On solution multiplicity in friction problems with normal compliance

Patrick HILD [1*] and Nour-Dine SAKKI [2]

[1] *Laboratoire de Mathématiques de Besançon,*
Université de Franche-Comté / CNRS UMR 6623,
16 route de Gray, 25030 Besançon, France.

[2] *Laboratoire de Mathématiques,*
Université de Savoie / CNRS UMR 5127,
73376 Le Bourget du Lac, France.

Abstract

This work is concerned with the normal compliance model and friction in linear elastostatics. We consider the two-dimensional sliding problem and we seek those interface parameters leading to infinitely many solutions of the continuous and the finite element discretized problem. The determination of the interface parameters uses a specific eigenvalue problem. In the discrete case the framework is illustrated with an elementary example and some finite element computations.

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Running title : Solution multiplicity in the compliance model.

1. Introduction and problem set-up

Contact mechanics involves highly nonlinear phenomena especially when friction effects are taken into account. The most common model of friction is due to Coulomb at the end of the eighteenth century and it is generally used together with the Signorini (or unilateral) contact conditions. Such a macroscopic frictional contact model with a very simple formulation nevertheless is strongly nonlinear and its understanding from a mathematical point of view is not complete yet. A more recent model motivated by phenomenological laws on the contact interface such as the presence of small asperities, oxides and impurities has lead to the so-called normal compliance model (with or without friction) introduced and studied in [13] and [12]. Note that this model

*Corresponding author. Tel: +33.3.81.66.63.49; fax: +33.3.81.66.66.23. E-mail adresses: hild@math.univ-fcomte.fr (P. Hild), nsakk@univ-savoie.fr (N.-D. Sakki)

can also be seen as a regularization of the Signorini contact conditions in which some penetration is allowed. In elastostatics several works concerning existence and/or uniqueness of solutions have been achieved: in [14] for the sliding case and in [9, 10] for the general case.

This work deals with nonuniqueness of solutions to the normal compliance model with friction. We focus on a simple and particular case of the model in which the normal and tangential constraints depend in a linear way on the penetration on the contact zone. According to [10] this problem admits a unique solution provided that the interface parameters denoted c_n and c_t are small enough. In section 2, we consider a solution $\boldsymbol{u}$ of this normal compliance problem which slips in a given direction and penetrates into the foundation. Following the ideas of [5] we introduce a specific eigenvalue problem. We prove that if c_n is an eigenvalue then an infinity of solutions to the normal compliance problem with friction exist in a neighborhood of $\boldsymbol{u}$. Section 3 deals with the finite dimensional case. We translate the results of the previous section when finite elements are used. Finally we illustrate the theory and we show some nonuniqueness cases with an elementary example and some finite element computations.

We consider an elastic body occupying a domain Ω in $\mathbb{R}^2$. The boundary Γ of Ω is assumed to be Lipschitz and is divided as follows: $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_C$ where Γ_D, Γ_N and Γ_C are three open disjoint parts with $\text{meas}(\Gamma_D) > 0$. The given displacements $\mathbf{U}$ are prescribed on the portion Γ_D, the part Γ_N is subjected to a density of surface forces denoted $\boldsymbol{F} \in (L^2(\Gamma_N))^2$ and Ω is being acted upon by the body forces $\boldsymbol{f} \in (L^2(\Omega))^2$. On the part Γ_C the body can come into contact with a rigid foundation. We denote by $\boldsymbol{n} = (n_1, n_2)$ the unit outward normal vector on the boundary Γ and by $\boldsymbol{t} = (n_2, -n_1)$ the unit tangent vector.

The frictional contact problem with normal compliance in elastostatics is to find the displacement field $\boldsymbol{u}$ such that equations (1.1)–(1.5) hold:

$$\mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} = \mathbf{0} \quad \text{in } \Omega, \tag{1.1}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \mathcal{C}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{1.2}$$

$$\boldsymbol{u} = \mathbf{U} \quad \text{on } \Gamma_D, \tag{1.3}$$

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{F} \quad \text{on } \Gamma_N, \tag{1.4}$$

where $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the linearized strain tensor defined by $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}^T\boldsymbol{u})/2$ and $\mathcal{C} = c_{ijkh}(\boldsymbol{x}) \in (L^\infty(\Omega))^{16}$ is the fourth order symmetric and elliptic tensor of linear elasticity.

We decompose the stress vector $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n}$ on the boundary Γ into normal stress and tangential stress denoted $\sigma_n(\boldsymbol{u})$ and $\sigma_t(\boldsymbol{u})$, respectively, so that $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \sigma_n(\boldsymbol{u})\boldsymbol{n} + \sigma_t(\boldsymbol{u})\boldsymbol{t}$. In the same way, the displacement field $\boldsymbol{u}$ on the boundary Γ is written $\boldsymbol{u} = u_n\boldsymbol{n} + u_t\boldsymbol{t}$ where u_n and u_t denote the normal and tangential displacements, respectively.

