# Starting flow analysis for Bingham fluids

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**Abstract**

The aim of this paper is to study some flow properties of Bingham fluids in one, two and three space dimensions. We focus on the behavior of the flow when the external forces vary. A special attention is devoted to the appearance of the flow when the loads increase sufficiently. The results are first established in an abstract setting and then applied to the Bingham fluid model.

**Keywords:** Bingham model, Viscoplastic fluid, Starting flow, variational inequality.

**AMS classification:** 49J40, 76A05.

# 1 Introduction

In fluid mechanics involving viscoplastic behavior a current choice is to consider as constitutive relation the Bingham model [1] exhibiting viscosity and yield stress. This model was investigated in the metal forming process in order to describe wire drawing (see [3]), in oil field plug-cementing process (see [7] and the references quoted therein) and in landslides modelling (see [4]). An important property of the Bingham model concerns the existence of rigid zones which are located in the interior of the flow. As the external loads decrease the rigid zones become larger and may completely block the flow if the forces become lower than a certain value which stands for a maximal blocking force.

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From a mathematical point of view the variational formulation of the Bingham problem is obtained in [5]. In the latter reference the authors consider the Bingham model in various contexts (from the sophisticated three dimensional evolution problem generalizing the Navier-Stokes model to the simpler two dimensional stationary problem describing the laminar flow in a cylindrical pipe) and prove also several existence and/or uniqueness results as well as some properties on the solutions, especially in the two dimensional case. Besides an important study concerning the properties of the solutions for the two dimensional stationary problem modelling the laminar flow in a cylindrical pipe was carried out in [15, 16, 17].

This paper deals with stationary problems in one, two and three space dimensions. The main aim of this work is to study the behavior of the flow when the external forces are near the maximal blocking force (i.e., when the fluid begins to run).

Our paper is outlined as follows. In section 2, we consider an abstract setting for a specific class of variational inequalities with unknowns in a Hilbert space $V$. We introduce the definition of a maximal blocking force $f$ and we consider the solution $u_\varepsilon, \varepsilon > 0$ corresponding to a force equal to $(1+\varepsilon)f$. We show that the sequence $u_\varepsilon/\varepsilon$ converges strongly in $V$ as $\varepsilon$ vanishes. A characterization of the limit is given as a projection onto a closed convex cone of a solution to an auxiliary problem governed by a variational equality. Necessary and sufficient conditions are given for the limit to be different from zero. We conclude the section with the study and the characterization of the limit of $u_\varepsilon/\varepsilon$ as $\varepsilon$ tends to infinity. Section 3 deals with the Bingham fluid model. We begin with the three dimensional problem. A characterization of the maximal blocking force is given and the results of the latter section are applied to the fluid model. Similar results are obtained for the two dimensional problem describing the laminar flow in a cylindrical pipe. In the latter case the incompressibility condition $div(u) = 0$ as well as the non linear term $(u\cdot\nabla)u$ disappear. We finish our study by analyzing also two one-dimensional cases.

# 2 An abstract setting

Let $(V, (\cdot,\cdot))$ be a real Hilbert space whose associated norm is denoted by $\|.\|$. The notation $V'$ stands for the dual space of $V$. We consider the variational inequality:

$$u \in V \ : \qquad a(u, v-u) + j(v) - j(u) \geq \langle l, v-u\rangle, \qquad \forall\, v \in V, \tag{1}$$

where $a : V \times V \to \mathbb{R}$ is a bilinear continuous $V$-elliptic (i.e., $\exists\ \alpha > 0$ such that $a(v,v) \geq \alpha(v,v) = \alpha\|v\|^2, \forall\ v \in V$) application, $j : V \to ]-\infty+\infty]$ is a proper, convex, lower semicontinuous (l.s.c.) function and $l : V \to \mathbb{R}$ is a linear continuous form on $V$. The duality pairing between $V'$ and $V$ is denoted by $\langle\cdot,\cdot\rangle$. We note by $J : V' \to V$ the duality application $\langle l, v\rangle = (J(l), v),\ \forall\ l \in V', \forall\ v \in V$. With this notation the problem (1) can be written also

$$u \in V, \qquad a(u, v-u) + j(v) - j(u) \geq (f, v-u), \qquad \forall\, v \in V, \tag{2}$$

where $f = J(l)$. It is well known that the problem (1) admits a unique solution (see [9], [10], [14]). Besides it is easy to check that if $j$ is positively homogeneous (i.e., $j(\lambda v) = \lambda j(v)\ \forall \lambda > 0, \forall v \in V$) with $j(0) = 0$, then the problem (1) is equivalent to finding $u \in V$ such that

$$\begin{cases} a(u,u) + j(u) = \langle l, u\rangle, \\ a(u,v) + j(v) \geq \langle l, v\rangle,\ \forall v \in V. \end{cases}$$

If we assume that $j(0) = 0$ then $u = 0$ is solution of (1) iff $j(v) \geq \langle l, v\rangle, \forall v \in V$. With the notation $f = J(l)$ the previous condition is equivalent to $j(v) \geq (f, v), \forall v \in V$, or $f \in \partial j(0)$.

**Definition 2.1** *We say that $f$ is a blocking force if $j(v) \geq (f,v), \forall v \in V$ or equivalently $f \in \partial j(0)$.*

**Proposition 2.1** *Assume that $j : V \to ]-\infty, +\infty]$ is a nonnegative proper, convex, l.s.c. function such that $j(0) = 0$. Then the set $\partial j(0)$ of all blocking forces is nonempty closed and convex.*

**Proof.** Since $j(v) \geq 0$ for any $v \in V$ we deduce that $\partial j(0)$ contains $f = 0$. According to [6] the set $\partial j(0)$ is closed and convex. $\square$

Since our interest focuses on the appearance of a non trivial solution ($u \neq 0$) when the forces increase, it is natural to introduce the notion of maximal blocking force.

**Proposition 2.2** *Let the assumptions on $j$ of the previous proposition hold and suppose that $D(j)$ is symmetric with respect to the origin. Let $f \in \partial j(0), f|_{D(j)} \neq 0$ and set $M = \sup\{\lambda > 0 \mid \lambda f \in \partial j(0)\}$. Then $M < +\infty$ and $Mf \in \partial j(0)$.*

**Proof.** The set $\{\lambda > 0 \mid \lambda f \in \partial j(0)\}$ is nonempty since it contains $\lambda = 1$ (in fact it contains $]0, 1]$). Since $f|_{D(j)} \neq 0$, $D(j) = -D(j)$, there is $v_0 \in D(j)$ satisfying $(f, v_0) > 0$. If $\lambda_0$ is large enough we have $(\lambda_0 f, v_0) > j(v_0)$ so that $\lambda_0 f \notin \partial j(0)$. Consequently $\{\lambda > 0 \mid \lambda f \in \partial j(0)\} \subset ]0, \lambda_0[$ and $M \leq \lambda_0 < +\infty$. Let $(\lambda_n)_n$ be a sequence converging towards $M$ and verifying $\lambda_n f \in \partial j(0)$. Hence $\lambda_n f \to Mf$ and since $\partial j(0)$ is closed we deduce that $Mf \in \partial j(0)$. $\square$

**Definition 2.2** *Let $f$ be a blocking force and let $M$ be defined as in the Proposition* 2.2. *We call $\tilde{f} = Mf$ the maximal blocking force associated with $f$.*

In other words $\tilde{f}$ is a maximal blocking force iff $j(v) \geq (\tilde{f}, v)$, $\forall v \in V$ and $\forall \varepsilon > 0, \exists\ v_\varepsilon \in V$ such that $j(v_\varepsilon) < ((1+\varepsilon)\tilde{f}, v_\varepsilon)$. Denoting by $u_\varepsilon$ and $u$ the solutions of (2) corresponding to the forces $(1+\varepsilon)\tilde{f}$ and $\tilde{f}$ respectively we obtain another equivalent definition : $\tilde{f}$ is a maximal blocking force iff $u = 0$ and $u_\varepsilon \neq 0, \forall \varepsilon > 0$. We can easily prove the following result:

**Proposition 2.3** *Let the assumptions on $j$ of the Proposition* 2.1 *hold. Assume that $j$ is homogeneous (i.e., $j(\lambda v) = |\lambda| j(v), \forall \lambda \in \mathbb{R}, \forall v \in V$) and $f \in \partial j(0)$, $f|_{D(j)} \neq 0$. Then the maximal blocking force is given by $\tilde{f} = M_1 f$ where $M_1 = \inf_{(f,v)\neq 0} \frac{j(v)}{|(f,v)|}$.*

**Proof.** Remark that $f \in \partial j(0)$ iff $|(f,v)| \leq j(v)$ $\forall v \in V$ and observe that $M = M_1$, where $M = \sup\{\lambda > 0 \mid \lambda f \in \partial j(0)\}$. $\square$

It is also important to determine if there is $v \in V$, $v \neq 0$ such that $j(v) = (f,v)$, where $f$ is a blocking force. The answer to this question is given in the following proposition.

