Two results on solution uniqueness and multiplicity for the linear elastic friction problem with normal compliance

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Abstract

This study is concerned with the frictional contact problem governed by the normal compliance law in linear elasticity. The paper presents two contributions dealing with the stationary problem: we first obtain improved bounds ensuring uniqueness of a solution. Second we exhibit examples in which infinitely many solutions to the problem exist.

Keywords : friction, normal compliance, penalized contact, linear elasticity, uniqueness of solutions, multiplicity of solutions.

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1. Introduction and notation

In solid mechanics, contact involves highly nonlinear phenomena especially when friction effects are taken into account. The most common model of friction is due to Coulomb at the end of the eighteenth century ([8]) and it is generally used together with the Signorini (or unilateral) contact conditions [25]. Such a simple macroscopic frictional contact model is strongly nonlinear in the dynamic, quasi-static and static cases and its understanding from a mathematical point of view is not complete yet. A more recent approach motivated by phenomenological laws on the contact interface such as the presence of small asperities, oxides and impurities has lead to the so-called normal compliance model (with or without friction) introduced and studied in [19] and [18] (see also [20, 21, 22] for other early studies). Note that this model can also be seen as a regularization of the Signorini contact conditions in which some penetration is allowed or also as a kind of (sophisticated) penalized contact problem. In the simple case of elastostatics several studies concerning existence and/or uniqueness of solutions have been achieved: in addition to the previous references, an important work can be found in [16, 17]. For elastic quasi-static problems, we refer the reader to e.g., [2, 3, 4, 5, 24].

This work deals with the two and three dimensional normal compliance models with friction in static linear elasticity. Section 2 is concerned with the model and its corresponding weak formulation. Section 3 deals with uniqueness: we refine the uniqueness results established in [16] and [17]. We prove in Theorem 3.1 two kind of uniqueness results both in two and three space dimensions: first we obtain global uniqueness (which does not depend on the normal variables c_n, m_n, see (2.5)) when the tangential coefficient c_t (see (2.6)) is small enough, in particular when c_n tends to infinity. Second we obtain in many cases global uniqueness results which remain valid for any given c_t under the condition that c_n is large enough. Section 4 is concerned with

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nonuniqueness. We exhibit examples of nonunique solutions to the two-dimensional problem in the simplest case where $m_n = m_t = 1$. Let us mention that a first study in [14] tried to find some nonunique solutions to this problem but without success. In [14], multiple solutions to the finite element problem are found but these solutions do not solve the continuous problem. Note that the first finite dimensional nonuniqueness examples of friction problems with unilateral contact conditions or with normal compliance have been exhibited in [15] by using a system of springs. In Theorem 4.3 of the present paper we show that for appropriately chosen geometry, Poisson ratio and loads, the continuous problem admits an infinity of solutions. An explicit elementary example ends the paper.

We now introduce some useful notation and several functional spaces. In what follows, bold letters like $\boldsymbol{u}, \boldsymbol{v}$, indicate vector valued quantities, while the capital ones (e.g., $\mathbf{V}_0, \mathbf{V}_U, \ldots$) represent functional sets involving vector fields. As usual, we denote by $(L^q(.))^d$ and by $(H^s(.))^d$, $1 \leq q \leq \infty$, $s \geq 0$, $d = 1, 2, 3$, the Lebesgue and Sobolev spaces in one, two and three space dimensions (see [1]). The usual norm of $(L^q(D))^d$ (resp. $(H^s(D))^d$) is denoted by $\| \cdot \|_{L^q(D)}$ (resp. $\| \cdot \|_{H^s(D)}$) and we keep the same notation for any value of d. In the same spirit and in order to lighten the notations we also skip the notation d in $(L^q(D))^d$ and $(H^s(D))^d$ and we simply write $L^q(D)$ and $H^s(D)$. Finally the symbol $| \cdot |$ will denote the Euclidean norm in $\mathbb{R}^d$.

2. Problem statement, weak formulation and various constants

We consider an elastic body occupying a domain Ω in $\mathbb{R}^d$, $d = 2, 3$. The boundary Γ of Ω is assumed to be Lipschitz and is divided as follows: $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_C$ where Γ_D, Γ_N and Γ_C are three open disjoint parts with $\mathrm{meas}(\Gamma_D) > 0$ and $\mathrm{meas}(\Gamma_C) > 0$. The given displacements $\mathbf{U}$ are prescribed on the portion Γ_D and we suppose that a function $\mathbf{U} \in H^1(\Omega)$ exists such that $\mathbf{U} = \mathbf{0}$ on Γ_C. The part Γ_N is subjected to a density of surface forces denoted $\mathbf{F} \in L^2(\Gamma_N)$ and Ω is being acted upon by the body forces $\boldsymbol{f} \in L^2(\Omega)$. On the part Γ_C the body can come into contact with a foundation and the frictional contact interaction between the body and the foundation is governed by the so called normal compliance law with friction. We denote by $\boldsymbol{n}$ the unit outward normal vector on the boundary Γ.

The frictional contact problem with normal compliance in elastostatics is to find the displacement field $\boldsymbol{u}$ such that equations (2.1)–(2.6) hold:

$$\mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} = \mathbf{0} \quad \text{in } \Omega, \tag{2.1}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \mathcal{C}\, \boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{2.2}$$

$$\boldsymbol{u} = \mathbf{U} \quad \text{on } \Gamma_D, \tag{2.3}$$

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \mathbf{F} \quad \text{on } \Gamma_N, \tag{2.4}$$

where $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the linearized strain tensor defined by $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{\nabla}^T \boldsymbol{u})/2$ and $\mathcal{C} = c_{ijkh}(\boldsymbol{x}) \in L^\infty(\Omega), 1 \leq i, j, k, h \leq d$ is the fourth order symmetric and elliptic tensor of linear elasticity.

We decompose the stress vector $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n}$ on the boundary Γ into a normal stress and a vector of tangential stresses denoted $\sigma_n(\boldsymbol{u})$ and $\boldsymbol{\sigma}_t(\boldsymbol{u})$, respectively, so that $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \sigma_n(\boldsymbol{u})\boldsymbol{n} + \boldsymbol{\sigma}_t(\boldsymbol{u})$. Similarly the displacement field $\boldsymbol{u}$ on the boundary Γ is written $\boldsymbol{u} = u_n \boldsymbol{n} + \boldsymbol{u}_t$ where u_n and $\boldsymbol{u}_t$ denote the normal displacement and the vector of tangential displacements, respectively.

Throughout this paper, we assume that the frictional contact behavior on the part Γ_C is governed by the normal compliance model introduced and studied by Oden and Martins (see

[19], [18]) in which the stresses follow the power law,

$$\begin{aligned}\sigma_n(\boldsymbol{u}) &= -c_n(u_n)_+^{m_n},\\ \boldsymbol{\sigma}_t(\boldsymbol{u}) &= -c_t\ (u_n)_+^{m_t}\frac{\boldsymbol{u}_t}{|\boldsymbol{u}_t|} \quad \text{if sliding occurs,}\end{aligned}$$

where $(.)_+$ stands for the positive part so that $(u_n)_+$ represents the penetration of the body into the foundation. In the following the notation x_+^m stands for $(x_+)^m$. The constants $m_n \geq 1$, $m_t \geq 1$ as well as the positive functions c_n and c_t in $L^\infty(\Gamma_C)$ stand for interface parameters characterizing the contact behavior between the body and the foundation. We consider these conditions on Γ_C:

$$\sigma_n(\boldsymbol{u}) = -c_n(u_n)_+^{m_n}, \tag{2.5}$$

and

$$\begin{cases} \boldsymbol{u}_t = \boldsymbol{0} \implies |\boldsymbol{\sigma}_t(\boldsymbol{u})| \leq c_t(u_n)_+^{m_t},\\ \boldsymbol{u}_t \neq \boldsymbol{0} \implies \boldsymbol{\sigma}_t(\boldsymbol{u}) = -c_t(u_n)_+^{m_t}\dfrac{\boldsymbol{u}_t}{|\boldsymbol{u}_t|}. \end{cases} \tag{2.6}$$

Remark 2.1 *Note that the true friction law involves the tangential contact velocities and not the tangential displacements. However, a problem analogous to the one discussed here is obtained by time discretization of the quasi-static frictional contact evolution problem. In this case* $\mathbf{U}$, $\boldsymbol{f}$ *and* $\mathbf{F}$ *stand for* $\mathbf{U}((i+1)\Delta t)$, $\boldsymbol{f}((i+1)\Delta t)$ *and* $\mathbf{F}((i+1)\Delta t)$ *respectively and* $\boldsymbol{u}_t$ *has to be replaced by* $\boldsymbol{u}_t((i+1)\Delta t) - \boldsymbol{u}_t(i\Delta t)$, *where* Δt *denotes the time step. For simplicity and without any loss of generality only the static case (also called incremental problem in [16]) described above will be considered in the following.*

It is easy to see that the case $m_n = m_t$ corresponds to the Coulomb friction model where the friction coefficient is c_t/c_n. Although the condition (2.5) can be seen (from a theoretical point of view) as a penalized version of the unilateral contact conditions there are some mechanical arguments which justify the validity of the model (see e.g., [19, 18, 16]). The unilateral contact conditions can be recovered when $c_n \to \infty$.

