Mathematical Models and Methods in Applied Sciences

EXTENSION OF THE MORTAR FINITE ELEMENT METHOD TO A VARIATIONAL INEQUALITY MODELING UNILATERAL CONTACT

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The purpose of this paper is to extend the mortar finite element method to handle the unilateral contact model between two deformable bodies. The corresponding variational inequality is approximated using finite element meshes which do not fit on the contact zone. The mortar technique allows to match these independent discretizations of each solid and takes into account the unilateral contact conditions in a convenient way. By using an adaptation of Falk's lemma and a bootstrap argument, we give an upper bound of the convergence rate similar to the one already obtained for compatible meshes.

1. Introduction and Notations

In mechanics, the problems of contact between deformable bodies occur in many applications. A great interest shows itself in the elaboration of numerical algorithms taking into account the contact constraints in an efficient way. The numerical solution was considered for instance by Kikuchi and Oden,[20] Zhong,[27]. The mathematical framework for such problems consists of a variational inequality whose approximation has been discussed by many authors. In particular, the analysis of the low order finite element method was studied by Kikuchi and Oden,[20] in the case of one deformable body in contact with a rigid support (Signorini problem). Haslinger, Hlaváček and Nečas,[17] considered the case of two deformable bodies with matching meshes across the contact surface.

The mortar element domain decomposition method introduced by Bernardi, Maday and Patera,[9] seems to fit naturally to contact problems. Indeed, this technique offers a great facility for coupling different variational approximations and, in consequence, for using meshes that do not match at the interfaces of the solids. Moreover, this method allows to consider independent discretizations within each body that are well adapted to their particularities (geometries, loadings, constitutive equations, etc . . .). In many realistic configurations, the mortar concept leads to a significant reduction of the engineering time devoted to the generation of meshes because it allows to build globally unstructured/locally structured meshes.

In the theoretical area, a lot of mathematical results can be found providing the optimality of the mortar approximation in a spectral and finite element framework. The second order following variational problems have been studied: the Poisson problem (Bernardi, Maday and Patera,[9] Ben Belgacem,[6]), Stokes equations (Debit and Maday,[12]), the linearized elasticity system (Le Tallec and Sassi,[22]) and, more recently, the inversion of Maxwell's operator (Ben Abdallah, Ben Belgacem and Maday,[3]) or the bilateral contact problem (Ben Belgacem, Hild and Laborde,[5]). The method is all the more attractive and competitive because it leads to an interesting speed up in a parallel implementation (Ben Belgacem and Maday,[4]). Numerical experiments confirm the effectiveness of such a technique, especially in Computational Fluids Dynamics (Achdou and Pironneau,[2] Mavriplis,[24] for instance).

So far the method has been applied to partial differential equations. In the present paper, we extend the mortar finite element procedure to approximate the variational inequality modeling unilateral contact.

The paper is organized as follows. In the second section, we introduce the model describing the unilateral contact without friction between two deformable elastic bodies. The associated weak formulation is exhibited.

Then, in the third section, we consider a finite element method to solve the problem using independent meshes within each body. The discrete unilateral contact conditions constitute the key point of the approximation model. On the contact zone, we choose the trace of one of the two meshes and we consider the continuous piecewise affine functions defined on this discretization of the interface. The unilateral contact conditions are then approximated using a projection operator onto this interface finite element space. Such a formulation allows us to obtain a simple discrete contact condition.

In the fourth section, we carry out a numerical analysis of the approximation method. We provide an adaptation of Falk's lemma that will be used with a bootstrap argument to derive an upper bound of the convergence rate. In the case of incompatible meshes, the mortar method gives the same order of convergence as Haslinger and Hlaváček,[16] Haslinger, Hlaváček and Nečas,[17] for compatible meshes.

First of all, we specify some notations we shall use. Let a Lipschitz domain $\Omega \subset \mathbb{R}^2$ be given ; the generic point of Ω is denoted $\boldsymbol{x}$. The classical Lebesgue space of square integrable functions $L^2(\Omega)$ is endowed with the inner product:

$$(\varphi, \psi) = \int_\Omega \varphi\psi \, d\boldsymbol{x}.$$

We will make a constant use of the standard Sobolev space $H^m(\Omega)$, $m \geq 1$, provided with the norm:

$$\|\psi\|_{H^m(\Omega)} = \Big(\sum_{0 \leq |\alpha| \leq m} \|\partial^\alpha \psi\|^2_{L^2(\Omega)} \Big)^{\frac{1}{2}},$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi–index in $\mathbb{N}^2$ and the symbol ∂^α represents a partial derivative. We adopt the convention $H^0(\Omega) = L^2(\Omega)$. As in Grisvard,[14] (Definition

1.3.2.1) the fractionally Sobolev space $H^\tau(\Omega), \tau \in \mathbb{R}_+ \setminus \mathbb{N}$, is defined by the norm

$$\|\psi\|_{H^\tau(\Omega)} = \Big(\|\psi\|^2_{H^m(\Omega)} + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{(\partial^\alpha \psi(\boldsymbol{x}) - \partial^\alpha \psi(\boldsymbol{y}))^2}{|\boldsymbol{x}-\boldsymbol{y}|^{2+2\theta}} \, d\boldsymbol{x} \, d\boldsymbol{y} \Big)^{\frac{1}{2}},$$

where $\tau = m + \theta, m$ being the integer part of τ and $\theta \in]0,1[$. The closure in $H^\tau(\Omega)$ of $\mathcal{D}(\Omega)$ is denoted $H^\tau_0(\Omega)$, where $\mathcal{D}(\Omega)$ is the space of indefinitely differentiable functions whose support is contained in Ω.

As usually practised when dealing with finite element methods, we shall only consider here, polygonally shaped domains. This assumption is not restrictive and allows to avoid techniques required for the treatment of curved boundaries (Bernardi,[7] Haslinger and Hlaváček,[16] Haslinger, Hlaváček and Nečas,[17]) which is beyond the scope of this paper. The boundary $\partial\Omega$ is the union of a finite number of segments $\Gamma_j, 0 \leq j \leq J$. The common corner of Γ_j and Γ_{j+1} is $\boldsymbol{c}_j$. In such a case, the space $H^\tau(\Omega)$ defined above coincides not only with the set of restrictions to Ω of all functions of $H^\tau(\mathbb{R}^2)$ (see Ref.[14], Theorem 1.4.5.2) but also with the Sobolev space defined by hilbertian interpolation of standard spaces $(H^m(\Omega))_{m\in\mathbb{N}}$ and the norms resulting from the different definitions of $H^\tau(\Omega)$ are equivalent (see Triebel,[26]).

To handle trace functions we introduce, for any $\tau \in \mathbb{R}_+ \setminus \mathbb{N}$, the Hilbert space $H^\tau(\Gamma_j)$ associated with the norm

$$\|\psi\|_{H^\tau(\Gamma_j)} = \Big(\|\psi\|^2_{H^m(\Gamma_j)} + \int_{\Gamma_j} \int_{\Gamma_j} \frac{(\partial^m \psi(\boldsymbol{x}) - \partial^m \psi(\boldsymbol{y}))^2}{|\boldsymbol{x}-\boldsymbol{y}|^{1+2\theta}} \, d\Gamma d\Gamma \Big)^{\frac{1}{2}},$$

where m is the integer part of τ, θ its decimal part. In the previous integral, $\partial^m\psi$ stands for the m–order derivative of ψ along the segment Γ_j and $d\Gamma$ denotes the linear measure on Γ_j. The space $H^{-\tau}(\Gamma_j)$ stands for the topological dual space of $H^\tau(\Gamma_j)$. Following Adams,[1] (Theorem 7.48) the previous norm is equivalent to the norm corresponding to the definition of $H^\tau(\Gamma_j)$ by hilbertian interpolation of the spaces $H^{m+1}(\Gamma_j)$ and $H^m(\Gamma_j)$ with index $(1-\theta)$.