Throughout this paper, we assume that the frictional contact behavior on the part Γ_C is governed by the normal compliance model introduced and studied by Oden and

Martins (see [13]) in which the stresses follow the power law,

$$\begin{aligned} &\sigma_n(\boldsymbol{u}) = -c_n(u_n)_+^{m_n}, \\ &\sigma_t(\boldsymbol{u}) = -c_t \ \mathrm{sgn}(u_t) \ (u_n)_+^{m_t} \quad \text{if sliding occurs,} \end{aligned}$$

where sgn denotes the sign function and $(.)_+$ stands for the positive part so that $(u_n)_+$ represents the penetration of the body into the foundation. The constants $m_n \geq 1$, $m_t \geq 1$ as well as the positive functions c_n and c_t in $L^\infty(\Gamma_C)$ stand for interface parameters characterizing the contact behavior between the body and the rigid foundation. Then the conditions of normal compliance with friction on Γ_C are:

$$\begin{cases} \sigma_n(\boldsymbol{u}) = -c_n(u_n)_+^{m_n}, \\ |\sigma_t(\boldsymbol{u})| \leq c_t(u_n)_+^{m_t} \quad \text{if} \quad u_t = 0, \\ \begin{cases} |\sigma_t(\boldsymbol{u})| = c_t(u_n)_+^{m_t} \quad \text{if} \quad u_t \neq 0, \\ u_t \ \sigma_t(\boldsymbol{u}) \leq 0. \end{cases} \end{cases} \tag{1.5}$$

Let us introduce the set of admissible displacements:

$$\mathbf{V}_U = \Big\{ \boldsymbol{v} \in (H^1(\Omega))^2; \ \boldsymbol{v} = \mathbf{U} \text{ on } \Gamma_D \Big\}.$$

The weak form of problem (1.1)–(1.5) consists to find $\boldsymbol{u} \in \mathbf{V}_U$ such that:

$$a(\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}) + j_n(\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}) + j_t(\boldsymbol{u}, \boldsymbol{v}) - j_t(\boldsymbol{u}, \boldsymbol{u}) \geq L(\boldsymbol{v}-\boldsymbol{u}), \quad \forall \boldsymbol{v} \in \mathbf{V}_U \tag{1.6}$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \ d\Omega, \qquad L(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}.\boldsymbol{v} \ d\Omega + \int_{\Gamma_N} \boldsymbol{F}.\boldsymbol{v} \ d\Gamma,$$

$$j_n(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_C} c_n(u_n)_+^{m_n} v_n \ d\Gamma, \qquad j_t(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_C} c_t(u_n)_+^{m_t} |v_t| \ d\Gamma,$$

for any $\boldsymbol{u}$ and $\boldsymbol{v}$ in $(H^1(\Omega))^2$ and $1 \leq m_n, m_t < +\infty$. The existence of solutions to problem (1.6) was proved by Klarbring, Mikelic and Shillor in [9]. In the latter reference (where $\mathbf{U} = \mathbf{0}$) the authors prove also that if the loads $\boldsymbol{f}, \boldsymbol{F}$ and the interface parameters c_n and c_t are small enough, then the problem (1.6) admits a unique solution in a ball centered at the origin and whose radius depends on the interface parameters and the loading. When $m_n = m_t = 1$, the authors improve in [10] the previous result and establish that the solution to (1.6) is globally unique when c_n and c_t are small enough.

If we search for solutions of (1.1)–(1.5) with slip in a given direction (i.e., $u_t > 0$ or $u_t < 0$) the conditions (1.5) become

$$\begin{cases} \sigma_n(\boldsymbol{u}) = -c_n(u_n)_+^{m_n}, \\ \begin{cases} -\sigma_t(\boldsymbol{u}) = c_t(u_n)_+^{m_t} \quad \text{if} \quad u_t > 0, \\ \sigma_t(\boldsymbol{u}) = c_t(u_n)_+^{m_t} \quad \text{if} \quad u_t < 0. \end{cases} \end{cases} \tag{1.7}$$

Set

$$\mathbf{V} = \Big\{\boldsymbol{v} \in (H^1(\Omega))^2;\ \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_D\Big\}.$$

The variational formulation (1.6) with conditions (1.7) is to find $\boldsymbol{u} \in \mathbf{V}_U$ such that $\operatorname{sgn}(u_t) = +1$ or $\operatorname{sgn}(u_t) = -1$ on Γ_C and

$$a(\boldsymbol{u}, \boldsymbol{v}) + \int_{\Gamma_C} c_n (u_n)_+^{m_n} v_n \, d\Gamma + \ \operatorname{sgn}(u_t) \int_{\Gamma_C} c_t (u_n)_+^{m_t} v_t \, d\Gamma = L(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \mathbf{V}. \quad (1.8)$$

In the next section we consider a solution to the sliding problem (1.8) when $m_n = m_t = 1$ and we look for sufficient conditions for nonuniqueness keeping in mind that the solution is unique when c_n and c_t are small enough (see [10]).

2. Eigenvalues and multiplicity of solutions

2.1. *Sufficient conditions for multiple solutions: a spectral approach*

The sufficient conditions leading to infinitely many solutions of problem (1.8) require the introduction of an eigenvalue problem which consists, for a fixed real number λ, of finding $(\mu, \boldsymbol{\varphi}) \in \mathbb{C} \times ((H^1(\Omega))^2 - \{\mathbf{0}\})$ such that:

$$\begin{cases} \mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{\varphi}) = \mathbf{0} & \text{in } \Omega, \\ \boldsymbol{\sigma}(\boldsymbol{\varphi}) = \mathcal{C}\, \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) & \text{in } \Omega, \\ \boldsymbol{\varphi} = \mathbf{0} & \text{on } \Gamma_D, \\ \boldsymbol{\sigma}(\boldsymbol{\varphi})\boldsymbol{n} = \mathbf{0} & \text{on } \Gamma_N, \\ \sigma_t(\boldsymbol{\varphi}) = -\lambda \sigma_n(\boldsymbol{\varphi}) & \text{on } \Gamma_C, \\ \sigma_n(\boldsymbol{\varphi}) = -\mu \varphi_n & \text{on } \Gamma_C, \end{cases} \quad (2.1)$$

where φ_n denotes the normal component of $\boldsymbol{\varphi}$ on Γ_C.

