges of E_h^C having x as node. Since v_h is continuous at x and is piecewise $\mathbb{P}_1$, for $j = 1, 2$, we may write

$$v_{h|E_j}(t) = v_h(x) + (\partial_t v_{h|E_j})(t - x) \text{ on } E_j.$$

Integrating this identity on $\Gamma_x \cap E_j$, using the definition of $\pi_h v_h(x)$ and denoting by 2ρ the diameter of Γ_x, we get

$$\begin{aligned} v_h(x) - \pi_h v_h(x) &= v_h(x) - \frac{1}{2\rho} \int_{x-\rho}^{x+\rho} v_h(t)\, dt \\ &= -\frac{1}{2\rho} \int_{-\rho}^{0} (\partial_t v_{h|E_1}) t\, dt - \frac{1}{2\rho} \int_{0}^{\rho} (\partial_t v_{h|E_2}) t\, dt \\ &= \frac{\rho}{4} (\partial_t v_{h|E_1} - \partial_t v_{h|E_2}). \end{aligned}$$

Using the estimate $\rho \le h_x$ and using the fact that x is not a vertex of Ω, we obtain

$$|v_h(x) - \pi_h v_h(x)| \lesssim h_x |\partial_t v_{h|E_1} - \partial_t v_{h|E_2}| \le h_x \left|\nabla v_{h|K_1} - \nabla v_{h|K_n}\right|,$$

where K_1 (resp. K_n) is the triangle containing E_1 (resp. E_2). We then conclude using Lemma 3.4 below and Cauchy-Schwarz inequality.

Now fix $x \in \mathcal{N}_h^E$. Since v_h is a polynomial of degree ≤ 1 on E_x, we have

$$v_h(m_x) = \frac{1}{|E_x|} \int_{E_x} v_h(x)\, d\sigma(x).$$

Consequently by the definition of π_h, we have $v_h(m_x) = \pi_h v_h(m_x)$ and

$$v_h(x) - \pi_h v_h(x) = 2(v_h(m_x) - \pi_h v_h(m_x)) - (v_h(z_x) - \pi_h v_h(z_x)) = -(v_h(z_x) - \pi_h v_h(z_x)).$$

As z_x belongs to $\mathcal{N}_h^C$, the estimate (13) already proved for z_x leads to the conclusion in (14).

■

It is easy to see that the result (13) is generally false when $x \in \mathcal{N}_h^N$ and that (13) is trivially true when $x \in \mathcal{N}_h^D$.

Lemma 3.4 *Let $x \in \mathcal{N}_h \setminus \mathcal{N}_h^{int}$ and denote by K_i, $i = 1, \cdots, n$ the set of elements of T_h having x as node and such that $K_i \cap K_{i+1} = \bar{E}_i$ ($E_i \in E_h$), $i = 1, \cdots, n-1$. Then for any $v_h \in W_h$ and all $k = 1, \cdots, n-1$, we have*

$$\left|\nabla v_{h|K_k} - \nabla v_{h|K_n}\right| \leq \sum_{i=1}^{n-1} \left| [\![\frac{\partial v_h}{\partial n_{E_i}}]\!]_{E_i} \right|.$$

Proof: For an arbitrary edge $\bar{E}_i = K_i \cap K_{i+1}$, there exists a matrix of rotation B_i such that

$$\nabla v_{h|K_i} = B_i \begin{pmatrix} \frac{\partial v_{h|K_i}}{\partial n_{E_i}} \\ \frac{\partial v_{h|K_i}}{\partial t_{E_i}} \end{pmatrix}; \nabla v_{h|K_{i+1}} = B_i \begin{pmatrix} \frac{\partial v_{h|K_{i+1}}}{\partial n_{E_i}} \\ \frac{\partial v_{h|K_{i+1}}}{\partial t_{E_i}} \end{pmatrix}.$$

The continuity of v_h through E_i implies that $\frac{\partial v_{h|K_i}}{\partial t_{E_i}} = \frac{\partial v_{h|K_{i+1}}}{\partial t_{E_i}}$ and therefore by difference we get

$$\nabla v_{h|K_i} - \nabla v_{h|K_{i+1}} = B_i \begin{pmatrix} [\![\frac{\partial v_h}{\partial n_{E_i}}]\!]_{E_i} \\ 0 \end{pmatrix}.$$

Summing this identity from $i = k$ to $n-1$, we obtain

$$\nabla v_{h|K_k} - \nabla v_{h|K_n} = \sum_{i=k}^{n-1} B_i \begin{pmatrix} [\![\frac{\partial v_h}{\partial n_{E_i}}]\!]_{E_i} \\ 0 \end{pmatrix}.$$

The conclusion follows from the property $\|B_i\|_2 = 1$ where $\|.\|_2$ denotes the matrix 2-norm. ∎

Corollary 3.5 *Let $v_h \in W_h$ and $E \in E_h$ such that $\bar{E} \cap \Gamma_N = \emptyset$. Set*

$$X_E := \bigcup_{G \in E_h^{int} \cup E_h^N : \bar{G} \cap \bar{E} \neq \emptyset} \bar{G}.$$

Then

$$\|v_h - \pi_h v_h\|_E \lesssim h_E \sum_{F \in E_h^{int} \cup E_h^N : \bar{F} \cap X_E \neq \emptyset} \|J_{E,n}(v_h)\|_F. \tag{15}$$

Proof: The corollary directly follows from Lemma 3.3 and the equivalence

$$\|v_h\|_E \sim h_E^{1/2} \sum_{x \in \mathcal{N}_h : x \in \bar{E}} |v_h(x)|, \forall v_h \in W_h.$$

We finally mention that if $\bar{E} \cap \mathcal{N}_h^E = \emptyset$ (and also $\bar{E} \cap \Gamma_N = \emptyset$) then the estimate (15) merely becomes

$$\|v_h - \pi_h v_h\|_E \lesssim h_E \sum_{F \in E_h^{int} \cup E_h^N : \bar{F} \cap \bar{E} \neq \emptyset} \|J_{E,n}(v_h)\|_F.$$

∎

4 Error estimators

4.1 Definition of the residual error estimators

The exact element residual is defined by

$$R_K := f + \Delta u_h = f \text{ on } K.$$

As usual this exact residual is replaced by some finite dimensional approximation called approximate element residual

$$r_K \in \mathbb{P}_k(K).$$

A current choice is to take $r_K := \int_K f(x)\,dx/|K|$ since for $f \in H^1(\Omega)$, scaling arguments yield $\|R_K - r_K\|_K \lesssim h_K|f|_{1,K}$ and is then negligible with respect to the estimator η.

Definition 4.1 (Residual error estimator) *The local and global residual error estimators are defined by*

$$\begin{aligned}
\eta_K^2 &:= \eta_{sK}^2 + \eta_{nsK}^2,\\
\eta_{sK}^2 &:= h_K \left(h_K\|r_K\|_K^2 + \sum_{E\in E_K^{int}\cup E_K^N} \|J_{E,n}(u_h)\|_E^2 + \sum_{E\in E_K^C} \| \left(\frac{\partial u_h}{\partial n_E}\right)^- \|_E^2 \right),\\
\eta_{nsK}^2 &:= \sum_{E\in E_K^C} \int_E \tilde{\pi}_h u_h \left(\frac{\partial u_h}{\partial n_E}\right)^+,\\
\eta^2 &:= \sum_{K\in T_h} \eta_K^2,
\end{aligned}$$

where $\tilde{\pi}_h u_h$ is defined as the unique element in $\mathcal{K}_h$ defined at each node as follows:

$$\begin{aligned}
\tilde{\pi}_h u_h(x) &:= \pi_h u_h(x), \forall x \in \mathcal{N}_h \setminus \mathcal{N}_h^E,\\
\tilde{\pi}_h u_h(x) &:= u_h(x), \forall x \in \mathcal{N}_h^E.
\end{aligned}$$

Moreover we set

$$\eta_s := \Big(\sum_{K\in T_h} \eta_{sK}^2\Big)^{1/2} \qquad \textit{and} \qquad \eta_{ns} := \Big(\sum_{K\in T_h} \eta_{nsK}^2\Big)^{1/2}.$$

The local and global approximation terms are defined by

$$\zeta_K^2 := h_K^2 \sum_{K'\subset\omega_K} \|R_{K'} - r_{K'}\|_{K'}^2, \qquad \zeta^2 := \sum_{K\in T_h} \zeta_K^2.$$

In the above definition we used the indexes s and ns to underline the fact that the estimator η_{sK} is quite standard, while η_{nsK} is not. Recall that the estimator η_{nsK} measures the non fulfillment of the complementary condition $u\partial_n u = 0$ on Γ.