**Proposition 2.4** *Assume that $j : V \to ]-\infty, +\infty]$ is a nonnegative, proper, convex, l.s.c., homogeneous function. Suppose also that $j(v) = 0$ iff $v = 0$. Consider $f \in \partial j(0), f|_{D(j)} \neq 0$ and let $\tilde{f} = Mf$ be the maximal blocking force. Then*

(1) *if $0 < \lambda < M$ and $v \neq 0$ we have $(\lambda f, v) < j(v)$;*
(2) *there is $v_0 \in V - \{0\}$ such that $(Mf, v_0) = j(v_0)$ iff $\inf_{(f,v)\neq 0} \frac{j(v)}{|(f,v)|}$ is attained.*

**Proof.** (1) Since $0 < \lambda < M$ we have $(\lambda f, v) \leq j(v)$ $\forall v \in V$. Suppose that there is $v_0 \neq 0$ such that $\lambda (f, v_0) = j(v_0)$. Since $j(v_0) > 0$, thus $(f, v_0) > 0$ and therefore we have

$$M > \lambda = \frac{j(v_0)}{|(f,v_0)|} \geq \inf_{(f,v)\neq 0} \frac{j(v)}{|(f,v)|} = M_1 = M,$$

which is not possible. So $\lambda(f,v) < j(v)$ $\forall v \in V - \{0\}, \forall\, 0 < \lambda < M$.
(2) Assume that there is $v_0 \neq 0$ such that $(Mf, v_0) = j(v_0)$. Since $j(v_0) > 0$ we have:

$$M = \frac{j(v_0)}{|(f,v_0)|} \geq \inf_{(f,v)\neq 0} \frac{j(v)}{|(f,v)|} = M_1 = M,$$

and therefore $\inf_{(f,v)\neq 0} \frac{j(v)}{|(f,v)|} = \frac{j(v_0)}{|(f,v_0)|} = M$. Conversely, if there is $v_0 \neq 0$ such that $M_1 = \inf_{(f,v)\neq 0} \frac{j(v)}{|(f,v)|} = \frac{j(v_0)}{|(f,v_0)|}$ we deduce that $M|(f,v_0)| = j(v_0)$. If $(f,v_0) > 0$ we obtain $(Mf, v_0) = j(v_0)$. If $(f, v_0) < 0$ we get $(Mf, \tilde{v_0}) = j(\tilde{v_0})$ with $\tilde{v_0} = -v_0 \neq 0$. $\square$

**Proposition 2.5** *In the finite dimensional case ($\dim V < +\infty$), under the hypotheses of the previous proposition, if $f$ is a blocking force, $f|_{D(j)} \neq 0$ and $\tilde{f} = Mf$ is the corresponding maximal blocking force, then there is $v_0 \in V - \{0\}$ such that $(Mf, v_0) = j(v_0)$.*

**Proof.** Since $j$ is homogeneous, we have $M = \inf_{\|v\|=1} \frac{j(v)}{|(f,v)|}$. Consider a sequence $(v_n)_n$ verifying $\|v_n\| = 1, (f, v_n) \neq 0$ and $M \leq \frac{j(v_n)}{|(f,v_n)|} < M + \frac{1}{n}$, $\forall n$. Since $\{v \in V \mid \|v\| = 1\}$ is a compact set we can extract a subsequence $v_{n_k} \to w_0$ as $k \to +\infty$, with $\|w_0\| = 1$. By the lower semicontinuity of $j$ we deduce that $j(w_0) \leq \liminf_{k\to+\infty} j(v_{n_k}) \leq \liminf_{k\to+\infty} \left(M + \frac{1}{n_k}\right) |(f, v_{n_k})| = M|(f, w_0)|$. Since $w_0 \neq 0$

thus $j(w_0) > 0$ we deduce that $(f, w_0) \neq 0$ and $\frac{j(w_0)}{|(f,w_0)|} \leq M$. Finally the infimum $M$ is attained and that there is $v_0 \in V - \{0\}$ such that $(Mf, v_0) = j(v_0)$.

$\square$

**Remark 2.1** *A particular case of the previous proposition is obtained when $V = V_h$ is a finite element space (see e.g. [2]).*

Let $f$ be a blocking force. We introduce the set

$$C = \{v \in V \mid j(v) = (f, v)\}. \tag{3}$$

**Proposition 2.6** *Assume that $j : V \to ]-\infty, +\infty]$ is a proper, convex, l.s.c. function with $j(0) = 0$ and $f$ is a blocking force. Then $C$ is a nonempty closed convex set. Moreover, if $j$ is positively homogeneous, then $C$ is a nonempty closed convex cone.*

**Proof.** Clearly $0 \in C$. Let $v_1, v_2 \in C$ and $\lambda \in [0, 1]$. We have

$$j(\lambda v_1 + (1-\lambda)v_2) \leq \lambda j(v_1) + (1-\lambda)j(v_2) = (f, \lambda v_1 + (1-\lambda)v_2).$$

Since $f$ is a blocking force, we get

$$(f, \lambda v_1 + (1-\lambda)v_2) \leq j(\lambda v_1 + (1-\lambda)v_2).$$

Hence $\lambda v_1 + (1-\lambda)v_2 \in C$ or $C$ is convex. Let $(v_n)_n$ be a sequence in $C$ converging towards $v$. Then

$$j(v) \leq \liminf_{n\to+\infty} j(v_n) = \liminf_{n\to+\infty} (f, v_n) = (f, v).$$

Since $f$ is a blocking force we deduce that $j(v) = (f, v)$ or $v \in C$ which implies that $C$ is closed. If $j$ is positively homogeneous, $v \in C$, $\lambda > 0$ we have $j(\lambda v) = \lambda j(v) = (f, \lambda v)$, or $\lambda v \in C$ and thus $C$ is a convex cone.

$\square$

Next we give an equivalent definition of $C$.

**Proposition 2.7** *Assume that $j : V \to ]-\infty, +\infty]$ is a proper, convex, l.s.c. function with $j(0) = 0$ and $f$ is a blocking force. Then $C = \big\{v \in V \mid f \in \partial j(v)\big\}$.*

**Proof.** If $v \in V$ satisfies $j(v) = (f, v)$ then for any $w \in V$ we write $j(w) - j(v) \geq (f, w) - (f, v) = (f, w - v)$ which implies that $f \in \partial j(v)$. Conversely if $f \in \partial j(v)$ then $j(w) - j(v) \geq (f, w - v) = (f, w) - (f, v)$ for any $w \in V$. Choosing $w = 0$ in the previous inequality we deduce that $j(v) \leq (f, v)$. Since $f$ is a blocking force we obtain $j(v) = (f, v)$, or $v \in C$.

$\square$

**Remark 2.2** *Consider a blocking force $f$ and let $\tilde{f} = Mf$ be the corresponding maximal blocking force. By the Proposition 2.4 we deduce that $C_\lambda = \{v \in V \mid (\lambda f, v) = j(v)\} = \{0\}$, $\forall\, 0 < \lambda < M$.*

We denote by $(u_\varepsilon)_{\varepsilon>0}$ the solutions of the variational inequalities:

$$u_\varepsilon \in V \;:\qquad a(u_\varepsilon, v - u_\varepsilon) + j(v) - j(u_\varepsilon) \geq (f_\varepsilon, v - u_\varepsilon),\ \forall v \in V, \tag{4}$$

where $f_\varepsilon = (1+\varepsilon)f, \forall \varepsilon > 0$. We set

$$w_\varepsilon = \frac{u_\varepsilon}{\varepsilon}, \quad \forall \varepsilon > 0.$$

The following theorem establishes the convergence of $(w_\varepsilon)_{\varepsilon>0}$ when $\varepsilon \searrow 0$ and gives a characterization of the limit.

**Theorem 2.1** *Assume that $j : V \to ]-\infty, +\infty]$ is a proper, convex, l.s.c., positively homogeneous function with $j(0) = 0$ and $f$ is a blocking force. Then $(w_\varepsilon)_{\varepsilon>0}$ converges strongly in $V$ when $\varepsilon \searrow 0$ and we have*

$$\lim_{\varepsilon \searrow 0} w_\varepsilon = w,$$

*where $w$ is the solution of the variational inequality :*

$$w \in C \;:\qquad a(w, v - w) \geq (f, v - w),\ \forall v \in C,$$

*with $C = \{v \in V \mid j(v) = (f, v)\}$. In particular, if the bilinear form $a(\cdot,\cdot)$ is symmetric, we have*

$$\lim_{\varepsilon \searrow 0} w_\varepsilon = Proj_C(u),$$

*where $Proj_C : V \to C$ denotes the projection operator on the closed convex cone $C$ with respect to the inner product given by the bilinear form $a(\cdot,\cdot)$ and $u$ is the solution of the variational equality:*

$$u \in V \;:\qquad a(u, v) = (f, v),\ \forall v \in V.$$

**Proof.** The problem (4) can be written in an equivalent form : find $u_\varepsilon \in V$ such that

$$\begin{cases} a(u_\varepsilon, u_\varepsilon) + j(u_\varepsilon) = (f_\varepsilon, u_\varepsilon), \\ a(u_\varepsilon, v) + j(v) \geq (f_\varepsilon, v),\ \forall v \in V. \end{cases} \tag{5}$$