In order to introduce the variational formulation of problem (2.1) we define the bilinear form $b_\lambda(.,.)$ given by:

$$b_\lambda(\boldsymbol{u}, \boldsymbol{v}) = \lambda \int_{\Gamma_C} u_n v_t \, d\Gamma - \int_{\Gamma_C} u_n v_n \, d\Gamma.$$

The weak formulation of eigenvalue problem (2.1) consists, for a fixed real number λ, to find $\mu \in \mathbb{C}$ and $\mathbf{0} \neq \boldsymbol{\varphi} \in \mathbf{V}$ such that:

$$a(\boldsymbol{\varphi}, \boldsymbol{v}) = \mu b_\lambda(\boldsymbol{\varphi}, \boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \mathbf{V}. \quad (2.2)$$

As a matter of fact one can easily check that if $\mu \in \mathbb{C}$ and a nonzero eigenfunction $\boldsymbol{\varphi}$ satisfy (2.1), then the pair $(\mu, \boldsymbol{\varphi})$ is also a solution of (2.2). Conversely, any sufficiently regular solution $(\mu, \boldsymbol{\varphi})$ of (2.2) satisfies (2.1).

Besides it is easy to see that if one chooses $\lambda = 0$ then problem (2.1) admits only negative eigenvalues μ. Moreover for any λ, the value $\mu = 0$ cannot solve problem

(2.1). Let us mention that a similar approach has already been introduced for the Coulomb friction model in [5] and developed in [6]. Such an eigenvalue approach has recently lead to explicit examples of nonuniqueness for the continuous Coulomb friction model with unilateral contact in [8]. In the latter case the eigenvalue problem does not depend on a constant λ and the conditions on Γ_C in (2.1) become $\varphi_n = 0$ and $\sigma_t(\boldsymbol{\varphi}) = \mu\sigma_n(\boldsymbol{\varphi})$ where the eigenvalue μ represents now a friction coefficient.

In the remainder of the paper we are interested in the problem with slip, (1.8), with $m_t = m_n = 1$. Moreover, we assume that c_n and c_t are positive constants independent of $\boldsymbol{x}$, and let $\boldsymbol{u}$ be an equilibrium position of the frictional contact problem with normal compliance (1.8).

The following proposition establishes sufficient conditions for infinitely many solutions of problem (1.8) located on a continuous branch originating at $\boldsymbol{u}$.

Proposition 2.1 *Let $\boldsymbol{u}$ be a solution of the frictional contact problem with normal compliance* (1.8) *and assume that there exists α and β such that:*

$$u_n(\boldsymbol{x}) \geq \alpha > 0, \quad |u_t(\boldsymbol{x})| \geq \beta > 0, \quad \forall \boldsymbol{x} \in \Gamma_C.$$

Consider the eigenvalue problem (2.1) *with $\lambda = -\ sgn(u_t)\dfrac{c_t}{c_n}$.*

If c_n is an eigenvalue of problem (2.1) *with $\boldsymbol{\varphi}$ as corresponding eigenvector and if $\varphi_n \in L^\infty(\Gamma_C)$, $\varphi_t \in L^\infty(\Gamma_C)$, then there exists $\delta_0 > 0$ such that $\boldsymbol{u} + \delta\boldsymbol{\varphi}$ is also a solution of the normal compliance contact problem* (1.8) *for any $|\delta| \leq \delta_0$.*

Proof. According to the hypotheses of the proposition, we get $\boldsymbol{u} \in \mathbf{V}_U$ and $\delta\boldsymbol{\varphi} \in \mathbf{V}$ for any $\delta \in \mathbb{R}$. Hence $\boldsymbol{u} + \delta\boldsymbol{\varphi} \in \mathbf{V}_U$ for any $\delta \in \mathbb{R}$. Moreover

$$a(\boldsymbol{u}, \boldsymbol{v}) + c_n \int_{\Gamma_C} (u_n)_+ v_n \, d\Gamma + \ \mathrm{sgn}(u_t) c_t \int_{\Gamma_C} (u_n)_+ v_t \, d\Gamma = L(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbf{V}, \quad (2.3)$$

$$a(\boldsymbol{\varphi}, \boldsymbol{v}) + c_n \int_{\Gamma_C} \varphi_n v_n \, d\Gamma + \ \mathrm{sgn}(u_t) c_t \int_{\Gamma_C} \varphi_n v_t \, d\Gamma = 0, \qquad \forall \boldsymbol{v} \in \mathbf{V}. \quad (2.4)$$

Since $u_n(\boldsymbol{x}) \geq \alpha > 0$ and $\varphi_n \in L^\infty(\Gamma_C)$, we deduce that there exists $\delta_0 > 0$ such that

$$(u_n)_+ + \delta\varphi_n = (u_n + \delta\varphi_n)_+ \tag{2.5}$$

for any $|\delta| \leq \delta_0$. Similarly since $|u_t(\boldsymbol{x})| \geq \beta > 0$ on Γ_C and $\varphi_t \in L^\infty(\Gamma_C)$, we see that

$$\mathrm{sgn}(u_t) = \ \mathrm{sgn}(u_t + \delta\varphi_t) \tag{2.6}$$

if $|\delta|$ is small enough. Multiplying equation (2.4) with δ and adding it with (2.3) completes the proof. □

Remark 2.2 *The $L^\infty(\Gamma_C)$ regularity assumptions for φ_n and φ_t are made to avoid any possible singularities of $H^{1/2}(\Gamma_C)$ functions and to obtain estimates* (2.5) *and* (2.6).