4.2 Upper error bound

Theorem 4.2 *Let $u \in \mathcal{K}$ be the solution of (2) and $u_h \in \mathcal{K}_h$ the solution of (4), and denote the error by*

$$e := u - u_h.$$

Then we have

$$|e|_{1,\Omega} \lesssim (1 + C(h))(\eta + \zeta),$$

where $C(h) = 0$ if $\mathcal{N}_h^E = \emptyset$, otherwise $C(h) = \sqrt{-\ln(h)}$ (h is supposed small enough).

Proof: Applying (2) with $v = u_h$ we have

$$|e|_{1,\Omega}^2 = (\nabla u, \nabla(u - u_h)) - (\nabla u_h, \nabla(u - u_h)) \leq (f, u - u_h) - (\nabla u_h, \nabla(u - u_h)),$$

where from now on $(\cdot, \cdot)$ means the $L^2(\Omega)$ inner product (for scalar or vector-valued functions according to the context). Therefore for any $v_h \in \mathcal{K}_h$ we may write

$$|e|_{1,\Omega}^2 \leq (f, u - u_h) - (\nabla u_h, \nabla(u - v_h)) - (\nabla u_h, \nabla(v_h - u_h)).$$

Applying the inequality (4) we obtain

$$|e|_{1,\Omega}^2 \leq (f, u - v_h) - (\nabla u_h, \nabla(u - v_h)), \forall v_h \in \mathcal{K}_h.$$

Now applying elementwise integration by parts we arrive at

$$(16)\quad |e|_{1,\Omega}^2 \leq (f, u - v_h) - \sum_{E \in E_h^{int}} \int_E J_{E,n}(u_h)(u - v_h) - \sum_{E \in E_h^N \cup E_h^C} \int_E \frac{\partial u_h}{\partial n_E}(u - v_h), \forall v_h \in \mathcal{K}_h.$$

At this stage we fix the choice of v_h. Namely v_h is the unique element of $\mathcal{K}_h$ defined at each node x as follows:

$$v_h(x) := \begin{cases} \pi_h u(x) & \text{if } x \in \mathcal{N}_h^{int}, \\ u_h(x) + \pi_h e(x) & \text{if } x \in \mathcal{N}_h^N, \\ \pi_h u_h(x) + \pi_h^\star e^+(x) - \pi_h^\star e^-(x) & \text{if } x \in \mathcal{N}_h^C, \\ u_h(x) & \text{if } x \in \mathcal{N}_h^E, \end{cases}$$

where for $v \in H^1(\Omega)$ and $x \in \mathcal{N}_h^C$ we define

$$\pi_h^\star v(x) := \min \left\{ \pi_h v(x), \frac{1}{|E_1|} \int_{E_1} v(x)\, d\sigma(x), \frac{1}{|E_2|} \int_{E_2} v(x)\, d\sigma(x) \right\},$$

$E_j, j = 1, 2$ being the two edges of Γ_C having x as extremity. We then set $\pi_h^\star v(x) = \pi_h v(x)$ at the nodes in $\mathcal{N}_h \backslash \mathcal{N}_h^C$. From the nodal values $\pi_h^\star v(x)$ we may define $\pi_h^\star v \in V_h$ (note that $\pi_h^\star$ is not linear). Note that the following positivity preserving property holds: $v \geq 0$ on $\Gamma_C \Rightarrow \pi_h^\star v(x) \geq 0, \forall x \in \mathcal{N}_h^C$. With this choice we deduce that $v_h(x) \geq 0$ for all $x \in \mathcal{N}_h^C$ because for such a x

$$\pi_h^\star e^-(x) \leq \pi_h e^-(x) \leq \pi_h u_h(x).$$

This last estimate follows from the estimate $e^- \leq u_h$ on Γ_C, which is a consequence of the non negativeness of u and u_h on Γ_C, and from the positivity preserving property of π_h at the nodes in $\mathcal{N}_h^C$. Remark furthermore that $\pi_h^\star e^+(x) \geq 0$ for all $x \in \mathcal{N}_h^C$.

With the above choice we are able to estimate each term of the right-hand side of (16). We start with the integral term. Cauchy-Schwarz's inequality implies

$$|(f, u - v_h)| \leq \sum_{K \in T_h} \|f\|_K \|u - v_h\|_K. \tag{17}$$

Therefore it remains to estimate $\|u - v_h\|_K$ for any triangle K. From the definition of v_h we have

$$\begin{aligned} u - v_h &= u - \sum_{x \in \mathcal{N}_h^{int} : x \in K} \pi_h u(x) \lambda_x \\ &- \sum_{x \in \mathcal{N}_h^N : x \in K} (u_h(x) + \pi_h e(x)) \lambda_x \\ &- \sum_{x \in \mathcal{N}_h^C : x \in K} (\pi_h u_h(x) + \pi_h^\star e^+(x) - \pi_h^\star e^-(x)) \lambda_x \\ &- \sum_{x \in \mathcal{N}_h^E : x \in K} u_h(x) \lambda_x \text{ on } K. \end{aligned} \tag{18}$$

This identity may be equivalently written

$$\begin{aligned} u - v_h &= e - \pi_h e + \sum_{x \in \mathcal{N}_h^{int} : x \in K} (u_h(x) - \pi_h u_h(x)) \lambda_x \\ &+ \sum_{x \in \mathcal{N}_h^C : x \in K} (u_h(x) - 2\pi_h u_h(x) + \pi_h u(x) - \pi_h^\star e^+(x) + \pi_h^\star e^-(x)) \lambda_x \\ &+ \sum_{x \in \mathcal{N}_h^E : x \in K} (\pi_h u(x) - \pi_h u_h(x)) \lambda_x \text{ on } K. \end{aligned} \tag{19}$$

By the triangular inequality and the fact that $\|\lambda_x\|_K \sim h_K$ for any node x of K, we then obtain

$$\begin{aligned} \|u - v_h\|_K &\lesssim \|e - \pi_h e\|_K + h_K \sum_{x \in \mathcal{N}_h^{int} \cup \mathcal{N}_h^C : x \in K} |u_h(x) - \pi_h u_h(x)| \\ &+ h_K \sum_{x \in \mathcal{N}_h^C : x \in K} |\pi_h u(x) - \pi_h u_h(x) - \pi_h^\star e^+(x) + \pi_h^\star e^-(x)| \\ &+ h_K \sum_{x \in \mathcal{N}_h^E : x \in K} |\pi_h u(x) - \pi_h u_h(x)|, \forall K \in T_h. \end{aligned}$$

Using Lemmas 3.2 and 3.3 as well as Lemma 4.4 below, we obtain

$$\|u - v_h\|_K \lesssim h_K \|\nabla e\|_{\tilde{\omega}_K} + h_K \sum_{K' \subset \omega_K} \eta_{sK'} + C_K h_K \sqrt{-\ln(h_K)} \|\nabla e\|_\Omega, \tag{20}$$

where $\tilde{\omega}_K := \cup_{E\in E_h:\bar{E}\cap K\neq\emptyset}\,\omega_E$, $C_K = 0$ if $K\cap\mathcal{N}_h^E = \emptyset$, otherwise $C_K = 1$.