The equality in (5) becomes $a(w_\varepsilon, w_\varepsilon) + \frac{1}{\varepsilon}(j(w_\varepsilon) - (f, w_\varepsilon)) = (f, w_\varepsilon)$, and therefore

$$\alpha \|w_\varepsilon\|^2 \leq a(w_\varepsilon, w_\varepsilon) \leq (f, w_\varepsilon) \leq \|f\|\|w_\varepsilon\|,$$

which implies that $(w_\varepsilon)_{\varepsilon>0}$ is bounded and $\|w_\varepsilon\| \leq \frac{\|f\|}{\alpha}$, $\forall \varepsilon > 0$. Therefore we can extract a subsequence $\varepsilon_k \searrow 0$ such that $w_k := w_{\varepsilon_k}$ converges weakly towards

$w$ in $V$. Since $j$ is convex l.s.c., $j$ is also weakly l.s.c. and therefore $j(w) = \liminf_{k\to+\infty} j(w_k)$. According to (5) we have $j(w_k) \le (1+\varepsilon_k)(f, w_k)$, $\forall k$ and therefore $j(w) \le \liminf_{k\to+\infty}(1+\varepsilon_k)(f, w_k) = (f, w)$. Since $f$ is a blocking force we write $j(w) \ge (f, w)$ and therefore we deduce that $j(w) = (f, w)$ or $w \in C$. Now we can prove that $(w_k)_k$ converges strongly in $V$ to $w$. We introduce the operator $A \in \mathcal{L}(V)$ such that $(Au, v) = a(u, v)$, $\forall u, v \in V$. The inequality (4) can be expressed in an equivalent way using the operator $A$ as follows:

$$f_\varepsilon \in Au_\varepsilon + \partial j(u_\varepsilon).$$

By dividing by $\varepsilon$ and noting that $\partial j(\lambda v) = \partial j(v)$, $\forall \lambda > 0$ (since $j$ is positively homogeneous), we obtain

$$f + \frac{1}{\varepsilon} f \in Aw_\varepsilon + \frac{1}{\varepsilon}\partial j(w_\varepsilon).$$

Since $w \in C$, by the *Proposition* 2.7 we deduce that $f \in \partial j(w)$ and we get

$$f \in Aw_\varepsilon + \frac{1}{\varepsilon}(\partial j(w_\varepsilon) - \partial j(w)).$$

After multiplication with $w_\varepsilon - w$ and using the monotonicity of $\partial j$ we obtain

$$a(w_\varepsilon, w_\varepsilon - w) = (Aw_\varepsilon, w_\varepsilon - w) \le (f, w_\varepsilon - w),$$

or

$$a(w_\varepsilon - w, w_\varepsilon - w) \le (f, w_\varepsilon - w) - (Aw, w_\varepsilon - w).$$

In particular, taking $\varepsilon = \varepsilon_k$ and using the $V$-ellipticity of $a(\cdot,\cdot)$ yields

$$\alpha \|w_k - w\|^2 \le a(w_k - w, w_k - w) \le (f - Aw, w_k - w),\ \forall k.$$

Since $(w_k)_k$ converges weakly towards $w$ we deduce that $\lim_{k\to+\infty} \|w_k - w\| = 0$. Next we give a characterization of the limit $w$. Consider $v \in C$, which implies that $f \in \partial j(v)$. As before we have

$$f \in Aw_k + \frac{1}{\varepsilon_k}(\partial j(w_k) - \partial j(v)).$$

After multiplication with $w_k - v$ we find

$$a(w_k, w_k - v) = (Aw_k, w_k - v) \le (f, w_k - v),$$

and by passing to the limit for $k \to +\infty$ we deduce that $w$ is the unique solution of the variational inequality

$$w \in C\ :\qquad a(w, v - w) \ge (f, v - w),\ \forall v \in C. \tag{6}$$

The uniqueness of the limit allows us to prove the strong convergence $\lim_{\varepsilon \searrow 0} w_\varepsilon = w$. Suppose now that $a(\cdot,\cdot)$ is symmetric and consider also the solution $u$ of the problem

$$u \in V \ : \qquad a(u,v) = (f,v), \ \forall v \in V.$$

We deduce that the limit $w$ verifies

$$w \in C \ : \qquad a(w-u, w-v) \leq 0, \ \forall v \in C,$$

which is equivalent to

$$w \in C \ : \qquad a(w-u, w-u) \leq a(v-u, v-u), \ \forall v \in C,$$

and therefore $w = Proj_C(u)$ (with respect to the inner product $a(\cdot,\cdot)$). As above, by the uniqueness of the limit we come to the conclusion that strong convergence holds : $\lim_{\varepsilon \searrow 0} w_\varepsilon = Proj_C(u)$.

$\square$

**Remark 2.3** *The above proof states the convergence of* $(w_\varepsilon)_{\varepsilon>0}$ *towards* $w$ *when* $\varepsilon$ *vanishes. The method used in the proof does not provide any convergence rates. As we will see later in Proposition 2.9 this is possible when* $\varepsilon \to +\infty$.

**Remark 2.4** *Suppose that* $a(\cdot,\cdot)$ *is symmetric. Since* $j$ *is positively homogeneous, by the Proposition* 2.6 *we know that* $C$ *is a convex cone and thus, by taking* $v = 0$ *and* $v = 2w$ *in* (6) *we get* $a(w,w) = (f,w)$ *and* $a(w,v) \geq (f,v), \ \forall v \in C$. *Consequently we have*

$$a(w,w) = (f,w) = a(u,w) \leq a(u,u)^{1/2} a(w,w)^{1/2},$$

*and finally we deduce that* $a(w,w) \leq a(u,u)$ *and* $(f,w) = a(w,w) \leq a(u,u) = (f,u)$.

**Remark 2.5** *Note that the most interesting case in the previous theorem is when* $f$ *is a maximal blocking force. Indeed, if* $f$ *is a blocking force but not a maximal blocking force, then for* $\varepsilon > 0$ *small enough* $f_\varepsilon = (1+\varepsilon) f$ *is also a blocking force and* $w_\varepsilon = \frac{u_\varepsilon}{\varepsilon} = 0$. *Consequently we have* $\lim_{\varepsilon \searrow 0} w_\varepsilon = 0$ *and the set* $C$ *reduces to* $\{0\}$ *(see the Remark* 2.2*).*

In the next proposition we give a necessary and sufficient condition for the limit $w$ to be 0.

**Proposition 2.8** *Assume that* $j : V \to ]-\infty, +\infty]$ *is a proper, convex, l.s.c., homogeneous function,* $j(v) > 0 \ \forall v \neq 0$ *and* $f$ *is a maximal blocking force. The following conditions are equivalent:*
(1) $C = \{0\}$*;*
(2) $\lim_{\varepsilon \searrow 0} w_\varepsilon = 0$*;*
(3) $\inf_{(f,v)\neq 0} \frac{j(v)}{|(f,v)|}$ *is not attained.*

**Proof.** Since $\lim_{\varepsilon \searrow 0} w_\varepsilon \in C$ it is straightforward that (1) implies (2). Conversely, suppose that $w = \lim_{\varepsilon \searrow 0} w_\varepsilon = 0$ and consider $v \in C$. By the inequality (6) we have

$$0 = a(w, w - v) \leq (f, w - v) = -(f, v) = -j(v).$$

Therefore $v = 0$ and $C = \{0\}$. The equivalence between (1) and (3) follows from the *Proposition* 2.4. $\square$

**Remark 2.6** *If $V$ is finite dimensional then none of the equivalent conditions of the previous proposition are fulfilled.*

**Remark 2.7** *The techniques introduced in this section can be extended in order to study the following inequality (7) (instead of (2)):*

$$u \in V \;:\; \qquad a(u, v - u) + b(u, u, v - u) + j(v) - j(u) \geq (f, v - u),\ \forall v \in V, \quad (7)$$

*where $b : V \times V \times V \to \mathbb{R}$ is a continuous trilinear form verifying $b(v, v, v) = 0, \forall v \in D(j)$.*

Indeed, we need to assume that for any 'small' $\varepsilon > 0$ the problem (7) admits a solution $u_\varepsilon$ corresponding to the force $f_\varepsilon = (1 + \varepsilon)f$. As before, by taking $v = 2u_\varepsilon$, $v = 0$ in (7) and since $b(u_\varepsilon, u_\varepsilon, u_\varepsilon) = 0$, the variational inequality (7) can be written: find $u_\varepsilon \in V$ such that

$$\begin{cases} a(u_\varepsilon, u_\varepsilon) + j(u_\varepsilon) = (f_\varepsilon, u_\varepsilon), \\ a(u_\varepsilon, v) + b(u_\varepsilon, u_\varepsilon, v) + j(v) \geq (f_\varepsilon, v),\ \forall v \in V. \end{cases} \tag{8}$$

By using (8) and the blocking condition we deduce as previously that $w_\varepsilon = \frac{u_\varepsilon}{\varepsilon}$ is bounded and we can extract a sequence $\varepsilon_k \searrow 0$ verifying $w_{\varepsilon_k} \rightharpoonup w \in C = \{v \in V \mid (f, v) = j(v)\}$. In order to prove that $(w_{\varepsilon_k})_k$ converges strongly, we introduce an operator $B \in \mathcal{L}(V \times V, V)$ satisfying $b(u, v, w) = (B(u, v), w), \forall u, v, w \in V$ and we observe that the variational inequality (7) can be written

$$Au_\varepsilon + B(u_\varepsilon, u_\varepsilon) + \partial j(u_\varepsilon) \ni f_\varepsilon,$$

or

$$Aw_\varepsilon + \varepsilon B(w_\varepsilon, w_\varepsilon) + \frac{1}{\varepsilon}\partial j(w_\varepsilon) \ni \left(1 + \frac{1}{\varepsilon}\right) f. \tag{9}$$