2.2. *Existence of eigenvalues*

This section is devoted to the existence of eigenvalues of problem (2.1). Let us recall that $\mu = 0$ is not an eigenvalue in (2.1) since $a(.,.)$ is $\mathbf{V}$-elliptic.

So we define $P : L^2(\Gamma_C) \to \mathbf{V}$ as follows: for any $f \in L^2(\Gamma_C)$,

$$a(P(f), \boldsymbol{v}) = \lambda \int_{\Gamma_C} f v_t \, d\Gamma - \int_{\Gamma_C} f v_n \, d\Gamma, \qquad \forall \boldsymbol{v} \in \mathbf{V}.$$

The operator P is linear and continuous, and

$$\|P(f)\|_1 \leq C_\lambda \|f\|_{L^2(\Gamma_C)},$$

where $\|.\|_1$ denotes the $(H^1(\Omega))^2$-norm. Consider now the normal trace operator with values in $L^2(\Gamma_C)$: $Q = IoTr$ such that $Q(\boldsymbol{v}) = v_n$ where Tr denotes the normal trace operator from $\mathbf{V}$ into $H^{\frac{1}{2}}(\Gamma_C)$ and I is the canonical embedding from $H^{\frac{1}{2}}(\Gamma_C)$ into $L^2(\Gamma_C)$. According to the embedding theorem (see [1], Theorem 7.57), the mapping I is compact and we deduce that $T = PQ$ is also compact. Then, $(\mu, \boldsymbol{\varphi})$ is a solution of the eigenvalue problem (2.1) if and only if

$$T(\boldsymbol{\varphi}) = \frac{1}{\mu}\, \boldsymbol{\varphi}.$$

Indeed, if $(\mu, \boldsymbol{\varphi})$ satisfies (2.2) then for any $\boldsymbol{v} \in \mathbf{V}$ one obtains

$$\begin{aligned} a(\boldsymbol{\varphi}, \boldsymbol{v}) &= \mu\Big(\lambda \int_{\Gamma_C} \varphi_n v_t \, d\Gamma - \int_{\Gamma_C} \varphi_n v_n \, d\Gamma\Big) \\ &= \mu\Big(\lambda \int_{\Gamma_C} Q(\boldsymbol{\varphi}) v_t \, d\Gamma - \int_{\Gamma_C} Q(\boldsymbol{\varphi}) v_n \, d\Gamma\Big) \\ &= \mu a(PQ(\boldsymbol{\varphi}), \boldsymbol{v}) \\ &= \mu a(T(\boldsymbol{\varphi}), \boldsymbol{v}). \end{aligned}$$

Therefore $T(\boldsymbol{\varphi}) = (1/\mu)\boldsymbol{\varphi}$. The compactness of T allows us to obtain the classical result concerning its spectrum.

Proposition 2.3 *The eigenvalues of problem* (2.2) *consist of a countable set of complex numbers* $\{\mu_n\}_{n \in I}$ *with* $\{\mu_n\} \neq 0$. *Each eigenvalue* $\{\mu_n\}$ *is of finite algebraic multiplicity. If the set* I *is infinite then* $\lim_{n\to\infty} |\mu_n| = +\infty$.

Remark 2.4 *A more detailed study would be to find if positive eigenvalues to the continuous problem* (2.2) *exist. In fact we are actually not able to exhibit configurations in which such positive eigenvalues exist (whereas it is possible in the discrete case, see the next section). Besides, note that if a positive eigenvalue* μ *of problem* (2.2) *exists then one can find a distribution of loads* $\boldsymbol{F}, \boldsymbol{f}$ *and a displacement field* $\mathbf{U}$ *such that a solution* $\boldsymbol{u}$ *of* (1.1)–(1.4), (1.7) *for the particular interface parameters* $c_n = \mu$ *and* $c_t = \mu\lambda \; sgn(\lambda)$ *satisfies the hypotheses of Proposition 2.1. To this end let us suppose that* Ω *is an homogeneous and isotropic body whose constitutive law in* (1.2) *reduces to*

$$\boldsymbol{\sigma}(\boldsymbol{v}) = L\, tr(\boldsymbol{\varepsilon}(\boldsymbol{v}))\mathbf{I} + 2G\, \boldsymbol{\varepsilon}(\boldsymbol{v}) \qquad in\ \Omega, \tag{2.7}$$

where tr denotes the trace operator, $\mathbf{I}$ *represents the identity matrix and* L, G *are the positive Lamé coefficients.*