Let us introduce the set of admissible displacements:

$$\mathbf{V}_U = \left\{\boldsymbol{v} \in H^1(\Omega);\ \boldsymbol{v} = \mathbf{U} \text{ on } \Gamma_D\right\}$$

and the space

$$\mathbf{V}_0 = \left\{\boldsymbol{v} \in H^1(\Omega);\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D\right\} = \mathbf{V}_U - \{\mathbf{U}\}.$$

The weak form of problem (2.1)–(2.6) consists to find $\boldsymbol{u} \in \mathbf{V}_U$ such that (see e.g., [16]):

$$a(\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}) + j_n(\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}) + j_t(\boldsymbol{u}, \boldsymbol{v}) - j_t(\boldsymbol{u}, \boldsymbol{u}) \geq L(\boldsymbol{v}-\boldsymbol{u}), \quad \forall \boldsymbol{v} \in \mathbf{V}_U \tag{2.7}$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{v})\ d\Omega, \qquad L(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}.\boldsymbol{v}\ d\Omega + \int_{\Gamma_N} \mathbf{F}.\boldsymbol{v}\ d\Gamma,$$

$$j_n(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_C} c_n(u_n)_+^{m_n} v_n\ d\Gamma, \qquad j_t(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_C} c_t(u_n)_+^{m_t}|\boldsymbol{v}_t|\ d\Gamma,$$

for any $\boldsymbol{u}$ and $\boldsymbol{v}$ in $H^1(\Omega)$. In order to give a sense to $j_n(\boldsymbol{u}, \boldsymbol{v})$ and $j_t(\boldsymbol{u}, \boldsymbol{v})$, we need to assume that

$$1 \leq m_n, m_t < +\infty \text{ if } d = 2, \qquad 1 \leq m_n, m_t \leq 3 \text{ if } d = 3. \tag{2.8}$$

The hypothesis (2.8) allows us to use the imbedding $H^{1/2}(\Gamma_C) \hookrightarrow L^q(\Gamma_C)$ for each $q \in [1, +\infty)$ if $d = 2$ and for each $q \in [1, 4]$ if $d = 3$ (see e.g., [1]) and we denote by C_q the corresponding imbedding constant:

$$\|v\|_{L^q(\Gamma_C)} \leq C_q \|v\|_{H^{1/2}(\Gamma_C)}, \qquad \forall v \in H^{1/2}(\Gamma_C). \tag{2.9}$$

When $d = 2$ then it has been proved in [6] that there exists a constant D such that

$$C_q = Dq^{1/2}, \qquad 1 \leq q < +\infty. \tag{2.10}$$

Besides we denote by C_{tr} the constant of the trace operator $H^1(\Omega) \hookrightarrow H^{1/2}(\Gamma_C)$:

$$\|v\|_{H^{1/2}(\Gamma_C)} \leq C_{tr} \|v\|_{H^1(\Omega)}, \qquad \forall v \in H^1(\Omega). \tag{2.11}$$

We denote by c_K and c_c the V_0-ellipticity and continuity constants of the bilinear form $a(.,.)$:

$$\begin{aligned} c_K \|\boldsymbol{v}\|^2_{H^1(\Omega)} &\leq a(\boldsymbol{v}, \boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \mathbf{V}_0, \\ a(\boldsymbol{v}, \boldsymbol{w}) &\leq c_c \|\boldsymbol{v}\|_{H^1(\Omega)} \|\boldsymbol{w}\|_{H^1(\Omega)}, \qquad \forall \boldsymbol{v}, \boldsymbol{w} \in H^1(\Omega). \end{aligned} \tag{2.12}$$

In the sequel we will use the following norm for the volume and surface loads:

$$\|L\| = \sup_{\{\boldsymbol{v} \in \mathbf{V}_0, \|\boldsymbol{v}\|_{H^1(\Omega)} = 1\}} |L(\boldsymbol{v})|$$

and we set

$$\chi(\mathbf{U}) = \inf_{\boldsymbol{v} \in \mathbf{V}_U, \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_C} \|\boldsymbol{v}\|_{H^1(\Omega)}$$

so that $\chi(\mathbf{0}) = 0$. We will also need the following quantity in some of our estimates:

$$\underline{c}_n = \operatorname{essinf}_{\Gamma_C} c_n,$$

in other words $\underline{c}_n$ is the greatest number c such that $c \leq c_n$ a.e. on Γ_C. Finally in order to lighten the notations, we introduce the following nonnegative quantities B_1 and B_2 which equal zero if and only if the external "loadings" $\boldsymbol{f}, \mathbf{F}, \mathbf{U}_{|\Gamma_C}$ vanish:

$$B_1(c_K, c_c, L, \mathbf{U}) = \frac{\|L\|}{c_K} + \left(\frac{c_c}{c_K} + 1 \right) \chi(\mathbf{U}),$$

$$B_2(c_K, c_c, c_n, L, \mathbf{U}) = \frac{\|L\|^2}{\underline{c}_n c_K} + 2 \frac{c_c \|L\|}{\underline{c}_n c_K} \chi(\mathbf{U}) + \frac{c_c}{\underline{c}_n} \left(\frac{c_c}{c_K} + 1 \right) \chi^2(\mathbf{U}).$$

If $d = 2$ we have, according to Cauchy-Schwarz inequality, for each $\boldsymbol{u}, \boldsymbol{v} \in H^1(\Omega)$ and since $C_2 = 1$:

$$\begin{aligned} |j_n(\boldsymbol{u}, \boldsymbol{v})| &\leq \|c_n\|_{L^\infty(\Gamma_C)} \|(u_n)_+\|^{m_n}_{L^{2m_n}(\Gamma_C)} \|v_n\|_{L^2(\Gamma_C)} \\ &\leq \|c_n\|_{L^\infty(\Gamma_C)} \|u_n\|^{m_n}_{L^{2m_n}(\Gamma_C)} \|v_n\|_{L^2(\Gamma_C)} \\ &\leq C_{tr}^{m_n+1} C_{2m_n}^{m_n} \|c_n\|_{L^\infty(\Gamma_C)} \|\boldsymbol{u}\|^{m_n}_{H^1(\Omega)} \|\boldsymbol{v}\|_{H^1(\Omega)} \end{aligned}$$

and a similar bound is obtained for $|j_t(\boldsymbol{u}, \boldsymbol{v})|$:

$$|j_t(\boldsymbol{u}, \boldsymbol{v})| \leq C_{tr}^{m_t+1} C_{2m_t}^{m_t} \|c_t\|_{L^\infty(\Gamma_C)} \|\boldsymbol{u}\|^{m_t}_{H^1(\Omega)} \|\boldsymbol{v}\|_{H^1(\Omega)}.$$

If $d = 3$, we get by Hölder inequality for $1 \le m_n \le 3$:

$$\begin{aligned}
|j_n(\boldsymbol{u}, \boldsymbol{v})| &\le \|c_n\|_{L^\infty(\Gamma_C)} \|(u_n)_+\|^{m_n}_{L^{\frac{4m_n}{3}}(\Gamma_C)} \|v_n\|_{L^4(\Gamma_C)} \\
&\le \|c_n\|_{L^\infty(\Gamma_C)} \|u_n\|^{m_n}_{L^{\frac{4m_n}{3}}} \|v_n\|_{L^4(\Gamma_C)} \\
&\le C_{tr}^{m_n+1} C_4 C_{4m_n/3}^{m_n} \|c_n\|_{L^\infty(\Gamma_C)} \|\boldsymbol{u}\|^{m_n}_{H^1(\Omega)} \|\boldsymbol{v}\|_{H^1(\Omega)}
\end{aligned}$$

and here again similar bounds are obtained for $|j_t(\boldsymbol{u}, \boldsymbol{v})|$ when $1 \le m_t \le 3$:

$$|j_t(\boldsymbol{u}, \boldsymbol{v})| \le C_{tr}^{m_t+1} C_4 C_{4m_t/3}^{m_t} \|c_n\|_{L^\infty(\Gamma_C)} \|\boldsymbol{u}\|^{m_t}_{H^1(\Omega)} \|\boldsymbol{v}\|_{H^1(\Omega)}.$$