We shall also need to use the space $H^{\frac{1}{2}}_{00}(\Gamma_j)$, for any $j, 1 \leq j \leq J$. For this purpose, let us define the map ρ_j as the distance to the extreme points of Γ_j

$$\rho_j(\boldsymbol{x}) = dist \; (\boldsymbol{x}, \{\boldsymbol{c}_{j-1}, \boldsymbol{c}_j\}), \qquad \forall \boldsymbol{x} \in \Gamma_j.$$

Hence, the space $H^{\frac{1}{2}}_{00}(\Gamma_j)$ is assigned with the norm

$$\|\varphi\|_{H^{\frac{1}{2}}_{00}(\Gamma_j)} = \Big(\|\varphi\|^2_{H^{\frac{1}{2}}(\Gamma_j)} + \int_{\Gamma_j} \frac{\varphi(\boldsymbol{x})^2}{\rho_j(\boldsymbol{x})} \, d\Gamma \Big)^{\frac{1}{2}}.$$

This space is also obtained as the hilbertian interpolation of $H^1_0(\Gamma_j)$ and $L^2(\Gamma_j)$ with index $\frac{1}{2}$ (see Lions and Magenes,[23] Theorems 11.1 and 11.7).

Finally the trace operator $T : \psi \mapsto (\psi_{|\Gamma_j})_{1\leq j \leq J}$, maps continuously $H^\tau(\Omega)$ onto $\prod_{j=1}^J H^{\tau-\frac{1}{2}}(\Gamma_j)$ when $\tau > \frac{1}{2}$. The space $H^{\frac{1}{2}}(\partial\Omega)$ (i.e. the set involving the traces of all the functions of $H^1(\Omega)$) is the subspace of $\prod_{j=1}^J H^{\frac{1}{2}}(\Gamma_j)$ satisfying some specific integral matching conditions at the corners $\boldsymbol{c}_j$ (see Ref.[14], Theorem 1.5.2.3). The dual space of $H^{\frac{1}{2}}(\partial\Omega)$ is denoted $H^{-\frac{1}{2}}(\partial\Omega)$.

Bold Latin letters like $\boldsymbol{u}, \boldsymbol{v}$, indicate vector quantities, while the capital ones (e.g. $\boldsymbol{V}, \boldsymbol{K}, \ldots$) are functional sets involving vector fields. The symbol σ stands for the stress tensor and ε is the strain tensor.

2. Setting of the Problem and Weak Formulation

We consider the deformation of two elastic bodies occupying, in the initial unconstrained configuration, two subsets $\overline{\Omega}^\ell$ of the space $\mathbb{R}^2$, $\ell = 1, 2$. The domain Ω^ℓ is only on one side of its boundary denoted $\partial\Omega^\ell$. The latter is smooth enough and consists of $\Gamma^\ell_{\boldsymbol{u}}, \Gamma^\ell_{\boldsymbol{g}}$ and Γ^ℓ_c. The body $\overline{\Omega}^\ell$ is fixed along $\Gamma^\ell_{\boldsymbol{u}}$ and subjected to surface traction forces $\boldsymbol{g}^\ell \in (L^2(\Gamma^\ell_{\boldsymbol{g}}))^2$ on $\Gamma^\ell_{\boldsymbol{g}}$; the body forces are denoted $\boldsymbol{f}^\ell \in (L^2(\Omega^\ell))^2$. In the initial configuration, both bodies have a common portion $\Gamma_c = \Gamma^1_c = \Gamma^2_c$ which will be considered as the candidate contact surface for the sake of simplicity. In other words, the contact zone cannot enlarge during the deformation process (Haslinger and Hlaváček,[15] Haslinger, Hlaváček and Nečas,[17]). The contact is assumed to be frictionless and will be effective on a portion of Γ_c that is not known in advance. The measure of $\Gamma^\ell_{\boldsymbol{u}}$ does not vanish and the outward unit normal vector of $\partial\Omega^\ell$ is denoted $\boldsymbol{n}^\ell$.

The unilateral contact problem consists of finding the displacement field $\boldsymbol{u} = (\boldsymbol{u}^\ell)_\ell = (\boldsymbol{u}|_{\Omega^1}, \boldsymbol{u}|_{\Omega^2})$, and the stress tensor field $\sigma = (\sigma^\ell)_\ell$ satisfying the following conditions (2.1)–(2.5) for $\ell = 1, 2$:

$$\begin{aligned}
\boldsymbol{div}\, \sigma^\ell(\boldsymbol{u}^\ell) + \boldsymbol{f}^\ell &= 0 \qquad \text{in } \Omega^\ell, \\
\sigma^\ell(\boldsymbol{u}^\ell)\boldsymbol{n}^\ell - \boldsymbol{g}^\ell &= 0 \qquad \text{on } \Gamma^\ell_{\boldsymbol{g}}, \\
\boldsymbol{u}^\ell &= 0 \qquad \text{on } \Gamma^\ell_{\boldsymbol{u}}.
\end{aligned} \tag{2.1}$$

The symbol $\boldsymbol{div}$ denotes the divergence operator of a tensor function and is defined as

$$\boldsymbol{div}\, \sigma = \Big(\frac{\partial \sigma_{ij}}{\partial x_j}\Big)_i.$$

The summation convention of repeated indices is adopted. The stress tensor is linked to the displacement by the constitutive law of linear elasticity

$$\sigma^\ell(\boldsymbol{u}^\ell) = A^\ell(\boldsymbol{x})\, \varepsilon(\boldsymbol{u}^\ell), \tag{2.2}$$

where $A^\ell(\boldsymbol{x}) = (a^\ell_{ij,kh}(\boldsymbol{x}))_{1\le i,j,k,h\le 2} \in (L^\infty(\Omega^\ell))^{16}$ is a fourth order tensor satisfying the usual symmetry and ellipticity conditions in Elasticity.

In the sequel, when no confusion may occur, we shall simply write σ^ℓ instead of $\sigma^\ell(\boldsymbol{u}^\ell)$. The situations we are investigating are restricted to infinitesimal deformations (small perturbations hypothesis). In such a case the nonlinear (quadratic) term is neglected, so that the strain tensor $\varepsilon(\boldsymbol{v})$ produced by a displacement field $\boldsymbol{v}$ is given by

$$\varepsilon(\boldsymbol{v}) = \frac{1}{2}(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T),$$

where the symbol ^{T} indicates a transposition.

Finally, we give the following contact conditions on Γ_c:

$$(\sigma^1 \boldsymbol{n}^1).\boldsymbol{n}^1 = (\sigma^2 \boldsymbol{n}^2).\boldsymbol{n}^2 = \sigma_{\boldsymbol{n}}, \tag{2.3}$$

$$[\boldsymbol{u}.\boldsymbol{n}] \leq 0, \qquad \sigma_{\boldsymbol{n}} \leq 0, \qquad \sigma_{\boldsymbol{n}}[\boldsymbol{u}.\boldsymbol{n}] = 0, \tag{2.4}$$

$$\sigma^1_{\boldsymbol{t}} = \sigma^2_{\boldsymbol{t}} = 0. \tag{2.5}$$

The notation $[\boldsymbol{u}.\boldsymbol{n}]$ represents the jump $(\boldsymbol{u}^1.\boldsymbol{n}^1 + \boldsymbol{u}^2.\boldsymbol{n}^2)$ of the normal displacement across the contact zone Γ_c and $\sigma^\ell_{\boldsymbol{t}} = \sigma^\ell \boldsymbol{n}^\ell - \sigma_{\boldsymbol{n}} \boldsymbol{n}^\ell$ is the tangential constraint. Conditions (2.4) allow each body to leave the other one on a portion of Γ_c: the contact is unilateral between the two solids. Equation (2.3) expresses the action and the reaction principle and finally (2.5) represents a contact without friction.