Assume for the sake of simplicity that Γ_C *is a straight line segment located on* Ox_1*-axis so that* $\boldsymbol{n} = (0, -1)$ *and* $\boldsymbol{t} = (-1, 0)$*. Set*

$$\mathbf{U}(\boldsymbol{x}) = \left(sgn(\lambda)\beta + \frac{\alpha\lambda\mu}{G} x_2, -\alpha - \frac{\alpha\mu}{L + 2G} x_2 \right)$$

for all $\boldsymbol{x} = (x_1, x_2) \in \Gamma_D$*, with* $\alpha > 0$*,* $\beta > 0$ *and let* $\boldsymbol{f} = \mathbf{0}$*. One can easily check that* $\boldsymbol{u}(\boldsymbol{x}) = \mathbf{U}(\boldsymbol{x})$*, for all* $\boldsymbol{x} \in \Omega$ *(and* $\boldsymbol{F}$ *given by* $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n}$ *on* Γ_N*) is a solution of* (1.1)–(1.4), (1.7) *and also of* (1.6)*. On* Γ_C*, we have* $\sigma_n(\boldsymbol{u}) = -\alpha\mu = -c_n(u_n)_+$*,* $\sigma_t(\boldsymbol{u}) = \alpha\lambda\mu = c_t\alpha\, sgn(\lambda) = -c_t(u_n)_+\, sgn(u_t)$*,* $u_n = \alpha > 0$ *and* $|u_t| = \beta > 0$*, we deduce that the assumptions of Proposition 2.1 are fulfilled.*

3. The discrete case

Our aim is to translate the analysis made in the previous section to the finite dimensional case and to carry out some numerical experiments to illustrate the theory. Note that such a study has already been performed for the Coulomb friction problem with Signorini contact conditions in [7].

3.1. *Finite element approximation*

In this section we consider the finite element approximation of the compliance problem (1.8) and of the eigenvalue problem (2.2) with its corresponding convergence analysis. We discretize the domain Ω by using a family of triangulations $(\mathcal{T}_h)_h$ where h denotes the discretization parameter. The finite dimensional sets approximating $\mathbf{V}_U$ and $\mathbf{V}$ are (see [4]):

$$\mathbf{V}_{U,h} = \left\{ \boldsymbol{v}_h;\ \boldsymbol{v}_h \in (C(\overline{\Omega}))^2,\ \boldsymbol{v}_h|_T \in (P_k(T))^2\ \ \forall T \in \mathcal{T}_h,\ \boldsymbol{v}_h = \mathbf{U}_h \text{ on } \Gamma_D \right\},$$

where $\mathbf{U}_h$ denotes a convenient approximation of $\mathbf{U}$ and

$$\mathbf{V}_h = \left\{ \boldsymbol{v}_h;\ \boldsymbol{v}_h \in (C(\overline{\Omega}))^2,\ \boldsymbol{v}_h|_T \in (P_k(T))^2\ \ \forall T \in \mathcal{T}_h,\ \boldsymbol{v}_h = \mathbf{0} \text{ on } \Gamma_D \right\}.$$

The notation $C(\overline{\Omega})$ represents the space of continuous functions on $\overline{\Omega}$ and $P_k(T)$ stands for the space of polynomial functions of degree k on T. Note that in the sections 3.3 and 3.4, we simply choose $k = 1$.

The finite element approximation of problem (1.8) (with $m_n = m_t = 1$) consists of finding $\boldsymbol{u}_h \in \mathbf{V}_{U,h}$ such that $\text{sgn}(u_{ht}) = +1$ or $\text{sgn}(u_{ht}) = -1$ on Γ_C and

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) + \int_{\Gamma_C} c_n (u_{hn})_+ v_{hn}\, d\Gamma + \ \text{sgn}(u_{ht}) \int_{\Gamma_C} c_t (u_{hn})_+ v_{ht}\, d\Gamma = L(\boldsymbol{v}_h) \qquad (3.1)$$

for all $\boldsymbol{v}_h \in \mathbf{V}_h$.

We now focus on the finite element approximation of the eigenvalue problem (2.2) which consists, for a real fixed number λ, to find $(\mu, \boldsymbol{\varphi}_h) \in \mathbb{C} \times (\mathbf{V}_h - \{\mathbf{0}\})$ such that

$$a(\boldsymbol{\varphi}_h, \boldsymbol{v}_h) = \mu b_\lambda(\boldsymbol{\varphi}_h, \boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h \in \mathbf{V}_h. \tag{3.2}$$

If μ^{-1} stands for an eigenvalue of $T = PQ$ (defined in section 2) and I is the identity map, there exists a least integer α such that $\mathrm{Ker}((\mu^{-1}I - T)^\alpha) = \mathrm{Ker}((\mu^{-1}I - T)^{\alpha+1}) = E$ with $\dim(E) = m < \infty$ (the algebraic multiplicity of μ^{-1} is m and α stands for the ascent of $\mu^{-1}I - T$). Denote by E the set of generalized eigenvectors of T corresponding to μ^{-1} and let T^* be the adjoint operator of T defined on the dual space $\mathbf{V}^*$. Then $\overline{\mu}^{-1}$ is an eigenvalue of T^* with algebraic multiplicity m and α is also the ascent of $\overline{\mu}^{-1}I - T^*$. The notation $E^* = \mathrm{Ker}((\overline{\mu}^{-1}I - T^*)^\alpha)$ stands for the space of generalized eigenvectors of T^* associated with $\overline{\mu}^{-1}$.