3. Uniqueness results

The existence of solutions to problem (2.7) when $d = 2$ and $d = 3$ under assumptions (2.8) was proved in [16] using an abstract theorem in [9]. In references [16, 17] (where $\mathbf{U} = \mathbf{0}$) the authors state and prove that if the loads $\boldsymbol{f}, \mathbf{F}$ and the interface parameters c_n and c_t are small enough (more precisely if $\alpha_1 \|c_n\|_{L^\infty(\Gamma_C)} + \alpha_2 \|c_t\|_{L^\infty(\Gamma_C)}$ is small enough where the coefficients α_i depend on the external loads and on $c_K, C_q, C_{tr}, m_n, m_t$), then the problem (2.7) admits a unique solution in a ball centered at the origin and whose radius depends on the interface parameters and the loadings.

When $m_n = m_t = 1$, the authors improve in [17] the previous result and establish that the solution to (2.7) is globally unique when $\|c_n\|_{L^\infty(\Gamma_C)} + \|c_t\|_{L^\infty(\Gamma_C)}$ is small enough and without any restrictions on the forces by using a result in [7]. Again this uniqueness result does not remain valid when $\|c_n\|_{L^\infty(\Gamma_C)} \to \infty$.

Next we propose to improve these uniqueness results and to obtain new bounds when $d = 2$ and $d = 3$ (under the assumptions (2.8)). The uniqueness results we obtain in the two and three dimensional cases are of two different types: the first ones depend only on the "tangential variables" c_t and m_t whereas the second ones depend on the tangential and normal variables c_t, m_t, c_n, m_n and these results are complementary.

Theorem 3.1 *(i) Let $d = 2$ and $1 \le m_n, m_t < +\infty$. Then problem (2.7) admits at most one solution if:*

$$2D^{m_t+1} C_{tr}^{m_t+1} \|c_t\|_{L^\infty(\Gamma_C)} m_t (m_t+1)^{(m_t+1)/2} \frac{(B_1(c_K, c_c, L, \mathbf{U}))^{m_t-1}}{c_K} < 1. \tag{3.1}$$

Suppose in addition that $m_n - m_t + 2 > 0$. Then problem (2.7) admits at most one solution if:

$$4D^2 C_{tr}^2 \|c_t\|_{L^\infty(\Gamma_C)} \frac{m_t(m_n+1)}{c_K(m_n - m_t + 2)} \left(B_2(c_K, c_c, c_n, L, \mathbf{U})\right)^{\frac{m_t-1}{m_n+1}} < 1. \tag{3.2}$$

(ii) Let $d = 3$ and $1 \le m_n, m_t \le 3$. Then problem (2.7) admits at most one solution if:

$$2C^2_{8/(5-m_t)} C_4^{m_t-1} C_{tr}^{m_t+1} \|c_t\|_{L^\infty(\Gamma_C)} m_t \frac{(B_1(c_K, c_c, L, \mathbf{U}))^{m_t-1}}{c_K} < 1. \tag{3.3}$$

Suppose in addition that $m_n - 2m_t + 3 \ge 0$. Then problem (2.7) admits at most one solution if:

$$2C^2_{2(m_n+1)/(m_n-m_t+2)} C_{tr}^2 \|c_t\|_{L^\infty(\Gamma_C)} \frac{m_t}{c_K} \left(B_2(c_K, c_c, c_n, L, \mathbf{U})\right)^{\frac{m_t-1}{m_n+1}} < 1. \tag{3.4}$$

Moreover, if $m_t = 1$, any of the four constants in (3.1)–(3.4) can be divided into two.

Proof. Let $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ be two solutions of (2.7). Writing (2.7) with $\boldsymbol{u} = \boldsymbol{u}_1, \boldsymbol{v} = \boldsymbol{u}_2$ and also with $\boldsymbol{u} = \boldsymbol{u}_2, \boldsymbol{v} = \boldsymbol{u}_1$, adding the two inequalities and using (2.12), we get:

$$\begin{aligned} c_K \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{H^1(\Omega)} &\leq a(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \\ &\leq j_n(\boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j_n(\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \\ &\quad + j_t(\boldsymbol{u}_1, \boldsymbol{u}_2) + j_t(\boldsymbol{u}_2, \boldsymbol{u}_1) - j_t(\boldsymbol{u}_1, \boldsymbol{u}_1) - j_t(\boldsymbol{u}_2, \boldsymbol{u}_2). \end{aligned}$$

First we note that

$$j_n(\boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j_n(\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) = -\int_{\Gamma_C} c_n((u_{2n})_+^{m_n} - (u_{1n})_+^{m_n})(u_{2n} - u_{1n}) \, d\Gamma \leq 0,$$

according to a monotonicity argument. Besides, from the Theorem 41, p. 39 in [11] we deduce that

$$|x^m - y^m| \leq m|x - y| \left(x^{m-1} + y^{m-1}\right), \forall x \geq 0, \forall y \geq 0, \forall m \geq 1.$$

Hence

$$\begin{aligned} |a_+^m - b_+^m| &\leq m|a_+ - b_+| \left(a_+^{m-1} + b_+^{m-1}\right) \\ &\leq m|a - b| \left(a_+^{m-1} + b_+^{m-1}\right) \forall\, a, b \in \mathbb{R}, \forall m \geq 1. \end{aligned}$$

Therefore

$$\begin{aligned} c_K \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{H^1(\Omega)} &\leq j_t(\boldsymbol{u}_1, \boldsymbol{u}_2) + j_t(\boldsymbol{u}_2, \boldsymbol{u}_1) - j_t(\boldsymbol{u}_1, \boldsymbol{u}_1) - j_t(\boldsymbol{u}_2, \boldsymbol{u}_2) \\ &= \int_{\Gamma_C} c_t((u_{1n})_+^{m_t} - (u_{2n})_+^{m_t})(|\boldsymbol{u}_{2t}| - |\boldsymbol{u}_{1t}|) \, d\Gamma \qquad (3.5) \\ &\leq \|c_t\|_{L^\infty(\Gamma_C)} m_t \int_{\Gamma_C} |u_{2n} - u_{1n}| \left((u_{1n})_+^{m_t-1} + (u_{2n})_+^{m_t-1}\right) |\boldsymbol{u}_{2t} - \boldsymbol{u}_{1t}| \, d\Gamma. \end{aligned}$$

Let $i = 1$ or $i = 2$ and let $\boldsymbol{v} \in \mathbf{V}_U$ such that $\boldsymbol{v} = \mathbf{0}$ on Γ_C. Since $j_n(\boldsymbol{u}_i, \boldsymbol{u}_i) + j_t(\boldsymbol{u}_i, \boldsymbol{u}_i) \geq 0$, (2.7) implies

$$\begin{aligned} c_K \|\boldsymbol{v} - \boldsymbol{u}_i\|^2_{H^1(\Omega)} \leq a(\boldsymbol{v} - \boldsymbol{u}_i, \boldsymbol{v} - \boldsymbol{u}_i) &\leq -j_n(\boldsymbol{u}_i, \boldsymbol{u}_i) - j_t(\boldsymbol{u}_i, \boldsymbol{u}_i) + L(\boldsymbol{u}_i - \boldsymbol{v}) + a(\boldsymbol{v}, \boldsymbol{v} - \boldsymbol{u}_i) \\ &\leq L(\boldsymbol{u}_i - \boldsymbol{v}) + a(\boldsymbol{v}, \boldsymbol{v} - \boldsymbol{u}_i) \\ &\leq \left(\|L\| + c_c \|\boldsymbol{v}\|_{H^1(\Omega)}\right) \|\boldsymbol{u}_i - \boldsymbol{v}\|_{H^1(\Omega)}. \qquad (3.6) \end{aligned}$$

A triangular inequality and (3.6) yield for $i = 1$ or $i = 2$:

$$\begin{aligned} \|\boldsymbol{u}_i\|_{H^1(\Omega)} &\leq \frac{\|L\|}{c_K} + \left(\frac{c_c}{c_K} + 1\right) \chi(\mathbf{U}) \\ &= B_1(c_K, c_c, L, \mathbf{U}). \qquad (3.7) \end{aligned}$$