The subsequent study is based on an equivalent variational formulation (related to the virtual work principle) which gives a mathematical sense to the previous formal equations. To this end, let us define the vector spaces

$$\boldsymbol{V}(\Omega^\ell) = \Big\{ \boldsymbol{v}^\ell \in \big(H^1(\Omega^\ell)\big)^2, \qquad \boldsymbol{v}^\ell = 0 \text{ on } \Gamma^\ell_{\boldsymbol{u}} \Big\}.$$

The current vector field of the product space $\boldsymbol{V}(\Omega^1) \times \boldsymbol{V}(\Omega^2)$ is denoted (as previously indicated) $\boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2)$. This space is endowed with the Hilbertian inner product:

$$(\boldsymbol{v}, \boldsymbol{w})_* = (\boldsymbol{v}^1, \boldsymbol{w}^1)_{(H^1(\Omega^1))^2} + (\boldsymbol{v}^2, \boldsymbol{w}^2)_{(H^1(\Omega^2))^2},$$

for all $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}(\Omega^1) \times \boldsymbol{V}(\Omega^2)$. The associated norm is denoted $\|.\|_*$. Then, the appropriate closed convex set $\boldsymbol{K}$ of admissible displacements is contained in $\boldsymbol{V}(\Omega^1) \times \boldsymbol{V}(\Omega^2)$ and incorporates the contact condition,

$$\boldsymbol{K} = \Big\{ \boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2) \in \boldsymbol{V}(\Omega^1) \times \boldsymbol{V}(\Omega^2), \qquad [\boldsymbol{v}.\boldsymbol{n}] \leq 0 \text{ a.e. on } \Gamma_c \Big\}.$$

Green's formula applied to problem (2.1)–(2.5) leads to the variational inequality: *find* $\boldsymbol{u} \in \boldsymbol{K}$ *such that*

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) \geq L(\boldsymbol{v} - \boldsymbol{u}), \qquad \forall \boldsymbol{v} \in \boldsymbol{K}. \tag{2.6}$$

In (2.6), we set:

$$a(\boldsymbol{u}, \boldsymbol{v}) = \sum_{\ell=1}^{2} \int_{\Omega^\ell} A^\ell(\boldsymbol{x})\, \varepsilon(\boldsymbol{u}^\ell).\, \varepsilon(\boldsymbol{v}^\ell)\, d\boldsymbol{x},$$

$$L(\boldsymbol{v}) = \sum_{\ell=1}^{2} \Big(\int_{\Omega^\ell} \boldsymbol{f}^\ell.\boldsymbol{v}^\ell\, d\boldsymbol{x} + \int_{\Gamma^\ell_{\boldsymbol{g}}} \boldsymbol{g}^\ell.\boldsymbol{v}^\ell\, d\Gamma^\ell \Big),$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}(\Omega^1) \times \boldsymbol{V}(\Omega^2)$. The symmetrical bilinear form $a(.,.)$ is coercive (via Korn's inequality) and continuous on $\boldsymbol{V}(\Omega^1) \times \boldsymbol{V}(\Omega^2)$. Moreover, the linear form $L(.)$ is continuous on $\boldsymbol{V}(\Omega^1) \times \boldsymbol{V}(\Omega^2)$.

Inequality (2.6) was widely studied in convex analysis and optimization theory. The existence and uniqueness of $\boldsymbol{u} \in \boldsymbol{K}$ solution of problem (2.6) results from Stampacchia's theorem.

3. The Mortar Finite Element Approximation

Towards the construction of the discrete finite element space based on the mortar concept, the first step consists of recalling some fundamental tools and results of the classical finite element approximation theory.

Let us recall that the bodies Ω^ℓ are polygonally shaped, $\ell = 1, 2$, and assume that Γ_c is a straight line segment to simplify. Let the approximation parameter $h = (h_1, h_2)$ be a given pair of real positive numbers that will decay to 0. With each Ω^ℓ, we associate a regular family of triangulations T_h^ℓ, whose elements are triangles denoted κ, the diameter of which does not exceed h_ℓ. We can write

$$\overline{\Omega}^\ell = \bigcup_{\kappa \in T_h^\ell} \overline{\kappa}.$$

The extension to rectangular elements is straightforward owing to some slight modifications. The extreme points $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ of the contact zone Γ_c can be common nodes of the meshes on both bodies. The contact zone Γ_c inherits two independent families of discretizations associated with T_h^1 and T_h^2. The mesh $\mathcal{T}_{c,h}^\ell$ on Γ_c is defined as the set of all the edges of $\kappa \in T_h^\ell$ on the contact zone. The set of the nodes associated with $\mathcal{T}_{c,h}^\ell$ is denoted ξ^ℓ. In general ξ^1 and ξ^2 are not identical because of the non compatibility of the meshes.

Denote $\mathbb{P}_1(\kappa)$ the space of the polynomials on κ whose global degree is lower or equal to one. With any κ, we associate the finite set Ξ_κ of the vertices of κ, so that $(\kappa, \mathbb{P}_1(\kappa), \Xi_\kappa)$ is a finite element of Lagrange type. The finite element space used in Ω^ℓ is then defined as

$$\boldsymbol{V}_h(\Omega^\ell) = \Big\{ \boldsymbol{v}_h^\ell \in (\mathcal{C}(\overline{\Omega}^\ell))^2, \quad \forall \kappa \in T_h^\ell, \quad \boldsymbol{v}_h^\ell|_\kappa \in (\mathbb{P}_1(\kappa))^2, \quad \boldsymbol{v}_h^\ell|_{\Gamma_{\boldsymbol{u}}^\ell} = 0 \Big\}.$$

If $\mathcal{I}_h^\ell$ stands for for the Lagrange interpolation operator ranging in $\boldsymbol{V}_h(\Omega^\ell)$, we have the following error estimate obtained from Ciarlet,[11] by hilbertian interpolation: for any couple of real numbers $(\mu, \nu) \in [0, 1] \times]1, 2]$, there exists a constant $C = C(\mu, \nu)$ satisfying:

$$\|\boldsymbol{v}^\ell - \mathcal{I}_h^\ell \boldsymbol{v}^\ell\|_{(H^\mu(\Omega^\ell))^2} \le C(\mu, \nu) h_\ell^{\nu-\mu} \|\boldsymbol{v}^\ell\|_{(H^\nu(\Omega^\ell))^2}, \qquad \forall \boldsymbol{v}^\ell \in (H^\nu(\Omega^\ell))^2, \quad (3.1)$$

with the convention $H^0 = L^2$. The interpolation property (3.1) seems to be false for negative exponents μ. The mortar element method was born precisely because such results were missing (Bernardi, Debit and Maday,[8]).

To express the contact constraints (2.4), we need to introduce some functional spaces over Γ_c. Let $W_h^\ell(\Gamma_c)$ be the range of $\boldsymbol{V}_h(\Omega^\ell)$ by the normal traces operator on Γ_c:

$$W_h^\ell(\Gamma_c) = \Big\{ \varphi_h = \boldsymbol{v}_h^\ell|_{\Gamma_c} . \boldsymbol{n}^\ell, \qquad \boldsymbol{v}_h^\ell \in \boldsymbol{V}_h(\Omega^\ell) \ \Big\},$$

which is called the mortar space. We introduce the space of the Lagrange multipliers that will be useful to express in a weak sense the contact conditions:

$$M_h^\ell(\Gamma_c) = \Big\{ \psi_h \in W_h^\ell(\Gamma_c), \quad \psi_h|_T \in \mathbb{P}_0(T), \quad \forall T \in \mathcal{T}_{c,h}^\ell, \text{ s.t. } \boldsymbol{c}_1 \text{ or } \boldsymbol{c}_2 \in T \Big\}.$$

Next, π_h^ℓ stands for the projection operator on $W_h^\ell(\Gamma_c)$ defined for any function $\varphi \in \mathcal{C}(\overline{\Gamma}_c)$ as

$$\begin{aligned} \pi_h^\ell \varphi &\in W_h^\ell(\Gamma_c), \\ (\pi_h^\ell \varphi)(\boldsymbol{c}_i) &= \varphi(\boldsymbol{c}_i) \qquad \text{for } i = 1 \text{ and } 2, \\ \int_{\Gamma_c} (\varphi - \pi_h^\ell \varphi)\psi_h \, d\Gamma &= 0 \qquad\qquad \forall \psi_h \in M_h^\ell(\Gamma_c). \end{aligned} \tag{3.2}$$

We suppose that the (1D) family of triangulations $\mathcal{T}_{c,h}^\ell$ are uniformly regular so that the inverse inequalities in the Sobolev spaces are available (Ciarlet,[11]). The properties of π_h^ℓ are enumerated in Ben Belgacem,[6] and we just recall them.