If A and B are two closed subspaces of $\mathbf{V}$, we define the gap between A and B by

$$\delta(A,B) = \max\left(\sup_{\boldsymbol{u} \in A, \|\boldsymbol{u}\|_1 = 1} \inf_{\boldsymbol{v} \in B} \|\boldsymbol{u} - \boldsymbol{v}\|_1, \ \sup_{\boldsymbol{u} \in B, \|\boldsymbol{u}\|_1 = 1} \inf_{\boldsymbol{v} \in A} \|\boldsymbol{u} - \boldsymbol{v}\|_1 \right).$$

When μ stands for an eigenvalue of (2.2) of algebraic multiplicity m, there exists, as h tends to zero, exactly m eigenvalues of (3.2) denoted $\mu_{1,h}, \mu_{2,h}, \ldots, \mu_{m,h}$ converging to μ. Denote by E_h be the direct sum of the generalized eigenspaces associated with $\mu_{1,h}, \mu_{2,h}, \ldots, \mu_{m,h}$ and set

$$\varepsilon_h = \sup_{\boldsymbol{u} \in E, \|\boldsymbol{u}\|_1 = 1} \inf_{\boldsymbol{v}_h \in \mathbf{V}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_1 \quad \text{and} \quad \varepsilon_h^* = \sup_{\boldsymbol{u} \in E^*, \|\boldsymbol{u}\|_1 = 1} \inf_{\boldsymbol{v}_h \in \mathbf{V}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_1.$$

The following theorem, proved by Kolata in [11] (see also [3]), states a convergence result for the finite element approximation (3.2).

Theorem 3.1 *If h is small enough, the following estimates hold:*

$$\left|\mu - \frac{1}{m}\sum_{i=1}^{m} \mu_{i,h}\right| \le C\varepsilon_h \varepsilon_h^*, \quad |\mu - \mu_{i,h}| \le C(\varepsilon_h \varepsilon_h^*)^{\frac{1}{\alpha}}, \ 1 \le i \le m, \quad \delta(E, E_h) \le C\varepsilon_h,$$

where the constant C does not depend on h.

Let us remark that the theorem does not prove that the solutions of the finite element eigenvalue problem converge towards a solution of the continuous eigenvalue problem. In fact the computed eigenvalues can have a limit which is not an eigenvalue of the continuous problem (spurious modes).

The next proposition is the finite dimensional version of Proposition 2.1.

Proposition 3.2 *Let $\boldsymbol{u}_h$ be a solution of the discrete frictional contact problem with normal compliance* (3.1) *and assume that there exists α and β such that:*

$$u_{hn}(\boldsymbol{x}) \ge \alpha > 0, \quad |u_{ht}(\boldsymbol{x})| \ge \beta > 0, \quad \forall \boldsymbol{x} \in \Gamma_C.$$

Consider the eigenvalue problem (3.2) *with* $\lambda = -\ sgn(u_{ht})\dfrac{c_t}{c_n}$.

If c_n is an eigenvalue of problem (3.2) with $\boldsymbol{\varphi}_h$ as corresponding eigenvector then there exists $\delta_0 > 0$ such that $\boldsymbol{u}_h + \delta\boldsymbol{\varphi}_h$ is also a solution of the normal compliance contact problem (3.1) for any $|\delta| \leq \delta_0$.

Proof. Obviously $\boldsymbol{u}_h \in \mathbf{V}_{U,h}$ and $\delta\boldsymbol{\varphi}_h \in \mathbf{V}_h$ for any $\delta \in \mathbb{R}$ so that $\boldsymbol{u}_h + \delta\boldsymbol{\varphi}_h \in \mathbf{V}_{U,h}$ for any $\delta \in \mathbb{R}$. From the definitions of (3.2) and (3.1) and using the assumptions of the proposition, we obtain

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) + c_n \int_{\Gamma_C} (u_{hn})_+ v_{hn}\, d\Gamma + \operatorname{sgn}(u_{ht}) c_t \int_{\Gamma_C} (u_{hn})_+ v_{ht}\, d\Gamma = L(\boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \mathbf{V}_h,$$

$$a(\boldsymbol{\varphi}_h, \boldsymbol{v}_h) + c_n \int_{\Gamma_C} \varphi_{hn} v_{hn}\, d\Gamma + \operatorname{sgn}(u_{ht}) c_t \int_{\Gamma_C} \varphi_{hn} v_{ht}\, d\Gamma = 0, \quad \forall \boldsymbol{v}_h \in \mathbf{V}_h.$$

if δ is small enough, there is

$$(u_{hn})_+ + \delta\varphi_{hn} = (u_{hn} + \delta\varphi_{hn})_+ \quad \text{and} \quad \operatorname{sgn}(u_{ht}) = \operatorname{sgn}(u_{ht} + \delta\varphi_{ht}).$$

That ends the proof as in Proposition 2.1. □

3.2. *Algebraic formulation*

In what follows, we consider the commonly used Hooke's constitutive law in (2.7) corresponding to homogeneous isotropic materials in (1.2). Note that the positive Lamé coefficients L and G can be written $L = (E\nu)/((1-2\nu)(1+\nu))$ and $G = E/(2(1+\nu))$ where $E > 0$ and $0 \leq \nu < 1/2$ represent Young's modulus and Poisson's ratio, respectively.

We are interested in the matrix formulation of problem (3.2). First we number as follows the basis functions of $\mathbf{V}_h$: the normal displacement basis functions on Γ_C from 1 to p, the tangential displacement basis functions on Γ_C from $p+1$ to $2p$ and the basis functions of interior nodes from $2p+1$ to $m = \dim(\mathbf{V}_h)$. With this notation problem (3.2) can be written as follows

$$K\mathbf{\Phi} = K \begin{pmatrix} \Phi_n \\ \Phi_t \\ \Phi_i \end{pmatrix} = \begin{pmatrix} -\mu M \Phi_n \\ \lambda\mu M \Phi_n \\ 0 \end{pmatrix}, \tag{3.3}$$

where K denotes the stiffness matrix of order m, M is the mass matrix on Γ_C of order p and $\mathbf{\Phi}$ (resp. Φ_n) denotes the vector associated with $\boldsymbol{\varphi}_h$ (resp. φ_{hn}). We adopt the following notation (in the same spirit as for $\mathbf{\Phi}$)

$$K^{-1} = \begin{pmatrix} \tilde{K}_{nn} & \tilde{K}_{nt} & \tilde{K}_{ni} \\ \tilde{K}_{nt} & \tilde{K}_{tt} & \tilde{K}_{ti} \\ \tilde{K}_{ni} & \tilde{K}_{ti} & \tilde{K}_{ii} \end{pmatrix}.$$