From (2.7) and since $a(\boldsymbol{u}_i, \boldsymbol{u}_i) + j_t(\boldsymbol{u}_i, \boldsymbol{u}_i) \geq 0$ we have for any $\boldsymbol{v} \in \mathbf{V}_U$ such that $\boldsymbol{v} = \mathbf{0}$ on Γ_C:

$$\begin{aligned} j_n(\boldsymbol{u}_i, \boldsymbol{u}_i) \leq a(\boldsymbol{u}_i, \boldsymbol{v}) + L(\boldsymbol{u}_i - \boldsymbol{v}) &\leq c_c \|\boldsymbol{u}_i\|_{H^1(\Omega)} \|\boldsymbol{v}\|_{H^1(\Omega)} + \frac{\|L\|}{c_K} \left(\|L\| + c_c \|\boldsymbol{v}\|_{H^1(\Omega)}\right) \\ &\leq \frac{\|L\|^2}{c_K} + \left(2\frac{c_c\|L\|}{c_K} + c_c \left(\frac{c_c}{c_K} + 1\right) \|\boldsymbol{v}\|_{H^1(\Omega)}\right) \|\boldsymbol{v}\|_{H^1(\Omega)} \end{aligned}$$

where we use (3.6) and (3.7). Besides

$$\underline{c}_n \|(u_{in})_+\|^{m_n+1}_{L^{m_n+1}(\Gamma_C)} \leq \int_{\Gamma_C} c_n (u_{in})_+^{m_n+1}\, d\Gamma = j_n(\boldsymbol{u}_i, \boldsymbol{u}_i)$$

where we recall that $\underline{c}_n = \operatorname{essinf} c_n$. Combining the last two estimates we obtain for $i = 1$ or $i = 2$

$$\begin{aligned}\|(u_{in})_+\|_{L^{m_n+1}(\Gamma_C)} &\leq \left(\frac{\|L\|^2}{\underline{c}_n c_K} + 2\frac{c_c\|L\|}{\underline{c}_n c_K}\chi(\mathbf{U}) + \frac{c_c}{\underline{c}_n}\left(\frac{c_c}{c_K}+1\right)\chi^2(\mathbf{U})\right)^{\frac{1}{m_n+1}} \\ &= (B_2(c_K, c_c, c_n, L, \mathbf{U}))^{\frac{1}{m_n+1}}. \end{aligned} \tag{3.8}$$

(i) We begin with the case $d = 2$. The case $m_t = 1$ is straightforward. Let $m_t > 1$. Let $1/r + 1/r + 1/q = 1$, so $r = 2q/(q-1)$ with $q > 1$. We choose q such that $q(m_t - 1) \geq 1$. From (3.5) we get by Hölder inequality

$$\begin{aligned}
&c_K\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{H^1(\Omega)} \\
&\leq \|c_t\|_{L^\infty(\Gamma_C)} m_t \|u_{2n} - u_{1n}\|_{L^r(\Gamma_C)} \left(\|(u_{1n})_+^{m_t-1}\|_{L^q(\Gamma_C)} + \|(u_{2n})_+^{m_t-1}\|_{L^q(\Gamma_C)}\right) \|\boldsymbol{u}_{2t} - \boldsymbol{u}_{1t}\|_{L^r(\Gamma_C)} \\
&\leq C_r^2 C_{tr}^2 \|c_t\|_{L^\infty(\Gamma_C)} m_t \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{H^1(\Omega)} \left(\|(u_{1n})_+\|^{m_t-1}_{L^{q(m_t-1)}(\Gamma_C)} + \|(u_{2n})_+\|^{m_t-1}_{L^{q(m_t-1)}(\Gamma_C)}\right) \\
&\leq D^2 C_{tr}^2 \|c_t\|_{L^\infty(\Gamma_C)} m_t r \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{H^1(\Omega)} \left(\|u_{1n}\|^{m_t-1}_{L^{q(m_t-1)}(\Gamma_C)} + \|u_{2n}\|^{m_t-1}_{L^{q(m_t-1)}(\Gamma_C)}\right) \\
&\leq D^{m_t+1} C_{tr}^{m_t+1} \|c_t\|_{L^\infty(\Gamma_C)} m_t r (q(m_t-1))^{(m_t-1)/2} \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{H^1(\Omega)} \left(\|\boldsymbol{u}_1\|^{m_t-1}_{H^1(\Omega)} + \|\boldsymbol{u}_2\|^{m_t-1}_{H^1(\Omega)}\right) \\
&\leq 2D^{m_t+1} C_{tr}^{m_t+1} \|c_t\|_{L^\infty(\Gamma_C)} m_t r (q(m_t-1))^{(m_t-1)/2} \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{H^1(\Omega)} (B_1(c_K, c_c, L, \mathbf{U}))^{m_t-1}
\end{aligned}$$

where we use the Sobolev inequality (2.9) and (2.10),the trace inequality (2.11) and (3.7). Keeping in mind that $q > 1$ and $q(m_t - 1) \geq 1$, we deduce that $q^{(m_t+1)/2}/(q-1)$ attains its minimal value when $q = (m_t + 1)/(m_t - 1)$. So we deduce that uniqueness holds when $d = 2$ and $m_t > 1$ if

$$2D^{m_t+1} C_{tr}^{m_t+1} \|c_t\|_{L^\infty(\Gamma_C)} m_t (m_t+1)^{(m_t+1)/2} \frac{(B_1(c_K, c_c, L, \mathbf{U}))^{m_t-1}}{c_K} < 1.$$

Now we propose to obtain for certain values of m_n and m_t some different estimates involving m_t, m_n, c_t, c_n which ensure uniqueness by using (3.8). We consider again (3.5) and we suppose that $m_n - m_t + 2 > 0$. Note that this assumption takes into account the case $m_t = m_n$. We use in (3.5) Hölder inequalities with $1/r + 1/r + 1/q = 1$ and

$$q = \frac{m_n+1}{m_t-1}, \qquad r = \frac{2(m_n+1)}{m_n - m_t + 2}$$

which requires that $m_t > 1$. Therefore

$$\begin{aligned}
&c_K\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{H^1(\Omega)} \\
&\leq \|c_t\|_{L^\infty(\Gamma_C)} m_t \|u_{2n} - u_{1n}\|_{L^r(\Gamma_C)} \left(\|(u_{1n})_+^{m_t-1}\|_{L^q(\Gamma_C)} + \|(u_{2n})_+^{m_t-1}\|_{L^q(\Gamma_C)}\right) \|\boldsymbol{u}_{2t} - \boldsymbol{u}_{1t}\|_{L^r(\Gamma_C)} \\
&\leq D^2 C_{tr}^2 \|c_t\|_{L^\infty(\Gamma_C)} m_t r \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{H^1(\Omega)} \left(\|(u_{1n})_+\|^{m_t-1}_{L^{m_n+1}(\Gamma_C)} + \|(u_{2n})_+\|^{m_t-1}_{L^{m_n+1}(\Gamma_C)}\right)
\end{aligned}$$

where we use again (2.9), (2.10) and (2.11). From (3.8) we deduce

$$c_K\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{H^1(\Omega)} \leq 4D^2 C_{tr}^2 \|c_t\|_{L^\infty(\Gamma_C)} \frac{m_t(m_n+1)}{(m_n - m_t + 2)} (B_2(c_K, c_c, c_n, L, \mathbf{U}))^{\frac{m_t-1}{m_n+1}} \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{H^1(\Omega)}.$$

Hence uniqueness holds when $d = 2$ and $m_t > 1$ (the case $m_t = 1$ is straightforward) if

$$4D^2 C_{tr}^2 \|c_t\|_{L^\infty(\Gamma_C)} \frac{m_t(m_n+1)}{c_K(m_n - m_t + 2)} \left(B_2(c_K, c_c, c_n, L, \mathbf{U})\right)^{\frac{m_t-1}{m_n+1}} < 1.$$