Using estimates (20), (17) and the fact that a node in $\mathcal{N}_h^E$ belongs at most to a bounded number of elements (independently of h) leads to

$$|(f, u - v_h)| \lesssim (1 + C(h))(\eta + \zeta)(\eta + \zeta + \|\nabla e\|_\Omega). \tag{21}$$

Let us now pass to the estimate of the interior boundary terms in (16): as before the application of Cauchy-Schwarz's inequality leads to

$$\left|\sum_{E\in E_h^{int}} \int_E J_{E,n}(u_h)(u - v_h)\right| \leq \sum_{E\in E_h^{int}} \|J_{E,n}(u_h)\|_E \|u - v_h\|_E. \tag{22}$$

Therefore using the expression (19) of $u - v_h$, the triangular inequality and the fact that $\|\lambda_x\|_E \sim h_E^{1/2}$ for any extremity x of $\bar{E}$, we then obtain

$$\begin{aligned}
\|u - v_h\|_E &\lesssim \|e - \pi_h e\|_E + h_E^{1/2} \sum_{x\in\mathcal{N}_h^{int}\cup\mathcal{N}_h^C:x\in\bar{E}} |u_h(x) - \pi_h u_h(x)| \\
&+ h_E^{1/2} \sum_{x\in\mathcal{N}_h^C:x\in\bar{E}} |\pi_h u(x) - \pi_h u_h(x) - \pi_h^\star e^+(x) + \pi_h^\star e^-(x)| \\
&+ h_E^{1/2} \sum_{x\in\mathcal{N}_h^E:x\in\bar{E}} |\pi_h u(x) - \pi_h u_h(x)|, \forall E \in E_h^{int}.
\end{aligned}$$

As before using Lemmas 3.2 and 3.3 as well as Lemma 4.4 below, we arrive at

$$\|u - v_h\|_E \lesssim h_E^{1/2}\|\nabla e\|_{\tilde{\omega}_E} + h_E^{1/2}\sum_{K\subset\omega_E}\eta_{sK} + C_E h_E^{1/2}\sqrt{-\ln(h_E)}\|\nabla e\|_\Omega,$$

where $\tilde{\omega}_E := \cup_{F\in E_h:\bar{F}\cap\bar{E}\neq\emptyset}\,\omega_F$, $C_E = 0$ if $\bar{E}\cap\mathcal{N}_h^E = \emptyset$, otherwise $C_E = 1$.

Inserting this estimate in (22) we arrive at

$$\left|\sum_{E\in E_h^{int}} \int_E J_{E,n}(u_h)(u - v_h)\right| \lesssim (1 + C(h))(\eta + \zeta)(\eta + \zeta + \|\nabla e\|_\Omega). \tag{23}$$

The estimate of the exterior boundary term is split up into terms on Γ_N and on Γ_C. The term on Γ_N is estimated exactly as interior boundary terms so that

$$\left|\sum_{E\in E_h^N} \int_E J_{E,n}(u_h)(u - v_h)\right| \lesssim (1 + C(h))(\eta + \zeta)(\eta + \zeta + \|\nabla e\|_\Omega). \tag{24}$$

On the contrary the terms on Γ_C are more carefully analyzed. Namely we write

$$-\sum_{E\in E_h^C} \int_E \frac{\partial u_h}{\partial n_E}(u - v_h) = I_+ + I_-, \tag{25}$$

where

$$
\begin{aligned}
I_+ &:= \sum_{E \in E_h^C} \int_E \left(\frac{\partial u_h}{\partial n_E}\right)^+ (v_h - u), \\
I_- &:= \sum_{E \in E_h^C} \int_E \left(\frac{\partial u_h}{\partial n_E}\right)^- (u - v_h).
\end{aligned}
$$

The term I_- is estimated exactly as the interior boundary terms where $J_{E,n}(u_h)$ is replaced with $\left(\frac{\partial u_h}{\partial n_E}\right)^-$. So we obtain

$$
|I_-| \lesssim (1 + C(h))(\eta + \zeta)(\eta + \zeta + \|\nabla e\|_\Omega). \tag{26}
$$

To estimate the term I_+ we consider an edge $E \in E_h^C$. Using the expression (18) and since $\pi_h^\star e^-(x) \geq 0,\ \forall x \in \mathcal{N}_h^C$, we get

$$
\begin{aligned}
v_h - u &= -u + \sum_{x \in \mathcal{N}_h^C : x \in \bar{E}} (\pi_h u_h(x) + \pi_h^\star e^+(x) - \pi_h^\star e^-(x))\lambda_x + \sum_{x \in \mathcal{N}_h^E : x \in \bar{E}} u_h(x)\lambda_x \\
&\leq -u + \sum_{x \in \mathcal{N}_h^C : x \in \bar{E}} (\pi_h u_h(x) + \pi_h^\star e^+(x))\lambda_x + \sum_{x \in \mathcal{N}_h^E : x \in \bar{E}} u_h(x)\lambda_x \text{ on } E.
\end{aligned}
$$

If $\bar{E} \cap \mathcal{N}_h^E = \emptyset$ then

$$
v_h - u \leq \tilde{\pi}_h u_h + \pi_h^\star e^+ - u \text{ on } E, \tag{27}
$$

where we have set

$$
\pi_h^\star e^+ := \sum_{x \in \mathcal{N}_h : x \in \bar{E}} \pi_h^\star e^+(x)\lambda_x.
$$

If $\bar{E} \cap \mathcal{N}_h^E = x$, then denote by $x' = \bar{E} \cap \mathcal{N}_h^C$. Therefore we have on E

$$
v_h - u \leq -u + (\pi_h u_h(x') + \pi_h^\star e^+(x'))\lambda_{x'} + (u_h(x) + \pi_h^\star e^+(x'))\lambda_x, \tag{28}
$$

since $\pi_h^\star e^+(x') \geq 0$. Integrating the above estimates (27) (resp. (28)) and using the properties

$$
\int_E \pi_h^\star e^+ \leq \int_E e^+ \qquad \Big(\text{resp.} \quad \pi_h^\star e^+(x') \leq \frac{1}{|E|}\int_E e^+\Big), \tag{29}
$$

which follow from the definition of $\pi_h^\star e^+(x)$, we deduce that for any $E \in E_h^C$:

$$
\int_E (v_h - u) \leq \int_E (\tilde{\pi}_h u_h + e^+ - u).
$$

Now using the property $e^+ = e + e^-$ and the estimate $e^- \leq u_h$ on Γ_C we arrive at

$$
\int_E (v_h - u) \leq \int_E \tilde{\pi}_h u_h.
$$

Using this estimate and the fact that $\left(\frac{\partial u_h}{\partial n_E}\right)^+$ is a nonnegative constant on each edge E of Γ_C, we conclude that

$$0 \leq I_+ \lesssim \sum_{E \in E_h^C} \int_E \tilde{\pi}_h u_h \left(\frac{\partial u_h}{\partial n_E}\right)^+ = \eta_{ns}^2. \tag{30}$$

Going back to the identity (16), using the estimates (21), (23), (24), (26) and (30), as well as the identity (25), we conclude that

$$|e|_{1,\Omega}^2 \lesssim (1 + C(h))(\eta + \zeta)(\eta + \zeta + \|\nabla e\|_\Omega).$$

The conclusion follows from Young's inequality. ∎

Remark 4.3 This remark is concerned with the choice of v_h on Γ in (16). From the above proof, the following three main properties have to be satisfied:
1. $v_h(x) \geq 0$, for all $x \in \mathcal{N}_h^C \cup \mathcal{N}_h^E$,
2. estimation of $u_h(x) + \pi_h u(x) - \pi_h u_h(x) - v_h(x)$, for all $x \in \mathcal{N}_h$,
3. for all $E \in E_h^C$, estimate the quantity $\int_E (v_h - u)$ by a term representing the error estimator.