Multiplying (3.3) with K^{-1} and writing the first p equations leads to the following eigenvalue problem: for a fixed real number λ, find $1/\mu$ and Φ_n satisfying:

$$(\lambda \tilde{K}_{nt} - \tilde{K}_{nn}) M \Phi_n = \frac{1}{\mu} \Phi_n. \tag{3.4}$$

Finding μ and Φ_n from (3.4), the eigenfunction $\mathbf{\Phi}$ is computed using (3.3), thus solving (3.2).

3.3. *An elementary example*

We now illustrate with a simple example the eigenvalue problem in (3.4). This means that we determine critical coefficients c_n such that an infinity of solutions located on a continuous branch exist (with slip only in one direction).

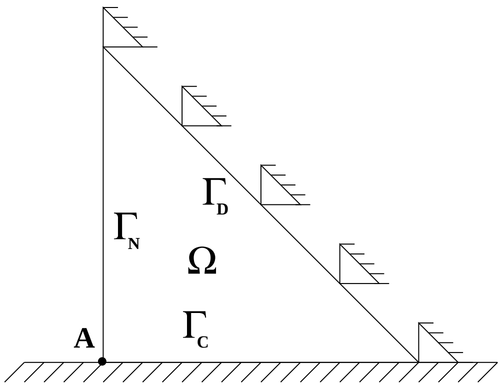

Figure 1: Problem involving a single element

We consider the triangular element, depicted in Figure 1 in which the two degrees of freedom are the normal and tangential displacement at point A. The stiffness matrix K becomes:

$$K = \frac{1}{2}\begin{pmatrix} L+3G & L+G \\ L+G & L+3G \end{pmatrix},$$

and the 1-by-1 mass matrix on the contact zone Γ_C is $\ell/3$ where ℓ denotes the length of Γ_C. We get

$$(\lambda\tilde{K}_{nt} - \tilde{K}_{nn})M = -\frac{(L(\lambda+1) + G(\lambda+3))\ell}{6G(L+2G)}.$$

In this case there exists a unique eigenvalue $1/\mu$ in (3.4) which is given by

$$\mu = -\frac{6G(L+2G)}{(L(\lambda+1) + G(\lambda+3))\ell}. \tag{3.5}$$

Let us prove with a direct calculation of the solutions to (3.1) (with $\mathbf{U} = \boldsymbol{f} = \mathbf{0}$ to simplify) that there exists, indeed, an infinity of solutions to the discrete problem (3.1) when $c_n = \mu$. Recalling that $\boldsymbol{n} = (0,-1)$ and $\boldsymbol{t} = (-1,0)$, let U_n (resp. U_t) denote the normal (resp. tangential) component of $\boldsymbol{u}_h$ at point A, i.e. the two unknowns in (3.1). Let the notations F_1 and F_2 represent the (constant) surface loads on Γ_N in the horizontal and vertical directions, respectively. Equation (3.1) can be written:

$$\left\{\begin{array}{l} \dfrac{L+3G}{2}U_n + \dfrac{L+G}{2}U_t + \dfrac{\ell c_n}{3}(U_n)_+ = -\dfrac{\ell F_2}{2}, \\ \dfrac{L+G}{2}U_n + \dfrac{L+3G}{2}U_t + \dfrac{\mathrm{sgn}(U_t)\ell c_t}{3}(U_n)_+ = -\dfrac{\ell F_1}{2}. \end{array}\right. \tag{3.6}$$

Our aim is not to carry out a complete discussion of all the solutions of (3.6) but only to find the multiple solutions which correspond to Proposition 3.2. Hence we solve (3.6) with $U_n > 0$ and $\lambda = -c_t \text{ sgn } (U_t)/c_n$. Then problem (3.6) admits an infinity of solutions provided that the following two conditions hold:

$$c_n = -\frac{6G(L+2G)}{(L(\lambda+1)+G(\lambda+3))\ell},$$

which is precisely the eigenvalue μ of (3.2) obtained in (3.5) and

$$F_2 = \frac{L+G}{L+3G} F_1.$$

Since c_n is positive, an infinity of solutions can occur only if $\lambda \leq -(L+3G)/(L+G) < 0$ which implies that $U_t > 0$ as well as $F_1 < 0$ and $F_2 < 0$. Under the above-mentioned assumption, the system of equations (3.6) admits an infinity of solutions verifying:

$$\left\{ \begin{array}{l} U_t \in \left]0, -\dfrac{\ell F_1}{L+3G}\right[, \\ U_n = \dfrac{-3\ell F_1 - 3(L+3G)U_t}{3(L+G)+2c_t\ell}. \end{array} \right.$$

This result corresponds to an infinity of solutions located on a continuous branch. In other words, if $\mu = c_n$ then there exists for some loads an infinity of solutions to the problem (3.1).