(ii) Consider the case where $d = 3$. Let $1 < m_t \le 3$. Here again the case $m_t = 1$ is straightforward. Write $1/r + 1/r + 1/q = 1$ with $r = 8/(5 - m_t)$ (note that $r \in (2, 4]$) and $q = 4/(m_t - 1)$ and use Hölder inequality in (3.5):

$$\begin{aligned}
& c_K \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{H^1(\Omega)} \\
\le\ & \|c_t\|_{L^\infty(\Gamma_C)} m_t \|u_{2n} - u_{1n}\|_{L^r(\Gamma_C)} \left(\|(u_{1n})_+^{m_t-1}\|_{L^q(\Gamma_C)} + \|(u_{2n})_+^{m_t-1}\|_{L^q(\Gamma_C)}\right) \|\boldsymbol{u}_{2t} - \boldsymbol{u}_{1t}\|_{L^r(\Gamma_C)} \\
\le\ & C_r^2 C_{tr}^2 \|c_t\|_{L^\infty(\Gamma_C)} m_t \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{H^1(\Omega)} \left(\|(u_{1n})_+\|^{m_t-1}_{L^4(\Gamma_C)} + \|(u_{2n})_+\|^{m_t-1}_{L^4(\Gamma_C)}\right) \\
\le\ & C_r^2 C_{tr}^2 \|c_t\|_{L^\infty(\Gamma_C)} m_t \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{H^1(\Omega)} \left(\|u_{1n}\|^{m_t-1}_{L^4(\Gamma_C)} + \|u_{2n}\|^{m_t-1}_{L^4(\Gamma_C)}\right) \\
\le\ & C^2_{8/(5-m_t)} C_4^{m_t-1} C_{tr}^{m_t+1} \|c_t\|_{L^\infty(\Gamma_C)} m_t \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{H^1(\Omega)} \left(\|\boldsymbol{u}_1\|^{m_t-1}_{H^1(\Omega)} + \|\boldsymbol{u}_2\|^{m_t-1}_{H^1(\Omega)}\right) \\
\le\ & 2C^2_{8/(5-m_t)} C_4^{m_t-1} C_{tr}^{m_t+1} \|c_t\|_{L^\infty(\Gamma_C)} m_t \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{H^1(\Omega)} (B_1(c_K, c_c, L, \mathbf{U}))^{m_t-1}.
\end{aligned}$$

Hence uniqueness holds when $d = 3$ and $1 < m_t \le 3$

$$2C^2_{8/(5-m_t)} C_4^{m_t-1} C_{tr}^{m_t+1} \|c_t\|_{L^\infty(\Gamma_C)} m_t \frac{(B_1(c_K, c_c, L, \mathbf{U}))^{m_t-1}}{c_K} < 1.$$

As in the two dimensional case we obtain hereafter for certain values of m_n and m_t by using (3.8) some different estimates involving m_t, m_n, c_t, c_n which ensure uniqueness. We consider again (3.5) and we suppose that $m_n - 2m_t + 3 \ge 0$. Note that this assumption takes into account the case $m_t = m_n (\le 3)$. By Hölder inequality with $1/r + 1/r + 1/q = 1$ (with $m_t > 1$ keeping in mind that the case $m_t = 1$ is straightforward) and

$$q = \frac{m_n + 1}{m_t - 1}, \qquad r = \frac{2(m_n + 1)}{m_n - m_t + 2},$$

(which implies $r \le 4$) we get:

$$\begin{aligned}
& c_K \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{H^1(\Omega)} \\
\le\ & \|c_t\|_{L^\infty(\Gamma_C)} m_t \|u_{2n} - u_{1n}\|_{L^r(\Gamma_C)} \left(\|(u_{1n})_+^{m_t-1}\|_{L^q(\Gamma_C)} + \|(u_{2n})_+^{m_t-1}\|_{L^q(\Gamma_C)}\right) \|\boldsymbol{u}_{2t} - \boldsymbol{u}_{1t}\|_{L^r(\Gamma_C)} \\
\le\ & C_r^2 C_{tr}^2 \|c_t\|_{L^\infty(\Gamma_C)} m_t \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{H^1(\Omega)} \left(\|(u_{1n})_+\|^{m_t-1}_{L^{m_n+1}(\Gamma_C)} + \|(u_{2n})_+\|^{m_t-1}_{L^{m_n+1}(\Gamma_C)}\right).
\end{aligned}$$

Hence uniqueness holds when

$$2C^2_{2(m_n+1)/(m_n-m_t+2)} C_{tr}^2 \|c_t\|_{L^\infty(\Gamma_C)} \frac{m_t}{c_K} \left(B_2(c_K, c_c, c_n, L, \mathbf{U})\right)^{\frac{m_t-1}{m_n+1}} < 1.$$

□

Remark 3.2 *1. In comparison with the existing results in [16, 17] we prove that the solutions are unique when $\|c_t\|_{L^\infty(\Gamma_C)}$ is small enough. In cases (3.1) and (3.3) the sufficient conditions of uniqueness do not depend on c_n.*

2. Suppose to simplify that c_t and c_n are constant functions. According to (3.2) and (3.4), for certain values of m_n and m_t the solution is unique when

$$\frac{c_t}{c_n^{\frac{m_t-1}{m_n+1}}} \text{ is small enough.}$$

This means that for any fixed c_t and $m_t > 1$, global uniqueness holds if c_n is large enough (when the contact model tends to the unilateral contact model).

3. All the uniqueness results in (3.1)–(3.4) are global.

4. If $m_t = 1$, the all the uniqueness conditions are independent on the external "loads" $\boldsymbol{f}, \mathbf{F}, \mathbf{U}$.

5. When $m_t = m_n$ satisfy (2.8) then the four cases (3.1)–(3.4) provide uniqueness conditions.

6. If there is a constant μ such that $c_t = \mu c_n$ (we assume here that c_n and c_t are constant) and $m_t = m_n = m$ which corresponds to the Coulomb friction model, then the most interesting bounds are (3.2) and (3.4). Define B_3 such that $B_3(c_K, c_c, L, \mathbf{U}) = c_n B_2(c_K, c_c, c_n, L, \mathbf{U})$. Then uniqueness holds if

$$2D^2 C_{tr}^2 \frac{m(m+1)}{c_K} \left(B_3(c_K, c_c, L, \mathbf{U})\right)^{\frac{m-1}{m+1}} c_n^{\frac{2}{m+1}} \mu < 1$$

when $d = 2$ and if

$$2C_{m+1}^2 C_{tr}^2 \frac{m}{c_K} \left(B_3(c_K, c_c, L, \mathbf{U})\right)^{\frac{m-1}{m+1}} c_n^{\frac{2}{m+1}} \mu < 1.$$

when $d = 3$. Unfortunately the uniqueness condition disappears when $c_n \to \infty$. Note that when m behaves like $\ln(c_n)$ then the bound ensuring uniqueness behaves like $\mu(\ln(c_n))^2$ when $d = 2$. Although it grows slowly when μ is constant, it still depends on c_n.

7. For the unilateral contact model with Coulomb friction obtained when $\mu = c_t/c_n$ is constant, $m_n = m_t$ and $c_n \to \infty$ there exist some existence results for small μ whose proof is quite technical (see [10] and the references therein). No standard uniqueness results like the ones of Theorem 3.1 are available. Nevertheless there exist some partial uniqueness results recently obtained in [23]. On the contrary the nonuniqueness examples are (from the author's point of view) simpler to find for the unilateral contact model with Coulomb friction (see [12, 13]) than for the normal compliance model (see next section) since the latter model contains less nonlinearities.

In the next section we consider a solution to the friction problem with normal compliance (2.1)–(2.6) when $m_n = m_t = 1$ and $d = 2$ and we look for sufficient conditions for nonuniqueness and also for nonuniqueness examples keeping in mind that the solution is unique when $\|c_t\|_{L^\infty(\Gamma_C)}$ is small enough.