The first idea is to choose $v_h = \pi_h u$ but this choice does not guarantee point 1 at $\mathcal{N}_h^E$, point 2 at $\mathcal{N}_h^N$ since Lemma 3.3 is not available in this case, while point 3 seems not possible. The second possibility is to take $v_h = u_h + \pi_h e$ (since $u_h + \pi_h e = u_h + \pi_h u - \pi_h u_h$ is close to $\pi_h u$). This latter choice guarantees point 2 for $\mathcal{N}_h^N$, but does not fit point 1 for $\mathcal{N}_h^C \cup \mathcal{N}_h^E$. Point 3 suggests to use a new operator $\pi_h^\star$ which satisfies $\int_E \pi_h^\star v - v \leq 0$ for almost all edges in E_h^C (see (29)) but also the edge approximation property as in Lemma 3.2. After some tentatives, an appropriate choice seems to be $v_h = \pi_h u_h + \pi_h^\star e^+ - \pi_h^\star e^-$, since it guarantees points 1 and 2 at $\mathcal{N}_h^C$ and point 3. For the exceptional nodes $\mathcal{N}_h^E$, we choose $v_h = u_h$ since points 1 and 3 are satisfied and point 2 is almost optimal (see Lemma 4.4 below).

Lemma 4.4 *The next estimates hold:*
i) For any $x \in \mathcal{N}_h^C$, one has

$$|\pi_h u_h(x) - \pi_h u(x) + \pi_h^\star e^+(x) - \pi_h^\star e^-(x)| \lesssim \|\nabla e\|_{\omega_E}, \tag{31}$$

where $E \in E_h^C$ is such that $x \in \bar{E}$ and $\bar{E} \cap \mathcal{N}_h^E = \emptyset$.
ii) For any $x \in \mathcal{N}_h^E$, one has

$$|\pi_h u_h(x) - \pi_h u(x)| \lesssim \sqrt{-\ln(h)}\|\nabla e\|_\Omega. \tag{32}$$

Proof: Fix $x \in \mathcal{N}_h^C$ and $E \in E_h^C$ such that $x \in \bar{E}$ and $\bar{E} \cap \mathcal{N}_h^E = \emptyset$. From a scaling argument and since all norms are equivalent on any finite dimensional space, we have

$$|\pi_h u_h(x) - \pi_h u(x) + \pi_h^\star e^+(x) - \pi_h^\star e^-(x)| \lesssim h_E^{-1/2}\| - \pi_h e + \pi_h^\star e^+ - \pi_h^\star e^-\|_E.$$

By the triangular inequality we get

$$|\pi_h u_h(x) - \pi_h u(x) + \pi_h^\star e^+(x) - \pi_h^\star e^-(x)| \lesssim h_E^{-1/2}(\|e - \pi_h e\|_E + \|e^+ - \pi_h^\star e^+\|_E + \|e^- - \pi_h^\star e^-\|_E).$$

By Lemma 3.2 and since $|\nabla e^+| \le |\nabla e|$, $|\nabla e^-| \le |\nabla e|$ we conclude that

$$|\pi_h u_h(x) - \pi_h u(x) + \pi_h^\star e^+(x) - \pi_h^\star e^-(x)| \lesssim \|\nabla e\|_{\omega_E},$$

recalling that if $\pi_h^\star(v) := \sum_{x\in\mathcal{N}_h} \alpha_x^\star(v)\lambda_x$ we have for any $x \in \mathcal{N}_h^C$, $|\alpha_x^\star(v)| \lesssim h_x^{-1}\|v\|_{\omega_x} + \|\nabla v\|_{\omega_x}$. Since $\pi_h^\star$ preserves the constant functions (because for $c \in \mathbb{R}$, $\pi_h^\star(v+c)(x) = \pi_h^\star v(x)+c$ and therefore $\pi_h^\star(v+c) = \pi_h^\star v+c$) we come to the conclusion that $\|v - \pi_h^\star v\|_E \lesssim h_E^{1/2}\|\nabla v\|_{\omega_E}$. Hence estimate (31) holds.

Let us now fix $x \in \mathcal{N}_h^E$ and denote by E an edge in Γ_C such that $x \in \bar{E}$. Then by the fact that all norms are equivalent on any finite dimensional space, we have

$$|\pi_h u_h(x) - \pi_h u(x)| = |\pi_h e(x)| \lesssim h_E^{-1/2}\|\pi_h e\|_E.$$

Introducing e artificially and using the triangular inequality we obtain

$$|\pi_h u_h(x) - \pi_h u(x)| \lesssim h_E^{-1/2}(\|e - \pi_h e\|_E + \|e\|_E).$$

The first term is estimated using Lemma 3.2:

$$h_E^{-1/2}\|e - \pi_h e\|_E \lesssim \|\nabla e\|_{\bar{\omega}_x} \tag{33}$$

where $\bar{\omega}_x := \cup_{F\in E_h: x\in\bar{F}}\, \omega_F$. To estimate the second term we remark that by Hölder's inequality we have

$$\|e\|_E \le \|e\|_{L^p(E)} h_E^{1/q},$$

for any $p, q > 2$ such that $1/p + 1/q = 1/2$. Since E is a subset of Γ_C we write

$$\|e\|_E \le \|e\|_{L^p(\Gamma_C)} h_E^{1/q}.$$

At this stage we use the trace theorem $H^1(\Omega) \hookrightarrow H^{1/2}(\Gamma_C)$ and the following embedding (see [2]): for any real number $p \in [1, \infty[$,

$$\|v\|_{L^p(\Gamma_C)} \le C\sqrt{p}\|v\|_{H^{\frac{1}{2}}(\Gamma_C)}, \quad \forall v \in H^{\frac{1}{2}}(\Gamma_C),$$

where C is independent of p. As a consequence

$$\|e\|_E \lesssim \|e\|_{1,\Omega}\sqrt{p}\; h_E^{-1/p} h_E^{1/2}.$$

Choosing $p = -\ln(h_E)$ (h_E is supposed sufficiently small) we finally get the estimate

$$\|e\|_E \lesssim \|\nabla e\|_\Omega\, h_E^{1/2}\sqrt{-\ln(h_E)},$$

by using Poincaré's inequality. This last estimate and (33) lead to (32). ∎

4.3 Lower error bound

Theorem 4.5 *For all elements K, the following local lower error bound holds:*

$$\eta_{sK} \lesssim \|\nabla e\|_{\omega_K} + \zeta_K. \tag{34}$$

Proof: The estimates of the element residual

$$h_K \|r_K\|_K \lesssim \|\nabla e\|_K + \zeta_K, \tag{35}$$

for $K \in T_h$ and of the normal jump

$$h_E^{1/2} \|J_{E,n}(u_h)\|_E \lesssim \|\nabla e\|_{\omega_K} + \zeta_K, \tag{36}$$

for $E \in E_K^{int} \cup E_K^N$ are standard [16] since for any $w \in V := \{v \in H^1(\Omega) : v = 0$ on $\Gamma_D \cup \Gamma_C\}$, w and $-w$ belong to $\mathcal{K}$ and therefore

$$\int_\Omega \nabla u \cdot \nabla w \, dx = \int_\Omega f w \, dx, \quad \forall w \in V.$$

We slightly modify this argument to estimate the negative part of the normal derivatives on Γ_C. Namely for an arbitrary edge $E \in E_h^C$, we introduce the edge bubble function b_E associated with E defined on the element K containing E (i.e., $b_E := 4\lambda_{a_1}\lambda_{a_2}$, where a_1, a_2 are the two extremities of E). Then we set $b_E := 0$ on $\bar{\Omega} \setminus K$. We recall that $(\partial u_h / \partial n_E)^- \in \mathbb{P}_0(E)$ and set

$$w_E := \left(\frac{\partial u_h}{\partial n_E}\right)^- b_E \in \mathcal{K},$$

We first remark that

$$\int_E \frac{\partial u_h}{\partial n_E} w_E \;=\; \int_E \frac{\partial u_h}{\partial n_E} \left(\frac{\partial u_h}{\partial n_E}\right)^- b_E = -\int_E \left(\left(\frac{\partial u_h}{\partial n_E}\right)^-\right)^2 b_E.$$