3.4. *A computational example*

Next we study the convergence of the eigenvalue problem (3.4) using triangular and quadrilateral linear finite elements and different meshes with the geometry depicted in Figure 2. The chosen triangular meshes are of three different types: Delaunay, regular (by considering an initial mesh made of squares and dividing into two triangles each square following a prescribed diagonal), and regular when choosing the other diagonal.

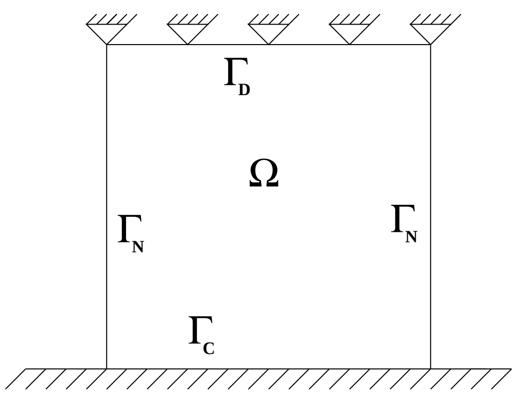

Figure 2: Geometry of the problem

Let us recall that if a positive eigenvalue of problem (3.4) exists then infinitely many solutions resulting from Proposition 3.2 can be explicitly constructed according to Remark 2.4.

The material characteristics of Ω are $E = 1$ and $\nu = 0.3$ (or equivalently $L = 15/26$ and $G = 5/13$) and the length of the edges of Ω is $\ell = 1$. Solving (3.4), we observe numerically that if λ lies approximately in $]-\infty, -1.96[\cup]1.96, +\infty[$ (in fact such intervals depend slightly on the mesh size) then there always exists at least one positive eigenvalue μ. Choosing $\lambda = 5$ the computations are performed using triangular and quadrilateral linear elements. We see that the lowest positive eigenvalue converges quite well to a limiting value as the discretization parameter vanishes. The obtained limit is approximately 0.75 (see Figure 3). Such a limit corresponds to values for c_n and c_t close to 0.75 and 3.75 respectively.

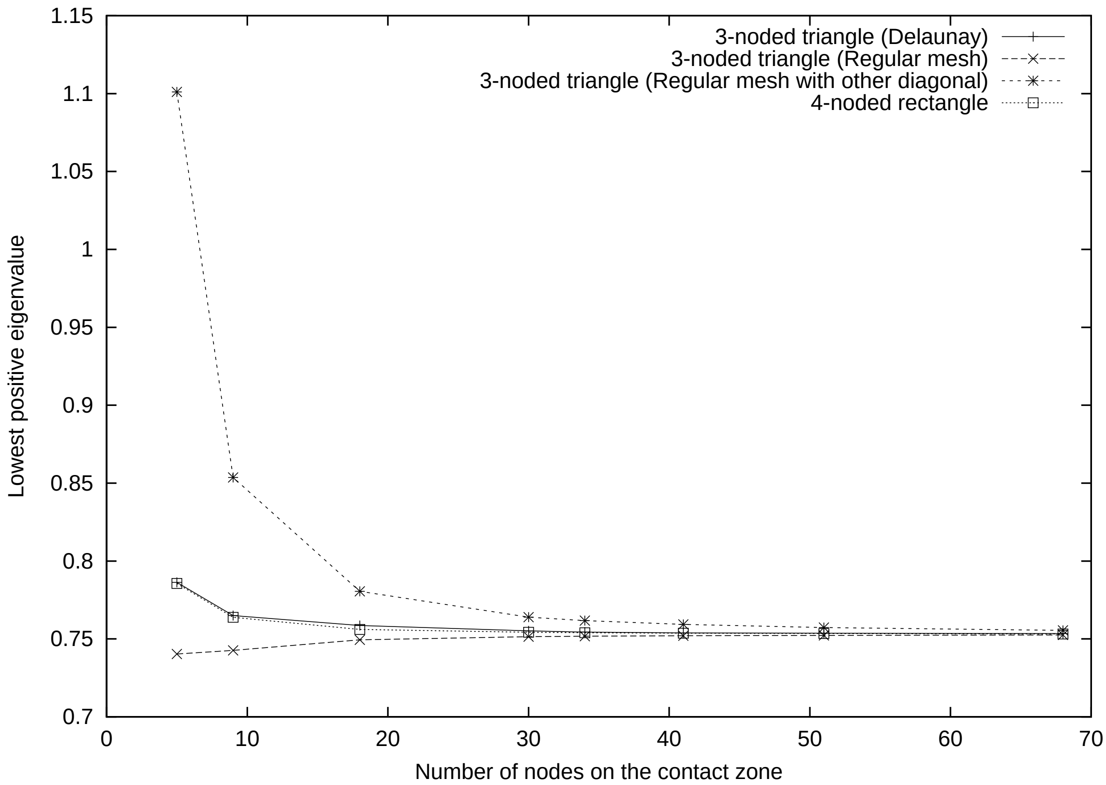

Figure 3: Convergence of the lowest positive eigenvalue

4. Conclusion

In this work we study the links between a specific eigenvalue problem and the existence of infinitely many solutions of the discrete and continuous friction problems with normal compliance. The results are established in the simple case of static linear elasticity in two space dimensions. A question actually under investigation is to determine explicit examples of nonuniqueness corresponding to the continuous framework in this paper. Moreover several extensions of this work could lead to new interesting problems. In particular the study of the relation between these nonuniqueness results and the uniqueness of the solutions in quasistatic frictional contact problems for viscoelastic materials (see, e.g., [2, 15]) as well as the generalization to the three dimensional case.

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