Since $w \in C$ we have $f \in \partial j(w)$. After multiplication with $w_\varepsilon - w$ and using the monotonicity of $\partial j$ we find

$$a(w_\varepsilon, w_\varepsilon - w) + \varepsilon b(w_\varepsilon, w_\varepsilon, w_\varepsilon - w) \leq (f, w_\varepsilon - w),$$

which implies

$$a(w_\varepsilon - w, w_\varepsilon - w) \leq (f, w_\varepsilon - w) - a(w, w_\varepsilon - w) + \varepsilon \|b\| \|w_\varepsilon\|^2 (\|w_\varepsilon\| + \|w\|).$$

Choosing $\varepsilon = \varepsilon_k$ we deduce that $\lim_{k\to+\infty} w_{\varepsilon_k} = w$ strongly in $V$. We take $v \in C$, or $f \in \partial j(v)$. After multiplication of (9) by $w_\varepsilon - v$ we deduce as in the proof of *Theorem* 2.1 that $w$ solves the problem

$$w \in C \ : \qquad a(w, w-v) \leq (f, w-v), \ \forall v \in C,$$

which proves that $\lim_{\varepsilon\searrow 0} w_\varepsilon = w$.

By similar arguments we can analyze the behavior of $w_\varepsilon$ when $\varepsilon \to +\infty$.

**Theorem 2.2** *Assume that $j : V \to ]-\infty, +\infty]$ is proper, convex, l.s.c., positively homogeneous with $j(0) = 0$ and $f \in V$ (not necessarily a blocking force). Then $(w_\varepsilon)_{\varepsilon>0}$ converges strongly in $V$ when $\varepsilon \to +\infty$ and we have*

$$\lim_{\varepsilon\to+\infty} w_\varepsilon = w,$$

*where $w$ is the solution of the variational inequality :*

$$w \in \overline{D(j)} \ : \qquad a(w, v-w) \geq (f, v-w), \ \forall v \in \overline{D(j)}.$$

*In particular, if the bilinear form $a(\cdot,\cdot)$ is symmetric then we have*

$$\lim_{\varepsilon\to+\infty} w_\varepsilon = Proj_{\overline{D(j)}}(u),$$

*where $u$ is the solution of the problem*

$$u \in V \ : \ a(u,v) = (f,v), \forall v \in V,$$

*and $Proj_{\overline{D(j)}}$ stands for the projection operator on the closed convex set $\overline{D(j)}$ with respect to the inner product given by $a(\cdot,\cdot)$.*

**Proof.** Since $j$ is proper, convex, l.s.c., there is $\beta \in \mathbb{R}$ and $v_0 \in V$ such that

$$j(v) \geq \beta + (v, v_0), \ \ \forall v \in V. \tag{10}$$

As before we have

$$a(w_\varepsilon, w_\varepsilon) + \frac{1}{\varepsilon} j(w_\varepsilon) = \left(1 + \frac{1}{\varepsilon}\right)(f, w_\varepsilon). \tag{11}$$

Consequently we have for $\varepsilon > 1$

$$\begin{aligned}\alpha\|w_\varepsilon\|^2 &\leq a(w_\varepsilon, w_\varepsilon) \leq \left(1 + \frac{1}{\varepsilon}\right)\|f\|\|w_\varepsilon\| - \frac{\beta}{\varepsilon} + \frac{\|w_\varepsilon\|\|v_0\|}{\varepsilon} \\ &\leq (2\|f\| + \|v_0\|)\|w_\varepsilon\| + |\beta|,\end{aligned}$$

which implies that $(w_\varepsilon)_{\varepsilon>1}$ is bounded. We can extract a sequence $w_k := w_{\varepsilon_k}$ with $\varepsilon_k \to +\infty$ such that $(w_k)_k$ converges weakly towards $w$ in $V$. By using (4) with $\varepsilon = \varepsilon_k$ and $v = \varepsilon_k w_l$ one gets

$$a(u_k, \varepsilon_k w_l - u_k) + j(\varepsilon_k w_l) - j(u_k) \geq (1 + \varepsilon_k)(f, \varepsilon_k w_l - u_k), \ \forall k, l,$$

where $u_k = \varepsilon_k w_k$ or

$$a(w_k, w_l - w_k) + \frac{1}{\varepsilon_k}(j(w_l) - j(w_k)) \geq \left(1 + \frac{1}{\varepsilon_k}\right)(f, w_l - w_k), \ \forall k, l,$$

and we obtain

$$a(w_k, w_k) \leq \left(1 + \frac{1}{\varepsilon_k}\right)(f, w_k - w_l) + a(w_k, w_l) + \frac{1}{\varepsilon_k}(j(w_l) - j(w_k)), \ \forall k, l. \quad (12)$$

By the equality (11) written for $\varepsilon = \varepsilon_l$ we see that $j(w_l) < +\infty$ and according to (10) we get

$$\limsup_{k\to+\infty} \frac{1}{\varepsilon_k}(j(w_l) - j(w_k)) \ \leq \ \limsup_{k\to+\infty} \frac{1}{\varepsilon_k}(j(w_l) - \beta - (w_k, v_0)) = 0.$$

After passing to the limit for $k \to +\infty$ in (12) we deduce that

$$\limsup_{k\to+\infty} a(w_k, w_k) \leq (f, w - w_l) + a(w, w_l), \ \forall l.$$

By passing to the limit for $l \to +\infty$ in the above inequality, we come to the conclusion that $\limsup_{k\to+\infty} a(w_k, w_k) \leq a(w, w)$. Finally $\lim_{k\to+\infty} a(w_k - w, w_k - w) = 0$ and thus $(w_k)_k$ converges strongly towards $w$. In order to identify the limit, take $v \in D(j)$ and write

$$a(w_k, v - w_k) + \frac{1}{\varepsilon_k}(j(v) - j(w_k)) \geq \left(1 + \frac{1}{\varepsilon_k}\right)(f, v - w_k).$$

As before we check that $\limsup_{k\to+\infty} \frac{1}{\varepsilon_k}(j(v) - j(w_k)) \leq 0$ and therefore, after passing to the limit for $k \to +\infty$ we obtain

$$a(w, v - w) \geq (f, v - w), \ \forall v \in D(j).$$

By the continuity we have also

$$a(w, v - w) \geq (f, v - w), \ \forall v \in \overline{D(j)}.$$

Equality (11) leads to $j(w_k) < +\infty, \forall k$ and thus $w = \lim_{k\to+\infty} w_k \in \overline{D(j)}$. Therefore $(w_k)_k$ converges strongly to the unique solution of

$$w \in \overline{D(j)} \ : \qquad a(w, v - w) \geq (f, v - w), \ \forall v \in \overline{D(j)}.$$

The strong convergence $\lim_{\varepsilon\to+\infty} w_\varepsilon = w$ follows from the uniqueness of the limit. If $a(\cdot,\cdot)$ is symmetric we have

$$w \in \overline{D(j)} \ : \qquad a(w, v-w) \geq (f, v-w) = a(u, v-w), \ \forall v \in \overline{D(j)},$$

or

$$w \in \overline{D(j)} \ : \qquad a(u-w, v-w) \leq 0, \ \forall v \in \overline{D(j)}.$$

Finally $w = Proj_{\overline{D(j)}}(u)$ (with respect to the inner product given by $a(\cdot,\cdot)$). By the uniqueness of the limit we obtain that $\lim_{\varepsilon\to+\infty} w_\varepsilon = Proj_{\overline{D(j)}}(u)$. $\square$

**Proposition 2.9** *With the notations of the previous theorem, if $j : V \to ]-\infty+\infty]$ is proper, convex, l.s.c., homogeneous and bounded (i.e., $\exists c > 0$ such that $|j(v)| \leq c\|v\|$, $\forall v \in D(j)$) then we have the estimate*

$$\|w_\varepsilon - w\| \leq \frac{c + \|f\|}{\alpha\varepsilon}, \ \ \forall \varepsilon > 0.$$

**Proof.** Indeed, since $j$ is l.s.c. and bounded, $D(j)$ is closed and thus $w = \lim_{\varepsilon\to+\infty} w_\varepsilon \in D(j)$. By using

$$a(w_\varepsilon, w - w_\varepsilon) + \frac{1}{\varepsilon}(j(w) - j(w_\varepsilon)) \geq \left(1 + \frac{1}{\varepsilon}\right)(f, w - w_\varepsilon) = \left(1 + \frac{1}{\varepsilon}\right) a(u, w - w_\varepsilon),$$

we obtain after multiplication with $\varepsilon$

$$\varepsilon a(u - w_\varepsilon, w - w_\varepsilon) \leq j(w) - j(w_\varepsilon) - a(u, w - w_\varepsilon), \ \forall \varepsilon > 0.$$