Using an integration by parts on K such that $E \subset K$ we have

$$\int_K \nabla u_h \cdot \nabla w_E = \int_E \frac{\partial u_h}{\partial n_E} w_E.$$

The last two identities lead to

$$\int_E \left(\left(\frac{\partial u_h}{\partial n_E}\right)^-\right)^2 b_E = -\int_K \nabla u_h \cdot \nabla w_E,$$

or equivalently

$$\int_E \left(\left(\frac{\partial u_h}{\partial n_E}\right)^-\right)^2 b_E = \int_K \nabla e \cdot \nabla w_E - \int_K \nabla u \cdot \nabla w_E. \tag{37}$$

Besides as $w_E \in \mathcal{K}$ by the inequality (3) we may write

$$\int_K \nabla u \cdot \nabla w_E \geq \int_K f w_E.$$

This inequality in (37) yields

$$\int_E \left(\left(\frac{\partial u_h}{\partial n_E} \right)^- \right)^2 b_E \leq \int_K \nabla e \cdot \nabla w_E - \int_K f w_E.$$

Applying Cauchy-Schwarz's inequality we obtain

$$\| \left(\frac{\partial u_h}{\partial n_E} \right)^- \|_E^2 \sim \int_E \left(\left(\frac{\partial u_h}{\partial n_E} \right)^- \right)^2 b_E \leq \|\nabla e\|_K \|\nabla w_E\|_K + \|f\|_K \|w_E\|_K.$$

From a standard inverse inequality [16] we arrive at

$$h_E^{1/2} \| \left(\frac{\partial u_h}{\partial n_E} \right)^- \|_E \lesssim \|\nabla e\|_K + h_K \|f\|_K.$$

Writing $f = R_K - r_K + r_K$ and using the estimate in (35) we finally conclude that

$$h_E^{1/2} \| \left(\frac{\partial u_h}{\partial n_E} \right)^- \|_E \ \lesssim \ \|\nabla e\|_K + \zeta_K. \tag{38}$$

We obtain (34) by putting together the estimates (35), (36) and (38). ∎

For the non standard residual estimator we may prove the following non optimal estimate:

Theorem 4.6 *For all element K such that $K \cap E_h^C \neq \emptyset$, the following local lower error bound holds:*

$$\begin{aligned} \eta_{nsK}^2 \ &\lesssim \ \sum_{E \in E_K^C} \Big((\|\nabla e\|_{\omega_K} + \zeta_K) h_K^{-1/2} \|\tilde{\pi}_h u_h\|_E \\ &+ \ \| \frac{\partial u}{\partial n_E} \|_E \Big(h_K^{1/2} \sum_{K \subset \tilde{\omega}_E} (\|\nabla e\|_{\omega_K} + \zeta_K) + \|e\|_E \Big) \Big), \end{aligned} \tag{39}$$

where $\tilde{\omega}_E := \cup_{F \in E_h : \bar{F} \cap \bar{E} \neq \emptyset} \, \omega_F$.

Proof: We only need to prove (39) if $\left(\frac{\partial u_h}{\partial n_E} \right)^+ > 0$ on E, and in that case $\left(\frac{\partial u_h}{\partial n_E} \right)^+ = \frac{\partial u_h}{\partial n_E}$. Therefore using the equivalence of norm in any finite dimensional space, we may write

$$\int_E \left(\frac{\partial u_h}{\partial n_E} \right)^+ \tilde{\pi}_h u_h = \int_E |\frac{\partial u_h}{\partial n_E} \tilde{\pi}_h u_h| \sim \int_E |\frac{\partial u_h}{\partial n_E} \tilde{\pi}_h u_h| b_E = \int_E \frac{\partial u_h}{\partial n_E} \tilde{\pi}_h u_h \, b_E,$$

where b_E stands for the edge bubble function defined on the triangle K containing E. Let x be the node of K which is not located in $\bar{E}$. We define $\hat{\pi}_h u_h \in \mathbb{P}_1(K)$ such that $\hat{\pi}_h u_h(x) := 0$ and $\hat{\pi}_h u_h := \tilde{\pi}_h u_h$ on E. Setting $v := \hat{\pi}_h u_h \, b_E$, which belongs to $H^1(K)$ and applying Green's formula we obtain

$$\int_E \left(\frac{\partial u_h}{\partial n_E}\right)^+ \tilde{\pi}_h u_h \sim \int_K \nabla u_h \cdot \nabla v.$$

Again Green's formula on K yields

$$\int_K \nabla u \cdot \nabla v = \int_K fv + \int_E \partial_n u v.$$

The last two identities show that

$$\int_E \left(\frac{\partial u_h}{\partial n_E}\right)^+ \tilde{\pi}_h u_h \sim -\int_K \nabla e \cdot \nabla v + \int_K fv + \int_E \partial_n u v.$$

Now we transform the last term in the previous expression as follows:

$$\int_E \partial_n u v \quad = \quad \int_E \partial_n u (\tilde{\pi}_h u_h - u_h) b_E + \int_E \partial_n u (u_h - u) b_E,$$

reminding that $u\partial_n u = 0$ on Γ_C. This identity in the previous one yields using a standard inverse inequality

$$\begin{aligned}\int_E \left(\frac{\partial u_h}{\partial n_E}\right)^+ \tilde{\pi}_h u_h \quad &\lesssim \quad (\|\nabla e\|_K + h_K \|f\|_K) h_E^{-1/2} \|\tilde{\pi}_h u_h\|_E \\ &+ \quad \|\partial_n u\|_E (\|\tilde{\pi}_h u_h - u_h\|_E + \|u_h - u\|_E).\end{aligned}$$

Since we readily check that

$$\|\tilde{\pi}_h u_h - u_h\|_E \le \frac{2}{\sqrt{3}} \|\pi_h u_h - u_h\|_E,$$

by Corollary 3.5 and Theorem 4.5, we get

$$\|\tilde{\pi}_h u_h - u_h\|_E \quad \lesssim \quad h_E^{1/2} \sum_{K \subset \tilde{\omega}_E} \eta_{sK} \lesssim h_E^{1/2} \sum_{K \subset \tilde{\omega}_E} (\|\nabla e\|_{\omega_K} + \zeta_K).$$

This estimate in the previous one leads to the conclusion. ∎

Corollary 4.7 *The following global lower error bound holds:*

$$\eta_{ns}^2 \lesssim \left(\sum_{K \in T_h : K \cap E_h^C \neq \emptyset} h^{-1} \Big(\sum_{K' \subset \tilde{\omega}_K} (\|\nabla e\|_{\omega_{K'}}^2 + \zeta_{K'}^2) \Big) \right)^{1/2} + \|\partial_n u\|_{\Gamma_C} \|e\|_{\Gamma_C},$$

where $\tilde{\omega}_K := \cup_{E \in E_h : \bar{E} \cap K \neq \emptyset} \, \omega_E$.

Proof: Summing the estimate (39), using discrete Cauchy-Schwarz's inequality we come to the conclusion since one readily sees that

$$\|\tilde{\pi}_h u_h\|_{\Gamma_C} \lesssim \|u_h\|_{\Gamma_C} \lesssim \|f\|_{\Omega}.$$

■

Remark 4.8 If theoretically one has $\|\nabla e\|_K \lesssim h_K \|u\|_{2,\omega_K}$ and $\|e\|_K \lesssim h_K^2 \|u\|_{2,\omega_K}$, then the previous Corollary gives the rough estimate

$$\eta_{ns}^2 \lesssim h^{1/2} \|u\|_{2,B_h}, \tag{40}$$

where B_h is a small neighbourhood of Γ_C satisfying $|B_h| \sim h$. If we suppose that the H^2-norm is uniformly distributed in Ω then (40) guarantees the convergence to 0 of the estimator η_{ns} (quicker than $h^{1/2}$). We finally mention that the following numerical experiments show that the convergence rates of η_{ns} and η_s are similar and that η_{ns} is small in comparison with η_s independently of the regularity of u.