Lemma 3.1 *The projection operator π_h^ℓ ranges continuously $H_{00}^{\frac12}(\Gamma_c)$ into $H_{00}^{\frac12}(\Gamma_c)$ i.e.*

$$\|\pi_h^\ell \varphi\|_{H_{00}^{\frac12}(\Gamma_c)} \le C \|\varphi\|_{H_{00}^{\frac12}(\Gamma_c)}, \qquad \forall \varphi \in H_{00}^{\frac12}(\Gamma_c), \tag{3.3}$$

and satisfies the following error estimate. Let $\frac12 < \mu \le 2$, then

$$\|\varphi - \pi_h^\ell \varphi\|_{H_{00}^{\frac12}(\Gamma_c)} \le C' h_\ell^{\mu - \frac12} \|\varphi\|_{H^\mu(\Gamma_c)}, \qquad \forall \varphi \in H^\mu(\Gamma_c). \tag{3.4}$$

Both constants C and C' are independent of h_ℓ.

We are in a position to define the discrete admissibility convex cone $\boldsymbol{K}_h$:

$$\boldsymbol{K}_h = \Big\{ \boldsymbol{v}_h = (\boldsymbol{v}_h^1, \boldsymbol{v}_h^2) \in \boldsymbol{V}_h(\Omega^1) \times \boldsymbol{V}_h(\Omega^2), \quad \boldsymbol{v}_h^1.\boldsymbol{n}^1 + \pi_h^1(\boldsymbol{v}_h^2.\boldsymbol{n}^2) \le 0 \text{ on } \Gamma_c \Big\}.$$

Notice that the condition incorporated in $\boldsymbol{K}_h$ makes sense because it is expressed in the space $W_h^1(\Gamma_c)$. Following the terminology of Bernardi, Maday and Patera,[9], $W_h^2(\Gamma_c)$ stands for the mortar space. Of course, it is possible to give a symmetrical definition of the convex by taking as mortar space $W_h^1(\Gamma_c)$ and using the projection π_h^2. Besides, it is straightforward to see that $\boldsymbol{K}_h \not\subset \boldsymbol{K}$, then the approximation is not "Hodge" conforming.

When compatible meshes are used, the discrete contact constraints can be expressed merely by the natural pointwise condition $[\boldsymbol{v}_h.\boldsymbol{n}] \le 0$ and the approximation becomes conforming ($\boldsymbol{K}_h \subset \boldsymbol{K}$). This situation was extensively studied by Haslinger and Hlaváček,[16] Haslinger, Hlaváček and Nečas,[17].

The finite element problem issued from (2.6) is the following variational inequality: *find $\boldsymbol{u}_h \in \boldsymbol{K}_h$ such that*

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) \ge L(\boldsymbol{v}_h - \boldsymbol{u}_h), \qquad \forall \boldsymbol{v}_h \in \boldsymbol{K}_h. \tag{3.5}$$

Using again Stamppachia's theorem, we conclude to the existence and uniqueness of the solution $\boldsymbol{u}_h \in \boldsymbol{K}_h$ which satisfies the stability condition

$$\|\boldsymbol{u}_h\|_* \leq C \sum_{\ell=1}^{2} \Big(\|\boldsymbol{f}^\ell\|_{(L^2(\Omega^\ell))^2} + \|\boldsymbol{g}^\ell\|_{(L^2(\Gamma_g^\ell))^2} \Big). \tag{3.6}$$

4. Error Estimation

In this section, we give an upper bound of the error resulting from the previous finite element approximation.

4.1. *The basic tool*

The starting point is an adaptation of Falk's lemma (Falk,[13] Ciarlet,[11] Haslinger, Hlaváček and Nečas,[17]); this result is given in the following lemma.

Lemma 4.1 *Assume that the solution $\boldsymbol{u} \in \boldsymbol{K}$ of problem* (2.6) *is such that $\boldsymbol{u}^1 \in (H^\nu(\Omega^1))^2$ and $\boldsymbol{u}^2 \in (H^\nu(\Omega^2))^2$ with $\nu > \frac{3}{2}$. Let $\boldsymbol{u}_h \in \boldsymbol{K}_h$ be the solution of* (3.5). *Then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_* \leq C \Big\{ \inf_{\boldsymbol{v}_h \in \boldsymbol{K}_h} \Big(\|\boldsymbol{u} - \boldsymbol{v}_h\|_* + \Big| \int_{\Gamma_c} \sigma_{\boldsymbol{n}} [\boldsymbol{v}_h.\boldsymbol{n}] \, d\Gamma \Big|^{\frac{1}{2}} \Big) + \inf_{\boldsymbol{v} \in \boldsymbol{K}} \Big| \int_{\Gamma_c} \sigma_{\boldsymbol{n}} [(\boldsymbol{v} - \boldsymbol{u}_h).\boldsymbol{n}] \, d\Gamma \Big|^{\frac{1}{2}} \Big\}.$$

Proof. Let α be the ellipticity constant of the bilinear form $a(.,.)$ on $\boldsymbol{V}(\Omega^1) \times \boldsymbol{V}(\Omega^2)$. We have

$$\alpha \|\boldsymbol{u} - \boldsymbol{u}_h\|_*^2 \leq a(\boldsymbol{u}, \boldsymbol{u}) - a(\boldsymbol{u}, \boldsymbol{u}_h) - a(\boldsymbol{u}_h, \boldsymbol{u}) + a(\boldsymbol{u}_h, \boldsymbol{u}_h).$$

Then, noticing that:

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{u}) &\leq a(\boldsymbol{u}, \boldsymbol{v}) - L(\boldsymbol{v} - \boldsymbol{u}), \qquad \forall \boldsymbol{v} \in \boldsymbol{K}, \\ a(\boldsymbol{u}_h, \boldsymbol{u}_h) &\leq a(\boldsymbol{u}_h, \boldsymbol{v}_h) - L(\boldsymbol{v}_h - \boldsymbol{u}_h), \qquad \forall \boldsymbol{v}_h \in \boldsymbol{K}_h, \end{aligned}$$

we deduce the following inequality

$$\begin{aligned} \alpha \|\boldsymbol{u} - \boldsymbol{u}_h\|_*^2 \leq a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}_h) &- L(\boldsymbol{v} - \boldsymbol{u}_h) + a(\boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{u}) \\ &- L(\boldsymbol{v}_h - \boldsymbol{u}) + a(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{u}). \end{aligned} \tag{4.1}$$

Applying twice Green's formula gives on the one side

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}_h) - L(\boldsymbol{v} - \boldsymbol{u}_h) = \int_{\Gamma_c} \sigma_{\boldsymbol{n}} [(\boldsymbol{v} - \boldsymbol{u}_h).\boldsymbol{n}] \, d\Gamma,$$

and on the other side, thanks to $\sigma_{\boldsymbol{n}}[\boldsymbol{u}.\boldsymbol{n}] = 0$,

$$a(\boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{u}) - L(\boldsymbol{v}_h - \boldsymbol{u}) = \int_{\Gamma_c} \sigma_{\boldsymbol{n}} [(\boldsymbol{v}_h - \boldsymbol{u}).\boldsymbol{n}] \, d\Gamma = \int_{\Gamma_c} \sigma_{\boldsymbol{n}} [\boldsymbol{v}_h.\boldsymbol{n}] \, d\Gamma.$$