4. Nonuniqueness results

4.1. *Sufficient conditions of existence of at least two solutions*

We consider the case where $d = 2$ with $m_n = m_t = 1$. According to bounds (3.1) and (3.2) in Theorem 3.1 and the fact that if $m_t = 1$ the constants in (3.1) and (3.2) can be divided into two, Problem (2.7) admits a unique solution when

$$2D^2 C_{tr}^2 c_K^{-1} \|c_t\|_{L^\infty(\Gamma_C)} < 1. \tag{4.1}$$

Let $\boldsymbol{n} = (n_x, n_y)$ and $\boldsymbol{t} = (-n_y, n_x)$ be the unit outward normal and tangent vectors on Γ. On Γ we write for any displacement field $\boldsymbol{v}$:

$$\boldsymbol{v} = v_n \boldsymbol{n} + \boldsymbol{v}_t = v_n \boldsymbol{n} + v_t \boldsymbol{t} \quad \text{and} \quad \boldsymbol{\sigma}(\boldsymbol{v})\boldsymbol{n} = \sigma_n(\boldsymbol{v})\boldsymbol{n} + \boldsymbol{\sigma}_t(\boldsymbol{v}) = \sigma_n(\boldsymbol{v})\boldsymbol{n} + \sigma_t(\boldsymbol{v})\boldsymbol{t}.$$

First we consider a solution $\boldsymbol{u}$ of (2.1)–(2.6) where u_t is of constant sign on Γ_C (either positive or negative) such that:

$$\begin{cases} \mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} = \mathbf{0} \quad \text{in } \Omega, \\ \boldsymbol{\sigma}(\boldsymbol{u}) = \mathcal{C}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \\ \boldsymbol{u} = \mathbf{U} \quad \text{on } \Gamma_D, \\ \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \mathbf{F} \quad \text{on } \Gamma_N, \\ \sigma_n(\boldsymbol{u}) = -c_n(u_n)_+ \quad \text{on } \Gamma_C, \\ \sigma_t(\boldsymbol{u}) = -c_t(u_n)_+ sgn(u_t) \quad \text{on } \Gamma_C, \end{cases} \tag{4.2}$$

where $sgn(u_t) = u_t/|u_t|$. Consider now the problem of finding the displacement field $\boldsymbol{\varphi}$ such that:

$$\begin{cases} \mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{\varphi}) = \mathbf{0} \quad \text{in } \Omega, \\ \boldsymbol{\sigma}(\boldsymbol{\varphi}) = \mathcal{C}\,\boldsymbol{\varepsilon}(\boldsymbol{\varphi}) \quad \text{in } \Omega, \\ \boldsymbol{\varphi} = \mathbf{0} \quad \text{on } \Gamma_D, \\ \boldsymbol{\sigma}(\boldsymbol{\varphi})\boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma_N, \\ \sigma_n(\boldsymbol{\varphi}) = -c_n\varphi_n \quad \text{on } \Gamma_C, \\ \sigma_t(\boldsymbol{\varphi}) = -c_t\varphi_n sgn(u_t) \quad \text{on } \Gamma_C, \end{cases} \tag{4.3}$$

where u_t is the tangential displacement of the field $\boldsymbol{u}$ solving (4.2). Note that the set of solutions to (4.3) is a function space.

Proposition 4.1 *Let $\boldsymbol{u}$ be a solution of the frictional contact problem with normal compliance (2.1)–(2.6) satisfying (4.2) and let $\boldsymbol{\varphi}$ be a solution to problem (4.3). If $(u_n)_+ + \varphi_n = (u_n + \varphi_n)_+$ and $sgn(u_t) = sgn(u_t + \varphi_t)$ on Γ_C then $\boldsymbol{u}$ and $\boldsymbol{u} + \boldsymbol{\varphi}$ solve (2.1)–(2.6).*

Proof. Straightforward. Of course the proposition is interesting only when a nonzero $\boldsymbol{\varphi}$ is considered. □

4.2. *Some explicit examples in the case* $d = 2$ *and* $m_n = m_t = 1$

Now we show that the statement of Proposition 4.1 can be illustrated in the case when Ω is a trapezoid (in which the edges represent Γ_D, Γ_N and Γ_C) and the displacement fields $\boldsymbol{u}$ and $\boldsymbol{\varphi}$ are linear. So we look after fields $\boldsymbol{u}$ and $\boldsymbol{\varphi}$ satisfying the assumptions of Proposition 4.1 in order to exhibit some examples of non-unique solutions to the frictional contact problem (2.1)–(2.6).

We consider a trapezoid of vertexes $A = (0,0)$, $B = (\theta, 0)$, $C = (x_c, y_c)$ and $D = (x_c + \theta(1 - x_c), (1-\theta)y_c)$ with $y_c > 0$ and $0 < \theta < 1$. We define $\Gamma_D = (C, D)$, $\Gamma_N = (A, C) \cup (B, D)$, $\Gamma_C = (A, B)$ so that the lines AC and BD are parallel (see Figure 1). The body Ω lies on a foundation, the half-space delimited by the straight line (A, B) as suggested in Figure 1. Consequently $\boldsymbol{n} = (0, -1)$ and $\boldsymbol{t} = (1, 0)$ on Γ_C.

We suppose that the body Ω is governed by Hooke's law concerning homogeneous isotropic materials so that (2.2) becomes

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \frac{E\nu}{(1-2\nu)(1+\nu)} tr(\boldsymbol{\varepsilon}(\boldsymbol{u}))\mathbf{I} + \frac{E}{1+\nu}\boldsymbol{\varepsilon}(\boldsymbol{u}) \tag{4.4}$$

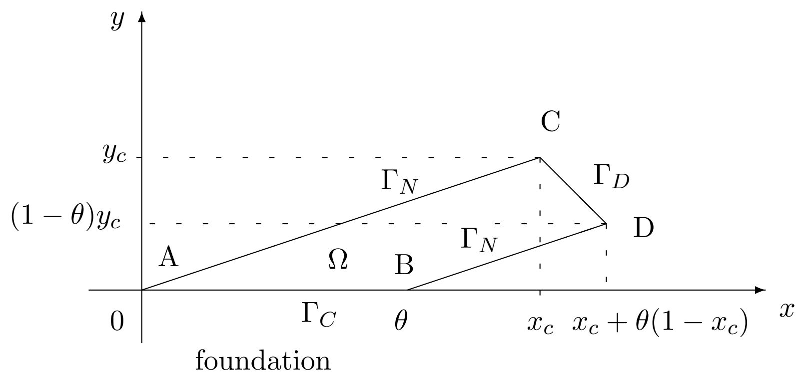

Figure 1: The geometry of the body Ω

where $\mathbf{I}$ represents the identity matrix, tr is the matrix trace operator, E and ν denote Young's modulus and Poisson ratio, respectively with $E > 0$ and $0 \leq \nu < 1/2$.

Let $(x = (1, 0), y = (0, 1))$ stand for the canonical basis of $\mathbb{R}^2$. We suppose that the volume forces $\boldsymbol{f} = (f_x, f_y) = (0, 0)$ are absent in Ω and that the surface forces on Γ_N are denoted by $\mathbf{F}$. Let $\mathbf{U}$ represent the given displacements on Γ_D.

4.2.1. *Determination of* $\boldsymbol{\varphi}$

We begin with the determination of a (nonzero) linear displacement field $\boldsymbol{\varphi} = (\varphi_x, \varphi_y)$ in (4.3). Since $\boldsymbol{\varphi} = \mathbf{0}$ on $\Gamma_D = (C, D)$, we get

$$\varphi_x = \alpha\Big(y_c x + (1 - x_c)y - y_c\Big), \tag{4.5}$$

$$\varphi_y = \beta\Big(y_c x + (1 - x_c)y - y_c\Big), \tag{4.6}$$

with $(\alpha, \beta) \neq (0, 0)$.

Obviously $\mathbf{div}\,(\boldsymbol{\sigma}(\boldsymbol{\varphi})) = \mathbf{0}$. Inserting now the expressions (4.5)–(4.6) of $\boldsymbol{\varphi}$ in the constitutive law (4.4) yields

$$\boldsymbol{\sigma}(\boldsymbol{\varphi}) = \begin{pmatrix} \dfrac{E(\alpha y_c(\nu - 1) + \nu\beta(x_c - 1))}{(1 + \nu)(-1 + 2\nu)} & \dfrac{E(\beta y_c + \alpha(1 - x_c))}{2(1 + \nu)} \\ \dfrac{E(\beta y_c + \alpha(1 - x_c))}{2(1 + \nu)} & \dfrac{E(\nu\alpha y_c + \beta(1 - x_c)(1 - \nu))}{(1 + \nu)(1 - 2\nu)} \end{pmatrix}. \tag{4.7}$$

Now we consider the Neumann conditions: $\boldsymbol{\sigma}(\boldsymbol{\varphi})\boldsymbol{n} = \mathbf{0}$ on Γ_N. Since the unit outward normal vector on Γ_N is $\boldsymbol{n} = \pm(-y_c/\sqrt{x_c^2 + y_c^2}, x_c/\sqrt{x_c^2 + y_c^2})$, the stress vector on Γ_N becomes

$$\boldsymbol{\sigma}(\boldsymbol{\varphi})\boldsymbol{n} = \begin{pmatrix} \dfrac{E(\alpha(2\nu y_c^2 - 2y_c^2 - x_c^2 + 2x_c^2\nu + x_c - 2x_c\nu) + \beta(-2y_c\nu + x_c y_c))}{2(1 - 2\nu)(1 + \nu)\sqrt{x_c^2 + y_c^2}} \\ \dfrac{E(\alpha(y_c x_c - y_c + 2\nu y_c) + \beta(-y_c^2 + 2y_c^2\nu + 2\nu x_c^2 - 2\nu x_c - 2x_c^2 + 2x_c))}{2(1 - 2\nu)(1 + \nu)\sqrt{x_c^2 + y_c^2}} \end{pmatrix}.$$