5 Numerical experiments

In this section, we solve numerically three examples of Signorini problems with linear triangular finite elements. As previously mentioned we denote by $e := u - u_h$ the exact error and by η the estimator. Among others we are interested in computing the convergence rates α and β of the errors η and $|e|_{1,\Omega}$. We compute these convergence rates by considering families of uniform meshes made of triangular elements and supposing that η and $|e|_{1,\Omega}$ behave as Ch^α and Dh^β respectively, where C, D denote positive constants. We are especially interested in determining the effectivity index

$$\gamma := \frac{\eta}{|e|_{1,\Omega}}.$$

This ratio measures the reliability of our proposed estimator.

We skip over the study concerning optimized computations obtained using together the error estimator and a mesh adaptivity procedure which is beyond the scope of this paper.

In the following we denote by N_C, the number of elements of the mesh on Γ_C. Since we use uniform meshes, this parameter measures the size of the mesh.

5.1 Test 1 : an example where the exact solution u is not explicitly known

We consider the problem depicted in Figure 1 where Ω is the square $]0,1[\times]0,1[$. The Dirichlet condition $u = -1$ is applied on the left edge $\{0\}\times]0,1[$ and the condition $u = 1$ is applied on the right edge $\{1\}\times]0,1[$. The part Γ_C of the boundary submitted to Signorini conditions is $]0.25, 0.75[\times\{0\}$. The remaining boundary parts of $\Gamma = \partial\Omega$

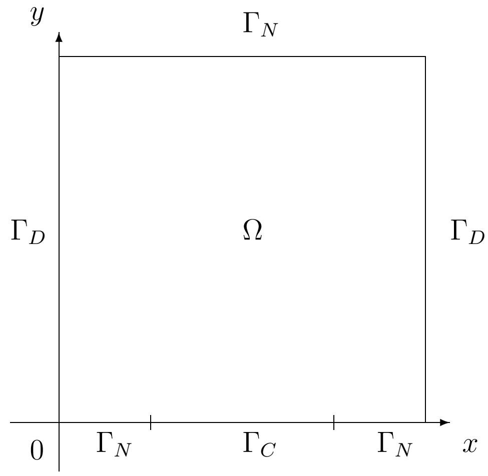

Figure 1: The geometry of the problem

are submitted to homogeneous Neumann conditions $\partial_n u = 0$. We further take $f = 0$ in Ω.

As far as we know this problem does not admit an explicit solution. Consequently, in order to obtain error estimates for $|e|_{1,\Omega}$, we must compute a reference solution denoted by u_{ref} corresponding to a mesh which is as fine as possible. The most refined mesh comprises 66049 nodes, 131072 triangles and 256 elements on each edge (128 on Γ_C). It furnishes the reference solution u_{ref} which is represented in Figure 2. Therefore we assume in the computations that $e = u_{ref} - u_h$.

We observe numerically on Γ_C that $u = 0$ (and $\partial_n u > 0$) on $]0.25, x_0] \times \{0\}$ and that $\partial_n u = 0$ (and $u > 0$) on $[x_0, 0.75[\times\{0\}$ where x_0 is approximately 0.37. From this observation, we may expect that the exact solution u belongs to $H^s(\Omega)$, for all $s < 3/2$, since u may be considered as a solution of a mixed (Dirichlet-Neumann) problem [8] (see also [4, 11] for Signorini problems). By classical a priori error estimates for finite element methods we may then expect that $|e|_{1,\Omega} \leq Ch^{s-1}$, for all $s < 3/2$. Note that if the Signorini condition is replaced by an homogeneous Neumann condition then the solution becomes $u(x, y) = 2x - 1$, $\forall(x, y) \in \Omega$. Since this function is negative on $]0.25, 0.5[\times\{0\}$ we see that the Signorini conditions modify the latter solution, in particular on the left part of Γ_C.

In Table 1 we report the square of the estimator η^2 together with the standard and nonstandard contributions (we recall that $\eta_s^2 := \sum_{K\in T_h} \eta_{sK}^2$, $\eta_{ns}^2 := \sum_{K\in T_h} \eta_{nsK}^2$ and $\eta^2 = \eta_s^2 + \eta_{ns}^2$). In this table we see that the contribution η_{ns}^2 corresponding roughly speaking to the non-fulfillment of the complementarity condition $u\,\partial_n u = 0$ on Γ_C is always negligible (lower than 2%) in comparison with η_s^2 and that η_{ns}/η converges (approximately towards 0.08) as h vanishes. Moreover we note that the norm terms included in η_s involving the negative part of $\partial u_h/\partial n$ are always equal to zero, at least in this example. If we compute the convergence rates of η, η_s, η_{ns} respectively on the two most refined meshes ($N_C = 32$ and $N_C = 64$) we find 0.508, 0.508, 0.521 respectively which seems to indicate that the convergence rates of η_s and η_{ns} are close. We also remark that if instead of $\tilde{\pi}_h u_h$ we take u_h in the computation of η_{ns} the values of this modified nonstandard term are similar.

The "exact" error $|e|_{1,\Omega}$, the estimator η and the effectivity index γ are reported in Table 2. The average convergence rate α for the estimator η and the average convergence rate β for the exact error $|e|_{1,\Omega}$ are $\alpha = 0.567$ and $\beta = 0.588$ and are therefore very close. Note that the convergence rate for $|e|_{1,\Omega}$ is even better than "theoretically" expected. We also observe that the effectivity index varies between 3.9 and 5.3 and more precisely between 3.9 and 4.3 for the intermediate meshes (the value 4.9 corresponds to the coarsest mesh with only 4 elements on each edge of Ω and the value 5.3 is obtained with the finest mesh comprising 128 elements on each edge of Ω. One can reasonably think that the latter value is a bit overestimated since the finite element solution is in this case "artificially" closer to the reference solution u_{ref}. Of course we cannot compute the effectivity index for $N_C = 128$ since $|e|_{1,\Omega} = 0$ in this case). Finally, Figure 3 depicts both the exact and the a posteriori errors as a function of $1/h$, this figure confirms the equivalence between $|e|_{1,\Omega}$ and η.

Figure 4 represents the distribution of the local indicators η_K for a specific mesh comprising 32 elements on each edge of Ω. We see that the local indicators η_K increase near the singularity located at $(0.25, 0)$.