Observing that when M is the norm of $a(.,.)$, we derive

$$\begin{aligned} a(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{u}) &\leq M \|\boldsymbol{u}_h - \boldsymbol{u}\|_* \|\boldsymbol{v}_h - \boldsymbol{u}\|_* \\ &\leq M\big(\frac{\alpha}{2M}\|\boldsymbol{u}_h - \boldsymbol{u}\|_*^2 + \frac{M}{2\alpha}\|\boldsymbol{v}_h - \boldsymbol{u}\|_*^2\big). \end{aligned}$$

The proof is achieved by using this inequality in (4.1). □

Remark 4.1 In the previous statement, we recognize the approximation error

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{K}_h} \Big(\|\boldsymbol{u} - \boldsymbol{v}_h\|_* + \Big| \int_{\Gamma_c} \sigma_{\boldsymbol{n}} [\boldsymbol{v}_h.\boldsymbol{n}] \; d\Gamma \Big|^{\frac{1}{2}} \Big).$$

The boundary integral is due to the nature of the problem and does not disappear even when the approximation is conforming. The consistency error is represented by

$$\inf_{\boldsymbol{v} \in \boldsymbol{K}} \Big| \int_{\Gamma_c} \sigma_{\boldsymbol{n}} [(\boldsymbol{v} - \boldsymbol{u}_h).\boldsymbol{n}] \; d\Gamma \Big|^{\frac{1}{2}}.$$

Indeed, this latter expression is generated by the non conformity of the finite element method. Otherwise, for matching meshes on the contact zone, we have $\boldsymbol{K}_h \subset \boldsymbol{K}$ and the consistency error disappears.

Remark 4.2 The proof without regularity assumptions of the previous lemma would require to consider functions belonging to dual Sobolev spaces (typically $H^{-\frac{1}{2}}(\partial\Omega^1)$ and $H^{-\frac{1}{2}}(\partial\Omega^2)$) whose non local character is well–known. To skip over such a technical concern, and keeping in mind that the estimate results issued in an important number of papers are proven from a H^2 regularity condition, we make weaker smoothness assumptions on the displacement field $\boldsymbol{u}$, see the discussion to the end of the section.

4.2. *Analysis of the best approximation error*

We attempt to derive an upper bound of the first infimum in Lemma 4.1.

Lemma 4.2 *Let* $\boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2) \in \boldsymbol{K}$ *such that* $\boldsymbol{v}^1 \in (H^\nu(\Omega^1))^2$ *and* $\boldsymbol{v}^2 \in (H^\nu(\Omega^2))^2$ *with* $\frac{3}{2} < \nu \leq 2$. *Assume that the normal stresses defined in* (2.3) *on the contact zone* Γ_c *are identical:*

$$(\sigma^\ell(\boldsymbol{v}^\ell)\boldsymbol{n}^\ell).\boldsymbol{n}^\ell = \sigma_{\boldsymbol{n}}(\boldsymbol{v}), \qquad \ell = 1, 2.$$

Then, there exists $\boldsymbol{v}_h \in \boldsymbol{K}_h$ *that satisfies the estimates*

$$\|\boldsymbol{v} - \boldsymbol{v}_h\|_* \leq C(\boldsymbol{v})(h_1^{\nu-1} + h_2^{\nu-1}), \tag{4.2}$$

$$\Big| \int_{\Gamma_c} \sigma_{\boldsymbol{n}}(\boldsymbol{v}) [(\boldsymbol{v}_h - \boldsymbol{v}).\boldsymbol{n}] \; d\Gamma \Big|^{\frac{1}{2}} \leq C(\boldsymbol{v})(h_1^{\frac{\nu}{2}-\frac{1}{4}} + h_2^{\nu-1}), \tag{4.3}$$

where the constant $C(\boldsymbol{v})$ *depends linearly on* $\|\boldsymbol{v}^1\|_{(H^\nu(\Omega^1))^2}$ *and* $\|\boldsymbol{v}^2\|_{(H^\nu(\Omega^2))^2}$.

Proof. (i) Let $\boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2) \in \boldsymbol{K}$ satisfying the regularity assumptions of the lemma. By classical trace theorems in Sobolev spaces, it comes out that $[\boldsymbol{v}.\boldsymbol{n}] \in H^{\nu-\frac{1}{2}}(\Gamma_c)$. Using expansion operators, we can build up a vector field $\boldsymbol{r} \in (H^\nu(\Omega^1))^2 \cap \boldsymbol{V}(\Omega^1)$ such that $\boldsymbol{r}.\boldsymbol{n}^1 = [\boldsymbol{v}.\boldsymbol{n}]$ on Γ_c, and verifying the stability relation

$$\|\boldsymbol{r}\|_{(H^\nu(\Omega^1))^2} \leq C\|[\boldsymbol{v}.\boldsymbol{n}]\|_{H^{\nu-\frac{1}{2}}(\Gamma_c)} \leq C(\|\boldsymbol{v}^1\|_{(H^\nu(\Omega^1))^2} + \|\boldsymbol{v}^2\|_{(H^\nu(\Omega^2))^2}).$$

Then, setting $\boldsymbol{w} = (\boldsymbol{w}^1, \boldsymbol{w}^2) = (\boldsymbol{v}^1 - \boldsymbol{r}, \boldsymbol{v}^2) \in \boldsymbol{V}(\Omega^1) \times \boldsymbol{V}(\Omega^2)$, it is clear that $[\boldsymbol{w}.\boldsymbol{n}] = 0$ on Γ_c and

$$\boldsymbol{w} \in (H^\nu(\Omega^1))^2 \times (H^\nu(\Omega^2))^2.$$

We are therefore in a similar situation to that of bilateral contact. Processing like in Ref.[5] we approximate $\boldsymbol{w}$ by $\boldsymbol{w}_h = (\boldsymbol{w}_h^1, \boldsymbol{w}_h^2) \in \boldsymbol{V}_h(\Omega^1) \times \boldsymbol{V}_h(\Omega^2)$ satisfying

$$\boldsymbol{w}_h^1.\boldsymbol{n}^1 + \pi_h^1(\boldsymbol{w}_h^2.\boldsymbol{n}^2) = 0,$$

and the estimate

$$\|\boldsymbol{w} - \boldsymbol{w}_h\|_* \leq C(\boldsymbol{v})(h_1^{\nu-1} + h_2^{\nu-1}),$$

with $C(\boldsymbol{v})$ depending linearly on $\|\boldsymbol{v}^1\|_{(H^\nu(\Omega^1))^2}$ and $\|\boldsymbol{v}^2\|_{(H^\nu(\Omega^2))^2}$. To achieve, we define

$$\boldsymbol{v}_h = (\boldsymbol{v}_h^1, \boldsymbol{v}_h^2) = (\boldsymbol{w}_h^1 + \mathcal{I}_h^1\boldsymbol{r}, \boldsymbol{w}_h^2).$$

Because $(\mathcal{I}_h^1\boldsymbol{r}).\boldsymbol{n}^1$ has the same sign as $(\boldsymbol{r}.\boldsymbol{n}^1) = [\boldsymbol{v}.\boldsymbol{n}]$, thus negative, it comes out that

$$\boldsymbol{v}_h^1.\boldsymbol{n}^1 + \pi_h^1(\boldsymbol{v}_h^2.\boldsymbol{n}^2) = (\mathcal{I}_h^1\boldsymbol{r}).\boldsymbol{n}^1 \leq 0.$$

It follows that $\boldsymbol{v}_h$ belongs to $\boldsymbol{K}_h$. Besides, thanks to (3.1), we have

$$\begin{aligned}
\|\boldsymbol{v} - \boldsymbol{v}_h\|_* &\leq \|\boldsymbol{w} - \boldsymbol{w}_h\|_* + \|\boldsymbol{r} - \mathcal{I}_h^1\boldsymbol{r}\|_{(H^1(\Omega^1))^2} \\
&\leq C(\boldsymbol{v})(h_1^{\nu-1} + h_2^{\nu-1}) + Ch_1^{\nu-1}\|\boldsymbol{r}\|_{(H^\nu(\Omega^1))^2} \\
&\leq C(\boldsymbol{v})(h_1^{\nu-1} + h_2^{\nu-1}),
\end{aligned}$$

which yields the first estimate (4.2) of the lemma.