Taking into account that $a(w-u, w-w_\varepsilon) \leq 0$ and using the hypotheses on $j$ (convex, homogeneous and bounded) we obtain

$$\varepsilon a(w - w_\varepsilon, w - w_\varepsilon) \leq j(w) - j(w_\varepsilon) - a(u, w - w_\varepsilon) \leq c\|w - w_\varepsilon\| + \|f\|\|w - w_\varepsilon\|,$$

and therefore $\alpha\varepsilon\|w - w_\varepsilon\|^2 \leq (c + \|f\|)\|w - w_\varepsilon\|$, or $\|w - w_\varepsilon\| \leq \frac{c+\|f\|}{\varepsilon\alpha}$, $\forall \varepsilon > 0$. $\square$

# 3 The Bingham model

We consider the equations describing the stationary flow of an incompressible Bingham fluid of constant density $\rho = 1$ in a domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$. The notation $u$ stands for the velocity field, $\sigma$ denotes the Cauchy stress tensor field, $p = -\operatorname{trace}(\sigma)/3$ represents the pressure and $\sigma'$ given by $\sigma = \sigma' - pI$ is the deviatoric part of the stress tensor ($\operatorname{trace}(\sigma') = 0$). Let $b$ denote the body forces. The momentum balance law in the Eulerian coordinates and the incompressibility condition are:

$$(u \cdot \nabla)u - \operatorname{div} \sigma' + \nabla p = b \quad \text{ in } \Omega, \tag{13}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega. \tag{14}$$

If we denote by $D(u) = (\nabla u + \nabla^T u)/2$ the rate deformation tensor, the constitutive equation of the Bingham fluid can be written as follows:

$$\sigma' = \eta D(u) + g\frac{D(u)}{|D(u)|} \qquad \text{if } D(u) \neq 0, \tag{15}$$

$$|\sigma'| \leq g \qquad \text{if } D(u) = 0, \tag{16}$$

where $0 < \eta_0 \leq \eta = \eta(x) \leq \eta_1, \forall x \in \Omega$ is the viscosity distribution and $g = g(x) > 0, \forall x \in \Omega$ is a function which stands for the yield limit distribution in $\Omega$. We suppose that $\Gamma = \partial\Omega$ is divided into two disjoint parts so that $\Gamma = \Gamma_0 \cup \Gamma_1$ with $\text{meas}(\Gamma_0) > 0$. We complete the equations with the boundary conditions

$$u = 0 \quad \text{on} \quad \Gamma_0, \qquad \sigma \cdot n = 0 \quad \text{on} \quad \Gamma_1, \tag{17}$$

where $n$ stands for the unit outward normal on $\Gamma$.

## 3.1 The three-dimensional case

We introduce the Hilbert space $V = \{v \in H^1(\Omega)^3 \mid v|_{\Gamma_0} = 0\}$ and we consider the symmetric continuous bilinear form $a : V \times V \to \mathbb{R}$ given by

$$a(u, v) = \int_\Omega \eta \sum_{1 \leq i,j \leq 3} D_{ij}(u) D_{ij}(v)\, dx = \int_\Omega \eta D(u) : D(v)\, dx, \ \forall u, v \in V.$$

According to the Korn lemma (see [5]) the form $a(\cdot, \cdot)$ is $V$-elliptic

$$a(v, v) = \int_\Omega \eta |D(v)|^2\, dx \geq \eta_0 \int_\Omega |D(v)|^2\, dx \geq \frac{\eta_0}{C_K} \|v\|^2 = \alpha \|v\|^2, \ \forall v \in V.$$

We introduce also the closed subspace $K = \{v \in V \mid \operatorname{div} v = 0\}$ and we consider the proper convex function $j : V \to [0, +\infty]$ given by

$$j(v) = \int_\Omega g |D(v)|\, dx, \ \text{if } v \in K \text{ and } j(v) = +\infty \text{ if } v \notin K,$$

where $g \in L^2(\Omega)$, $g > 0$. It is easy to check that $j$ is l.s.c., homogeneous, $j(0) = 0$ and $|j(v)| \leq \|g\|_{L^2(\Omega)} \|v\|, \forall v \in V$. Moreover, if $j(v) = 0$, since $g > 0$, we deduce that $D(v) = 0$ and by the Korn lemma it comes that $v = 0$. We consider also a continuous linear form $l : V \to \mathbb{R}$ (if $b \in L^2(\Omega)^3$ we take $\langle l, v\rangle = \int_\Omega b(x) \cdot v(x)\, dx, \ \forall v \in V$). If we neglect the term $(u \cdot \nabla)u$, the variational formulation of the problem (13)–(17) becomes (see [5]): find $u \in K$ such that for all function $v \in K$

$$\int_\Omega \eta D(u) : (D(v) - D(u))dx + \int_\Omega g\big(|D(v)| - |D(u)|\big)dx \geq \int_\Omega b \cdot (v - u)dx.$$

By using the previous notations we obtain an equivalent variational inequality in which $K$ is replaced by $V$:

$$u \in V \ : \qquad a(u, v-u) + j(v) - j(u) \geq \langle l, v-u \rangle, \ \forall v \in V, \tag{18}$$

and therefore we can apply the results of section 2. First of all let us identify the blocking condition and the maximal blocking forms $l \in V'$. The blocking condition is given by

$$\langle l, v \rangle \leq \int_\Omega g|D(v)| \, dx, \ \forall v \in K, \tag{19}$$

or equivalently $\langle l, v \rangle \leq j(v), \forall v \in V$. We introduce also

$$\mathcal{H} = \left\{ \tau \in L^2(\Omega)^{3\times 3} \mid \tau_{ij} = \tau_{ji}, \forall 1 \leq i,j \leq 3, \ \sum_{i=1}^{3} \tau_{ii} = 0 \right\},$$

$$\mathcal{H}_0 = \{\tau \in L^2(\Omega)^{3\times 3} \mid \tau_{ij} = \tau_{ji}, \forall 1 \leq i,j \leq 3, \ \operatorname{div} \tau = 0, \ \tau \cdot n|_{\Gamma_1} = 0\},$$

and

$$\mathcal{A}_l = \{\tau \in \mathcal{H} \mid \exists p \in L^2(\Omega), \operatorname{div}(-pI + \tau) = -l, \ (-pI + \tau) \cdot n|_{\Gamma_1} = 0\}.$$

Another characterization of the set $\mathcal{A}_l$ is given by

$$\mathcal{A}_l = \left\{ \tau \in \mathcal{H} \mid \int_\Omega \tau : D(v) \, dx = \langle l, v \rangle, \ \forall v \in K \right\}.$$

We use the following result proved in [12].

**Proposition 3.1** *The Bingham fluid is blocked, i.e.,* (19) *holds iff there is* $\tau \in \mathcal{A}_l$ *such that* $|\tau(x)| \leq g(x)$ *a.e.* $x \in \Omega$.

**Proposition 3.2** *Assume that* $l \in V', l|_K \neq 0$ *is a blocking form (i.e.,* $\exists \tau \in \mathcal{A}_l$ *such that* $|\tau(x)| \leq g(x)$ *a.e.* $x \in \Omega$*). Then the maximal blocking form corresponding to* $l$ *is given by* $\tilde{l} = M_2 l$ *where*

$$M_2 = \sup_{\xi \in \mathcal{H}_0, q \in L^2(\Omega)} \operatorname*{essinf}_{x \in \Omega} \frac{g(x)}{|\tau(x) + \xi(x) + q(x)I|}.$$

**Proof.** We have to show that $M = M_2$ where $M = \sup\{\lambda > 0 \mid \lambda\langle l, v \rangle \leq j(v) \ \forall v \in V\}$. For all $\xi \in \mathcal{H}_0, q \in L^2(\Omega)$ we have $\int_\Omega (\xi + qI) : D(v) \, dx = 0, \forall v \in K$ and thus $\langle l, v \rangle = \int_\Omega (\tau + \xi + qI) : D(v) \, dx, \forall v \in K$. Note that $\operatorname{essinf}_{x\in\Omega} g(x)|\tau(x) + \xi(x) + q(x)I|^{-1} < +\infty$, otherwise $\tau(x) + \xi(x) + q(x)I = 0$ a.e. $x \in \Omega$ and $l|_K = 0$. We obtain for $v \in K$

$$\begin{aligned} \operatorname*{essinf}_{x\in\Omega} \frac{g(x)}{|\tau(x) + \xi(x) + q(x)I|} \langle l, v \rangle &= \operatorname*{essinf}_{x\in\Omega} \frac{g(x)}{|\tau(x) + \xi(x) + q(x)I|} \\ &\quad \int_\Omega (\tau + \xi + qI) : D(v) \, dy \\ &\leq \int_\Omega g|D(v)| \, dy = j(v), \forall v \in K, \end{aligned}$$

and therefore $\operatorname{essinf}_{x\in\Omega}\big(g(x)|\tau(x)+\xi(x)+q(x)I|^{-1}\big)l$ is a blocking form $\forall\xi\in\mathcal{H}_0, \forall q\in L^2(\Omega)$, which implies that $M_2 \le M$. Conversely, for all $\varepsilon > 0$ there is $\lambda_\varepsilon > M-\varepsilon$ such that $\lambda_\varepsilon l$ is a blocking form. By using the *Proposition* 3.1 we deduce that $\lambda_\varepsilon l = -\operatorname{div}(-p_1I+\tau_1)$, with $\tau_1\in\mathcal{H}, p_1\in L^2(\Omega), (-p_1I+\tau_1)\cdot n|_{\Gamma_1}=0$ and $|\tau_1(x)|\le g(x)$, a.e. $x\in\Omega$. Since $l$ is a blocking form, we set $\xi = \frac{1}{\lambda_\varepsilon}(-p_1I+\tau_1)-(-pI+\tau)$ and we deduce that $\xi\in\mathcal{H}_0$. We set $q=\frac{p_1}{\lambda_\varepsilon}-p\in L^2(\Omega)$ and thus we can write $\frac{\tau_1}{\lambda_\varepsilon}=\tau+\xi+qI$. We obtain for a.e. $x\in\Omega$

$$|\tau_1(x)| = \lambda_\varepsilon\left|\frac{\tau_1(x)}{\lambda_\varepsilon}\right| = \lambda_\varepsilon|\tau(x)+\xi(x)+q(x)I| \le g(x).$$