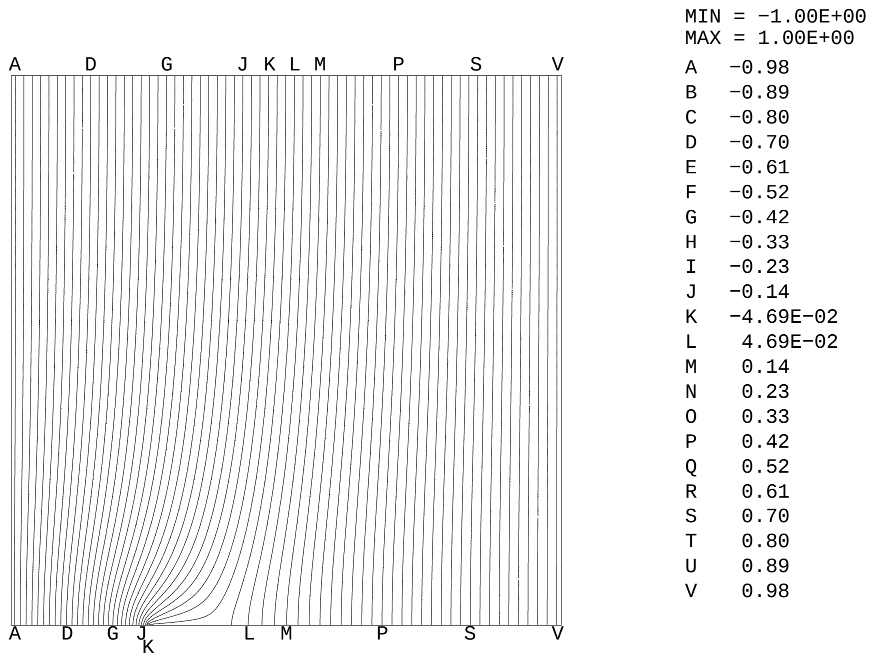

Figure 2: The reference solution (256×256 finite element mesh) and its corresponding isovalues

	η^2	η_s^2	η_{ns}^2	η_{ns}^2/η^2(in %)
$N_C = 2$	3.6742	3.6245	4.97046 10^{-2}	1.352%
$N_C = 4$	1.2940	1.2789	1.51296 10^{-2}	1.169%
$N_C = 8$	0.60746	0.60233	5.13779 10^{-3}	0.846%
$N_C = 16$	0.29630	0.29424	2.06056 10^{-3}	0.695%
$N_C = 32$	0.14556	0.14462	9.45764 10^{-4}	0.649%
$N_C = 64$	0.07198	0.07152	4.59264 10^{-4}	0.638%

Table 1: Standard and nonstandard contributions in the estimator

	η	$\|e\|_{1,\Omega}$	$\gamma = \eta/\|e\|_{1,\Omega}$
$N_C = 2$	1.9168	0.38904	4.9269
$N_C = 4$	1.1375	0.28701	3.9634
$N_C = 8$	0.77879	0.19894	3.9146
$N_C = 16$	0.54433	0.13475	4.0395
$N_C = 32$	0.38152	0.08777	4.3468
$N_C = 64$	0.26829	0.05054	5.3084

Table 2: The estimator, the exact error and the effectivity index

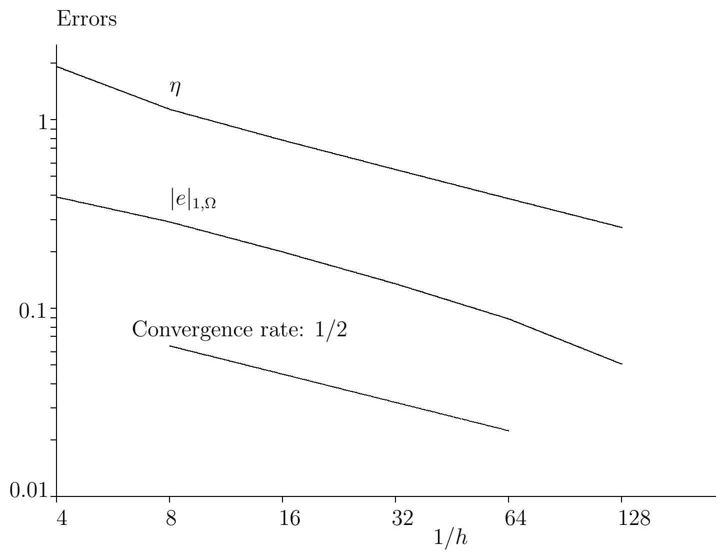

Figure 3: The exact and a posteriori errors $|e|_{1,\Omega}$ and η as a function of $1/h$ (log-log scale)

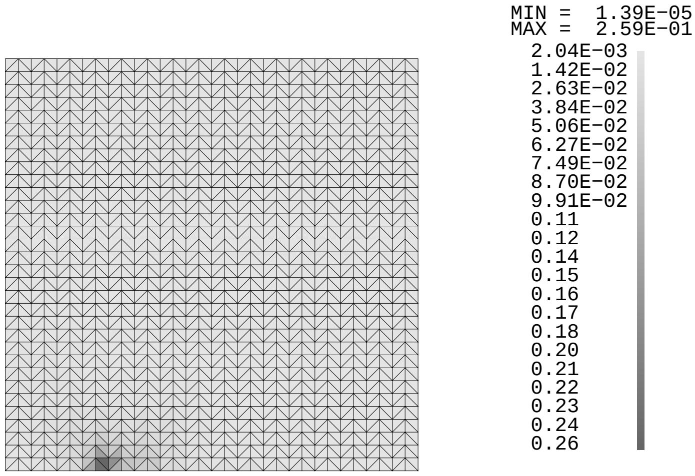

Figure 4: The map of local indicators η_K (case of the 32×32 finite element mesh)

5.2 Test 2 : an example where the exact solution u is explicitly known

We consider the domain Ω as the 3/4 of the unit disk, whose geometry is suggested in Figure 5. It corresponds to the unit disk in which a quarter (the lower right part) has been removed. We suppose that $\Gamma_C =]0,1[\times\{0\}$, $\Gamma_N = \{0\}\times]-1,0[$ and Γ_D is the remaining part of the boundary. On Γ_N we set $\partial_n u = 0$ and impose the non-homogeneous Dirichlet condition $u(r,\theta) = \cos(2\theta/3)$ on Γ_D, where (r,θ) stand for the polar coordinates. Moreover we take $f = 0$ in Ω.

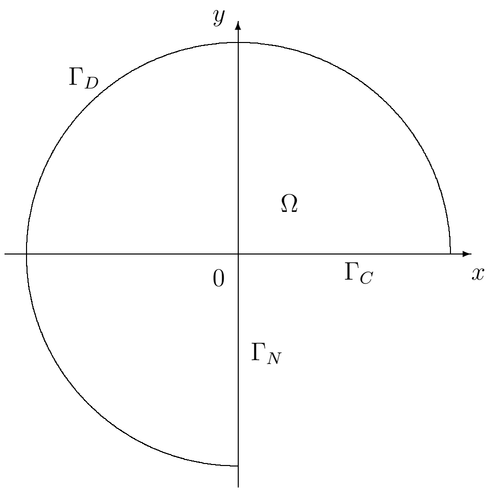

Figure 5: Problem set-up

It can be checked that the exact solution of this Signorini problem is

$$u(r,\theta) = r^{\frac{2}{3}} \cos\frac{2\theta}{3}$$

and that $\partial_n u = 0$ and $u > 0$ on Γ_C. Such a solution belongs to $H^s(\Omega)$, for all $s < 5/3$. Therefore $|e|_{1,\Omega} \leq Ch^{s-1}$, for all $s < 5/3$, thanks to classical a priori error estimates for finite element methods.

We compute the exact (here it is not necessary to determine a reference solution) and the a posteriori errors and we report these quantities together with the effectivity index in Table 3. For this example the contribution η_{ns}^2 is always zero, a phenomenon that we cannot explain but which confirms that η_{ns} seems to be negligible. We observe that the average convergence rates α and β for the estimator η and the exact error $|e|_{1,\Omega}$ are given by: $\alpha = 0.652$ and $\beta = 0.662$, and are still very close in this example. Note that β is close to $2/3$ as theoretically expected. Moreover the effectivity indexes show few variations since they stay between 10.5 and 11.1. We finally observe that these values are greater than those from the previous example. Figure 6 represents both the exact and the a posteriori errors as a function of $1/h$, where we still see their equivalence.

Figure 7 represents the distribution of the local estimators η_K for a specific mesh. Again we see that the estimator increases near the origin in which a singularity is located.