(ii) To determine an upper bound of the integral term, we write

$$\int_{\Gamma_c} \sigma_{\boldsymbol{n}}(\boldsymbol{v})[(\boldsymbol{v}_h - \boldsymbol{v}).\boldsymbol{n}]\, d\Gamma = \int_{\Gamma_c} \sigma_{\boldsymbol{n}}(\boldsymbol{v})[\boldsymbol{w}_h.\boldsymbol{n}]\, d\Gamma + \int_{\Gamma_c} \sigma_{\boldsymbol{n}}(\boldsymbol{v})((\mathcal{I}_h^1\boldsymbol{r}).\boldsymbol{n}^1 - \boldsymbol{r}.\boldsymbol{n}^1)\, d\Gamma.$$

The first term is evaluated in a standard way like in Ref.[9], and yields

$$\Big|\int_{\Gamma_c} \sigma_{\boldsymbol{n}}(\boldsymbol{v})[\boldsymbol{w}_h.\boldsymbol{n}]\, d\Gamma\Big| \leq C(\boldsymbol{v})(h_1^{2(\nu-1)} + h_2^{2(\nu-1)}).$$

The second one is handled as follows

$$\begin{aligned}
\Big|\int_{\Gamma_c} \sigma_{\boldsymbol{n}}(\boldsymbol{v})((\mathcal{I}_h^1\boldsymbol{r}).\boldsymbol{n}^1 - \boldsymbol{r}.\boldsymbol{n}^1)\, d\Gamma\Big| &\leq \|\sigma_{\boldsymbol{n}}(\boldsymbol{v})\|_{L^2(\Gamma_c)}\|(\mathcal{I}_h^1\boldsymbol{r}).\boldsymbol{n}^1 - \boldsymbol{r}.\boldsymbol{n}^1\|_{L^2(\Gamma_c)} \\
&\leq C(\boldsymbol{v})h_1^{\nu-\frac{1}{2}}\|\sigma_{\boldsymbol{n}}(\boldsymbol{v})\|_{L^2(\Gamma_c)}\|\boldsymbol{r}.\boldsymbol{n}^1\|_{H^{\nu-\frac{1}{2}}(\Gamma_c)} \\
&\leq C(\boldsymbol{v})h_1^{\nu-\frac{1}{2}}.
\end{aligned}$$

Hence the lemma. □

The estimate given in (4.3) is not optimal, due to the fact that the interpolation operator does not provide good approximation results with respect to Sobolev norms with negative exponents. In the forthcoming study, we prove that this estimate slows down the convergence rate of the method.

4.3. *First estimate of the consistency error*

We are interested in providing a first (rough) bound of the consistency error that will be used as a starting point of the study of the global error estimate. Due to the use of an inverse inequality, we need to assume that the size of the meshes is such that the ratio h_1/h_2 is bounded.

Lemma 4.3 *Assume that the solution $\boldsymbol{u} \in \boldsymbol{K}$ of problem* (2.6) *is such that $\boldsymbol{u}^1 \in (H^\nu(\Omega^1))^2$ with $\frac{3}{2} < \nu \leq 2$ and let $\boldsymbol{u}_h \in \boldsymbol{K}_h$ be the solution of* (3.5). *Then*

$$\inf_{\boldsymbol{v}\in\boldsymbol{K}} \Big| \int_{\Gamma_c} \sigma_{\boldsymbol{n}}(\boldsymbol{u})[(\boldsymbol{v}-\boldsymbol{u}_h).\boldsymbol{n}]\, d\Gamma \Big|^{\frac{1}{2}} \leq C(\boldsymbol{u}) h_1^{\frac{\nu}{2}-\frac{1}{2}},$$

where the constant C depends only on $\|\boldsymbol{u}^1\|_{(H^\nu(\Omega^1))^2}$.

Proof. Choosing $\boldsymbol{v} \in \boldsymbol{K}$ such that

$$\boldsymbol{v}^1.\boldsymbol{n}^1|_{\Gamma_c} = \boldsymbol{u}_h^1.\boldsymbol{n}^1|_{\Gamma_c} \quad \text{and} \quad \boldsymbol{v}^2.\boldsymbol{n}^2|_{\Gamma_c} = \pi_h^1(\boldsymbol{u}_h^2.\boldsymbol{n}^2)|_{\Gamma_c},$$

which is possible by using the extension operators, we have:

$$\int_{\Gamma_c} \sigma_{\boldsymbol{n}}[(\boldsymbol{v}-\boldsymbol{u}_h).\boldsymbol{n}]\, d\Gamma = \int_{\Gamma_c} \sigma_{\boldsymbol{n}}(\pi_h^1(\boldsymbol{u}_h^2.\boldsymbol{n}^2) - \boldsymbol{u}_h^2.\boldsymbol{n}^2)\, d\Gamma, \tag{4.4}$$

and then:

$$\int_{\Gamma_c} \sigma_{\boldsymbol{n}}[(\boldsymbol{v}-\boldsymbol{u}_h).\boldsymbol{n}]\, d\Gamma = \int_{\Gamma_c} (\sigma_{\boldsymbol{n}} - \psi_h)(\pi_h^1(\boldsymbol{u}_h^2.\boldsymbol{n}^2) - \boldsymbol{u}_h^2.\boldsymbol{n}^2)\, d\Gamma,$$

for all $\psi_h \in M_h^1(\Gamma_c)$. By duality, we obtain

$$\Big| \int_{\Gamma_c} \sigma_{\boldsymbol{n}}[(\boldsymbol{v}-\boldsymbol{u}_h).\boldsymbol{n}]\, d\Gamma \Big| \leq \inf_{\psi_h\in M_h^1(\Gamma_c)} \|\sigma_{\boldsymbol{n}} - \psi_h\|_{(H_{00}^{\frac{1}{2}}(\Gamma_c))'} \|\pi_h^1(\boldsymbol{u}_h^2.\boldsymbol{n}^2) - \boldsymbol{u}_h^2.\boldsymbol{n}^2\|_{H_{00}^{\frac{1}{2}}(\Gamma_c)}.$$

By using (3.4), we deduce, for any small positive ε

$$\Big| \int_{\Gamma_c} \sigma_{\boldsymbol{n}}[(\boldsymbol{v}-\boldsymbol{u}_h).\boldsymbol{n}]\, d\Gamma \Big| \leq C h_1^{\nu-1} \|\sigma_{\boldsymbol{n}}\|_{H^{\nu-\frac{3}{2}}(\Gamma_c)} h_1^{\varepsilon} \|\boldsymbol{u}_h^2.\boldsymbol{n}^2\|_{H^{\frac{1}{2}+\varepsilon}(\Gamma_c)}.$$

Using the inverse inequality implies

$$\Big| \int_{\Gamma_c} \sigma_{\boldsymbol{n}}[(\boldsymbol{v}-\boldsymbol{u}_h).\boldsymbol{n}]\, d\Gamma \Big| \leq C h_1^{\nu-1} \Big(\frac{h_1}{h_2}\Big)^{\varepsilon} \|\boldsymbol{u}^1\|_{(H^\nu(\Omega^1))^2} \|\boldsymbol{u}_h\|_{*}.$$

The result is achieved due to the stability of the discrete solution given by (3.6) and the assumption on the ratio h_1/h_2. □

4.4. *Bootstrap and global error estimate*

Up to now, the preliminary lemmas do not lead to the expected convergence rate. This is caused by the rough evaluation of the consistency error. However, combining these first results with a bootstrap procedure, we are able to improve the previous lemma and then to derive the following lemma. Using again this bootstrap argument, we obtain Theorem 4.1 hereafter.