As a consequence

$$M-\varepsilon < \lambda_\varepsilon \le \operatorname*{essinf}_{x\in\Omega}\frac{g(x)}{|\tau(x)+\xi(x)+q(x)I|} \le M_2.$$

We deduce that $M_2 \ge M$ and finally $M = M_2$.

$\square$

**Corollary 3.1** *Assume that $l\in V'$, $l|_K\neq 0$ is a blocking form. Then $l$ is a maximal blocking form iff $\exists\tau\in\mathcal{H}, \exists p\in L^2(\Omega)$ such that $-l = div(-pI+\tau)$, $(-pI+\tau)\cdot n|_{\Gamma_1} = 0$, $|\tau(x)|\le g(x)$, a.e. $x\in\Omega$ and*

$$\operatorname*{essinf}_{x\in\Omega}\frac{g(x)}{|\tau(x)+\xi(x)+q(x)I|}\le 1,\ \forall\xi\in\mathcal{H}_0, \forall q\in L^2(\Omega).$$

In (3) we introduce the set $C=\{v\in V \mid \langle l,v\rangle = j(v)\}$. In this case $C$ is a nonempty closed convex cone and it is given by $C=\{v\in K \mid \langle l,v\rangle = \int_\Omega g|D(v)|\,dx\}$. As before, we denote by $u_\varepsilon$ and $u$ the solutions of the problems:

$$u_\varepsilon\in V\ :\qquad a(u_\varepsilon, v-u_\varepsilon)+j(v)-j(u_\varepsilon)\ge(1+\varepsilon)\langle l, v-u_\varepsilon\rangle, \forall v\in V,$$

and

$$u\in V\ :\qquad a(u,v)=\langle l,v\rangle, \forall v\in V,$$

respectively. By applying the general results proved in section 2 (see *Theorems* 2.1, 2.2) we obtain the following theorem :

**Theorem 3.1** *Assume that $g>0$ belongs to $L^2(\Omega)$.*

*(1) If $l\in V'$ is a maximal blocking form (see Corollary 3.1) then $w_\varepsilon=\frac{u_\varepsilon}{\varepsilon}$ converges strongly in $V$ when $\varepsilon\searrow 0$ and we have $\lim_{\varepsilon\searrow 0}w_\varepsilon = Proj_C(u)$. Moreover $\lim_{\varepsilon\searrow 0}w_\varepsilon = 0$ iff $C=\{0\}$.*
*(2) If $l\in V'$ then $w_\varepsilon=\frac{u_\varepsilon}{\varepsilon}$ converges strongly in $V$ when $\varepsilon\to+\infty$ and we have $\lim_{\varepsilon\to+\infty}w_\varepsilon = Proj_K(u)$. Moreover we have*

$$\|w_\varepsilon - Proj_K(u)\| \le \frac{\|g\|_{L^2(\Omega)}+\|l\|_{V'}}{\alpha\varepsilon},\ \forall\varepsilon>0.$$

The next proposition describes the relation between $l$ and $g$ such that $C \neq \{0\}$.

**Proposition 3.3** *Assume that $l \in V'$, $l|_K \neq 0$, $g \in L^2(\Omega), g > 0$. The following statements are equivalent:*

(1) $\langle l, v\rangle \leq j(v), \forall v \in V$ *and there is* $v_0 \in K - \{0\}$ *such that* $\langle l, v_0\rangle = j(v_0)$*(which means that $l$ is a maximal blocking form and $C \neq \{0\}$);*
(2) *there is* $\beta : \Omega \to \mathbb{R}^+, \exists v_0 \in K - \{0\}, \exists \tau \in \mathcal{H}, \exists p \in L^2(\Omega)$ *such that* $-l = div(-pI + \tau)$, $(-pI + \tau) \cdot n|_{\Gamma_1} = 0$, $\tau(x) = \beta(x) D(v_0)(x)$ *and* $g(x) = \beta(x)|D(v_0)(x)|$ *a.e. $x$ with* $D(v_0)(x) \neq 0$, $|\tau(x)| \leq g(x)$ *a.e. $x$ with* $D(v_0)(x) = 0$.

**Proof.** Let us check that (2) implies (1). We have

$$
\begin{aligned}
\langle l, v_0\rangle = \langle -\operatorname{div}(-pI + \tau), v_0\rangle &= \int_\Omega \tau : D(v_0)\, dx \\
&= \int_\Omega \beta D(v_0) : D(v_0)\, dx \\
&= \int_\Omega g|D(v_0)|\, dx = j(v_0).
\end{aligned}
$$

Similarly we deduce that for $v \in K$

$$
\begin{aligned}
\langle l, v\rangle &= \langle -\operatorname{div}(-pI + \tau), v\rangle = \int_\Omega \tau : D(v)\, dx \\
&\leq \int_\Omega |\tau||D(v)|\, dx \leq \int_\Omega g|D(v)|\, dx = j(v).
\end{aligned}
$$

Conversely, assume that $l$ is a blocking form and there is $v_0 \in K - \{0\}$ such that $\langle l, v_0\rangle = j(v_0)$. By using *Proposition* 3.1 we know that there is $\tau \in \mathcal{H}$, $p \in L^2(\Omega)$ such that $-l = \operatorname{div}(-pI + \tau)$, $(-pI + \tau) \cdot n|_{\Gamma_1} = 0$ and $|\tau(x)| \leq g(x)$ a.e. $x \in \Omega$. We can write

$$
\begin{aligned}
\langle l, v_0\rangle &= \int_\Omega \tau : D(v_0)\, dx \leq \left|\int_\Omega \tau : D(v_0)\, dx\right| \\
&\leq \int_\Omega |\tau : D(v_0)|\, dx \leq \int_\Omega |\tau||D(v_0)|\, dx \\
&\leq \int_\Omega g|D(v_0)|\, dx = j(v_0).
\end{aligned}
$$

Since $\langle l, v_0\rangle = j(v_0)$ all the above inequalities are equalities. We deduce that there is $\beta : \Omega \to \mathbb{R}^+$ such that $\tau(x) = \beta(x) D(v_0)(x)$, $g(x) = \beta(x)|D(v_0)(x)|$ a.e. $x$ with $D(v_0)(x) \neq 0$. $\square$

Let us also consider the term $(u \cdot \nabla)u$. We set $V = H_0^1(\Omega)^3$. In this case the variational inequality (18) becomes

$$
u \in V \ : \qquad a(u, v - u) + b(u, u, v - u) + j(v) - j(u) \geq \langle l, v - u\rangle,\ \forall v \in V, \tag{20}
$$

where $b : V \times V \times V \to \mathbb{R}$ is given by

$$b(u, v, w) = \int_\Omega \sum_{1 \le i,j \le 3} u_i \frac{\partial v_j}{\partial x_i} w_j \, dx.$$

The trilinear application $b$ is continuous (see [5, 8]) and verifies $b(v, v, v) = 0,\ \forall v \in K$. According to [5] there exists for all $\varepsilon > 0$ a solution $u_\varepsilon$ of (20) corresponding to $l_\varepsilon = (1 + \varepsilon)l$, where $l$ is a maximal blocking form. By applying the result proved in the *Remark* 2.7 we deduce that $\lim_{\varepsilon \searrow 0} \frac{u_\varepsilon}{\varepsilon} = Proj_C(u)$, where $u \in V$ is such that $a(u, v) = \langle l, v\rangle, \forall v \in V$.