So the Neumann condition is equivalent to the linear system

$$M \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.8}$$

with

$$M = \begin{pmatrix} 2\nu y_c^2 - 2y_c^2 - x_c^2 + 2x_c^2\nu + x_c - 2x_c\nu & -2y_c\nu + x_c y_c \\ \\ y_c x_c - y_c + 2\nu y_c & -y_c^2 + 2y_c^2\nu + 2\nu x_c^2 - 2\nu x_c - 2x_c^2 + 2x_c \end{pmatrix}.$$

Since $(\alpha, \beta) \neq (0, 0)$, we deduce that $\det(M) = 0$. After some calculation we get

$$\det(M) = 2(-1+2\nu)(x_c^4\nu+2\nu y_c^2 x_c^2-2\nu y_c^2 x_c+\nu y_c^4-2x_c^3\nu+x_c^2\nu+\nu y_c^2-2y_c^2x_c^2-x_c^4+2x_c^3-x_c^2-y_c^4+2y_c^2x_c).$$

Hence this leads to the expression of the Poisson ratio:

$$\nu = \frac{(y_c^2 - x_c + x_c^2)^2}{((x_c - 1)^2 + y_c^2)(x_c^2 + y_c^2)}. \tag{4.9}$$

We insert the expression of ν in the definition of M so that

$$M = \begin{pmatrix} \frac{(y_c^2x_c+y_c^2-x_c^2+x_c^3)(y_c^2x_c-2y_c^2+x_c-2x_c^2+x_c^3)}{(x_c^2+y_c^2)(x_c^2-2x_c+1+y_c^2)} & \frac{y_c(y_c^2x_c-2y_c^2+x_c-2x_c^2+x_c^3)(y_c^2-2x_c+x_c^2)}{(x_c^2+y_c^2)(x_c^2-2x_c+1+y_c^2)} \\ \\ \frac{y_c(y_c^2x_c+y_c^2-x_c^2+x_c^3)(y_c^2-1+x_c^2)}{(x_c^2+y_c^2)(x_c^2-2x_c+1+y_c^2)} & \frac{y_c^2(y_c^2-1+x_c^2)(y_c^2-2x_c+x_c^2)}{(x_c^2+y_c^2)(x_c^2-2x_c+1+y_c^2)} \end{pmatrix}.$$

Denoting $\delta_1 = y_c^2x_c+y_c^2-x_c^2+x_c^3$, $\delta_2 = y_c^2x_c-2y_c^2+x_c-2x_c^2+x_c^3$, $\delta_3 = y_c^2-2x_c+x_c^2$, $\delta_4 = y_c^2-1+x_c^2$, the system (4.8) can be written in an equivalent way:

$$\delta_2(\delta_1\alpha + y_c\delta_3\beta) = 0$$

and

$$\delta_4(\delta_1\alpha + y_c\delta_3\beta) = 0.$$

The case $\delta_2 = \delta_4 = 0$ leads to $(x_c, y_c) = (1, 0)$ which contradicts $y_c > 0$. Therefore $\delta_1\alpha+y_c\delta_3\beta = 0$ and the condition $\boldsymbol{\sigma}(\boldsymbol{\varphi})\boldsymbol{n} = \boldsymbol{0}$ reduces to one of the two following cases: the first one corresponds to $\delta_1 = 0$ and we have necessarily $\beta = 0$ (supposing $\delta_3 = 0$ implies $y_c = 0$ which contradicts $y_c > 0$). This implies that $\varphi_n = 0$ on Γ_C and from (4.3) we deduce that $\boldsymbol{\varphi} = \boldsymbol{0}$ in Ω which is not interesting. The second case corresponds to $\delta_1 \neq 0$:

$$\alpha = -\frac{(x_c^2 - 2x_c + y_c^2)y_c}{x_c^3 - x_c^2 + y_c^2x_c + y_c^2}\beta. \tag{4.10}$$

A calculation of $\sigma_n(\boldsymbol{\varphi})$ on Γ_C by using (4.7) as well as the previous expressions of α and ν gives:

$$\begin{aligned} \sigma_n(\boldsymbol{\varphi}) = \frac{E\beta y_c^2(y_c^2 + (x_c - 1)^2)^2(x_c^2 + y_c^2)}{(x_c^3 - x_c^2 + x_c y_c^2 + y_c^2)(2(y_c^2 - x_c + x_c^2)^2 + y_c^2)} &= -c_n\varphi_n \\ &= c_n\beta y_c(x - 1). \end{aligned} \tag{4.11}$$

So we deduce that

$$x_c^3 - x_c^2 + x_c y_c^2 + y_c^2 < 0. \tag{4.12}$$

It is straightforward that $\nu \geq 0$ in (4.9). Besides, the condition $\nu < 1/2$ is equivalent to

$$(x_c^2 - x_c + y_c^2 + y_c)(x_c^2 - x_c + y_c^2 - y_c) < 0. \tag{4.13}$$

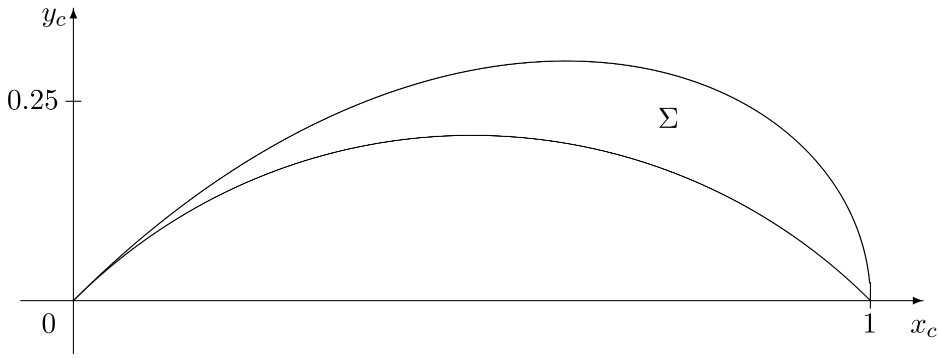

Figure 2: The open admissible region Σ for point $C = (x_c, y_c)$.

Putting together conditions (4.12) and (4.13) gives

$$x_c \in (0,1), \quad \sqrt{\frac{1}{4} + x_c - x_c^2} - \frac{1}{2} < y_c < x_c\sqrt{\frac{1-x_c}{1+x_c}}. \tag{4.14}$$

The admissible domain Σ in which are located the pairs (x_c, y_c) satisfying (4.14) is depicted in Figure 2. Note that this also implies that $\alpha\beta < 0$. Besides, we obtain on Γ_C:

$$\frac{\sigma_t(\boldsymbol{\varphi})}{\sigma_n(\boldsymbol{\varphi})} = -\frac{x_c}{y_c}. \tag{4.15}$$

Finally (4.6) and (4.11) allow us to define c_n on $\Gamma_C = (0,\theta) \times \{0\}$:

$$c_n = c_n(x) = \frac{Ey_c(y_c^2 + (x_c-1)^2)^2(x_c^2 + y_c^2)}{(x_c^3 - x_c^2 + x_c y_c^2 + y_c^2)(2(y_c^2 - x_c + x_c^2)^2 + y_c^2)(x-1)} \tag{4.16}$$

and we deduce the expression of c_t from (4.3) and (4.15): $c_t = -x_c c_n sgn(u_t)/y_c$ which requires that $u_t < 0$ on Γ_C. So

$$c_t = \frac{x_c}{y_c} c_n. \tag{4.17}$$

Note that any choice of (nonzero) β can be made in (4.6).