Lemma 4.4 *Assume that the solution $\boldsymbol{u} \in \boldsymbol{K}$ of problem* (2.6) *is such that $\boldsymbol{u}^1 \in (H^\nu(\Omega^1))^2$ and $\boldsymbol{u}^2 \in (H^\nu(\Omega^2))^2$ with $\frac{3}{2} < \nu \leq 2$ and let $\boldsymbol{u}_h \in \boldsymbol{K}_h$ be the solution of* (3.5). *Then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_* \leq C(\boldsymbol{u})(h_1^{\frac{3}{4}(\nu-1)} + h_2^{\nu-1}),$$

where the constant C depends only on $\|\boldsymbol{u}^1\|_{(H^\nu(\Omega^1))^2}$ and $\|\boldsymbol{u}^2\|_{(H^\nu(\Omega^2))^2}$.

Proof. Putting together the results of Lemmas 4.1, 4.2 and 4.3 yields a first bound of the error

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_* \leq C(\boldsymbol{u})(h_1^{\frac{\nu}{2}-\frac{1}{2}} + h_2^{\nu-1}). \tag{4.5}$$

Such an estimate allows to derive a better upper bound of the consistency error. Taking back the intermediary term (4.4), we notice that it can be written as

$$\begin{aligned}\int_{\Gamma_c} \sigma_{\boldsymbol{n}}[(\boldsymbol{v} - \boldsymbol{u}_h).\boldsymbol{n}]\, d\Gamma &= \int_{\Gamma_c} \sigma_{\boldsymbol{n}}\big\{\pi_h^1((\boldsymbol{u}_h^2 - \mathcal{I}_h^2\boldsymbol{u}^2).\boldsymbol{n}^2) - ((\boldsymbol{u}_h^2 - \mathcal{I}_h^2\boldsymbol{u}^2).\boldsymbol{n}^2)\big\}\, d\Gamma \\ &+ \int_{\Gamma_c} \sigma_{\boldsymbol{n}}\big\{((\boldsymbol{u}^2 - \mathcal{I}_h^2\boldsymbol{u}^2).\boldsymbol{n}^2) - \pi_h^1((\boldsymbol{u}^2 - \mathcal{I}_h^2\boldsymbol{u}^2).\boldsymbol{n}^2)\big\}\, d\Gamma \\ &+ \int_{\Gamma_c} \sigma_{\boldsymbol{n}}(\pi_h^1(\boldsymbol{u}^2.\boldsymbol{n}^2) - \boldsymbol{u}^2.\boldsymbol{n}^2)\, d\Gamma.\end{aligned}$$

For the clarity of the presentation we shall denote T_1, T_2 and T_3 the three different integral quantities involved in the previous sum. We begin by bounding the third one, it is made as in Ref.[9], and yields

$$|T_3| \leq C'(\boldsymbol{u})h_1^{2(\nu-1)},$$

with $C'(\boldsymbol{u})$ depending linearly on $\|\boldsymbol{u}^2\|^2_{(H^\nu(\Omega^2))^2}$. Using the nature of the operator π_h^1, it comes out that:

$$|T_2| = \Big|\int_{\Gamma_c} (\sigma_{\boldsymbol{n}} - \psi_h)\big\{((\boldsymbol{u}^2 - \mathcal{I}_h^2\boldsymbol{u}^2).\boldsymbol{n}^2) - \pi_h^1((\boldsymbol{u}^2 - \mathcal{I}_h^2\boldsymbol{u}^2).\boldsymbol{n}^2)\big\}\, d\Gamma\Big|,$$

for all $\psi_h \in M_h^1(\Gamma_c)$. By duality, we obtain

$$|T_2| \leq \inf_{\psi_h \in M_h^1(\Gamma_c)} \|\sigma_{\boldsymbol{n}} - \psi_h\|_{(H_{00}^{\frac{1}{2}}(\Gamma_c))'} \|((\boldsymbol{u}^2 - \mathcal{I}_h^2\boldsymbol{u}^2).\boldsymbol{n}^2) - \pi_h^1((\boldsymbol{u}^2 - \mathcal{I}_h^2\boldsymbol{u}^2).\boldsymbol{n}^2)\|_{H_{00}^{\frac{1}{2}}(\Gamma_c)}.$$

Observing that $\boldsymbol{u}^2.\boldsymbol{n}^2 - (\mathcal{I}_h^2\boldsymbol{u}^2).\boldsymbol{n}^2$ belongs to $H_{00}^{\frac{1}{2}}(\Gamma_c)$, and thanks to the stability (3.3), we have

$$|T_2| \leq Ch_1^{\nu-1}\|\sigma_{\boldsymbol{n}}\|_{H^{\nu-\frac{3}{2}}(\Gamma_c)}\|(\boldsymbol{u}^2 - \mathcal{I}_h^2\boldsymbol{u}^2).\boldsymbol{n}^2\|_{H_{00}^{\frac{1}{2}}(\Gamma_c)}.$$

Employing (3.1) and the inequality $2h_1^{\nu-1}h_2^{\nu-1} \leq h_1^{2(\nu-1)} + h_2^{2(\nu-1)}$ yields

$$|T_2| \leq C'(\boldsymbol{u})(h_1^{2(\nu-1)} + h_2^{2(\nu-1)}).$$

We are left with the first term T_1. Following the same points as for T_2 leads to

$$|T_1| \leq Ch_1^{\nu-1}\|\sigma_{\boldsymbol{n}}\|_{H^{\nu-\frac{3}{2}}(\Gamma_c)}h_1^{\varepsilon}\|\boldsymbol{u}_h^2.\boldsymbol{n}^2 - (\mathcal{I}_h^2\boldsymbol{u}^2).\boldsymbol{n}^2\|_{H^{\frac{1}{2}+\varepsilon}(\Gamma_c)}.$$

Applying the inverse inequality and inserting $\boldsymbol{u}^2.\boldsymbol{n}^2$ gives

$$\begin{aligned}|T_1| &\leq Ch_1^{\nu-1}\Big(\frac{h_1}{h_2}\Big)^{\varepsilon}\|\sigma_{\boldsymbol{n}}\|_{H^{\nu-\frac{3}{2}}(\Gamma_c)}\|\boldsymbol{u}_h^2.\boldsymbol{n}^2 - (\mathcal{I}_h^2\boldsymbol{u}^2).\boldsymbol{n}^2\|_{H^{\frac{1}{2}}(\Gamma_c)}\\ &\leq Ch_1^{\nu-1}\Big(\frac{h_1}{h_2}\Big)^{\varepsilon}\|\sigma_{\boldsymbol{n}}\|_{H^{\nu-\frac{3}{2}}(\Gamma_c)}(\|\boldsymbol{u}-\boldsymbol{u}_h\|_* + \|\boldsymbol{u}^2 - \mathcal{I}_h^2\boldsymbol{u}^2\|_{(H^1(\Omega^2))^2}).\end{aligned}$$

Using estimate (4.5) together with (3.1) so as the boundedness of the ratio h_1/h_2 conclude to

$$|T_1| \leq C'(\boldsymbol{u})(h_1^{\frac{3}{2}(\nu-1)} + h_2^{2(\nu-1)}).$$

Assembling estimates on different terms T_1, T_2 and T_3 gives the following error estimate

$$\inf_{\boldsymbol{v}\in\boldsymbol{K}}\Big|\int_{\Gamma_c}\sigma_{\boldsymbol{n}}(\boldsymbol{u})[(\boldsymbol{v}-\boldsymbol{u}_h).\boldsymbol{n}]\,d\Gamma\Big|^{\frac{1}{2}} \leq C(\boldsymbol{u})(h_1^{\frac{3}{4}(\nu-1)} + h_2^{\nu-1}).$$

Moreover, the best approximation error decays to zero faster as the consistency error. That concludes the proof. □

By applying as long as possible the bootstrap argument with the consistency error, we obtain the following result.