### 3.2 The laminar flow in an infinite cylinder with a 2D section

We consider the equations modelling the stationary laminar flow of a Bingham fluid in a cylindrical pipe $\Omega \times \mathbb{R}$ of cross section $\Omega \subset \mathbb{R}^2$ with a smooth boundary $\partial\Omega$. The fluid is under effect of a drop in pressure. Therefore the problem consists of finding a velocity field $\tilde{u} = (0, 0, u(x_1, x_2))$ in the $Ox_3$ direction. We have $(\tilde{u} \cdot \nabla)\tilde{u} = 0$. We introduce the Hilbert space $V = \{v \in H^1(\Omega) \mid v|_{\Gamma_0} = 0\}$ where $\Gamma = \partial\Omega = \Gamma_0 \cup \Gamma_1$ with $\mathrm{meas}(\Gamma_0) > 0$. In this case we have for $u, v \in V$

$$D(\tilde{u}) : D(\tilde{v}) = \frac{1}{2}\nabla u \cdot \nabla v, \quad |D(\tilde{v})| = \frac{1}{\sqrt{2}}|\nabla v|,$$

and therefore the bilinear form $a : V \times V \to \mathbb{R}$ is given by

$$a(u, v) = \int_\Omega \eta D(\tilde{u}) : D(\tilde{v}) \, dx = \frac{1}{2}\int_\Omega \eta \nabla u \cdot \nabla v \, dx,$$

where $0 < \eta_0 \le \eta = \eta(x) \le \eta_1$ a.e. $x \in \Omega$. Since $\mathrm{meas}(\Gamma_0) > 0$, by using the Poincaré inequality $\|v\|_{H^1(\Omega)} \le C_P \|\nabla v\|_{L^2(\Omega)}, \forall v \in V$, we deduce that $a(\cdot, \cdot)$ is $V-$elliptic, $a(v, v) \ge \alpha\|v\|^2, \forall v \in V$, where $\alpha = \frac{\eta_0}{2C_P^2}$. We consider also the convex function $j : V \to [0, +\infty]$, $j(v) = \int_\Omega g|D(\tilde{v})| \, dx = \frac{1}{\sqrt{2}}\int_\Omega g|\nabla v| \, dx, \forall v \in V$, where $g \in L^2(\Omega)$, $g > 0$. Note that in this case, since $\operatorname{div} \tilde{v} = div(0, 0, v(x_1, x_2)) = 0, \forall v \in V$ we have $D(j) = V$ and $j(v) \le \frac{1}{\sqrt{2}}\|g\|_{L^2(\Omega)}\|v\|, \forall v \in V$. We easily check that $j$ is continuous, homogeneous and $j(0) = 0$. Using the Poincaré inequality we deduce also that $j(v) = 0$ iff $v = 0$. Consider also a continuous linear form $l \in V'$. The variational formulation becomes : find $u \in V$ such that

$$\frac{1}{2}\int_\Omega \eta \nabla u \cdot (\nabla v - \nabla u) \, dx + \frac{1}{\sqrt{2}}\int_\Omega g\big(|\nabla v| - |\nabla u|\big) \, dx \ge \langle l, v - u\rangle, \ \forall v \in V.$$

We denote by $u_\varepsilon$ and $u$ the solutions of the problems

$$u_\varepsilon \in V \ : \qquad a(u_\varepsilon, v - u_\varepsilon) + j(v) - j(u_\varepsilon) \ge \langle l, v - u_\varepsilon\rangle, \forall v \in V$$

and

$$u \in V \ : \qquad a(u,v) = \langle l, v\rangle, \forall v \in V,$$

respectively. We start by identifying the blocking form. The blocking condition is given by

$$\langle l, v\rangle \leq \int_\Omega g_1 |\nabla v| \, dx, \forall v \in V, \tag{21}$$

where $g_1 = \frac{g}{\sqrt{2}}$. We introduce the notation

$$\mathcal{A}_l = \{F \in L^2(\Omega)^2 \mid \operatorname{div} F = -l, \ F \cdot n|_{\Gamma_1} = 0\}.$$

As before we have a second characterization of $\mathcal{A}_l$ given by

$$\mathcal{A}_l = \left\{ F \in L^2(\Omega)^2 \mid \int_\Omega F \cdot \nabla v \, dx = \langle l, v\rangle, \forall v \in V \right\}.$$

We use the following result proved in [12] :

**Proposition 3.4** *The Bingham fluid is blocked, i.e.,* (21) *holds, iff there is* $F \in \mathcal{A}_l$ *such that* $|F(x)| \leq g_1(x)$ *a.e.* $x \in \Omega$.

Exactly as in the three-dimensional case we can prove the following results :

**Proposition 3.5** *Assume that* $l \in V' - \{0\}$ *is a blocking form* ($\exists F \in L^2(\Omega)^2, l = -\,div F,\ F \cdot n|_{\Gamma_1} = 0,\ |F(x)| \leq g_1(x)$, *a.e.* $x \in \Omega$). *Then the maximal blocking form corresponding to* $l$ *is given by* $\tilde{l} = M_2 l$, *where*

$$M_2 = \sup_H \operatorname*{essinf}_{x \in \Omega} \frac{g_1(x)}{|F(x) + H(x)|},$$

*the supremum being taken on* $H \in L^2(\Omega)^2$, $div H = 0$, $H \cdot n|_{\Gamma_1} = 0$.

**Corollary 3.2** *Assume that* $l \in V' - \{0\}$ *is a blocking form. Then* $l$ *is a maximal blocking form iff* $\exists F \in L^2(\Omega)^2$ *such that* $l = -\,div F$, $F \cdot n|_{\Gamma_1} = 0$, $|F(x)| \leq g_1(x)$ *a.e.* $x \in \Omega$ *and*

$$\operatorname*{essinf}_{x \in \Omega} \frac{g_1(x)}{|F(x) + H(x)|} \leq 1, \ \forall H \in L^2(\Omega)^2, div H = 0, H \cdot n|_{\Gamma_1} = 0.$$

**Theorem 3.2** *Assume that* $g_1 > 0$ *belongs to* $L^2(\Omega)$.

(1) *If* $l \in V'$ *is a maximal blocking form (see Corollary 3.2) then* $w_\varepsilon = \frac{u_\varepsilon}{\varepsilon}$ *converges strongly in* $V$ *when* $\varepsilon \searrow 0$ *and we have* $\lim_{\varepsilon \searrow 0} w_\varepsilon = Proj_C(u)$. *Moreover* $\lim_{\varepsilon \searrow 0} w_\varepsilon = 0$ *iff* $C = \{0\}$.
(2) *If* $l \in V'$ *then* $w_\varepsilon = \frac{u_\varepsilon}{\varepsilon}$ *converges strongly in* $V$ *when* $\varepsilon \to +\infty$ *and we have* $\lim_{\varepsilon \to +\infty} w_\varepsilon = u$. *Moreover we have*

$$\|w_\varepsilon - u\| \leq \frac{\|g_1\|_{L^2(\Omega)} + \|l\|_{V'}}{\alpha \varepsilon}, \ \forall \varepsilon > 0.$$

**Proposition 3.6** *Assume that $l \in V' - \{0\}$, $g_1 \in L^2(\Omega), g_1 > 0$. The following statements are equivalent:*

(1) $\langle l, v\rangle \leq j(v), \forall v \in V$ *and there is $v_0 \in V - \{0\}$ such that $\langle l, v_0\rangle = j(v_0)$(which means that $l$ is a maximal blocking form and $C \neq \{0\}$);*
(2) *there is $\beta : \Omega \to \mathbb{R}^+, \exists v_0 \in V - \{0\}, \exists F \in L^2(\Omega)^2$ such that $l = -\, div F$, $F \cdot n|_{\Gamma_1} = 0$, $F(x) = \beta(x)\nabla v_0(x), g_1(x) = \beta(x)|\nabla v_0(x)|$ a.e. $x$ with $\nabla v_0(x) \neq 0$, $|F(x)| \leq g_1(x)$ a.e. $x$ with $\nabla v_0(x) = 0$.*

**Remark 3.1** *It has been recently proven in [13] that the condition (1) in the Proposition 3.6 is always satisfied if the functions lie in a BV space (instead of an Hilbert space). The determination of $v_0$ in Proposition 3.6, (1) becomes then equivalent to a shape optimisation problem whose corresponding numerical experiments are carried out in [11].*

We consider also two cases in one dimension.

## 3.3 The flow between two infinite planes

Now we consider the anti-plane flow in one dimension, i.e., in a region $\Omega \times \mathbb{R}^2$, with $\Omega = ]0, L[ \subset \mathbb{R}$. The choice of $\Gamma_0 = \partial\Omega = \{0, L\}$ corresponds to the flow between two infinite planes $x = 0$ and $x = L$ and consists of finding a velocity in the $Oy$ direction $\tilde{u} = (0, u(x), 0)$. In this case $V = H_0^1(]0, L[)$, $D(\tilde{u}) : D(\tilde{v}) = \frac{1}{2}u'v'$, $|D(\tilde{v})| = \frac{1}{\sqrt{2}}|v'|$, $a(u, v) = \frac{1}{2}\int_0^L \eta(x)u'(x)v'(x)\, dx$, $\forall u, v \in V$, where $0 < \eta_0 \leq \eta = \eta(x) \leq \eta_1, 0 < x < L$, $j : V \to [0, +\infty]$, $j(v) = \frac{1}{\sqrt{2}}\int_0^L g(x)|v'(x)|\, dx$, $\forall v \in V$, where $g > 0$, $g \in L^2(]0, L[)$. Obviously $a(\cdot, \cdot)$ is symmetric, bilinear, continuous and $j$ is continuous, $j(v) = 0$ iff $v = 0$ and $j(v) \leq \frac{1}{\sqrt{2}}\|g\|_{L^2(]0,L[)}\|v\|, \forall v \in V$. If $l \in V'$, the variational problem is: find $u \in V$ such that for all $v \in V$

$$\frac{1}{2}\int_0^L \eta(x)u'(x)(v'(x) - u'(x))\, dx + \frac{1}{\sqrt{2}}\int_0^L g(x)\big(|v'(x)| - |u'(x)|\big)\, dx \geq \langle l, v - u\rangle.$$