	η	$\|e\|_{1,\Omega}$	$\gamma = \eta/\|e\|_{1,\Omega}$
$N_C = 10$	0.47541	0.043546	10.917
$N_C = 20$	0.30731	0.029029	10.586
$N_C = 40$	0.19165	0.018050	10.617
$N_C = 80$	0.12246	0.010990	11.142

Table 3: The estimator, the exact error and the effectivity index

5.3 Test 3 : a more regular example

We consider the triangle Ω of vertexes $A = (0,0)$, $B = (1,0)$ and $C = (1/2,1/2)$ and we define $\Gamma_D =]B,C[$, $\Gamma_N =]A,C[$, $\Gamma_C =]A,B[$. The Dirichlet condition $u = 0.05$ is applied on Γ_D, the Neumann condition $\partial_n u = 0$ holds on Γ_N and we choose $f = 1$ in Ω. The solution (approximated with a fine mesh) is depicted in Figure 8 and we observe that it is more regular in comparison with the two first tests. As in the first example Γ_C is divided into two parts: the right side where $u > 0$ and the left one where $\partial_n u > 0$. So we determine the convergence of all the terms involved in the global estimator η and we report the results in Table 4 where we set

$$\eta_1 \quad = \Big(h \sum_{E \in E_h^{int}} \|J_{E,n}(u_h)\|_E^2\Big)^{1/2},$$

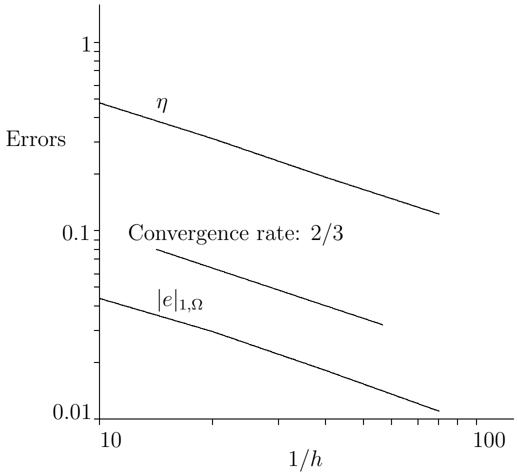

Figure 6: The exact and a posteriori errors $|e|_{1,\Omega}$ and η as a function of $1/h$ (log-log scale)

$$\eta_2 = \Big(h \sum_{E\in E_h^N} \|J_{E,n}(u_h)\|_E^2\Big)^{1/2},$$

$$\eta_3 = \Big(h \sum_{E\in E_h^C} \|\left(\frac{\partial u_h}{\partial n_E}\right)^- \|_E^2\Big)^{1/2}.$$

Note that the convergence rate of the term: $h(\sum_{K\in T_h} \|r_K\|_K^2)^{1/2} = h/2$ is 1.

	η_1	η_2	η_3	η_{ns}
$N_C = 20$	$3.40463\ 10^{-2}$	$2.75369\ 10^{-3}$	$4.11545\ 10^{-3}$	$1.18376\ 10^{-3}$
$N_C = 40$	$1.66169\ 10^{-2}$	$9.81873\ 10^{-4}$	$1.54214\ 10^{-3}$	$6.03231\ 10^{-4}$
$N_C = 80$	$8.67974\ 10^{-3}$	$3.49439\ 10^{-4}$	$5.64690\ 10^{-4}$	$3.15263\ 10^{-4}$
$N_C = 160$	$4.39990\ 10^{-3}$	$1.33811\ 10^{-4}$	$1.85859\ 10^{-4}$	$1.76036\ 10^{-4}$
Convergence rate	0.98	1.45	1.49	0.92

Table 4: Contributions in the estimator

We first observe that neither η_3 nor η_{ns} vanish in this test (in the first example we had $\eta_3 = 0$ and in the second one $\eta_{ns} = 0$). From these results we see that the convergence rate of η_1 is near 1 and that η_3 is close to η_2 (the convergence rates around 1.5 for η_2 and η_3 are due to the fact that the number of edges on the boundary parts is $\sim 1/h$ whereas the number of edges in Ω is $\sim 1/h^2$). The interesting phenomena are that the nonstandard error term admits (as η_s) a convergence rate close to 1 and that it remains small in comparison with η_s while it is of the same size than η_2 and η_3, so it is not negligible.

We also remark that the nonstandard error term is located on a small area (whose length does not really depend on h) near the transition point (i.e. the point where

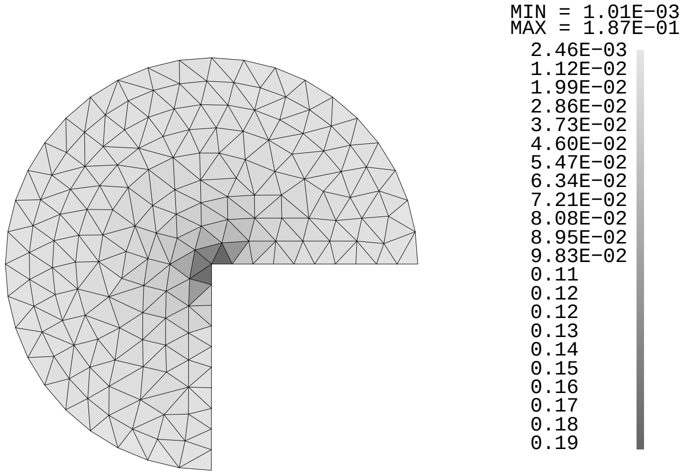

Figure 7: The map of local indicators η_K (case $N_C = 10$)

$u > 0$ at right and $\partial_n u > 0$ at left); the number of elements in which $\eta_{nsK} \neq 0$ is $2, 3, 5, 9$ when $N_C = 20, 40, 80, 160$ respectively and we observe that the values of η_{nsK} are of the same order on the triangles of the little area. Besides the number of elements K where $(\partial_n u_h)^- \neq 0$ in E_K^C is $9, 17, 34, 67$ when $N_C = 20, 40, 80, 160$. These elements are located on the right part of Γ_C and $(\partial_n u_h)^-$ grows when one reaches point B.

In conclusion, the above experiments show that our estimator is reliable, as theoretically expected. It is furthermore appropriate for adaptivity since it detects the region of large errors.

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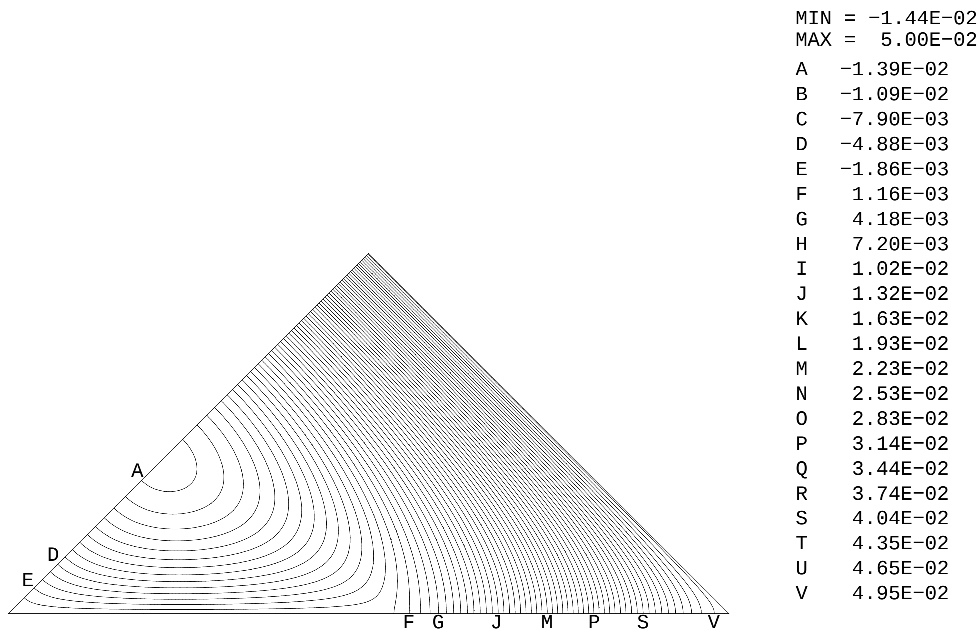

Figure 8: The solution and its corresponding isovalues

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