Theorem 4.1 *Assume that the solution* $\boldsymbol{u}\in\boldsymbol{K}$ *of problem* (2.6) *is such that* $\boldsymbol{u}^1 \in (H^\nu(\Omega^1))^2$ *and* $\boldsymbol{u}^2 \in (H^\nu(\Omega^2))^2$ *with* $\frac{3}{2} < \nu \leq 2$ *and let* $\boldsymbol{u}_h \in \boldsymbol{K}_h$ *be the solution of* (3.5)*. Then*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_* \leq C(\boldsymbol{u})(h_1^{\frac{\nu}{2}-\frac{1}{4}} + h_2^{\nu-1}),$$

where the constant C depends only on $\|\boldsymbol{u}^1\|_{(H^\nu(\Omega^1))^2}$ *and* $\|\boldsymbol{u}^2\|_{(H^\nu(\Omega^2))^2}$.

Under the $H^2\times H^2$ regularity conditions, the statement gives a convergence rate in $h_1^{\frac{3}{4}} + h_2$.

In the particular case of compatible meshes, the discrete unilateral condition is reduced to the natural pointwise condition, as above–mentioned in section 3. Then, if the exact solution is assumed to lie in $H^2\times H^2$, Theorem 4.1 gives the same rate of convergence in $|h|^{\frac{3}{4}}$ as in Haslinger, Hlaváček and Nečas,[17]. We denote by $|h|$ a given norm of h in $\mathbb{R}^2$. The present statement seems to be the first result for non–matching meshes in unilateral problems.

Let us now consider the case without an embedding condition. The existence and uniqueness results for this case are established in Ref.[17]. As a by-product of Theorem 4.1, we obtain an information on the dual unknown of the displacement field. The approximation of the stress field $\sigma = (\sigma^1, \sigma^2)$ by the discrete one $\sigma_h = (\sigma_h^1, \sigma_h^2)$ (where $\sigma^\ell = A^\ell \varepsilon(\boldsymbol{u}^\ell)$ and $\sigma_h^\ell = A^\ell \varepsilon(\boldsymbol{u}_h^\ell)$), is governed by the following error estimate. The notation $|\tau|_*$ stands for the standard $L^2(\Omega^1 \cup \Omega^2)$–norm defined on the space of tensor fields.

Corollary 4.1 *Let us assume that $\Gamma_{\boldsymbol{u}}^1 = \Gamma_{\boldsymbol{u}}^2 = \emptyset$. Let there exist solutions $\boldsymbol{u} \in \boldsymbol{K}$ and $\boldsymbol{u}_h \in \boldsymbol{K}_h$ of problems (2.6) and (3.5) respectively. Assume that the norms $\|\boldsymbol{u}_h\|_*$ remain bounded and let the smoothness hypothesies of Theorem 4.1 hold. Then*

$$|\sigma - \sigma_h|_* \leq C(\boldsymbol{u})(h_1^{\frac{\nu}{2}-\frac{1}{4}} + h_2^{\nu-1}),$$

where the constant $C(\boldsymbol{u})$ is independent on h.

This statement can be naturally formulated in terms of the strain tensor. Under the same assumptions, the quantity $|\varepsilon(\boldsymbol{u}-\boldsymbol{u}_h)|_*$ is dominated exactly as in Corollary 4.1. This property is immediately derived from the ellipticity condition satisfied by the elasticity coefficients. Let us recall that the quantity $|\varepsilon(\boldsymbol{v})|_*$ defines an equivalent norm to the H^1–norm $\|\boldsymbol{v}\|_*$ if $mes\Gamma_{\boldsymbol{u}}^\ell > 0$, $(\ell = 1, 2)$, owing to Korn inequality. Clearly, the corollary is still valid for the embedding case.

To our knowledge, there is no detailed study of the regularity aspects for unilateral contact problems. Nevertheless, the smoothness condition in Theorem 4.1 can be removed and it can be proved that the condition $\boldsymbol{u} \in H^1 \times H^1$ implies the following convergence property, (Hild,[18]):

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_* \to 0 \text{ as } h \to 0.$$

Further comments on the regularity of the exact solution to the contact problem can be found in Remark 4.4 hereafter.

Remark 4.3 The error estimate stated in Theorem 4.1 is close to the optimal properties obtained for finite element methods. Under the $H^{\frac{3}{2}+\varepsilon}$ regularity assumption, the loss in rate of convergence is given by the factor $|h|^{\frac{\varepsilon}{2}}$ for the present unilateral problem. Such a decrease is negligible for small values of $\varepsilon > 0$. In other respects, if $\boldsymbol{u}$ is assumed to be in $H^2 \times H^2$, the loss is more important: the factor of reduction is equal to $|h|^{\frac{1}{4}}$. Nevertheless, if we compare to the known result in a unilateral contact problem (conforming method), we have already noticed that Theorem 4.1 achieves the same error estimate in a more general situation.

Remark 4.4 The lack of regularity in the present contact problem comes from many causes. The first one classically corresponds to the mixed boundary conditions of Dirichlet type on $\Gamma_{\boldsymbol{u}}^\ell$ and Neumann type on $\Gamma_{\boldsymbol{g}}^\ell$. The smoothness of the solution to the linear elasticity problem is well-studied and the resulting property is close to the assumption used in the statement (Grisvard,[14] Kinderlehrer and Stampacchia,[21]). This first difficulty is not intrinsically connected to the contact model, and the problem without an embedding condition is considered in Corollary 4.1. A second cause of singularity results from the geometry of the polygonal domains. Again, this cause is not directly connected to the contact problem, and we have simply considered the case where Γ_c is a straight line segment. This case allows to avoid nonconvex domains (Grisvard,[14] Moussaoui and Khodja,[25]). A last cause of nonsmoothness is more fundamental in our problem; this is the Signorini condition. For the analogous problem defined by the Laplace operator and a Signorini type boundary condition, it was proved that the solution is more regular than H^2 (Brezzi, Hager and Raviart,[10] Moussaoui and Khodja,[25]). This somewhat surprising property is to compare to the weaker smoothness condition used in Theorem 4.1.

Remark 4.5 In the proof of Theorem 4.1, we have stopped the bootstrap argument because the global error is now equivalent to the best approximation error. However, it is possible to push further the technique and to bound the consistency error by $C(\boldsymbol{u})\ (h_1^{\frac{3\nu}{4}-\frac{5}{8}} + h_2^{\nu-1})$ under the assumptions of Theorem 4.1.

5. Conclusion and Perspectives

In order to solve the contact problem between two elastic bodies, we have considered a finite element method of order one using non–matching meshes at the contact zone. The extension of the mortar technique (with $\nu = 2$ in Theorem 4.1) yields a convergence rate in $h_1^{\frac{3}{4}} + h_2$, denoting h_1 and h_2 the discretization parameters associated with each of the two bodies. Therefore, the convergence rate of the method is the same as in the case of compatible meshes (Haslinger, Hlaváček and Nečas,[17]).

As for the variational equalities (Bernardi, Debit and Maday,[8]), the discrete matching constraint used in this paper should be preferred to a pointwise interpolation constraint. The latter leads to a global error bounded by $C(\boldsymbol{u}, \varepsilon)\ (h_1^{\frac{1}{2}-\varepsilon} + h_2)$, under H^2 regularity conditions (Hild,[18]).

In other respects, we consider in Ref.[19] the more general contact model taking into account the friction.

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