The blocking condition is given by

$$\langle l, v\rangle \leq \int_0^L g_1(x)|v'(x)|\, dx,\ \forall v \in V, \tag{22}$$

where $g_1 = \frac{g}{\sqrt{2}}$. As in the previous cases, we obtain

**Proposition 3.7** *The Bingham fluid is blocked, i.e., (22) holds, iff there is $F \in L^2(]0, L[)$ such that $l = -F'$ and $|F(x)| \leq g_1(x)$, a.e. $0 < x < L$.*

**Proposition 3.8** *Assume that $l \in V' - \{0\}$ is a blocking form ($\exists F \in L^2(]0, L[), l = -F'$, $|F(x)| \leq g_1(x)$, a.e. $0 < x < L$). Then the maximal blocking form corresponding to $l$ is given by $\tilde{l} = M_2 l$, where*

$$M_2 = \sup_{k\in\mathbb{R}} \operatorname*{essinf}_{x\in]0,L[} \frac{g_1(x)}{|F(x) + k|}.$$

**Corollary 3.3** *Assume that $l \in V' - \{0\}$ is a blocking form. Then $l$ is a maximal blocking form iff $\exists F \in L^2(]0,L[)$ such that $l = -F'$, $|F(x)| \leq g_1(x)$, a.e. $0 < x < L$ and*

$$\operatorname*{essinf}_{x \in ]0,L[} \frac{g_1(x)}{|F(x)+k|} \leq 1, \ \forall k \in \mathbb{R}.$$

As in the three and two dimensional cases we obtain results concerning the convergence of $w_\varepsilon = \frac{u_\varepsilon}{\varepsilon}$ when $\varepsilon \searrow 0$ and $\varepsilon \to +\infty$ and we have a characterization of the relation between $l$ and $g$ such that $\lim_{\varepsilon \searrow 0} w_\varepsilon \neq 0$ (which is equivalent to $C \neq \{0\}$). Let us analyze a particular case in detail. Assume that $g_1(x) = g_1, \forall\ 0 < x < L$ and $\langle l, v\rangle = \int_0^L lv(x)\ dx, \forall v \in V$ for some constant $l \in \mathbb{R} - \{0\}$. We assume also that $\eta(x) = \eta > 0, \forall\ 0 < x < L$. By using the *Proposition* 3.7 we see that $l$ is a blocking force iff there is $k \in \mathbb{R}$ such that $|lx - k| \leq g_1, \forall\ 0 < x < L$ or $|k| \leq g_1$ and $|lL - k| \leq g_1$ which is equivalent to $|l|L \leq 2g_1$. We deduce that $\frac{2g_1}{L}\operatorname{sign}(l)$ is the maximal blocking force corresponding to $l$. Besides we know that the maximal blocking force is given by $\tilde{l} = Ml$ with $M = \inf_{\langle l,v\rangle \neq 0} \frac{j(v)}{|\langle l,v\rangle|}$. We deduce that

$$\frac{2g_1}{L|l|} = \frac{2g_1 \operatorname{sign}(l)}{L}\frac{1}{l} = \frac{\tilde{l}}{l} = M = \inf_{\int_0^L v(x)\ dx \neq 0} \frac{\int_0^L g_1|v'(x)|\ dx}{|\int_0^L lv(x)\ dx|}.$$

Hence

$$\inf_{\int_0^L v(x)\ dx \neq 0} \frac{\int_0^L |v'(x)|\ dx}{|\int_0^L v(x)\ dx|} = \frac{2}{L}.$$

In particular $2\left|\int_0^L v(x)\ dx\right| \leq L\int_0^L |v'(x)|\ dx, \ \forall v \in V$. Suppose now that the above infimum is attained which means that there is $v_0 \in V - \{0\}$ such that $\langle \frac{2g_1}{L}, v_0\rangle = j(v_0)$. By using the analogous result to the *Propositions* 3.3 and 3.6 we deduce that there is $\beta : ]0,L[ \to \mathbb{R}^+$, $k \in \mathbb{R}$ such that

$$-\frac{2g_1}{L}x + k = \beta(x)v_0'(x), \quad \text{a.e. } x \in ]0,L[,\ v_0'(x) \neq 0,$$

$$g_1 = \beta(x)|v_0'(x)|, \ \text{a.e. } x \in ]0,L[,\ v_0'(x) \neq 0,$$

$$\left|-\frac{2g_1}{L}x + k\right| \leq g_1, \ \text{a.e. } x \in ]0,L[,\ v_0'(x) = 0.$$

We deduce that $\left|-\frac{2g_1}{L}x + k\right| = \beta(x)|v_0'(x)| = g_1$ a.e. $x \in ]0,L[$, $v_0'(x) \neq 0$ and therefore $v_0' = 0$ a.e. $x \in ]0,L[$, or $v_0 = 0$. Hence the infimum is not attained and the set $C = \{v \in V \mid \langle \frac{2g_1}{L}, v\rangle = j(v)\}$ reduces to $\{0\}$. In this case we have $\lim_{\varepsilon \searrow 0} \frac{u_\varepsilon}{\varepsilon} = 0$ and $\lim_{\varepsilon \to +\infty} \frac{u_\varepsilon}{\varepsilon} = u$, where $u(x) = \frac{l}{\eta}x(L-x), 0 < x < L$.

### 3.4 The flow between an infinite plane and a rigid roof

We suppose that $\Gamma_0 = \{0\}$ and $\Gamma_1 = \{L\}$, *i.e.*, $V = \{v \in H^1(]0, L[) \mid v(0) = 0\}$. Such a boundary condition corresponds to the flow on the plane $x = 0$ with a rigid roof at $x = L$. The blocking condition is given by

$$\langle l, v\rangle \leq \int_0^L g_1(x)|v'(x)|\, dx,\ \forall v \in V, \tag{23}$$

where $g_1 = \frac{g}{\sqrt{2}}$.

**Proposition 3.9** *The Bingham fluid is blocked, i.e.,* (23) *holds, iff there is* $F \in L^2(]0, L[)$ *such that* $l = -F'$, $F(L) = 0$ *and* $|F(x)| \leq g_1(x)$, *a.e.* $0 < x < L$.

**Proposition 3.10** *Assume that* $l \in V' - \{0\}$ *is a blocking form* ($\exists F \in L^2(]0, L[), l = -F'$, $F(L) = 0$, $|F(x)| \leq g_1(x)$, *a.e.* $0 < x < L$). *Then the maximal blocking form corresponding to* $l$ *is given by* $\tilde{l} = M_2 l$, *where*

$$M_2 = \operatorname*{essinf}_{x \in ]0,L[} \frac{g_1(x)}{|F(x)|}.$$

**Corollary 3.4** *Assume that* $l \in V' - \{0\}$ *is a blocking form. Then* $l$ *is a maximal blocking form iff* $\exists F \in L^2(]0, L[)$ *such that* $l = -F'$, $F(L) = 0$ *and*

$$\operatorname*{essinf}_{x \in ]0,L[} \frac{g_1(x)}{|F(x)|} = 1.$$

Let us analyze the case $g_1(x) = g_1, \forall\, 0 < x < L$ and $\langle l, v\rangle = \int_0^L lv(x)\, dx, \forall v \in V$ for some constant $l \in \mathbb{R} - \{0\}$, $\eta(x) = \eta > 0, \forall\, 0 < x < L$. By using the *Proposition* 3.9 we deduce that $l$ is a blocking force iff $|-l(x-L)| \leq g_1, \forall\, 0 < x < L$ or $|l|L \leq g_1$. In this case the maximal blocking force corresponding to $l$ is $\tilde{l} = \frac{g_1 \operatorname{sign}(l)}{L}$ and we have

$$\frac{g_1}{|l|L} = \frac{g_1 \operatorname{sign}(l)}{L}\frac{1}{l} = \frac{\tilde{l}}{l} = M = \inf_{\int_0^L v(x)\, dx \neq 0} \frac{\int_0^L g_1|v'(x)|\, dx}{|\int_0^L lv(x)\, dx|}.$$

Hence

$$\inf_{\int_0^L v(x)\, dx \neq 0} \frac{\int_0^L |v'(x)|\, dx}{|\int_0^L v(x)\, dx|} = \frac{1}{L}.$$

In particular $\left|\int_0^L v(x)\, dx\right| \leq L \int_0^L |v'(x)|\, dx,\ \forall v \in V$. As previously we can prove that the infimum is not attained and the set $C = \{v \in V \mid \langle \frac{g_1}{L}, v\rangle = j(v)\}$ reduces to $\{0\}$. In this case we obtain $\lim_{\varepsilon \searrow 0} \frac{u_\varepsilon}{\varepsilon} = 0$ and $\lim_{\varepsilon \to +\infty} \frac{u_\varepsilon}{\varepsilon} = u$, where $u(x) = \frac{l}{\eta} x(2L - x), 0 < x < L$.

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