Remark 4.2 *1. The choice of $\theta \in (0,1)$ is not fundamental in our study. We exclude the value $\theta = 1$ in our discussion (case where Ω is a triangle) since we want that c_n and c_t belong to $L^\infty(\Gamma_C)$. Note that*

$$\|c_n\|_{L^\infty(\Gamma_C)} = \frac{Ey_c(y_c^2 + (x_c-1)^2)^2(x_c^2 + y_c^2)}{(x_c^3 - x_c^2 + x_c y_c^2 + y_c^2)(2(y_c^2 - x_c + x_c^2)^2 + y_c^2)(\theta - 1)}, \ \|c_t\|_{L^\infty(\Gamma_C)} = \frac{x_c}{y_c}\|c_n\|_{L^\infty(\Gamma_C)}.$$

2. The above functions c_n and c_t are not constant although c_t/c_n is constant. It seems difficult to find a function $\boldsymbol{\varphi}$ satisfying (4.3) and involving functions c_n and c_t which are constant on Γ_C.

4.2.2. *Determination of $\boldsymbol{u}$*

Let us now focus on the field $\boldsymbol{u} = (u_x, u_y)$ solving problem (4.2). To simplify we search a linear field

$$u_x = ax + by + c,$$
$$u_y = dx + ey + f,$$

with $u_n(=-u_y)>0$ on Γ_C and keeping in ming that the condition $u_t(=u_x)<0$ on Γ_C was required in the previous section when determining $\boldsymbol{\varphi}$. Since $\sigma_n(\boldsymbol{u})$ and $\sigma_t(\boldsymbol{u})$ are constant on Γ_C, we get from the definitions of c_n and c_t in (4.16)–(4.17) and from (4.2):

$$u_y = d(x-1)+ey, \tag{4.18}$$

with $d>0$. Again to simplify we search a tangential displacement field with $c=-a$:

$$u_x = a(x-1)+by, \tag{4.19}$$

where $a>0$. Inserting the previous expressions of (u_x,u_y) in the constitutive law (4.4), we obtain that the condition $\sigma_n(\boldsymbol{u})=-c_n(u_n)_+$ is equivalent to

$$d = -\frac{(a(x_c^2+y_c^2-x_c)^2+ey_c^2)(x_c^3-x_c^2+x_cy_c^2+y_c^2)}{y_c((x_c-1)^2+y_c^2)(x_c^2-x_c+y_c^2+y_c)(x_c^2-x_c+y_c^2-y_c)} \tag{4.20}$$

which requires that

$$a(x_c^2+y_c^2-x_c)^2+ey_c^2<0. \tag{4.21}$$

Finally, condition $\sigma_t(\boldsymbol{u})=-c_t(u_n)_+ sgn(u_t)$ is equivalent to

$$b=-d-2\frac{x_c(a+e)}{y_c(x_c^2-x_c+y_c^2+y_c)(x_c^2-x_c+y_c^2-y_c)}. \tag{4.22}$$

In order to satisfy the assumptions of Proposition 4.1 we have to check that $(u_n)_+ + \varphi_n = (u_n+\varphi_n)_+$ and $sgn(u_t)=sgn(u_t+\varphi_t)$ which reduces to the inequalities $0\le d+\beta y_c$ and $0\le a+\alpha y_c$ or equivalently

$$-\frac{d}{y_c}\le\beta\le\frac{a(x_c^3-x_c^2+y_c^2x_c+y_c^2)}{y_c^2(x_c^2-2x_c+y_c^2)}. \tag{4.23}$$

We are now in a position to conclude our discussion with the following theorem.

Theorem 4.3 *Let be given the trapezoid Ω of vertexes $A=(0,0)$, $B=(\theta,0)$, $C=(x_c,y_c)$ and $D=(x_c+\theta(1-x_c),(1-\theta)y_c)$ with $y_c>0$ and $0<\theta<1$. Set $\Gamma_D=(C,D)$, $\Gamma_N=(A,C)\cup(B,D)$, $\Gamma_C=(A,B)$. Assume that the pair (x_c,y_c) satisfies (4.14) (i.e., point C belongs to the domain Σ depicted in Figure 2). Suppose that ν is given by (4.9) and let $E>0$. Assume that c_n and c_t are given by (4.16) and (4.17) respectively and that $m_n=m_t=1$.*

Let $\boldsymbol{u}$ be a displacement field defined in (4.18), (4.19) with $a>0$, e satisfying (4.21), d given by (4.20) and b given by (4.22). Assume that $\boldsymbol{f}=\mathbf{0}$ and let $\mathbf{F}$ and $\mathbf{U}$ obtained from $\boldsymbol{u}$ as follows: $\mathbf{U}=\boldsymbol{u}$ on Γ_D and $\mathbf{F}=\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n}$ on Γ_N. Let $\boldsymbol{\varphi}$ given by (4.5), (4.6) where α is defined in (4.10).

Then there exist an infinity of solutions to the problem (2.1)–(2.6). More precisely any displacement field $\boldsymbol{u}+\boldsymbol{\varphi}$ where β satisfies (4.23) solves (2.1)–(2.6).

Remark 4.4 *According to (4.4), we have for any $\boldsymbol{v}\in\mathbf{V}_0$*

$$a(\boldsymbol{v},\boldsymbol{v})\ge\frac{E}{1+\nu}\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|^2_{L^2(\Omega)}\ge\frac{KE}{1+\nu}\|\boldsymbol{v}\|^2_{H^1(\Omega)},$$

where K stands for a Korn type constant which does neither depend on E nor on ν. So we can choose $c_K=KE/(1+\nu)$. According to (4.1) we deduce that the data in Theorem 4.3 is such that $2D^2C_{tr}^2(EK)^{-1}(1+\nu)\|c_t\|_{L^\infty(\Gamma_C)}=2D^2C_{tr}^2x_c(y_c^2+x_c^2-2x_c+1)/(K(x_c^3-x_c^2+y_c^2+y_c^2x_c)(\theta-1))\ge 1.$

4.3. *A simple example*

Finally we illustrate Theorem 4.3 with a simple example and an illustration. Set $x_c = 1/2, y_c = 1/4$ and choose $\theta = 1/2$. Clearly $C = (x_c, y_c) \in \Sigma$. Let be given $E > 0$. According to Theorem 4.3, we obtain $D = (3/4, 1/8)$, $\nu = 9/25$, $c_n = 125E/(68(1-x)), c_t = 125E/(34(1-x))$. Then we choose $a = 1/10$. Clearly $e = -1/10$ satisfies (4.21) and we obtain $d = 1/25$ and $b = -11/25$. So $\boldsymbol{f} = \mathbf{0}$ in Ω, $\mathbf{F} = (-5E\sqrt{5}/68, 0)$ on Γ_N, $\mathbf{U}$ is linear on Γ_D with $\mathbf{U}(C) = (-4/25, -9/200)$ and $\mathbf{U}(D) = (-2/25, -9/400)$.

Any displacement field $\boldsymbol{u} + \boldsymbol{\varphi} = ((u+\varphi)_x, (u+\varphi)_y)$ defined by

$$(u+\varphi)_x = \Big(\frac{1}{10} - \frac{11\beta}{8}\Big)x + \Big(-\frac{11}{25} - \frac{11\beta}{4}\Big)y - \frac{1}{10} + \frac{11\beta}{8},$$

$$(u+\varphi)_y = \Big(\frac{1}{25} + \frac{\beta}{4}\Big)x + \Big(-\frac{1}{10} + \frac{\beta}{2}\Big)y - \frac{1}{25} - \frac{\beta}{4},$$

with $-4/25 \le \beta \le 4/55$ solves Problem (2.1)–(2.6) according to Theorem 4.3. The case $\beta = -4/25$ corresponds to slip with grazing contact (i.e., $\sigma_n(\boldsymbol{u}+\boldsymbol{\varphi}) = \sigma_t(\boldsymbol{u}+\boldsymbol{\varphi}) = (u+\varphi)_n = 0$ and $(u+\varphi)_t = 8(x-1)/25$) and the case $\beta = 4/55$ corresponds to stick with penetration (i.e., $(u+\varphi)_t = 0$ with $(u+\varphi)_n > 0$, $\sigma_t(\boldsymbol{u}+\boldsymbol{\varphi}) = c_t(u+\varphi)_n$). Figure 3 depicts the initial configuration $(ABDC)$ and three of the infinitely possible deformed configurations corresponding to $\beta = -4/25, \beta = 0, \beta = 4/55$. Finally we mention that this example involves important strains

Figure 3: Initial configuration $(ABDC)$ and three possible deformed configurations: $(A+, B+)$ corresponds to $\beta = 4/55$, $(A0, B0)$ corresponds to $\beta = 0$ and $(A-, B-)$ corresponds to $\beta = -4/25$.

(although the small strain hypothesis has been adopted). Of course this is in order to have a better graphical representation and it could be avoided.

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